Entropy, Reversible Diffusion Processes and Markov Uniqueness

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Abstract

Consider a symmetric bilinear form $\mathcal{E}_\varphi$ defined on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ by

$$\mathcal{E}_\varphi(f,g) = \int_{\mathbb{R}^d} \nabla f . \nabla g \ \varphi^2 \ dx \quad , \qquad \varphi \in H^1_{loc}(\mathbb{R}^d)$$

In this paper we study the stochastic process associated with the smallest closed markovian extension of $(\mathcal{E}_\varphi, \mathcal{C}_c^\infty)$, and give a new proof of Markov uniqueness (i.e. the uniqueness of a closed markovian extension) based on purely probabilistic arguments. We also give another purely analytic one. As a consequence, we show that all invariant measures are reversible, provided they are of finite energy. The problem of uniqueness of such measures is also partially solved.

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1 Introduction

Consider the symmetric bilinear form $\mathcal{E}_\varphi$ defined on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ by

$$\mathcal{E}_\varphi(f,g) = \int \nabla f.\nabla g \ \varphi^2 \ dx$$

for $\varphi \in H^1_{loc}(\mathbb{R}^d)$. This form is closable and its minimal extension $(\mathcal{E}_\varphi, H^1_o(\varphi^2 dx))$ is actually a Dirichlet form.

Several questions are then natural to ask :

1. does it exist a maximal closed markovian extension $(\mathcal{E}^+_\varphi, \mathcal{D}(\mathcal{E}^+_\varphi))$ of $(\mathcal{E}_\varphi, \mathcal{C}^\infty_c)$, and can we describe explicitly this extension ?

2. when is $(\mathcal{E}^+_\varphi, \mathcal{D}(\mathcal{E}^+_\varphi))$ equal to $(\mathcal{E}_\varphi, H^1_o(\varphi^2 dx))$? (this property is known as Markov uniqueness)

3. is it possible to describe the (eventually) associated Markov process, in particular in terms of a Girsanov transform of a standard Brownian motion ?

When $\varphi \equiv 1$, it is well known that Markov uniqueness holds with $\mathcal{D}(\mathcal{E}^+_\varphi) = H^1(\mathbb{R}^d)$, the usual Sobolev space, since $H^1(\mathbb{R}^d) = H^1_o(\mathbb{R}^d)$. When $\varphi \not\equiv 1$, the three questions above have been studied for a long time by many authors. Surprisingly, they only recently received a fully satisfactory answer (see [AHKS77] [AR89] [AKR90] [RZ92] [RZ94] [Son94b] [Son94a], ...) namely

$$\mathcal{D}(\mathcal{E}^+_\varphi) \text{ is a "pseudo" Sobolev space and Markov uniqueness holds.} \tag{1.1}$$

The stochastic structure of the associated process follows from general arguments in the theory of Dirichlet forms, and is studied in [ARZ93a].

The aim of all the works mentioned above was not only to discuss the finite dimensional setting of this problem, but to provide an adapted setting in order to extend the results to some infinite dimensional spaces (extending the natural abstract Wiener space setting).

In this paper, we shall stay in the simple finite dimensional case. Our aim is to link these questions to the construction (and the properties) of singular diffusions (more precisely Brownian motions with singular drifts) which have deserved attention for some times, in particular in Nelson's stochastic approach of the Schrödinger equation. The recent new developments of this theory allow to extend the results presented here to more general situations, like general $\mathbb{R}^d$-valued diffusion processes (with a non necessarily uniformly elliptic diffusion matrix), or infinite dimensional diffusion processes. One can also expect to get interesting results in the non symmetric case.

Let us describe briefly the contents of the present paper, and compare our results (or proofs) with those of the (impressive) existing literature.

In section 2, we introduce the main tools and results concerning the Dirichlet form associated with $\mathcal{E}_\varphi$. In particular, we take up the friendly challenge proposed to us by M.Röckner, and give a completely elementary (and purely analytical) proof of Markov uniqueness (assuming the maximality results due to [AKR90]) (see theorem 2.7). Actually, even in this simple case, the proofs proposed in the existing literature are a sophisticated mixture of deep functional analytical results and a touch of Probability theory ([RZ92],[RZ94], [Son94b]), or use the specific Gaussian structure of the underlying Brownian motion ([Son94b] and [Son94a]).

Section 3 collects some results on singular diffusion processes associated with the generator $S_\varphi = \frac{1}{2}\Delta + \frac{\nabla\varphi}{\varphi}.\nabla$. After the pioneering work by Nelson [Nel88] and Carlen [Car84], a new completely different approach was proposed by Föllmer ([Föl88], at least in implicit form). This approach is based on relative entropy, and was recently developed by Léonard and one of the authors (see [CL94], [CL95a] and [CL95b]). It allows in particular to build a Brownian motion with drift $\frac{\nabla\varphi}{\varphi}$, and to characterize it as a solution of a minimization problem (see theorem 3.5 and proposition 6.1). Time reversibility is obtained thanks to results due to Föllmer [Föl84] (see also [CP95] for extensions). The analytic building (in the symmetric case) of this process is explained in [AR91].

In section 4, we study the fine (in the usual potential theoretic sense) structure of the diffusion process builded in section 3. The main result is that the nodal set $\{\varphi = 0\}$ cannot be attained, starting from any point outside of it (up to a nice modification of φ). This result is a consequence of Nelson's estimate, as recalled in [MZ85], and an ad-hoc choice of φ. It can easily be extended to more general frames. We also give a "Gibbsian" description of the process, based on an extension of Ito 's formula due to Föllmer and Protter [FP94], which (in the non symmetric case) plays the role of Fukushima's one [Fuk80].

In section 5, we link the singular diffusion process to the Dirichlet form. The uniqueness of a quasi-regular extension of $(\mathcal{E}_\varphi, \mathcal{C}_c^\infty)$ is then an immediate consequence of old results on martingale problems explained in [Jac79]. The unattainability of the nodal set gives the key argument in Song's proof of regularity (i.e. of Markov uniqueness, assuming the maximality result [Son94b]). This proof is simplified in order to get a purely probabilistic one (the only analytical material required is really elementary and does not call to Dirichlet forms theory).

Finally, in section 6, we present some immediate consequences to the study of invariant measures of finite energy. These results extend (in the framework of the present paper) similar results of Bogachev and Röckner ([BR94a] and [BR94b]), and are actually the probabilistic counterpart of the methods of these authors.

This paper is deliberately written in a naive way (at least for all the analytical arsenal) in order to be easily understood by both specialists in Dirichlet forms or stochastic calculus. In particular we apologize in advance for insisting on some trivial points which are not necessarily familiar to some or others.

Extensions of these results to general diffusion processes in $\mathbb{R}^d$ are studied in [Fra95]. Related results in the infinite dimensional setting (in particular connections with Gibbs states) will be published elsewhere.

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2 Dirichlet forms

We first study by purely analytical means the Dirichlet form we introduce above. Though we shall sometimes recall basic definitions, we refer to Fukushima's book [Fuk80] for all general results on Dirichlet forms in $\mathbb{R}^d$.

To begin with, consider the classical form $\mathcal{E}$ defined for f and g in $\mathcal{C}_c^\infty$ by

$$\mathcal{E}(f,g) = \int_{\mathbb{R}^d} \nabla f . \nabla g \; dx \tag{2.2}$$

where ∇ denotes the usual gradient operator, "." stands for scalar product in $\mathbb{R}^d$ and dx is Lebesgue measure. As a bilinear symmetric form defined on a dense subset of $L^2(dx)$, $\mathcal{E}$ is closable, i.e. the completion of $\mathcal{C}_c^\infty$ with respect to the Euclidean norm $\sqrt{\mathcal{E}^1}$, where $\mathcal{E}^1(f) = \int f^2 \; dx + \int |\nabla f|^2 \; dx$, is a Hilbert subspace H_o^1 of $L^2(dx)$, with scalar product extending $\mathcal{E}^1$.

Of course it is well known that H_o^1 is equal to the usual Sobolev space H^1 defined by :

$$\begin{aligned} H^1 &= \{ f \in L^2(dx) \quad / \quad \nabla f \in (L^2(dx))^d \} \qquad (2.3) \\ & \quad \text{where } \nabla \text{ is the gradient operator in the set } \mathcal{D}' \text{ of Schwartz distributions, i.e.} \\ &= \{ f \in L^2(dx) \text{ s.t. there exists an element of } (L^2(dx))^d \text{ denoted by } \nabla f \\ & \quad \text{satisfying for all } g \in \mathcal{C}_c^\infty \text{ and } i = 1, \ldots, d \quad \int \nabla_i f \; g \; dx = - \int f \; \nabla_i g \; dx \} \end{aligned}$$

Furthermore the closure of $(\mathcal{E}, \mathcal{C}_c^\infty)$, denoted by $(\mathcal{E}, H^1)$ (or $(\mathcal{E}, H_o^1)$) is again defined by formula (2.2).

Another fundamental property of $(\mathcal{E}, H^1)$ is the Markov property, which is implied by the fact that $(\mathcal{E}, H^1)$ is the closure of a markovian form. If one wants to prove the Markov property of $(\mathcal{E}, H^1)$ without using the density of $\mathcal{C}_c^\infty$ in H^1, one can show directly that the unit contractions operate on H^1 by using the following characterization :

$$\begin{aligned} H^1 \;=\; & \{f \in L^2(dx) \text{ s.t. for all } k \in \mathbb{R}^d, \;\; dx_k\text{-almost all } x \in k^\perp \\ & (k^\perp \text{ being the orthogonal set of } k \text{ and } dx_k \text{ the Lebesgue measure on } k^\perp) \\ & \text{the function } s \to f(x+sk) \text{ has an absolutely continuous } ds\text{-version } \tilde{f}_x \\ & \text{s.t. } \frac{d\tilde{f}_x}{ds} \in L^2 \quad (\frac{d\tilde{f}_x}{ds} \text{ being considered as a function of } (s,x))\} \end{aligned} \tag{2.4}$$

Furthermore, if we define $\frac{\partial f}{\partial k} := \frac{d\tilde{f}_x}{ds}$, then $\nabla_i f = \frac{\partial f}{\partial e_i}$ where e_i is the i^{th} vector of the canonical basis.

It should also be possible to choose the following (third) description of H^1 :

$$\begin{aligned} H^1 \;=\; & \{f \in L^2(dx) \text{ s.t. for all } k \in \mathbb{R}^d, \text{ the } L^2(dx)\text{-limit of} \\ & \frac{1}{s}(f(\cdot + sk) - f(\cdot)) \text{ exists when } s \text{ goes to } 0\} \end{aligned} \tag{2.5}$$

Furthermore the above limit is equal to $\nabla f.k$.

Finally, Weyl's lemma shows (see [Fuk80]) that $(\mathcal{E}, H^1)$ is the maximal markovian extension of $(\mathcal{E}, \mathcal{C}_c^\infty)$, hence the only one since it is also the minimal one.

Now consider the more general form $\mathcal{E}_\varphi$ defined for f and g in $\mathcal{C}_c^\infty$ by

$$\mathcal{E}(f,g) = \int_{\mathbb{R}^d} \nabla f.\nabla g \; \varphi^2 \; dx \tag{2.6}$$

For this definition to make sense, we have to assume that $\varphi \in L^2_{loc}(dx)$, but if we want to mimic what is done in the classical case ($\varphi \equiv 1$), and for reasons which will be clear later, we shall assume that

$$\varphi \in H^1_{loc}(dx) \tag{2.7}$$

Actually 2.7 is necessary and sufficient for $\mathcal{E}_\varphi$ to be admissible in the sense of [AR89], because of the following result :

Proposition 2.1 *(see [ARZ93b] prop 1.5) Let ν be a definite positive measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ s.t. for every $i = 1, \cdots, d$ there exists $\beta_i \in L^2(d\nu)$ s.t.*

$$\forall f \in \mathcal{C}_c^\infty \quad \int \nabla_i f \; d\nu = \int f \; \beta_i \; d\nu$$

Then $\nu = \varphi^2 \; dx$ for some $\varphi \in H^1(dx)$.

The following properties of $\mathcal{E}_\varphi$ have been proved to be true :

Proposition 2.2

i. The form $(\mathcal{E}_\varphi, \mathcal{C}_c^\infty)$ is closable in $L^2(\varphi^2 dx)$, see [MR92]

ii. Existence and description of its maximal closed markovian extension, see [AKR90] and [Tak92]

iii. Markov uniqueness , i.e. uniqueness of a markovian closed extension, see [RZ92] and [RZ94]

Our aim is to give elementary proofs of parts *i* and *iii* of this proposition 2.2.

The first difficulty is the following : An element f of $L^2(\varphi^2 dx)$ does not necessarily belong to $\mathcal{D}'$ (because φ^2 is not bounded by below), and so the usual ∇f is not a priori well defined. Nevertheless, for such an f, $f\varphi^2$ belongs to $L^1_{loc}(dx)$ and so $\nabla(f\ \varphi^2)$ is well defined in $\mathcal{D}'$. Furthermore, if $f \in \mathcal{C}_c^\infty$

$$\nabla(f\ \varphi^2) = \varphi^2\ \nabla f + 2\ f\ \varphi\ \nabla\varphi \qquad (\in (L^1(dx))^d) \tag{2.8}$$

So it is natural to define the weighted Sobolev space $H^1(\varphi^2 dx)$ as follows :

Definition 2.3 *$H^1(\varphi^2 dx)$ is the set of elements $f \in L^2(\varphi^2 dx)$ s.t. there exists an element $\overline{\nabla} f = (\overline{\nabla}_i f)_{i\in\{1,\cdots,d\}} \in (L^2(\varphi^2 dx))^d$ satisfying*

$$\forall i \in \{1,\cdots,d\} \qquad \varphi^2\ \overline{\nabla}_i f = \nabla_i(f\ \varphi^2) - 2\ f\ \varphi\ \nabla_i\varphi \quad \text{in } \mathcal{D}'$$

that is , the following integration by parts formula

$$\int g\ \overline{\nabla}_i f\ \varphi^2\ dx = -\int f\ \nabla_i g\ \varphi^2\ dx - 2\int\ f\ g\ \frac{\nabla_i\varphi}{\varphi}\ \varphi^2\ dx \tag{2.9}$$

holds for all $i \in \{1,\cdots,d\}$ and $g \in \mathcal{C}_c^\infty$.

