SOLUTION OF A KINETIC STOCHASTIC EQUATION MODELING A SPRAY IN A TURBULENT GAS FLOW

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This work studies the modeling of the effect of turbulence on spray. The spray obeys a kinetic equation. The use of the method of characteristics gives under certain assumptions the solution of the problem. When the gas velocity is modeled by using a white-noise, the solution of the problem obeys a kinetic stochastic equation and may be written in terms of a Wiener chaos expansion.

1 Introduction and Modeling

1.1 Introduction

We consider in this work the modeling of the turbulent dispersion of a cloud of solid particles. The behaviour of many devices such as fludized bed coal combustion or Diesel engines indeed heavily depends upon the turbulent mixing between a gas phase and a dispersed phase. Our aim is to give a rigourous mathematical justification of several models commonly used to predict the turbulent dispersion of the dispersed phase: the turbulent gas flow is modeled by using a random field characterized by its correlation time t_d. The authors P.J.O'Rourke and Reeks describe the droplets dispersion for any time t_d (see [1, 2]), whereas S.Chandrasekhar, Z.Kotulski and P.L.Chow assume t_d to be 0, that is the random field is a gaussian white-noise (see [3, 4, 5]). We would like to emphasize also that Z.Kotulski and P.L.Chow use a stochastic spray equation (although restricted to the dispersion of passive particles) while S.Chandrasekhar restricts himself to the Fokker-Planck equation satisfied by the mean solution of the spray equation. Here, we show how these results are linked. On the one hand, we show for any t_d that the equation satisfied by the mean solution (of the spray equation) is obtained by adding a diffusion term to the spray equation (see [2]). On the other hand, we rigourously study the limiting behaviour of the spray in the white-noise limit t_d tends to 0. Furthermore a chaos expansion enables us to estimate the difference between the mean solution and the solution of the spray equation for a particular experiment, in the limit of the white-noise.

1.2 A mathematical modeling

In order to model two-phase fluid flows, one can use essentially to approaches: a fluid one or a kinetic one. In the fluid model, one assumes that all the particles have the same radius r and that at a given location of the physical space, the velocity distribution function of the cloud of particles has a known shape such as a Dirac distribution or a gaussian function. We wish however to consider more general velocity distribution functions and we rather use the kinetic model.

The particles are characterized by their position $\mathbf{x}$, their velocity $\mathbf{v}$, their radius r. The probability density function (pdf) is then a function $f = f(\mathbf{x}, \mathbf{v}, r, t)$ and the spray equation writes (see [6, 7]):

$$\partial_t f + \nabla_{\mathbf{x}}(f\mathbf{v}) + \nabla_{\mathbf{v}}(f\beta) + \nabla_r(fR) = Q(f, f). \tag{1.1}$$

The term $\beta = \beta(\mathbf{x}, \mathbf{v}, r, t)$ is the acceleration of a droplet at the location $(\mathbf{x}, \mathbf{v}, r)$ of the phase space and at time t. The term R is the regression velocity of the radius of the particle:

$$R = \frac{dr}{dt}.$$

Finally $Q(f, f)$ is a nonlinear expression that takes into account the collisions of particles and the break-up. Here, we are mainly interested in the turbulent dispersion of dilute clouds of particles so that we omit the term $Q(f, f)$ in (1.1). Furthermore, we assume that all the particles have the same radius r for the sake of simplicity. The pdf is then $f = f(\mathbf{x}, \mathbf{v}, t)$ and the spray equation reduces to the transport equation:

$$\partial_t f + \nabla_{\mathbf{x}}(f\mathbf{v}) + \nabla_{\mathbf{v}}(f\beta) = 0. \tag{1.2}$$

The accelaration β of a droplet with radius r and velocity $\mathbf{v}$ is taken as:

$$\beta = \beta_g + \mathbf{g} \tag{1.3}$$

where $\mathbf{g}$ is the acceleration of gravity and β_g is the drag force. We model it by using the Stokes drag force:

$$\beta_g = -\frac{9}{2}\frac{\mu_g}{r^2\rho_p}(\mathbf{v} - \mathbf{u}_g). \tag{1.4}$$

Here, μ_g denotes the viscosity of the gas, r is the radius of the particle, ρ_p is the mass density of the particle and $\mathbf{u}_g$ is the velocity of the gas. We assume below that μ_g and ρ_p are positive constants.

The turbulent gas flow field is modeled by using the ideas of J.K.Dukowicz (see [8]): the velocity field $\mathbf{u}_g$ of the gas is given by:

$$\mathbf{u}_g(\mathbf{x}, t) = \overline{\mathbf{u}_g}(\mathbf{x}, t) + \mathbf{u}(\mathbf{x}, t)$$

where $\overline{\mathbf{u}_g}(\mathbf{x},t)$ is the mean value of the gas velocity flow field and $\mathbf{u}(\mathbf{x},t)$ is the fluctuating part of this field. More precisely we assume that for $\mathbf{x}$ given, $\mathbf{u}(\mathbf{x},t)$ is a piecewise constant function:

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_i(\mathbf{x}) \quad if\ it_d \leq t < (i+1)t_d.$$

The components $(u_i^j)_{1\leq j\leq 3}$ of $\mathbf{u}_i$ are gaussian centered random variables with variance σ (that is $u_i^j = \mathcal{N}(0,\sigma^2)$): denoting by $dp(u_i^j)$ the probability density of u_i^j, we have:

$$\begin{cases} dp(u_i^j(\mathbf{x})) = G_\sigma(u_i^j(\mathbf{x}))du_i^j(\mathbf{x}) \\ G_\sigma(u_i^j(\mathbf{x})) = \dfrac{1}{\sqrt{2\pi}\sigma} exp\left(-\dfrac{u_i^j(\mathbf{x})^2}{2\sigma^2}\right) \end{cases} \tag{1.5}$$

This is a simple modeling of an isotropic homogeneous turbulent flow field. In fact the turbulent flow field a priori depends on the space coordinate $\mathbf{x}$. Since we study the behaviour of the cloud of particles, we are only interested in local properties and we may omit the $\mathbf{x}$-dependency of $\mathbf{u}_i$. Hence, given a turbulent flow field $\mathbf{u}(t)$ modeled by

$$\mathbf{u}(t) = \mathbf{u}_i \quad if \quad it_d \leq t < (i+1)t_d, \qquad u_i^j = \mathcal{N}(0,\sigma^2) \tag{1.6}$$

we are left to solving the following model problem:

Problem 1.1 (model problem) *Find f such that*

$$\begin{cases} \partial_t f + \nabla_{\mathbf{x}}(f\mathbf{v}) + \nabla_{\mathbf{v}}(f\beta) = 0 \\ \qquad \beta = \beta(\mathbf{v},t) = -c\mathbf{v} + c\mathbf{u}(t) \\ f(\mathbf{x},\mathbf{v},0) = f_0(\mathbf{x},\mathbf{v}), \quad f_0 \geq 0 \end{cases} \tag{1.7}$$

Here $c = \frac{9}{2}\frac{\mu_g}{r^2\rho_p}$. In the following, we write $\kappa = \sigma^2 t_d$. Since we want to insist on the dependency of the solution of (1.7) on the random variable $\mathbf{u}$ (and hence on κ and t_d) we shall write its solution as $f_{\mathbf{u}_{(\kappa,t_d)}}$.

The system (1.7) may be solved explicitly by using the method of characteristics. The scale of spatial resolution that should be used in a numerical simulation is of course smaller than the scale of the smallest eddies in the gas flow, which means that a direct simulation of turbulent flow field of the dispersed phase should be used. We want here to introduce some turbulence model for the dispersed phase, i.e., we want to determine the behaviour of the solution f in function of the gas turbulence parameters κ and t_d. We first consider the expectation over all the possible gas turbulent flow fields of the solutions $f_{\mathbf{u}_{(\kappa,t_d)}}$ of system (1.7). This expectation, denoted by $\overline{f}_{(\kappa,t_d)}$ satisfies a parabolic equation whose coefficients are explicitly computed in function of c, κ, and t_d. We consider next the formal limit for κ fixed of the function $\overline{f}_{(\kappa,t_d)}$ as t_d tends to zero and, so that $\overline{f}_{(\kappa,t_d)}$ tends to a solution $\overline{f}_{(\kappa,0)}$ of a parabolic equation. On the other hand, we consider the limit as t_d tends to zero of the solution $f_{\mathbf{u}_{(\kappa,t_d)}}$ of (1.7): the limit is denoted by $f_{\mathbf{u}_{(\kappa,0)}}$. Here, the gas turbulent flow field is modeled by a white-noise $\mathbf{u}_{(\kappa,0)}$ and, for a particular realization of the white-noise, the function $f_{\mathbf{u}_{(\kappa,0)}}$ is solution of a stochastic kinetic equation. Taking finally the expectation over all the possible white-noise of the solutions $f_{\mathbf{u}_{(\kappa,0)}}$ we obtain the function $\overline{f}_{(\kappa,0)}$ solution of the same equation as above. We obtain thus the following commutative diagram:

$$\begin{array}{ccc} f_{\mathbf{u}_{(\kappa,t_d)}} & \xrightarrow{\mathbb{E}_{(\kappa,t_d)}} & \overline{f}_{(\kappa,t_d)} \\ {\scriptstyle t_d\longrightarrow 0}\big\downarrow & & \big\downarrow{\scriptstyle t_d\longrightarrow 0} \\ f_{\mathbf{u}_{(\kappa,0)}} & \xrightarrow{\mathbb{E}_{(\kappa,0)}} & \overline{f}_{(\kappa,0)} \end{array} \tag{1.8}$$