Notice that if $f \in H^1(\varphi^2 dx)$ then $f\ \varphi^2 \in W^{1,1}_{loc}$ (usual Sobolev space).

We can now state :

Proposition 2.4 *The form $(\mathcal{E}_\varphi, H^1(\varphi^2 dx))$ defined by*

$$\forall f,g \in H^1(\varphi^2 dx) \qquad \mathcal{E}_\varphi(f,g) = \int \overline{\nabla} f . \overline{\nabla} g\ \varphi^2\ dx$$

is a closed extension of $(\mathcal{E}_\varphi, \mathcal{C}_c^\infty)$, which in turn is closable.

Proof: Take a Cauchy sequence $(f_n)_{n\geq 1}$ of $H^1(\varphi^2dx)$ equipped with the scalar product $\mathcal{E}^1_\varphi(f,g) = \mathcal{E}_\varphi(f,g) + \int f\ g\ \varphi^2\ dx$, and let f be the $L^2(\varphi^2dx)$ limit of $(f_n)_{n\geq 1}$. $(\overline{\nabla} f_n)_{n\geq 1}$ converges to h in $(L^2(\varphi^2dx))^d$, and by taking limits in (2.9) it is immediate that $f \in H^1(\varphi^2dx)$ with $h = \overline{\nabla} f$. Thus $(\mathcal{E}_\varphi, H^1(\varphi^2dx))$ is closed. Furthermore $\mathcal{C}^\infty_c \subset H^1(\varphi^2dx)$ and $\nabla f = \overline{\nabla} f$ for f in $\mathcal{C}^\infty_c$ according to (2.8). So $(\mathcal{E}_\varphi, H^1(\varphi^2dx))$ is an extension of $(\mathcal{E}_\varphi, \mathcal{C}^\infty_c)$.
∎

Since $(\mathcal{E}_\varphi, \mathcal{C}^\infty_c)$ admits a closed extension, it is closable and we shall of course denote its smallest closed extension by $(\mathcal{E}_\varphi, H^1_o(\varphi^2dx))$, which is actually a Dirichlet form since $(\mathcal{E}_\varphi, \mathcal{C}^\infty_c)$ is trivially markovian. The Markov property of $(\mathcal{E}_\varphi, H^1(\varphi^2dx))$ is not so immediate in view of its definition. As we did in the classical case, we shall get other descriptions of $(\mathcal{E}_\varphi, H^1(\varphi^2dx))$ similar to (2.4) and (2.5) to prove the Markov property. The description similar to (2.4) was obtained in [AR90] th 3.2 , [AKR90] prop 2.2 , [RZ92] and [ARZ93b] lemma 2.2 in this and more general contexts (see also [Son94b]).

Proposition 2.5 *$f \in H^1(\varphi^2dx)$ if and only if $f \in L^2(\varphi^2dx)$ and for all $k \in \mathbb{R}^d$, ν_k-almost all $x \in k^\perp$ (where ν_k is the image of φ^2dx by the projection $\mathbb{R}^d \to k^\perp$) the following holds :*

i. $\left\{ \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ s & \to & f(x+sk) \end{array} \right\}$ *has an absolutely continuous ds-version $\tilde{f}_x$ on the set* $\{s\ /\ \varphi^2(x+sk) > 0\}$.

ii. $\dfrac{d\tilde{f}_x}{ds} \in L^2(\varphi^2dx)$.

In this case if we define $\dfrac{\overline{\partial} f}{\partial k} := \dfrac{d\tilde{f}_x}{ds}$, *then* $\overline{\nabla}_i f = \dfrac{\overline{\partial} f}{\partial e_i}$.

Here are some important consequences, already noticed by Albeverio and Röckner :

Corollary 2.6

i. *$(\mathcal{E}_\varphi, H^1(\varphi^2dx))$ is markovian (hence a Dirichlet form). It is an easy consequence of the chain rule and proposition 2.5.*

ii. *If f and g belongs to $H^1(\varphi^2dx)$ and if fg, $f\overline{\nabla}_i g$ and $g\overline{\nabla}_i f$ are in $L^2(\varphi^2dx)$ for all $i \in \{1,\cdots,d\}$, then $fg \in H^1(\varphi^2dx)$ and $\overline{\nabla}(fg) = f\overline{\nabla} g + g\overline{\nabla} f$.*

iii. *The set of bounded compactly supported functions of $H^1(\varphi^2dx)$ is a dense subset of $H^1(\varphi^2dx)$ for the norm $\sqrt{\mathcal{E}^1_\varphi}$. It is an immediate consequence of i and ii, by using a truncature argument.*

We now are ready to state the main result of [RZ94] and to give an elementary proof of it.

Theorem 2.7 *$\mathcal{C}_c^\infty$ is $\mathcal{E}_\varphi^1$ dense in $H^1(\varphi^2 dx)$, i.e. $H_o^1(\varphi^2 dx) = H^1(\varphi^2 dx)$.*

Proof: According to corollary 2.6, it is enough to approximate functions in $H^1(\varphi^2 dx)$ which are bounded and compactly supported. In the sequel, f will denote such a function.

Let J_ε be a standart mollifier, i.e. for every $\varepsilon \in]0,1]$, $J_\varepsilon = \dfrac{1}{\varepsilon^d} J(\dfrac{x}{\varepsilon})$ where $J \in \mathcal{C}_c^\infty$ satisfies $0 \le J \le 1$, $\operatorname{supp} J \subset B(0,1)$ and $\displaystyle\int J(x)\, dx = 1$.

Since f is bounded with compact support, $J_\varepsilon * f$ is well defined, $\mathcal{C}^\infty$, compactly supported with $\operatorname{supp} J_\varepsilon * f \subset \operatorname{supp} f + \varepsilon B(0,1)$ and bounded by $\|f\|_\infty$ for all $\varepsilon > 0$. We shall prove that

$$\begin{array}{c}\text{for a given sequence } (\varepsilon_n)_n \text{ decreasing to } 0, \\ J_{\varepsilon_n} * f \text{ converges to } f \text{ in } H^1(\varphi^2 dx)\end{array} \tag{2.10}$$

The easy part is the $L^2(\varphi^2 dx)$-convergence.

Lemma 2.8 *One can find a sequence $(\varepsilon_n)_n$ decreasing to 0 , such that $J_{\varepsilon_n} * f$ converges to f in $L^2(\varphi^2 dx)$.*

Proof of 2.8 : First, $J_\varepsilon * f \in L^2(dx)$ and converges to f in $L^2(dx)$ when ε goes to 0. Thus we can find a sequence $(\varepsilon_n)_n$ such that $J_{\varepsilon_n} * f$ converges to f dx a.e. Finally, since

$$|\varphi(J_{\varepsilon_n} * f) - \varphi\, f|^2 \le 2\, \|f\|_\infty^2\, |\varphi|^2\, \mathbb{1}_{(\operatorname{supp} f + B(0,1))}$$

which belongs to $L^1(dx)$, one may apply Lebesgue's bounded convergence theorem to show that

$$\int |(J_{\varepsilon_n} * f) - f|^2\, \varphi^2\, dx \xrightarrow[n \to +\infty]{} 0$$

∎

We now have to prove that $\nabla_i(J_{\varepsilon_n} * f)$ converges to $\overline{\nabla}_i f$ in $L^2(\varphi^2 dx)$ for a given sequence $(\varepsilon_n)_n$, or equivalently that $\varphi \nabla_i (J_{\varepsilon_n} * f)$ converges to $\varphi \overline{\nabla}_i f$ in $L^2(dx)$. But

$$\begin{aligned} \left\| \varphi \nabla_i (J_\varepsilon * f) - \varphi \overline{\nabla}_i f \right\|_{L^2(dx)}^2 \;\le\; & \left\| \varphi \overline{\nabla}_i f - (J_\varepsilon * (\varphi \overline{\nabla}_i f)) \right\|_{L^2(dx)}^2 \\ + \; & \left\| J_\varepsilon * (\varphi \overline{\nabla}_i f) - \varphi \nabla_i (J_\varepsilon * f) \right\|_{L^2(dx)}^2 \end{aligned} \tag{2.11}$$

since $\varphi\overline{\nabla}_i f \in L^2(dx)$. But the first term of the above sum goes to 0 with ε. So we only have to study the second term. To this end we adapt the idea of [Fra94] th 2.1.8. We write

$$
\begin{aligned}
&\left\|\varphi\nabla_i(J_\varepsilon * f) - (J_\varepsilon * (\varphi\overline{\nabla}_i f))\right\|^2_{L^2(dx)} \\
&\quad = \int \left|\varphi(x) \int \nabla_i J_\varepsilon(x-y)\ f(y)\ dy - \int J_\varepsilon(x-y)\ \varphi(y)\ \overline{\nabla}_i f(y)\ dy\right|^2\ dx \\
&\quad = \int \left|\int \nabla_i J_\varepsilon(x-y)\ f(y)\ [\varphi(x)-\varphi(y)]\ dy\right. \\
&\qquad\quad \left. + \int [J_\varepsilon(x-y)\ f(y)\ \varphi(y) - J_\varepsilon(x-y)\ \varphi(y)\ \overline{\nabla}_i f(y)]\ dy\right|^2\ dx
\end{aligned}
\tag{2.12}
$$

and we apply the following lemma :

Lemma 2.9 *For all $g \in \mathcal{C}_c^\infty$*

$$\int \overline{\nabla}_i f\ \varphi\ g\ dx + \int f\ \nabla_i\varphi\ g\ dx + \int f\ \varphi\ \nabla_i g\ dx = 0$$

The proof of lemma 2.9 is postponed to the end of the section. Just remark that lemma 2.9 is meaningful since $\varphi\overline{\nabla}_i f$ and φf are in $L^2(dx)$, so in $L^1_{loc}(dx)$ while $f\nabla_i\varphi \in L^1_{loc}(dx)$ since f is bounded and $\nabla_i\varphi \in L^2(dx)$. Applying lemma 2.9 with $g(y) = J_\varepsilon(x-y)$ in (2.12), we get

$$
\begin{aligned}
&\left\|\varphi\nabla_i(J_\varepsilon * f) - (J_\varepsilon * (\varphi\overline{\nabla}_i f))\right\|^2_{L^2(dx)} \\
&= \int \left|\int \nabla_i J_\varepsilon(x-y)\ f(y)\ [\varphi(x)-\varphi(y)]\ dy + \int J_\varepsilon(x-y)\ f(y)\ \nabla_i\varphi(y)\ dy\right|^2 dx \\
&\leq 2\|J_\varepsilon * (f\nabla_i\varphi)\|^2_{L^2(dx)} + 2\int \left|\int \nabla_i J_\varepsilon(x-y)\ f(y)\ [\varphi(x)-\varphi(y)]\ dy\right|^2\ dx \\
&\leq 2\|f\nabla_i\varphi\|^2_{L^2(dx)} + 2\int \left|\int \nabla_i J_\varepsilon(x-y)\ f(y)\ [\varphi(x)-\varphi(y)]\ dy\right|^2\ dx
\end{aligned}
\tag{2.13}
$$

To control the second term, we shall use (2.5). Since we didn't assume $\varphi \in H^1$, we will replace φ here by a function $\hat{\varphi} \in H^1$ such that $\hat{\varphi} = \varphi$ on supp $f + B(0,1)$:

$$\hat{\varphi} = \varphi\xi \text{ with } \xi \in \mathcal{C}_c^\infty \text{ and } \mathbb{I}_{\mathrm{supp} f + B(0,2)} \leq \xi \leq \mathbb{I}_{\mathrm{supp} f + B(0,3)}$$

Then

$$\begin{aligned}
&\int \left| \int \nabla_i J_\varepsilon(x-y)\ f(y)\ [\varphi(x)-\varphi(y)]\ dy \right|^2\ dx \\
&= \int_{\mathbb{R}^d} \left| \int_{B(0,1)} \nabla_i J_\varepsilon(s)\ f(x+s\varepsilon)\ \frac{\hat{\varphi}(x)-\hat{\varphi}(x+s\varepsilon)}{\varepsilon}\ ds \right|^2\ dx \\
&\leq 2 \int_{\mathbb{R}^d} \left| \int_{B(0,1)} \nabla_i J_\varepsilon(s)\ f(x+s\varepsilon)\ (-\nabla\hat{\varphi}(x+s\varepsilon).s)\ ds \right|^2\ dx \\
&\quad + 2 \int_{\mathbb{R}^d} \left| \int_{B(0,1)} \nabla_i J_\varepsilon(s)\ f(x+s\varepsilon)\ \left[\frac{\hat{\varphi}(x)-\hat{\varphi}(x+s\varepsilon)}{\varepsilon} + \nabla\hat{\varphi}(x+s\varepsilon).s \right] ds \right|^2 dx
\end{aligned} \tag{2.14}$$

Of course

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left| \int_{B(0,1)} \nabla_i J_\varepsilon(s)\ f(x+s\varepsilon)\ (-\nabla\hat{\varphi}(x+s\varepsilon).s)\ ds \right|^2\ dx \\
&\leq C^{ste}\ \|\nabla_i J\|_\infty^2 \sum_{j=1}^d \|f\nabla_j\varphi\|^2_{L^2(dx)}
\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left| \int_{B(0,1)} \nabla_i J_\varepsilon(s)\ f(x+s\varepsilon)\ \left[\frac{\hat{\varphi}(x)-\hat{\varphi}(x+s\varepsilon)}{\varepsilon} + \nabla\hat{\varphi}(x+s\varepsilon).s \right]\ ds \right|^2\ dx \\
&\leq C^{ste}\ \|\nabla_i J\|_\infty^2 \|f\|_\infty^2 \int_{\mathbb{R}^d} \int_{B(0,1)} \left| \frac{\hat{\varphi}(x)-\hat{\varphi}(x+s\varepsilon)}{\varepsilon} + \nabla\hat{\varphi}(x+s\varepsilon).s \right|^2\ ds\ dx \qquad (2.16) \\
&\leq C^{ste}\ \|\nabla_i J\|_\infty^2 \|f\|_\infty^2 \int_{B(0,1)} \left\| \frac{\hat{\varphi}(x)-\hat{\varphi}(x+s\varepsilon)}{\varepsilon} + \nabla\hat{\varphi}(x+s\varepsilon).s \right\|^2_{L^2(dx)}\ ds
\end{aligned}$$

According to (2.5), the quantity under the integral goes to 0 as ε goes to 0. Furthermore, it is easy to see (by using for instance Fourier transforms) that the convergence is uniform in s on the closed unit ball. We thus have proved

Lemma 2.10

$$\left\| \varphi\nabla_i(J_\varepsilon * f) - (J_\varepsilon * (\varphi\overline{\nabla}_i f)) \right\|^2_{L^2(dx)} \leq C^{ste}\ \sum_{j=1}^d \|f\nabla_j\varphi\|^2_{L^2(dx)} + C^{ste}\ \|f\|_\infty^2 \theta(\varepsilon)$$

with $\theta(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0$, *and* θ *only depends on* $\hat{\varphi}$, *thus on* $supp\ f$.

Of course we want to show that the left hand side in lemma 2.10 goes to 0 with ε. To this end, we shall use one more approximation.

Take $f_\eta = J_\eta * f$ which is $\mathcal{C}_c^\infty$. By the same argument as in lemma 2.8 we can find a sequence $(\eta_k)_k$ such that

$$\sum_{j=1}^{d} \|(f - f_{\eta_k})\nabla_j\varphi\|^2_{L^2(dx)} \xrightarrow[k\to+\infty]{} 0 \tag{2.17}$$

Hence

$$\begin{aligned}
&\left\|\varphi\nabla_i(J_\varepsilon * f) - (J_\varepsilon * (\varphi\overline{\nabla}_i f))\right\|^2_{L^2(dx)} \\
&\leq 2\left\|\varphi\nabla_i(J_\varepsilon * (f - f_{\eta_k})) - (J_\varepsilon * (\varphi\overline{\nabla}_i(f - f_{\eta_k})))\right\|^2_{L^2(dx)} \\
&\quad + 2\left\|\varphi\nabla_i(J_\varepsilon * f_{\eta_k}) - (J_\varepsilon * (\varphi\overline{\nabla}_i f_{\eta_k}))\right\|^2_{L^2(dx)} \qquad (2.18)\\
&\leq C^{ste}\sum_{j=1}^{d}\|(f - f_{\eta_k})\nabla_j\varphi\|^2_{L^2(dx)} + C^{ste}\|(f - f_{\eta_k})\|^2_\infty\theta(\varepsilon) \\
&\quad + 2\left\|\varphi\nabla_i(J_\varepsilon * f_{\eta_k}) - (J_\varepsilon * (\varphi\overline{\nabla}_i f_{\eta_k}))\right\|^2_{L^2(dx)}
\end{aligned}$$

using lemma 2.10 with f replaced by $(f - f_{\eta_k})$ (we can do it without changing anything in the proof of this lemma since we actually have $\hat{\varphi} = \varphi$ on supp $f + B(0,2)$, thus on supp $(f - f_{\eta_k}) + B(0,1)$, which is enough for (2.14) to hold). By choosing first η_k and then ε, the first two terms in the above sum can be chosen arbitrarily small (since $\|(f - f_{\eta_k})\|^2_\infty \leq 2\|f\|^2_\infty$). It remains to show that for a fixed η_k the third term goes to 0 with ε i.e. that

$$\forall g \in \mathcal{C}_c^\infty \quad \|\varphi\nabla_i(J_\varepsilon * g) - (J_\varepsilon * (\varphi\nabla_i g))\|^2_{L^2(dx)} \xrightarrow[\varepsilon\to 0]{} 0 \tag{2.19}$$

But

$$\begin{aligned}
&\|\varphi\nabla_i(J_\varepsilon * g) - (J_\varepsilon * (\varphi\nabla_i g))\|^2_{L^2(dx)} \\
&= \int\left|\int J_\varepsilon(x-y)\nabla_i g(y)[\varphi(x) - \varphi(y)]\,dy\right|^2 dx \qquad (2.20)\\
&\leq C^{ste}\|J\|^2_\infty\|\nabla_i g\|^2_\infty\int_{B(0,1)}\|(\varphi\xi_g)(\cdot) - (\varphi\xi_g)(\cdot + s\varepsilon)\|^2_{L^2(dx)}\,ds
\end{aligned}$$

(by the same manipulations as in (2.14), with $\xi_g \in \mathcal{C}_c^\infty$ and $\xi_g = 1$ on supp $g + B(0,1)$). Since $\|(\varphi\xi_g)(\cdot) - (\varphi\xi_g)(\cdot + s\varepsilon)\|^2_{L^2(dx)}$ goes to 0 when

ε goes to 0, uniformly on s on the closed unit ball (again use Plancherel's theorem) , (2.19) is proved.

The proof of the theorem will be achieved once we prove lemma 2.9, and we shall proceed now with this proof.

Let $M \in \mathbb{N}^*$ and let $\psi_M \in \mathcal{C}_c^\infty(\mathbb{R})$ be such that

$$\begin{aligned}&\psi_M(t) = t \text{ for } t \in [-M, M] \ , \quad |\psi_M| \leq M+1 \ , \quad |\psi'_M| \leq 1 \\ &\text{and} \quad \text{supp } (\psi_M) \subset [-3M, 3M]\end{aligned}$$

Define φ_M by

$$\begin{aligned}\varphi_M &:= \psi_M(\frac{1}{\varphi}) \quad \text{if} \quad \varphi \neq 0 \\ \varphi_M &:= 0 \quad \text{if} \quad \varphi = 0\end{aligned}$$

It is an elementary exercise to prove that $\varphi_M \in H^1_{loc}$ and

$$\begin{aligned}\nabla\varphi_M &:= -\frac{\nabla\varphi}{\varphi^2}\psi'_M(\frac{1}{\varphi}) \quad \text{if} \quad \varphi \neq 0 \\ \nabla\varphi_M &:= 0 \quad \text{if} \quad \varphi = 0\end{aligned}$$

by using (2.4) since $\psi'_M(\frac{1}{\varphi}) = 0$ on $\{\varphi \leq \dfrac{1}{3M}\}$. Noticing that

$$|\nabla\varphi_M| \leq |\nabla\varphi| \ \frac{1}{\varphi^2} \ \mathbb{1}_{|\varphi| > \frac{1}{3M}}$$

we get $\varphi_M \in H^1_{loc}$, $\varphi \in H^1_{loc}$, $\varphi_M\nabla\varphi \in L^2_{loc}(dx)$ since φ_M is bounded, and $\varphi\nabla\varphi_M \in L^2_{loc}(dx)$ since

$$|\varphi\nabla\varphi_M| \leq |\nabla\varphi| \ \frac{1}{\varphi} \ \mathbb{1}_{|\varphi| > \frac{1}{3M}} \leq 3M|\nabla\varphi|$$

So according to corollary 2.6ii in the classical case

$$\varphi \ \varphi_M \in H^1_{loc} \quad \text{and} \quad \nabla(\varphi \ \varphi_M) = \varphi_M\nabla\varphi - \frac{\nabla\varphi}{\varphi} \ \psi'_M(\frac{1}{\varphi}) \tag{2.21}$$

where by convention $\dfrac{\nabla\varphi}{\varphi} = 0$ when $\varphi = 0$. But again

- φ and $\varphi \ \varphi_M$ are in H^1_{loc}
- $\varphi \ \varphi_M\nabla\varphi \in L^2_{loc}(dx)$ since $|\varphi \ \varphi_M| \leq \dfrac{M+1}{M}$ (hence bounded)
- $\varphi\nabla(\varphi \ \varphi_M) \in L^2_{loc}(dx)$ since $\varphi\nabla(\varphi \ \varphi_M) = \varphi \ \varphi_M\nabla\varphi - \nabla\varphi\psi'_M(\dfrac{1}{\varphi})$ and both terms are in $L^2_{loc}(dx)$

It follows that

$$\varphi^2\ \varphi_M \in H^1_{loc} \text{ and } \nabla(\varphi^2\ \varphi_M) = 2\ \varphi\ \varphi_M \nabla\varphi - \nabla\varphi\ \psi'_M(\frac{1}{\varphi})$$

In particular we can use the integration by parts formula (2.3) to get

$$\forall g \in \mathcal{C}^\infty_c \quad \int \nabla_i g\, \varphi_M\, \varphi^2\ dx = -2\int g\ \varphi_M\ \frac{\nabla_i\varphi}{\varphi}\ \varphi^2\ dx + \int g\ \frac{\nabla_i\varphi}{\varphi^2}\ \psi'_M(\frac{1}{\varphi})\, \varphi^2\ dx$$

In addition $\nabla\varphi_M \in L^2_{loc}(dx)$ since $|\varphi\nabla\varphi_M| \leq 3M|\nabla\varphi| \in L^2_{loc}(dx)$.

It follows

$$\varphi_M \in H^1_{loc}(\varphi^2 dx) \quad \text{and} \quad \overline{\nabla}\varphi_M = -\nabla\varphi_M = \frac{\nabla\varphi}{\varphi^2}\ \psi'_M(\frac{1}{\varphi}) \tag{2.22}$$

Though we did not define the local space $H^1_{loc}(\varphi^2 dx)$, the above sentence is clear. Thanks to (2.22) and corollary 2.6*ii* again, we have

- $\varphi_M \in H^1_{loc}(\varphi^2 dx)$, $f \in H^1(\varphi^2 dx)$ and is compactly supported
- $f\overline{\nabla}\varphi_M \in L^2(\varphi^2 dx)$ since f is bounded with compact support
- $\varphi_M\overline{\nabla} f \in L^2(\varphi^2 dx)$ since φ_M is bounded

So $f\varphi_M \in H^1(\varphi^2 dx)$ and $\overline{\nabla}(f\varphi_M) = \varphi_M\overline{\nabla} f + f\overline{\nabla}\varphi_M$. We thus may apply (2.9), which yields for $i \in \{1, \ldots, d\}$ and $g \in \mathcal{C}^\infty_c$

$$\begin{aligned} &\int \overline{\nabla}_i f\ g\ \varphi_M\ \varphi^2\ dx - \int f\ g\ \frac{\nabla_i\varphi}{\varphi^2}\ \psi'_M(\frac{1}{\varphi})\ \varphi^2\ dx \\ = &- \int f\ \varphi_M\ \nabla_i g\ \varphi^2\ dx - 2\int f\ g\ \varphi_M\ \frac{\nabla_i\varphi}{\varphi}\ \varphi^2\ dx \end{aligned} \tag{2.23}$$

Now let M go to infinity. We can pass to the limit in (2.23) thanks to the following facts

- φ_M converges dx-a.e. to $\frac{1}{\varphi}\ \mathbb{I}_{\varphi\neq 0}$ and $\psi'_M(\frac{1}{\varphi})$ converges dx-a.e. to 1
- $|\varphi\ \varphi_M| \leq \frac{M+1}{M} \leq 2$
  so that $|\overline{\nabla}_i f\ g\ \varphi_M\ \varphi^2| \leq 2\ |\varphi\ \overline{\nabla}_i f|\ \|g\|_\infty \mathbb{I}_{\text{supp } g} \in L^1(dx)$
- $|f\ g\ \nabla_i\varphi\ \psi'_M(\frac{1}{\varphi})| \leq |f\ \nabla_i\varphi|\ \|g\|_\infty \mathbb{I}_{\text{supp } g} \in L^1(dx)$
- $|f\ \varphi_M\ \nabla_i g\ \varphi^2 \leq 2\ |f\ \varphi|\ \|\nabla_i g\|_\infty \mathbb{I}_{\text{supp } g} \in L^1(dx)$
- $|f\ g\ \varphi_M\ \varphi\ \nabla_i\varphi| \leq 2\ |f\ \nabla_i\varphi|\ \|g\|_\infty \mathbb{I}_{\text{supp } g} \in L^1(dx)$

The formula obtained by the limiting procedure is exactly lemma 2.9

∎

Remark 2.11 : The previous proof, though technical and perhaps a little bit tedious, is purely analytic and elementary. It should be underlined that the proofs in [RZ94] or in [Son94b] required more sophisticated material and a touch of Probability theory in both cases.

Theorem 2.7 tells that $(\mathcal{E}_\varphi, H^1(\varphi^2 dx))$ is the minimal closed extension of $(\mathcal{E}_\varphi, \mathcal{C}_c^\infty)$. According to [AKR90] and [Tak92], it is also the maximal closed extension, hence the unique closed markovian extension, i.e. Markov uniqueness holds, provided $\varphi \in H^1_{loc}$ and $\varphi \neq 0$ dx-a.e. , or $\varphi \in H^1$. Another consequence is that $(\mathcal{E}_\varphi, H^1(\varphi^2 dx))$ is regular (i.e. $\mathcal{C}_c \cap H^1(\varphi^2 dx)$ is a core for the form), thus associated with a Hunt process, which is actually a diffusion process since $(\mathcal{E}_\varphi, H^1(\varphi^2 dx))$ is local. Let we recall some basic facts concerning these points.

There exists an unique $L^2(\varphi^2 dx)$ strongly continuous symmetric sub-markovian semi-group $(T_t^\varphi)_{t\geq 0}$ associated with $(\mathcal{E}_\varphi, H^1(\varphi^2 dx))$, i.e. if A_φ denotes the generator of $(T_t^\varphi)_{t\geq 0}$:

$$\begin{cases} \mathcal{D}(A_\varphi) \subset H^1(\varphi^2 dx) \\ \forall f \in \mathcal{D}(A_\varphi) \quad \forall g \in H^1(\varphi^2 dx) \quad \mathcal{E}_\varphi(f,g) = -\int A_\varphi f \ g \ \varphi^2 \ dx \end{cases}$$

Since $\varphi \in H^1_{loc}$ it is easy to see that A_φ coïncide on $\mathcal{C}_c^\infty$ with the partial differential operator S_φ

$$S_\varphi = \frac{1}{2}\Delta + \frac{\nabla\varphi}{\varphi}.\nabla$$

Actually $\varphi \in H^1_{loc}$ is a necessary and sufficient condition for $\mathcal{C}_c^\infty \subset L^2(\varphi^2 dx)$ and $S_\varphi(\mathcal{C}_c^\infty) \subset L^2(\varphi^2 dx)$ (see [RZ92]).

According to the properties of the form described above, there exists a $\varphi^2 dx$-symmetric Hunt process with (a.s.) continuous paths (i.e. a diffusion process) associated with $(T_t^\varphi)_{t\geq 0}$. If we denote by $(\Omega, \mathcal{F}_t, X_t, (P_x)_{x\in\mathbb{R}^d})$ the canonical realization of this process (with usual conditions), standard arguments show that there exists a $\varphi^2 dx$-negligible set $\mathcal{N}$ such that for all $x \notin \mathcal{N}$ and all $f \in \mathcal{C}_c^\infty$

$$t \longrightarrow f(X_t) - f(X_o) - \int_o^t A_\varphi f(X_s) \ ds$$

is an $\mathcal{F}_t$ martingale under P_x, i.e. for $\varphi^2 dx$-almost all x, P_x is a solution to the martingale problem $\mathcal{M}(S_\varphi, \mathcal{C}_c^\infty)$ (we follow the notations of [Jac79]) starting from x (or δ_x).

A consequence of Markov uniqueness is that two (markovian) solutions of $\mathcal{M}(S_\varphi, \mathcal{C}_c^\infty)$ are equivalent, i.e. their transition functions coïncide out of a set of capacity zero (for all this see [RZ92] §6 and [Fuk80]). Furthermore, Albeverio, Röckner and Zhang proved in [AR91] that $(P_x)_x$ is obtained by means of a Cameron-Martin-Girsanov-Maruyama transform, by using Fukushima's stochastic calculus.

A natural question to ask is thus the following : is it possible to prove (at least the major part of) the results of this section, by purely probabilistic methods ? The answer to this question is the aim of the next sections.

3 Singular diffusion processes via relative entropy

Because probabilists prefer to deal with Probability measures, we will assume in this section that

$$\varphi \in H^1 \quad \text{and} \quad \int \varphi^2 \, dx = 1 \tag{3.24}$$

The passage to a local condition will be done later on.

We consider the operator S_φ defined on $\mathcal{C}_c^\infty$ by

$$S_\varphi = \frac{1}{2}\Delta + \frac{\nabla\varphi}{\varphi}.\nabla \tag{3.25}$$

and shall study the martingale problem $\mathcal{M}(S_\varphi, \mathcal{C}_c^\infty)$ starting from $\varphi^2 dx$. From the stochastic calculus viewpoint, the difficulty lies in the fact that $\dfrac{\nabla\varphi}{\varphi}$ can be a very irregular drift, out of reach of standard result such as Novikov or Kazamaki criteria. But if we remark that

$$\int \left|\frac{\nabla\varphi}{\varphi}\right|^2 \varphi^2 \, dx < +\infty \tag{3.26}$$

(letting $\dfrac{\nabla\varphi}{\varphi} := 0$ on $\{\varphi = 0\}$) hypothesis 3.24 becomes a "finite energy condition" familiar to specialists of Nelson's stochastic mechanics, and actually almost all problems have already been solved by "entropic methods". Let we first recall the definition of relative entropy (or Kullback information).

Definition 3.1 *Let P and Q be two probability measures on the same space. The relative entropy of Q with respect to P, denoted by $H(Q,P)$, is*

$$\begin{cases} H(Q,P) = \int Z \ln Z \; dP & \textit{if } \; Q << P, \quad Z = \dfrac{dQ}{dP} \quad \textit{and} \quad Z \ln Z \in L^1(P) \\ H(Q,P) = +\infty & \textit{otherwise} \end{cases}$$

Our framework in this section will be the following :

- $\Omega = \mathcal{C}^o([0,T], \mathbb{R}^d)$ (for $T > 0$) is the Wiener space, equipped with its usual structure $(\Omega, X_t, \mathcal{F}_t)$.
- if μ is a probability measure on $\mathbb{R}^d$, W_μ denotes the Wiener measure such that $W_\mu \circ X_o^{-1} = \mu$. When $\mu = \varphi^2 dx$ we write W_φ instead of $W_{\varphi^2 dx}$.

- usual universal completions and right modifications are assumed.

In the sequel the following stopping times will be very important :

$$\tau_n = \inf\{t \geq 0 \quad / \quad \int_o^t \left|\frac{\nabla\varphi}{\varphi}\right|^2 (X_s)\ ds \geq n\} \quad , \quad \tau = \sup_n \tau_n \tag{3.27}$$

The first result we state is an existence theorem due to Carlen [Car84] (also see [FT84], [Car85], [MZ85], [Zhe85] and many others), and recently extended in [CL94], [CL95a], and [CL95b] to more general contexts. The precise statement below is a consequence of [CL94].

Theorem 3.2 *The process* $Z_t^\varphi = \exp(\int_o^{t\wedge\tau} \frac{\nabla\varphi}{\varphi}(X_s).dX_s - \frac{1}{2}\int_o^{t\wedge\tau} \left|\frac{\nabla\varphi}{\varphi}(X_s)\right|^2\ ds)$ *is a* $\mathcal{F}_t$*,* W_φ *martingale, and the probability measure* $Q_\varphi = Z_T^\varphi W_\varphi$ *satisfies*

i. Q_φ *is stationary i.e.* $Q_\varphi \circ X_t^{-1} = \varphi^2\ dx \quad \forall t \in [0,T]$

ii. $H(Q_\varphi, W_\varphi) = \frac{1}{2}E^{Q_\varphi}\left[\int_o^T \left|\frac{\nabla\varphi}{\varphi}(X_s)\right|^2 ds\right] = \frac{T}{2}\int |\nabla\varphi|^2\ dx < +\infty$

iii. Q_φ *is a solution to* $\mathcal{M}(S_\varphi, \mathcal{C}_c^\infty)$

iv. Q_φ *is a strongly markovian probability measure.*

Indeed, to apply [CL94], the only point to be checked is that $\varphi^2 dx$ satisfies the weak stationary Fokker Planck equation

$$S_\varphi{}^*\nu = 0$$

which is immediate.

The other point to be precised is the meaning of *iii.* We shall say that Q satisfies the martingale problem $\mathcal{M}(S_\varphi, \mathcal{C}_c^\infty)$ if for all $f \in \mathcal{C}_c^\infty$

$$\begin{cases} \int_0^T \left(\frac{\nabla\varphi}{\varphi}.\nabla f\right)(X_s)\ ds \text{ is } Q \text{ a. s. finite} \\ C_t^f = f(X_t) - f(X_0) - \int_o^t (S_\varphi f)(X_s)\ ds \text{ is a } Q \text{ (continuous) local martingale} \end{cases} \tag{3.28}$$

Since $\mathcal{C}_c^\infty$ is an algebra, it follows (see [Jac79] 13.42 or [DM87]) that $\left\langle C^f \right\rangle_t = \int_o^t \Gamma(f)(X_s)\ ds$, where $\Gamma(f) = S_\varphi\ f^2 - 2fS_\varphi f = |\nabla f|^2$, i. e. the local characteristics of $f(X_t)$, (which is a semi-martingale) are known. Now standard Girsanov theory tells us that Q_φ is a solution to $\mathcal{M}(S_\varphi, \mathcal{C}_c^\infty)$ with $\tau_n \wedge \inf\{t \geq 0 \ / \ |X_t| \geq n\}$ as a localizing sequence of stopping times.

The next result is an uniqueness theorem, which is a consequence of th 12.57(a) in [Jac79], according to the preceding discussion.

Theorem 3.3 *Let Q be a solution of $\mathcal{M}(S_\varphi, \mathcal{C}_c^\infty)$ such that $Q \circ X_o^{-1} = \varphi^2\, dx$. If $Q(\tau < +\infty) = 0$ then $Q = Q_\varphi$.*

In particular, Q_φ is the only stationary solution of $\mathcal{M}(S_\varphi, \mathcal{C}_c^\infty)$ such that $Q \circ X_o^{-1} = \varphi^2\, dx$.

Remark 3.4 : Actually, th 12.57 in [Jac79] is proved for a martingale problem involving a finite number of test functions (or more general processes) . It is easy to see that everything is preserved for a countable set of test functions approximating the coordinates functions so that theorem 3.3 holds.

Let we return for a moment to the Dirichlet form $(\mathcal{E}_\varphi, H_o^1(\varphi))$ which is regular and local. According to [Fuk80] chap 6, it is thus associated to a $\varphi^2 dx$-symmetric diffusion process Q_φ^o, which is in particular a stationary solution of $\mathcal{M}(S_\varphi, \mathcal{C}_c^\infty)$ as above.

Consequently, $Q_\varphi^o = Q_\varphi$ and Q_φ is thus $\varphi^2 dx$-symmetric. But as we said before, theorems 3.2 and 3.3 can be considerably improved, including cases where Fukushima's theory is no more valid. It is thus interesting to get a direct proof of the symmetry of Q_φ. Here again one can use "entropic" arguments, due to H.Föllmer [Föl84].

Indeed, let r denote the time reversal operator

$$\begin{array}{rccc} r: & \Omega & \longrightarrow & \Omega \\ & X & \longrightarrow & (t \to X_{T-t}) \end{array} \tag{3.29}$$

Relative entropy is preserved by r, i.e. $H(Q_\varphi \circ r^{-1}, W_\varphi \circ r^{-1}) = H(Q_\varphi, W_\varphi)$ and it follows that $Q_\varphi \circ r^{-1}$ has finite relative entropy with respect to W_{φ_T} (where $\varphi_T = W_\varphi \circ X_T^{-1}$) up to any time $t < T$. Furthermore, the "dual" drift $\hat{B}(t,x)$ (we are in the markovian case) satisfies the duality equation (see [Föl88] 2.13)

$$\hat{B}(t,x) + \frac{\nabla\varphi}{\varphi}(x) = \nabla(\ln\varphi^2)(x) = 2\frac{\nabla\varphi}{\varphi}(x) \tag{3.30}$$

Hence $\hat{B} = \dfrac{\nabla\varphi}{\varphi}$, so that $Q_\varphi \circ r^{-1}$ is again a solution of $\mathcal{M}(S_\varphi, \mathcal{C}_c^\infty)$ which is $\varphi^2 dx$-stationary. It follows that $Q_\varphi \circ r^{-1} = Q_\varphi$, i.e. Q_φ is $\varphi^2 dx$-symmetric.

But one can obtain another very nice property of Q_φ, stated in the next

Theorem 3.5 *Let $\mathcal{A}_{\varphi,H}$ denote the set of Q probability measure on Ω such that $Q \circ X_t^{-1} = \varphi^2 dx$ for all $t \in [0,T]$ and $H(Q, W_\varphi) < +\infty$. Then*

i. Q_φ is the unique markovian $\varphi^2 dx$-symmetric element of $\mathcal{A}_{\varphi,H}$.

ii. Furthermore, if $\int (\ln|\varphi|)\ \varphi^2\ dx < +\infty$ *then*

$$H(Q_\varphi, W_\varphi) = \inf_{Q\in\mathcal{A}_{\varphi,H}} H(Q, W_\varphi)$$

Before we proceed with the proof, let we recall the following properties of $\mathcal{A}_{\varphi,H}$ which are obtained in [CL94] (in particular th 5.3)

Proposition 3.6

i. There exist an unique $Q^*_\varphi \in \mathcal{A}_{\varphi,H}$ *such that* $H(Q^*_\varphi, W_\varphi) = \inf_{Q\in\mathcal{A}_{\varphi,H}} H(Q, W_\varphi)$

ii. Any markovian element of $\mathcal{A}_{\varphi,H}$ *is a solution to a martingale problem* $\mathcal{M}(S_{\varphi,B}, \mathcal{C}^\infty_c)$ *where* $S_{\varphi,B} = \frac{1}{2}\Delta + B.\nabla$ *and* $B \in L^2(\varphi^2 dx)$.

iii. Conversely, for any $B \in L^2(\varphi^2 dx)$ *such that* $S^*_{\varphi,B}(\varphi^2 dx) = 0$, *the measure* Q_B *defined by* $Q_B = Z^B_T\ W_\varphi$ *where*

$$\begin{cases} Z^B_T = \exp\left(\int_0^{t\wedge\tau_B} B(X_s).dX_s - \frac{1}{2}\int_0^{t\wedge\tau_B} |B|^2(X_s)\ ds\right) \\ \tau_B = \inf\{t \geq 0 \quad / \quad \int_0^t |B|^2(X_s)\ ds = +\infty\} \end{cases}$$

is an element of $\mathcal{A}_{\varphi,H}$.

iv. Q^*_φ *is markovian and its drift* B^*_φ *defined in ii. belongs to the* $L^2(\varphi^2 dx)$ *closure of the set* $\{\nabla f\ ,\ f \in \mathcal{C}^\infty_c\}$, *denoted by* $H^{-1}_o(\varphi)$.

v. The set $\{B \in L^2(\varphi^2 dx)\ \ /\ \ S^*_{\varphi,B}(\varphi^2 dx) = 0\}$ *is the affine space* $B^*_\varphi + (H^{-1}_o(\varphi))^\perp$, *where* $\perp$ *stands for the orthogonality in* $L^2(\varphi^2 dx)$.

We shall use part of this proposition in the proof of theorem 3.5.

Proof of theorem 3.5 :

i. We shall show that Q_φ is the unique markovian $\varphi^2 dx$-symmetric element of $\mathcal{A}_{\varphi,H}$. Let Q be such an element. It is immediate that the associated $S_{\varphi,B}$ is $\varphi^2 dx$-symmetric, by using the martingale problem. Since S_φ is also $\varphi^2 dx$-symmetric, we get by elementary computations

$$\forall f, g \in \mathcal{C}^\infty_c \qquad \int (g\nabla f - f\nabla g).(B - \frac{\nabla\varphi}{\varphi})\ \varphi^2\ dx = 0$$

But $g\nabla f - f\nabla g = \nabla(gf) - 2f\nabla g$ and $\int \nabla(gf).(B - \frac{\nabla\varphi}{\varphi})\ \varphi^2\ dx = 0$ according to proposition 3.6v, so that $\int f\nabla g.(B - \frac{\nabla\varphi}{\varphi})\ \varphi^2\ dx = 0$.

If we choose $g = \psi_M$ as in the proof of lemma 2.9, and pass to the limit, we get $B = \frac{\nabla\varphi}{\varphi}$. The results follows from the uniqueness theorem 3.3

ii. The definition 3.1 can be extended to positive measures (not necessarily bounded), and in particular if $\left|\int (\ln|\varphi|)\,\varphi^2\,dx\right| < +\infty$, one has for $Q \in \mathcal{A}_{\varphi,H}$

$$-\infty < H(Q, W_{dx}) = H(Q, W_\varphi) - H(\varphi^2 dx, dx) < +\infty$$

Furthermore

$$H(Q_\varphi^*, W_{dx}) = \inf_{Q\in\mathcal{A}_{\varphi,H}} H(Q, W_{dx}) \tag{3.31}$$

But here again H is time reversal invariant and so

$$H(Q_\varphi^*, W_{dx}) = H(Q_\varphi^* \circ r^{-1}, W_{dx} \circ r^{-1}) = H(Q_\varphi^* \circ r^{-1}, W_{dx})$$

since W_{dx} is dx-symmetric. So $Q_\varphi^* \circ r^{-1} \in \mathcal{A}_{\varphi,H}$ and realizes the infimum in (3.31), hence $Q_\varphi^* \circ r^{-1} = Q_\varphi^*$. The result follows from the first part of the theorem. ∎

Remark 3.7 :

i. According to proposition 3.6v, we could prove that Q_φ is the minimizing measure in $\mathcal{A}_{\varphi,H}$ for relative entropy if we would be able to show directly that $\frac{\nabla\varphi}{\varphi}$ belongs to $H_o^{-1}(\varphi)$. Notice that this result follows from theorem 2.7 under the assumption $\ln|\varphi| \in L^2(\varphi^2 dx)$ (see [BR94b] th 2.8 and section 6 of the present paper).

ii. Notice that hypothesis $\int (\ln|\varphi|)\,\varphi^2\,dx < +\infty$ is satisfied when $\int |x|^2\,\varphi^2\,dx < +\infty$, i.e. X_o has a second order moment. Indeed if $\int |x|^2\,\varphi^2\,dx < +\infty$, one can use the Gross Log-Sobolev inequality for the Gaussian law $d\nu$ (see [Gro93])

$$-\infty < \int f^2\,\ln|f|\,d\nu \leq \int |\nabla f|^2\,d\nu + \|f\|^2_{L^2(d\nu)} \ln \|f\|_{L^2(d\nu)}$$

with $f = \varphi\,\exp\frac{|x|^2}{4}$. Elementary computations lead to

$$-\infty < \int \varphi^2\,\ln|\varphi|\,dx \leq \int |\nabla\varphi|^2\,dx + C^{ste} < +\infty$$

If $\int |x|^2\,\varphi^2\,dx < +\infty$, the derivation of the previous section can also be obtained by replacing the Wiener measure by the Ornstein-Uhlenbeck one which in this case is more natural.

iii. Of course, the previous two remarks can be completed in the $\mathbb{R}^d$ framework. Indeed, according to the well known Sobolev embedding theorem (see [Ada75] chap 5)

- if $d = 1$, $\quad \varphi \in \mathcal{C}_b(\mathbb{R})$
- if $d = 2$, $\quad \varphi \in L^q(\mathbb{R}^2)$ for all $q \geq 2$
- if $d \geq 3$, $\quad \varphi \in L^q(\mathbb{R}^2)$ for all q such that $2 \leq q \leq \dfrac{2d}{d-2}$

So $\displaystyle\int |\ \ln|\varphi|\ |\ \mathbb{1}_{|\varphi|>1}\ \varphi^2\ dx < +\infty$.

The main problem is of course the behavior of φ in a neighborhood of $\{\varphi = 0\}$. In particular if $\varphi \in L^{p'}(\mathbb{R}^d)$ for some $p' < 2$, we also have

$$\int |\ \ln|\varphi|\ |\ \mathbb{1}_{|\varphi|\leq 1}\ \varphi^2\ dx < +\infty$$

This holds in particular if φ is compactly supported. In this case however, φ also satisfied the condition $\displaystyle\int |x|^2\ \varphi^2\ dx < +\infty$ in *ii.* which can be used in an infinite dimensional framework.

We shall return to these problems in section 6.

4 Singular diffusion processes : Markov kernels and decompositions

In this section again we assume that

$$\varphi \in H^1(\mathbb{R}^d), \quad \int \varphi^2\ dx = 1 \qquad (\ 3.24\ )$$

and we put $\dfrac{\nabla\varphi}{\varphi}(x) = 0$ on $\{\varphi = 0\}$.

We will first build a nice version of φ.

Indeed, it is well known that we can choose a version of φ which is finely continuous outside a polar Brownian set $\mathcal{N}_1$, or equivalently (see [Fuk80]) which is quasi-continuous for the usual (Newtonian) 1-capacity associated to the form $(\mathcal{E}, H^1)$. Such a version is called " précisée" in [MZ85]. But we need to modify again this version. First of all, we modify φ by choosing $\varphi(x) = 0$ on $\mathcal{N}_1$. Now denote by $\mathcal{N}$ the (nodal) set $\{\varphi = 0\}$ (in particular, $\mathcal{N} \supset \mathcal{N}_1$). It is known (see e.g. [BG68] th II4.8) that

$$t \longrightarrow \varphi(X_t) \text{ is } W_\mu\text{-a.s. right continuous for all } \mu \text{ such that } \mu(\mathcal{N}_1) = 0 \qquad (4.32)$$

It immediately follows that if $x \in \mathcal{N}^r$ (the set of regular points of $\mathcal{N}$) and $x \notin \mathcal{N}_1$, then $\varphi(x) = 0$. Hence $\mathcal{N}^r \subset \mathcal{N}$, i.e. $\mathcal{N}$ is finely closed, for the Brownian fine topology. But this version is not nice enough.

We next introduce as in (3.27)

$$\tau_k = \inf \left\{ t \geq 0 \quad / \quad \int_o^t \left| \frac{\nabla \varphi}{\varphi} \right|^2 (X_s)\, ds \geq k \right\} \tag{4.33}$$

$$\tau = \inf \left\{ t \geq 0 \quad / \quad \int_o^t \left| \frac{\nabla \varphi}{\varphi} \right|^2 (X_s)\, ds = +\infty \right\}$$

and the following two $(\mathcal{F}_t, W_x)$ supermartingales

$$\left\{ \begin{array}{l} Z_t^\varphi = \exp \left(\int_o^{t\wedge\tau} \frac{\nabla\varphi}{\varphi}(X_s).dX_s - \frac{1}{2}\int_o^{t\wedge\tau} \left| \frac{\nabla\varphi}{\varphi}(X_s)\right|^2 \, ds \right) \\ \\ \overline{Z}_t^\varphi = \exp \left(\int_o^{t} \frac{\nabla\varphi}{\varphi}(X_s).dX_s - \frac{1}{2}\int_o^{t} \left| \frac{\nabla\varphi}{\varphi}(X_s)\right|^2 \, ds \right) \mathbb{I}_{t<\tau} \end{array} \right. \tag{4.34}$$

It is clear that both Z^φ and $\overline{Z}^\varphi$ are (a.s.) right continuous and that $\overline{Z}^\varphi \leq Z^\varphi$. Moreover (see [CL94] and its correction) :

Proposition 4.1

i. *Z^φ is a.s. continuous, Z^φ and $\overline{Z}^\varphi$ only differ on the set*

$$\{(t,\omega) \quad / \quad t \geq \tau(\omega) \text{ and } \omega \in \cup_{k\geq 1}\{\tau = \tau_k\}\}$$

The only possible discontinuities of $\overline{Z}^\varphi$ are for $t = \tau(\omega)$ and $\omega \in \cup_{k\geq 1}\{\tau = \tau_k\}$.

ii. *$\overline{Z}^\varphi$ is a strong multiplicative functional, while in general Z^φ is not.*

iii. *$W_\varphi(\cup_{k\geq 1}\{\tau = \tau_k\}) = 0$, hence Z^φ and $\overline{Z}^\varphi$ coïncide W_φ-a.s., and are actually $(\mathcal{F}_t, W_\varphi)$ martingales (recall th 3.2)*

In the sequel, we denote by $\mathcal{N}'$ the subset of $\mathbb{R}^d$ such that

$$\mathbb{R}^d - \mathcal{N}' = \left\{ x \in \mathbb{R}^d \ / \ \begin{array}{c} Z^\varphi \text{ is a } (\mathcal{F}_t, W_x) \text{ martingale (i.e. } E^{W_x}(Z_T^\varphi) = 1) \\ \text{and } W_x(\cup_{k\geq 1}\{\tau = \tau_k\}) = 0 (\text{ i.e. } Z^\varphi = \overline{Z}^\varphi \ \ W_x\text{-p.s.}) \end{array} \right\} \tag{4.35}$$

(recall that we are working on $\Omega = \mathcal{C}^o([0,T], \mathbb{R}^d)$ i.e. we define $X_t = X_T$ for $t \geq T$) Since $x \longrightarrow W_x$ is measurable, $\mathcal{N}'$ is a Borel set. So according to th I.10.19 of [BG68] one can find an increasing sequence of compact subsets K_n of $\mathcal{N}'$ such that $T_{K_n} \searrow T_{\mathcal{N}'}$ where T_A is the hitting time of A. Applying the strong Markov property of W_x with stopping time T_{K_n}, and Lebesgue's bounded convergence theorem, it is not difficult to check that $\mathcal{N}'$ is finely closed (for the Brownian fine topology). Furthermore, $\mathcal{N}'$ is $\varphi^2 dx$-negligible thanks to proposition 4.1*iii*. Finally, define

$$\left\{ \begin{array}{l} \sigma_n = \inf\{t \geq 0 \ \ / \ \ \varphi(X_t) \notin [\frac{1}{n}, n]\} \\ \sigma = \sup_n \sigma_n \\ \sigma' = \sigma \wedge \inf\{t \geq 0 \ \ / \ \ X_t \in \mathcal{N}'\} \end{array} \right. \tag{4.36}$$

and

$$\begin{cases} Q_x = Z_T^\varphi \, W_x & \text{if} \quad x \notin \mathcal{N} \cup \mathcal{N}' \\ Q_x = \delta_{\{x\}} & \text{if} \quad x \in \mathcal{N} \cup \mathcal{N}' \end{cases} \tag{4.37}$$

where $\delta_{\{x\}}$ is the Dirac mass on the constant path $X_t = x \ \ \forall t \in [0, T]$. Our first result is

Theorem 4.2

i. $\forall x \notin \mathcal{N} \cup \mathcal{N}' \quad Q_x(\sigma' < +\infty) = 0$

ii. $(Q_x)_{x \in \mathbb{R}^d}$ *is a strong Markov family of probability measures, such that* $Q_\varphi = \int Q_x \ \varphi^2 \ dx$.

In particular we can modify φ by putting $\varphi(x) = 0$ for $x \in \mathcal{N}'$ without changing Q_x. The above theorem provides a nice realization of the disintegration of Q_φ, i.e. if we choose the good φ, the nodal set $\mathcal{N}$ cannot be reached (except if one starts from a nodal point), and outside of $\mathcal{N}$ Q_x has the good Girsanov density.

Proof of theorem 4.2 :

Since $\mathcal{N} \cup \mathcal{N}'$ is $\varphi^2 dx$-negligible, the equality $Q_\varphi = \int Q_x \ \varphi^2 \ dx$ is immediate. Indeed, Q_x satisfies the martingale problem $\mathcal{M}(\mathcal{C}_c^\infty, S_\varphi)$ for $x \notin \mathcal{N} \cup \mathcal{N}'$ and so does $\int Q_x \ \varphi^2 \ dx$. Furthermore, $Q_x(\tau < +\infty) = 0$ for $x \notin \mathcal{N} \cup \mathcal{N}'$, and we may apply theorem 3.3. In order to prove the rest of the theorem, we first prove

Proposition 4.3 $Q_\varphi(\sigma < +\infty) = 0$

This proposition is actually proved in [MZ85], and is the main ingredient in the proof of the existence by the authors. We shall however briefly give a proof, which is a slight modification of the one in [MZ85].

Proof: For $\varepsilon > 0$, the function $\varphi_\varepsilon = (\varphi \vee \varepsilon) \wedge \frac{1}{\varepsilon} \in H^1_{loc}$, $\nabla \varphi_\varepsilon \in L^2(dx)$ and one can find smooth functions φ_ε^k such that $\varepsilon \leq \varphi_\varepsilon^k \leq \frac{1}{\varepsilon}$ and which approximate φ_ε in H^1_{loc} and in $L^2(dx)$ for the gradient. We may apply Ito formula to $\ln \varphi_\varepsilon^k$ in both time directions, and sum up the two resulting equalities in order to get for all x :

$$2 \ (\ln \varphi_\varepsilon^k(X_t) - \ln \varphi_\varepsilon^k(X_o)) = K_t^{k,\varepsilon} + K_{T-t}^{k,\varepsilon} \circ r - K_T^{k,\varepsilon} \circ r$$

where

$$K_t^{k,\varepsilon} = \int_o^t \frac{\nabla \varphi_\varepsilon^k}{\varphi_\varepsilon^k}(X_s).dX_s \quad W_x - a.s.$$

and r denotes again the time reversal operator. By Fubini's theorem, the same equality holds W_{dx}-a.s. But since φ_ε^k is bounded from below by ε, and from above by $\frac{1}{\varepsilon}$, it is easily seen that

$$\sup_{0\leq t\leq T} |K_t^{k,\varepsilon} - K_t^\varepsilon| \text{ converges to } 0 \text{ in } L^2(W_{dx})$$

where

$$K_t^\varepsilon = \int_o^t \frac{\nabla\varphi_\varepsilon}{\varphi_\varepsilon}(X_s).dX_s$$

By taking, if necessary, a subsequence, we thus have

$$2\ (\ln\varphi_\varepsilon(X_t) - \ln\varphi_\varepsilon(X_o)) = K_t^\varepsilon + K_{T-t}^\varepsilon \circ r - K_T^\varepsilon \circ r \quad W_{dx} - a.s.$$

Since $Q_\varphi << W_\varphi << W_{dx}$ the same equality holds Q_φ a.s., but now K_t^ε is a Q_φ semi-martingale which decomposition is given by Girsanov's theory, and applying Doob's inequality and time reversibility of Q_φ, we obtain

$$\begin{aligned}
&Q_\varphi\left(\sup_{0\leq t\leq T} |\ln\varphi_\varepsilon(X_t) - \ln\varphi_\varepsilon(X_o)| > \lambda\right)\\
&\leq\ 2Q_\varphi\left(\sup_{0\leq t\leq T}|K_t^\varepsilon| > \frac{\lambda}{2}\right)\\
&\leq\ C^{ste}\frac{T}{\lambda}\left(\int_0^t \left|\frac{\nabla\varphi_\varepsilon}{\varphi_\varepsilon}\right|\varphi^2\ dx + \int_0^t\left|\frac{\nabla\varphi_\varepsilon}{\varphi_\varepsilon} - \frac{\nabla\varphi}{\varphi}\right|\varphi^2\ dx\right)
\end{aligned}$$

But $\nabla\varphi_\varepsilon = \nabla\varphi 1\!\!1_{\varepsilon\leq\varphi\leq\frac{1}{\varepsilon}}$, hence using Cauchy-Schwarz inequality

$$Q_\varphi\left(\sup_{0\leq t\leq T} |\ln\varphi_\varepsilon(X_t) - \ln\varphi_\varepsilon(X_o)| > \lambda\right) \leq C^{ste}\frac{T}{\lambda}\int|\nabla\varphi|^2\ dx$$

It follows that for all $a > 0$ and all n big enough

$$Q_\varphi(\frac{1}{a} > \varphi(X_o) > a\ ,\ \sigma_n < T) \leq \frac{C(T)}{\ln(an)}\int|\nabla\varphi|^2\ dx$$

Thus by taking limits, first in n, then in a (since $\varphi^2 dx$ is supported by $\{\varphi > 0\}$) proposition 4.3 is proved.

∎

Proposition 4.3 implies that for $\varphi^2 dx$-almost all x, $Q_x(\sigma < +\infty) = 0$. We want more, since we want that $Q_x(\sigma' < +\infty) = 0$ for the x's which do not belong to $\mathcal{N}\cup\mathcal{N}'$. To this end, first remark that

$$\text{if } x \notin \mathcal{N}\cup\mathcal{N}',\quad \inf\{t\geq 0\ /\ \ X_t\in\mathcal{N}'\} = +\infty \quad W_x - a.s.$$

which is an immediate consequence of the strong Markov property of W_x, since $\mathcal{N}'$ is finely closed. Since $Q_x << W_x$ for these x's, we only have to prove that

$$\text{if } x \notin \mathcal{N} \cup \mathcal{N}', \quad Q_x(\sigma < +\infty) = 0$$

Assume that $Q_x(\sigma < +\infty) = \alpha > 0$, which is equivalent to $Q_x(\sigma \wedge \tau < +\infty) = \alpha > 0$ since $\tau = +\infty$ $Q_x - a.s.$ according to (4.35). We can then find $t_o > 0$ and $n \in \mathbb{N}^*$ such that for all $t \le t_o$

$$Q_x(t < \sigma_n \wedge \tau_n < \sigma \wedge \tau < +\infty) \ge \frac{\alpha}{2} > 0 \tag{4.38}$$

Indeed, thanks to (4.35), $\tau > \tau_n$ and $\tau > 0$ $Q_x - a.s.$, and thanks to (3.27) $\sigma > \sigma_n > 0$ (for n large enough). Then we apply the multiplicativity of $\overline{Z}^\varphi$.

$$\begin{aligned}
&Q_x(t < \sigma_n \wedge \tau_n < \sigma \wedge \tau < +\infty) \\
&= \int \mathbb{I}_{t<\sigma_n\wedge\tau_n\wedge T}\ \overline{Z}^\varphi_t\ \overline{Z}^\varphi_{(\sigma\wedge\tau)-t}(\theta_t(\omega))\ \mathbb{I}_{\sigma\wedge\tau<+\infty}(\theta_t(\omega))\ dW_x \\
&= \int \mathbb{I}_{t<\sigma_n\wedge\tau_n\wedge T}\ \overline{Z}^\varphi_t\ \left(\int \overline{Z}^\varphi_{(\sigma\wedge\tau)-t}(\omega')\ \mathbb{I}_{\sigma\wedge\tau<+\infty}(\omega')\ dW_{X_t(\omega)}(\omega')\right)\ dW_x(\omega) \\
&\le \int \mathbb{I}_{t<\sigma_n\wedge\tau_n\wedge T}\ h(X_t)\ \overline{Z}^\varphi_{\sigma_n\wedge\tau_n}\ dW_x \\
&\qquad \text{where } h(x) = \int \overline{Z}^\varphi_{(\sigma\wedge\tau)-t}(\omega')\ \mathbb{I}_{\sigma\wedge\tau<+\infty}(\omega')\ dW_x(\omega') \\
&\le \int_{\mathbb{R}^d} h(y)\ p^n_t(x,y)\ dy
\end{aligned} \tag{4.39}$$

where $p^n_t(x,\cdot)$ is the density of $(\overline{Z}^\varphi_{\sigma\wedge\tau\wedge T}\ W_x) \circ X_t^{-1}$ with respect to Lebesgue's measure (which exists since $(\overline{Z}^\varphi_{\sigma\wedge\tau\wedge T}\ W_x)$ is equivalent to W_x).

If $\varphi \neq 0$ dx-a.s., the proof is finished, since $\varphi^2 dy$ is equivalent to dy and $h(y) = Q_y(\sigma \wedge \tau < +\infty)$ for $\varphi^2 dy$-almost all y, i.e. vanishes $\varphi^2 dy$-a.s. thanks to proposition 4.3. This leads to a contradiction with (4.38).

If $\{\varphi = 0\}$ is not of Lebesgue's measure equal to 0, (4.38) and (4.39) imply that

$$\forall t \le t_o \quad \int_{\mathcal{N}} h(y)\ p^n_t(x,y)\ dy \ge \frac{\alpha}{2} > 0$$

and since $h \le 1$

$$\forall t \le t_o \quad \int \overline{Z}^\varphi_{\sigma\wedge\tau\wedge T}\ \mathbb{I}_{X_t\in\mathcal{N}}\ dW_x \ge \frac{\alpha}{2} > 0 \tag{4.40}$$

Now recall that $\mathcal{N}$ is finely closed, and $x \in \mathcal{N}^c$ which is finely open i.e.

$$W_x(\inf\{t \ge 0 \ / \ \ X_t \in \mathcal{N}\} = 0) = 0$$

Since W_x and $\overline{Z}^\varphi_{\sigma\wedge\tau\wedge T}\ W_x$ are equivalent, the above also holds for $\overline{Z}^\varphi_{\sigma\wedge\tau\wedge T}\ W_x$, and we obtain a contradiction with (4.40).

We thus have proved theorem 4.2i. Part ii now is immediate since $\overline{Z}^\varphi = Z^\varphi$ is strong multiplicative for $x \notin \mathcal{N} \cup \mathcal{N}'$, and $\mathcal{N} \cup \mathcal{N}'$ cannot be attained if we start from such an x. ∎

In addition

Proposition 4.4

Let $\mathcal{N}'' = \mathcal{N} \cup \mathcal{N}' \cup \left\{ x \in \mathbb{R}^d \ / \ \exists t > 0 \ / \ E^{Q_x}\left(\int_o^t \left| \frac{\nabla \varphi}{\varphi} \right|^2 (X_s)\ ds \right) = +\infty \right\}$

Then $\mathcal{N}''$ *is* $\varphi^2 dx$*-negligible, finely closed (for both the Brownian or the* $(Q_x)_x$ *fine topology), and*

$$\text{if } x \notin \mathcal{N}'' \quad Q_x(\sigma' \wedge \inf\{s \geq 0 \ / \ X_s \in \mathcal{N}''\} < +\infty) = 0$$

Hence we can modify Q_x*, by setting* $Q_x = \delta_{\{x\}}$ *if* $x \in \mathcal{N}''$ *in (4.37).*

The proof is immediate and left to the reader.

During the proof of proposition 4.3, appeared $\ln \varphi$. In most of the works concerned with the problem we are dealing with (or similar existence and uniqueness problems), the results are derived by studying precisely the behavior of the process $\ln \varphi(X_t) - \ln \varphi(X_o)$ (see [Tak92], [Son94b] for the symmetric case, also [ARZ93a], and [MZ84], [Car85], [Zhe85] in the framework of Nelson's stochastic mechanics). In the symmetric case, one can use Fukushima's decomposition, in order to write

$$\ln \varphi(X_t) - \ln \varphi(X_o) = N_t^\varphi + A_t^\varphi \quad Q_\varphi - a.s.$$

where N^φ is a martingale (additive functional) of finite energy, and A^φ is (an additive functional) of zero energy, provided $\ln \varphi$ belongs to the domain of the form induced by $(Q_x)_x$ (in particular the integrability condition $\int (\ln \varphi)^2 \varphi^2 \ dx < +\infty$ is required, condition which is forgotten in some papers).

This decomposition can be recovered, in a probabilistic framework, by using the recent extension of Ito formula due to Föllmer and Protter [FP94], i.e.

Theorem 4.5 *([FP94] th 4.4, 4.5 and 5.8)*
Let F *be absolutely continuous with (* dx *) square integrable first partial derivatives. Then for any bounded nonnegative function* k *such that*

$$F(X_t) - F(X_o) = \int_o^t \nabla F(X_s).dX_s + \frac{1}{2}[\nabla F(X), X]_t \quad W_{k\ dx} - a.s.$$

where $[\nabla F(X), X]_t$ *is the quadratic covariation of* $\nabla F(X)$ *and* X*, and is equal to*

$$-\int_o^t \nabla F(X_s).dX_s + \left(\int_{T-t}^T \nabla F(X_s).dX_s \right) \circ r$$

where as before r *denotes the time reversal operator.*

Remark 4.6 : Our formulation is slightly different from the one in [FP94]. Indeed, in the construction of a stochastic approximation in Example (4.13) of [FP94], the normalizing $(2\pi s)^{-d/2}$ (in d dimensions) is missing, and the construction seems to fail unless one relaxes the initial condition.

Remark 4.7 : Now it is immediate that theorem 4.5 holds W_x-a.s. for dx almost all x, thus $W_{k\ dx}$-a.s. for any $k\ dx << dx$. If we apply theorem 4.5 to $\ln\varphi$ we get

$$\int_o^t \frac{\nabla\varphi}{\varphi}(X_s).dX_s = \ln\frac{\varphi(X_t)}{\varphi(X_o)} - \frac{1}{2}[\frac{\nabla\varphi}{\varphi}(X), X]_t \quad W_\varphi a.s.$$

hence

$$\frac{dQ_\varphi}{dW_\varphi} = \mathbb{1}_{T<\tau}\ \frac{\varphi(X_T)}{\varphi(X_o)}\ \exp\left(-\frac{1}{2}\left([\frac{\nabla\varphi}{\varphi}(X), X]_T + \int_o^T \left|\frac{\nabla\varphi}{\varphi}\right|^2 (X_s)\ ds\right)\right)$$

This formula is the Gibbsian description of Q_φ, and can be rewritten

$$\frac{dQ_\varphi}{dW_{dx}} = \mathbb{1}_{T<\tau}\ \varphi(X_T)\ \varphi(X_o)\ \exp\left(-\frac{1}{2}\left([\frac{\nabla\varphi}{\varphi}(X), X]_T + \int_o^T \left|\frac{\nabla\varphi}{\varphi}\right|^2 (X_s)\ ds\right)\right)$$

Time reversibility of Q_φ is (apparently) clear on this last formula (see [CRZ94] for an infinite dimensional example).

The last point to be discussed in this section is the connection with Fukushima's decomposition. Indeed $\int_o^t \frac{\nabla\varphi}{\varphi}(X_s).dX_s$ is a $\mathcal{F}_t, Q_\varphi$ semi-martingale (use Girsanov theory with τ_n as a localizing sequence of stopping times) with decomposition $M_t^\varphi + \int_o^t \left|\frac{\nabla\varphi}{\varphi}\right|^2 (X_s)\ ds$ i.e.

$$\ln\varphi(X_t) - \ln\varphi(X_o) = M_t^\varphi + \int_o^t \left|\frac{\nabla\varphi}{\varphi}\right|^2 (X_s)\ ds + \frac{1}{2}[\frac{\nabla\varphi}{\varphi}(X), X]_t \quad Q_\varphi a.s.$$

and it suffices to check that the sum of the last two terms is of zero energy, which does not seem to be so easy.

Extension to $\mathcal{C}^o(\mathbb{R}^+, \mathbb{R}^d)$:

Of course all we have done can easily be extended to the whole paths space $\mathcal{C}^o(\mathbb{R}^+, \mathbb{R}^d)$. It suffices to apply the above results, with the increasing sequence of times $T_n = n$, by defining $\mathcal{N}\cup\mathcal{N}' = \cup_n(\mathcal{N}\cup\mathcal{N}')_{T_n}$ and Q_x as the Föllmer measure associated with the martingale (but supermartingale up to and including $T = +\infty$) Z^φ for $x \notin \mathcal{N}\cup\mathcal{N}'$. It can happen that Q_x and W_x are singular, but $Q_x << W_x$ in restriction to $\mathcal{F}_T$ for all $T < +\infty$. (see [Jac79] or [Föl72] for a discussion about the Föllmer measure.)

5 Singular diffusion processes and the associated Dirichlet forms

In this section we shall discuss the relationship between the Dirichlet forms we introduced in section 2 and the singular diffusion processes we studied in the previous two sections. In particular we intend to give another proof of theorem 2.7, by using "quasi only" probabilistic tools. So before completing this new proof, we will not assume that $H_o^1(\varphi^2 dx) = H^1(\varphi^2 dx)$.

Let we assume for the moment that (3.24) holds.

We saw in theorem 3.3 that there exists one and only one stationary (hence symmetric) solution to $\mathcal{M}(\mathcal{C}_c^\infty, S_\varphi)$ (though we worked in $\mathcal{C}([0,T], \mathrm{I\!R}^d)$, theorem 3.3 extends to any solution of $\mathcal{M}(\mathcal{C}_c^\infty, S_\varphi)$ in the Skorohod space of cadlag paths). In particular if $(\mathcal{E}_\varphi, H)$ is a closed markovian extension of $(\mathcal{E}_\varphi, \mathcal{C}_c^\infty)$, which is associated with a right Markov process, one expects that this process is given by Q_φ. Actually

Proposition 5.1 *Let Q be a symmetric Probability measure associated with $(\mathcal{E}_\varphi, H)$ (a markovian closed extension of $(\mathcal{E}_\varphi, \mathcal{C}_c^\infty)$) i.e. for all $f \in L^2(\varphi^2 dx)$, $E^Q(f(X_t))$ is a $L^2(\varphi^2 dx)$ version of the semi-group of $(\mathcal{E}_\varphi, H)$. Then $Q = Q_\varphi$.*

Proof: The only thing to prove is that Q solves the martingale problem $\mathcal{M}(\mathcal{C}_c^\infty, S_\varphi)$. But it is clear that the generator A of Q is an extension of S_φ, and then Q satisfies $\mathcal{M}(\mathcal{D}(A), A)$. ∎

Unfortunately, it is not true that any markovian closed extension of $(\mathcal{E}_\varphi, \mathcal{C}_c^\infty)$ has an associated right Markov process. A necessary and sufficient condition has been recently obtained in [AM91].

Theorem 5.2 *([AM91] th 1.7, add a remark of [Fit89]) A symmetric Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is associated with a right symmetric process if and only if $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is quasi-regular (see [AM91] 1.14 - 1.16).*

(for some extensions to non symmetric forms, see [AMR92] and the pioneering work of Le Jan [Jan83]).

As a consequence we get

Corollary 5.3 *There exists an unique quasi-regular markovian closed extension $(\mathcal{E}_\varphi, H_o^1(\varphi^2 dx))$ of $(\mathcal{E}_\varphi, \mathcal{C}_c^\infty)$.*

The notion of quasi-regularity we shall not discuss here is a generalization of the classical regularity of [Fuk80] (i.e. $\mathcal{D}(\mathcal{E}) \cap \mathcal{C}_c$ is a core) and it seems that to prove quasi-regularity for $(\mathcal{E}_\varphi, H^1(\varphi^2 dx))$ is not easier than a direct proof of $H^1_o(\varphi^2 dx) = H^1(\varphi^2 dx)$, i.e. the above corollary is not really tractable for the proof of Markov uniqueness.

The proof we shall propose now is inspired by [Son94b] with some simplifications.

Theorem 5.4 *If (3.24) holds then* $H^1_o(\varphi^2 dx) = H^1(\varphi^2 dx)$

Proof of theorem 5.4 :

Let χ_n denote the 1-equilibrium potential of $\{\varphi \notin [\frac{1}{n}, n]\}$, for the form $(\mathcal{E}_\varphi, H^1_o(\varphi^2 dx))$, i.e. (see [Fuk80] th 3.3.1)

$$\chi_n \in H^1_o(\varphi^2 dx) \quad \text{and} \quad cap_\varphi(\{\varphi \notin [\frac{1}{n}, n]\}) = \mathcal{E}^1_\varphi(\chi_n, \chi_n)$$

Since $(\mathcal{E}_\varphi, H^1_o(\varphi^2 dx))$ is regular, it is known that one can choose

$$\chi_n(x) = E^{Q_x}(e^{-\sigma_n})$$

and that

$$cap_\varphi(\{\varphi \notin [\frac{1}{n}, n]\}) = E^{Q_\varphi}(e^{-\sigma_n})$$

(see [Fuk80] th 4.3.5)

Remark 5.5 : Though supp $\varphi^2 dx \neq \mathbb{R}^d$ a priori, the above results are still true (see e.g. [AM91])

According to proposition 4.3 and Lebesgue's dominated convergence theorem

$$\lim_{n\to+\infty} E^{Q_\varphi}(e^{-\sigma_n}) = 0 \tag{5.41}$$

Pick an $f \in H^1(\varphi^2 dx)$ which is bounded. We shall prove that f can be approximated by a sequence of elements of $H^1_o(\varphi^2 dx)$.

Since f and χ_n are bounded elements of $H^1(\varphi^2 dx)$, $f\chi_n \in H^1(\varphi^2 dx)$ and is bounded (see [Fuk80] th 1.4.2 *ii*). Furthermore

$$\overline{\nabla}(f\chi_n) = f\overline{\nabla}\chi_n + \chi_n\overline{\nabla}f \tag{5.42}$$

which is easily obtained by taking limits in 2.9 since $\chi_n \in H^1_o(\varphi^2 dx)$ (i.e. one can find a sequence of $\mathcal{C}^\infty_c$ which converges to χ_n in $\mathcal{E}^1_\varphi$ norm).

Lemma 5.6 $\lim_{n\to+\infty} \mathcal{E}^1_\varphi(\chi_n f, \chi_n f) = 0$ *(up to a subsequence)*

Proof: Since $\mathcal{E}^1_\varphi(\chi_n,\chi_n)$ goes to 0, one can find a subsequence such that χ_n and $\overline{\nabla}\chi_n$ goes to 0, both in $L^2(\varphi^2dx)$ and φ^2dx a.s. Thus since f is bounded,

$$\lim_{n\to+\infty}\int(\chi_n f)^2\varphi^2\ dx = 0$$

and

$$\int\left|\overline{\nabla}(\chi_n f)\right|^2\varphi^2\ dx \le 2\int f^2\left|\overline{\nabla}\chi_n\right|^2\varphi^2\ dx + \int\chi_n^2\left|\overline{\nabla}f\right|^2\varphi^2\ dx$$

which again goes to 0 thanks to Lebesgue's theorem (recall that $\chi_n^2\le 1$).
∎

In view of lemma 5.6, It is enough to approximate $(1-\chi_n)f$ by elements of $H^1_o(\varphi^2dx)$. The proof will rely on the following result.

Lemma 5.7 *Define*

$$H^1_n(\varphi) = \left\{g\in H^1(\varphi^2dx), g\ \textit{bounded}\quad/\quad g(x)=0\ \textit{for}\ dx\textit{-almost all}\ x\notin\{\varphi\in[\frac{1}{n},n]\}\right\}$$
$$H^1_n = \left\{g\in H^1, g\ \textit{bounded}\quad/\quad g(x)=0\ \textit{for}\ dx\textit{-almost all}\ x\notin\{\varphi\in[\frac{1}{n},n]\}\right\}$$

Then $H^1_n(\varphi)\subset H^1_n$, *and if* $g\in H^1_n(\varphi)$, $\nabla g = \overline{\nabla}g\,\mathbb{1}_{\varphi\in[\frac{1}{n},n]} = \overline{\nabla}g$.

The proof of lemma 5.7 is postponed to the end of the proof of theorem 5.4.

According to lemma 5.7, since $(1-\chi_n)f\in H^1_n(\varphi)$ one can find a sequence $(\psi^n_k)_{k\ge1}$ of test functions such that $|\psi^n_k|\le\|f\|_\infty$ and

$$\psi^n_k \xrightarrow[k\to+\infty]{} (1-\chi_n)f \text{ in } H^1$$

(take care that ψ^n_k does not need to vanish on $\{\varphi\notin[\frac{1}{n},n]\}$). But

$$(1-\chi_n)\psi^n_k\in H^1_o(\varphi^2dx)\cap H^1_n(\varphi)$$

Let we compute the distance between $(1-\chi_n)f$ and $(1-\chi_n)\psi^n_k$. First of all

$$\begin{aligned}
&\int\left[(1-\chi_n)\psi^n_k-(1-\chi_n)f\right]^2\varphi^2\ dx\\
&= \int\mathbb{1}_{\varphi\in[\frac{1}{n},n]}\ \left[(1-\chi_n)\psi^n_k-(1-\chi_n)f\right]^2\varphi^2\ dx\\
&\le 2\int\mathbb{1}_{\varphi\in[\frac{1}{n},n]}\ \left[\psi^n_k-(1-\chi_n)f\right]^2\varphi^2\ dx + 2\int|\chi_n\psi^n_k|^2\varphi^2\ dx\\
&\le 2\ n^2\ \|\psi^n_k-(1-\chi_n)f\|^2_{L^2(dx)} + 2\ \|f\|^2_\infty\ \|\chi_n\|^2_{L^2(\varphi^2dx)}
\end{aligned}$$

Next

$$\int \left|\overline{\nabla}((1-\chi_n)\psi_k^n) - \overline{\nabla}((1-\chi_n)f)\right|^2 \varphi^2\ dx$$

$$= \int \left|(1-\chi_n)\overline{\nabla}\psi_k^n - \overline{\nabla}((1-\chi_n)f) - \psi_k^n\overline{\nabla}\chi_n\right|^2\ \mathbb{1}_{\varphi\in[\frac{1}{n},n]}\ \varphi^2\ dx$$

$$= \int \left|(1-\chi_n)\left[\overline{\nabla}\psi_k^n - \overline{\nabla}((1-\chi_n)f)\right] - \chi_n\overline{\nabla}((1-\chi_n)f) - \psi_k^n\overline{\nabla}\chi_n\right|^2\ \mathbb{1}_{\varphi\in[\frac{1}{n},n]}\ \varphi^2\ dx$$

$$= \int \left|(1-\chi_n)\left[\overline{\nabla}\psi_k^n - \overline{\nabla}((1-\chi_n)f)\right] - \chi_n(1-\chi_n)\overline{\nabla}f + (\chi_n f - \psi_k^n)\overline{\nabla}\chi_n\right|^2\ \mathbb{1}_{\varphi\in[\frac{1}{n},n]}\ \varphi^2\ dx$$

$$\leq\ 3\ n^2\ \left\|\overline{\nabla}\psi_k^n - \overline{\nabla}((1-\chi_n)f)\right\|^2_{L^2(dx)} + 3\int|\chi_n\overline{\nabla}f|^2\varphi^2\ dx + 3\times 2\|f\|^2_\infty\ \left\|\overline{\nabla}\chi_n\right\|^2_{L^2(\varphi^2 dx)}$$

By choosing first n big enough, one can control all terms where k does not appear (thanks to lemma 5.6, (5.41) and Lebesgue's theorem). One can then choose k large enough for $\|\psi_k^n - (1-\chi_n)f\|^2_{H^1}$ to be as small as we want. We just have proved that

$$\mathcal{E}^1_\varphi((1-\chi_n)\psi_k^n - (1-\chi_n)f, (1-\chi_n)\psi_k^n - (1-\chi_n)f)$$

is arbitrary small, and since $(1-\chi_n)\psi_k^n \in H^1_o(\varphi^2 dx)$, the proof of theorem 5.4 is finished provided we show lemma 5.7 :

Proof: Let $u \in \mathcal{C}_c^\infty$ such that supp $u \subset \{\varphi \notin [\frac{1}{n}, n]\}$. Then thanks to (2.9) $\int u\ \overline{\nabla}g\ \varphi^2\ dx = 0$ if $g \equiv 0$ on $\{\varphi \notin [\frac{1}{n}, n]\}$. By standard arguments, this shows that

$$\overline{\nabla}g = \overline{\nabla}g\ \mathbb{1}_{\varphi\notin[\frac{1}{n},n]}\quad \varphi^2\ dx\ a.s. \text{ and }\ dx\ a.s.$$

Let $u \in \mathcal{C}_c^\infty$ and let u_n denote the function

$$u_n = \frac{u}{\left((\varphi\vee\frac{1}{n})\wedge n\right)^2}$$

Then $u_n \in H^1$ and

$$\nabla u_n = \frac{\nabla u}{\left((\varphi\vee\frac{1}{n})\wedge n\right)^2} - 2u\frac{\nabla\varphi}{\left((\varphi\vee\frac{1}{n})\wedge n\right)^3}\ \mathbb{1}_{\varphi\notin[\frac{1}{n},n]}$$

Let $(h_n^k)_{k\geq 1}$ be a sequence of test functions which converges to u_n in H^1. Apply (2.9) in order to get

$$\int h_n^k\ \overline{\nabla}g\ \varphi^2\ dx = -\int g\ \overline{\nabla}h_n^k\ \varphi^2\ dx - 2\int g\ h_n^k\ \frac{\nabla\varphi}{\varphi}\ \varphi^2\ dx \qquad (5.43)$$

But according to what precedes, all functions under integral signs vanish outside of $\{\varphi \in [\frac{1}{n}, n]\}$, and on $\{\varphi \in [\frac{1}{n}, n]\}$, $\varphi^2 dx$ and dx are equivalent. But

$$h_n^k \ \mathbb{1}_{\varphi\in[\frac{1}{n},n]}\varphi^2 \xrightarrow[k\to+\infty]{} u \ \mathbb{1}_{\varphi\in[\frac{1}{n},n]} \quad dx \ a.s.$$

$$\nabla h_n^k \ \mathbb{1}_{\varphi\in[\frac{1}{n},n]} \ \varphi^2 \xrightarrow[k\to+\infty]{} (\nabla u - 2u\frac{\nabla\varphi}{\varphi}) \ \mathbb{1}_{\varphi\in[\frac{1}{n},n]} \quad dx \ a.s.$$

and then by taking limits in (5.43), we get

$$\begin{aligned}\int u \ \overline{\nabla} g \ \mathbb{1}_{\varphi\in[\frac{1}{n},n]} \ dx &= \int \nabla u \ g \ \mathbb{1}_{\varphi\in[\frac{1}{n},n]} \ dx \\ &= \int \nabla u \ g \ dx\end{aligned}$$

The proof of lemma 5.7, thus of theorem 5.4, is finished. ∎

Remark 5.8 : Though the proof seems to be a little bit more complicated than we expected, we want to underline that all analytical material we used is really elementary and does not refer to the special form of the generator (i.e. to proposition 2.5 and its consequences). See e.g. [Fra95] for some extensions.

Theorem 5.4 can now be easily extended to the H^1_{loc} case, i.e.

Theorem 5.9 *If $\varphi \in H^1_{loc}$, then $H^1_o(\varphi^2 dx) = H^1(\varphi^2 dx)$.*

Proof: According to corollary 2.6*iii*, it suffices to approximate a $f \in H^1(\varphi^2 dx)$ which is bounded and compactly supported. Take χ and χ', two $\mathcal{C}_c^\infty$ functions satisfying $\mathbb{1}_{\mathrm{supp} f} \leq \chi \leq \chi' \leq 1$ and $\mathbb{1}_{\mathrm{supp}\chi} \leq \chi'$. Then $f = f\chi$ belongs to $H^1((\chi'\varphi)^2 dx)$. But $\chi'\varphi$ being compactly supported satisfies (3.24) (up to a normalizing constant) i.e. $H^1_o((\chi'\varphi)^2 dx) = H^1((\chi'\varphi)^2 dx)$ thanks to theorem 5.4. Let f_n be a sequence of $\mathcal{C}_c^\infty$ such that f_n converges to f in $H^1((\chi'\varphi)^2 dx)$ and put $g_n = \chi f_n$. Then

$$\int |f - g_n|^2 \ \varphi^2 \ dx = \int |\chi f - \chi f_n|^2 \ (\chi'\varphi)^2 \ dx \leq \int |f - f_n|^2 \ (\chi'\varphi)^2 \ dx$$

and

$$\begin{aligned}&\int |\overline{\nabla} f - \nabla g_n|^2 \ \varphi^2 \ dx \\ &\leq \ 2\int |\chi\overline{\nabla} f - \chi\nabla f_n|^2 \ (\chi'\varphi)^2 \ dx + 2\int |f_n|^2 \ |\nabla\chi|^2 \ (\chi'\varphi)^2 \ dx \\ &\leq \ 2\int |\overline{\nabla} f - \nabla f_n|^2 \ (\chi'\varphi)^2 \ dx + 2\int |f_n - f|^2 \ |\nabla\chi|^2 \ (\chi'\varphi)^2 \ dx \\ &\qquad \text{since } f\nabla\chi \equiv 0 \\ &\leq \ C^{ste}\|f_n - f\|^2_{H^1((\chi'\varphi)^2 dx)}\end{aligned}$$

Both quantities are then going to 0 with n, and the proof is finished. ∎

Remark 5.10 : We cannot change the reference measure dx into $d\nu$ (with ν the Gaussian standard law) except if we assume $\int |x|^2 \ \varphi^2 \ dx < +\infty$ (otherwise $\frac{\varphi}{\sqrt{\gamma}}$, where $\gamma \ dx = d\nu$, does not belongs to $H^1(\nu)$). It seems that this point is overcomed in the proof of [RZ92] th 4.1 .

Corollary 5.11 *If either $\varphi \in H^1$, or $\varphi \in H^1_{loc}$ and $\varphi \neq 0 \ dx a.e.$, Markov uniqueness holds.*

6 Invariant measures

In this section we shall use our previous results to study a large class of invariant measures of the Markov process induced on $\Omega = \mathcal{C}^o(\mathbb{R}^+, \mathbb{R}^d)$ by the family $(Q_x)_{x \in \mathbb{R}^d}$ given by (4.4). A similar study is done in [BR94b], by using purely analytical tools. Most of the results of this section are already stated in [BR94b], with different proofs here. But we underline that a large part of section 3 of [BR94b] is actually contained in [CL94] (for the finite dimensional case) and [CL95b] for the more general Markov case (in particular theorem 3.1 and the major part of theorem 3.5). We also stress the fact that the probabilistic approach can be easily extended to more general diffusion cases (see [Fra95]) or infinite dimensional spaces, as well as to non symmetric cases. Nevertheless, both approaches are exciting, and complementary.

We assume again that

$$\varphi \in H^1(\mathbb{R}^d) \quad \text{and} \quad \int \varphi^2 \ dx = 1$$

A positive measure μ on $\mathbb{R}^d$ is said to be an invariant or stationary measure (we will not specify for the Markov process) if

$$Q_\mu \circ X_t^{-1} = \mu \quad \text{for all } t \in \mathbb{R}^+, \quad \text{where } Q_\mu = \int Q_x \ d\mu(x)$$

(recall that $x \longrightarrow Q_x$ is a measurable flow in our construction in proposition 4.4).

Among the invariant measures, the set of the reversible ones is of particular interest. We already know that $\varphi^2 dx$ is reversible. We shall see that any reversible measure "looks like" $\varphi^2 dx$, and moreover that any stationary measure satisfying an integrability condition is reversible. To this end we shall use the entropy minimization result mentioned in theorem 3.5*ii*, but we first drop the condition $\int (\ln |\varphi|) \ \varphi^2 \ dx < +\infty$.

Proposition 6.1 $H(Q_\varphi, W_\varphi) = \inf_{Q \in \mathcal{A}_{\varphi,H}} H(Q, W_\varphi)$
where $\mathcal{A}_{\varphi,H} = \left\{ Q \ \ / \ \ Q \circ X_t^{-1} = \varphi^2 dx \quad \forall t \right\}$

Proof: Proposition 6.1 will be proved if we show that $\dfrac{\nabla\varphi}{\varphi}$ belongs to $H_o^{-1}(\varphi^2 dx)$ (the closure of $\{\nabla f \ / \ f \in \mathcal{C}_c^\infty\}$ in $L^2(\varphi^2 dx)$). But

$$g_\varepsilon = \ln((\varphi \vee \varepsilon) \wedge \frac{1}{\varepsilon}) \quad \text{and} \quad \nabla g_\varepsilon = \frac{\nabla\varphi}{\varphi} 1\!\!1_{\varepsilon \leq \varphi \leq \frac{1}{\varepsilon}}$$

By taking limits, and since $H_o^1(\varphi^2 dx) = H^1(\varphi^2 dx)$ (according to theorem 5.9) the desired result is obtained. ∎

Now let μ be a bounded invariant measure. μ splits into

$$\mu = \mu 1\!\!1_{\varphi=0} + \mu 1\!\!1_{\varphi>0} = \mu_o + \mu_+$$

The first elementary fact is

Proposition 6.2 *μ_o is a reversible measure, μ_+ is an invariant measure.*

The proof is left to the reader. In the sequel, we shall study μ_+.

Proposition 6.3 *$\mu_+ << dx$.*

Proof: $Q_{\mu_+} << W_{\mu_+}$, thus $Q_{\mu_+} \circ X_t^{-1} << W_{\mu_+} \circ X_t^{-1} << dx$ for all $t > 0$. ∎

Proposition 6.4 *Denote by ψ^2 the density of μ_+, i.e. $d\mu_+ = \psi^2 \ dx$. If $\int \left|\frac{\nabla\varphi}{\varphi}\right|^2 \psi^2 \ dx < +\infty$, then $\psi \in H^1$.*

Proof:

$$H(Q_{\mu_+}, W_{\mu_+}) = \frac{1}{2} E^{Q_{\mu_+}} \left[\int_o^T \left|\frac{\nabla\varphi}{\varphi}\right|^2 (X_s) \ ds \right] = \frac{T}{2} \int \left|\frac{\nabla\varphi}{\varphi}\right|^2 \psi^2 \ dx$$

(if we restrict the paths to the time interval $[0, T]$.) Thus we can use Föllmer's results on time reversal (see [Föl88] and [Föl84]), which shows that $\nabla\psi$ is dx a.e. well defined and $\dfrac{\nabla\psi}{\psi} \in L^2(\psi^2 dx)$ i.e. (since μ_+ is bounded) $\psi \in H^1$. ∎

(see [BR94b] for an analytical proof).

Finally, we can state

Theorem 6.5 *Any bounded invariant measure μ satisfying $\int \left|\frac{\nabla\varphi}{\varphi}\right|^2 d\mu < +\infty$ is a reversible measure.*

Proof: Since $\dfrac{\nabla\varphi}{\varphi} = 0$ on $\{\varphi = 0\}$, the above statement reduces to $\displaystyle\int \left|\frac{\nabla\varphi}{\varphi}\right|^2 \, d\mu_+ < +\infty$. It follows from proposition 6.4 that $d\mu_+ = \psi^2 dx$ with $\psi \in H^1$. Thanks to theorem 5.9, $H^1_o(\psi^2 dx) = H^1(\psi^2 dx)$, and by using the same argument than in the proof of proposition 6.1 we obtain that $\dfrac{\nabla\varphi}{\varphi} \in H^{-1}_o(\psi^2 dx)$. But this proves, thanks to proposition 3.6, that

$$H(\,Q_{\mu_+}, W_\psi) = \inf_{Q \in \mathcal{A}_{\psi,H}} H(\,Q, W_\psi)$$

But this infimum is attained at a single point, hence $Q_{\mu_+} = Q_\psi$ according to proposition 6.1, and we know that Q_ψ is time reversible. ∎

The above proof gives another information : $Q_{\mu_+} = Q_\psi$. So Q_{μ_+} solves the martingale problem $\mathcal{M}(\mathcal{C}^\infty_c, S_\varphi)$ and is time reversible. It follows immediately (see the proof of theorem 3.5i) that

$$\frac{\nabla\psi}{\psi} = \frac{\nabla\varphi}{\varphi} \qquad \psi^2 dx \ a.s.$$

Hence $\varphi^2 - \psi^2$ is locally constant on the interior of the set $\{\psi > 0\}$, i.e. constant on each connected component of the interior of $\{\psi > 0\}$. Of course, the topological nature of $\{\psi > 0\}$ is not trivial at all, but at least :

Proposition 6.6 *If $\{\varphi = 0\}$ is closed, then $\psi^2 = c\,\varphi^2$ on each connected component of $\{\varphi > 0\}$ (of course c may depend of the connected component). Conversely, if $\psi^2 = c\,\varphi^2$ on each connected component of $\{\varphi > 0\}$ then μ is reversible.*

Proposition 6.6 and the discussion that precedes also deals with the problem of uniqueness of invariant measures (satisfying the finite energy condition, and bounded). This problem is studied in [BR94b] section 6, with a less complete solution than ours (the special case $\dfrac{|\nabla\varphi|}{\varphi} \in L^1_{loc}(dx)$) and a mistake in the proof, since $\nabla \ln\varphi = \nabla \ln\psi \ \ dx a.e.$ does not imply $\varphi = c\,\psi \quad dx\ a.e.$ (with a given constant) as shown by example 6.1 of [BR94b].

Finally, we shall briefly discuss the problem of non necessarily bounded μ. Proposition 6.3 extends to this case without any trouble. If we consider the generalization of Kullback information to non necessarily finite measures, it is well known that

$$H(Q_\mu, W_\mu) = \int H(Q_x, W_x)\, d\mu$$

If $\dfrac{\nabla\varphi}{\varphi}$ is of finite $\psi^2 dx$-energy, it is not hard to extend Föllmer time reversal argument (the nature of the time reversed Brownian motion does not depend on the initial measure), hence $\psi \in L^2_{loc}(dx)$ and $\nabla\psi \in L^2(dx)$. The minimality argument which leads to proposition 6.1 is only connected with Riesz projection theorem on a Hilbert space (see [CL94]), so extends to a non necessarily bounded measure (provided $\mathcal{C}_c^\infty \subset L^2(\mu)$). Accordingly

Corollary 6.7 *Any invariant measure μ satisfying $\int \left|\frac{\nabla\varphi}{\varphi}\right|^2 d\mu < +\infty$ is a reversible measure. Furthermore $\mu = \mu 1\!\!1_{\varphi=0} + \mu 1\!\!1_{\varphi>0} = \mu_o + \mu_+$, where $\mu_+ = \psi^2 dx$, $\psi \in L^2_{loc}(dx)$ and $\nabla\psi \in L^2(dx)$, satisfy*

$$\frac{\nabla\psi}{\psi} = \frac{\nabla\varphi}{\varphi} \qquad \psi^2 dx \ a.s.$$

In particular, if $\{\varphi = 0\}$ is closed, then $\psi = c\,\varphi$ on each connected component of $\{\varphi > 0\}$. In this case, uniqueness of an invariant measure of finite energy is equivalent to $\{\varphi > 0\}$ connected.

Note that the finite energy condition implies that $(X_t - X_o)^2$ is of finite Q_φ expectation for all t. It should be interesting to find an example of an invariant measure which does not satisfy this condition. Unfortunately (or fortunately) we did not succeed in building such a counterexample.

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