2 Solution of the problem

In this section the parameters κ and t_d are fixed positive constants. Hence the (κ, t_d)-dependency will be omitted. Furthermore, we shall assume that f_0 is smooth with compact support (i.e. $f_0 \in \mathcal{C}_0^\infty(\mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_{\mathbf{v}})$) and we are looking for solutions of (1.7) whose moments are all finite. So, we introduce the space $\mathcal{L}^{1,2} = \{f \in L^1(\mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_{\mathbf{v}}, \|\mathbf{x}\|^1\|\mathbf{v}\|^2 d\mathbf{x}d\mathbf{v}) \times C[0,T]\}$. The functions of $\mathcal{L}^{1,2}$ have finite inertial moment, mass, impulsion and energy. Furthermore, for the sake of simplicity, we set for the mean velocity of the gas $\mathbf{u}_g = 0$, and we introduce the mean acceleration β_0 such that $\beta = -c\mathbf{v} + c\mathbf{u} = \beta_0 + c\mathbf{u}$.

2.1 Solution of the model problem

Theorem 2.1 (solution) *Let $\mathbf{u}(t)$ be defined by (1.6). Define the characteristics $(\mathbf{X}_\mathbf{u}, \mathbf{V}_\mathbf{u})$ as the solution of the following Cauchy problem:*

$$\left\{\begin{array}{ll} \partial_t \mathbf{X}_\mathbf{u}(t;\mathbf{x},\mathbf{v},s) & = \mathbf{V}_\mathbf{u}(t;\mathbf{x},\mathbf{v},s) \\ \partial_t \mathbf{V}_\mathbf{u}(t;\mathbf{x},\mathbf{v},s) & = \beta(\mathbf{V}_\mathbf{u}(t;\mathbf{x},\mathbf{v},s)) \\ & = -c\mathbf{V}_\mathbf{u}(t;\mathbf{x},\mathbf{v},s) + c\mathbf{u}(t) \end{array}\right. \qquad \left\{\begin{array}{l} \mathbf{X}_\mathbf{u}(s;\mathbf{x},\mathbf{v},s) = \mathbf{x} \\ \mathbf{V}_\mathbf{u}(s;\mathbf{x},\mathbf{v},s) = \mathbf{v} \end{array}\right. \tag{2.1}$$

then, the function

$$f_\mathbf{u}(\mathbf{x},\mathbf{v},t) = e^{3ct} f_0(\mathbf{X}_\mathbf{u}(0;\mathbf{x},\mathbf{v},t), \mathbf{V}_\mathbf{u}(0;\mathbf{x},\mathbf{v},t)) \tag{2.2}$$

is the unique solution in $\mathcal{L}^{1,2}$ of the model problem (1.7).

The characteristics defined in (2.1) are the solutions of the Langevin's equation $\frac{d\mathbf{X}}{dt} = \mathbf{v}$, $\frac{d\mathbf{V}}{dt} = \beta$. This is the path of a single droplet in the phase space.They will be called the direct characteristics and denoted by $\mathbf{X}_\mathbf{u}(t;\mathbf{x},\mathbf{v},0)$ and $\mathbf{V}_\mathbf{u}(t;\mathbf{x},\mathbf{v},0)$. The functions associated to the backward direction will be called reverse characteristics and denoted by $\mathbf{X}_\mathbf{u}(0;\mathbf{x},\mathbf{v},t)$ and $\mathbf{V}_\mathbf{u}(0;\mathbf{x},\mathbf{v},t)$. We first solve the Langevin's equation:

Lemma 2.2 (characteristics) *Let $\mathbf{u}(t)$ be defined by (1.6) and let $t > 0$. The direct characteristics defined in (2.1) are explicitly*

$$\left\{\begin{array}{l} \mathbf{X_u}(t;\mathbf{x},\mathbf{v},0) = \mathbf{x} + \dfrac{1-e^{-ct}}{c}\mathbf{v} + \displaystyle\sum_{i=0}^{i=N-1}\{t_d - \frac{1-e^{-ct_d}}{c}e^{-c(t-it_d)}\mathbf{u}_i\} \\ \qquad\qquad\qquad\qquad\qquad\qquad + \{(t-Nt_d) - \dfrac{1-e^{-c(t-Nt_d)}}{c}\}\mathbf{u}_N \\ \mathbf{V_u}(t;\mathbf{x},\mathbf{v},0) = e^{-ct}\mathbf{v} + \displaystyle\sum_{i=0}^{i=N-1}(1-e^{-ct_d})e^{-c(t-it_d)}\mathbf{u}_i + (1-e^{-c(t-Nt_d)})\mathbf{u}_N \end{array}\right. \tag{2.3}$$

where $N = [t/t_d]$ denotes the integer part of t/t_d. The transformation

$$\Phi_{\mathbf{u}}(t): \ (\mathbf{x},\mathbf{v}) \ \longrightarrow \ (\mathbf{X_u}(t;\mathbf{x},\mathbf{v},0), \mathbf{V_u}(t;\mathbf{x},\mathbf{v},0))$$

is a bijection of $\mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_{\mathbf{v}}$ whose inverse writes in terms of the reverse characteristics:

$$\left\{\begin{array}{l} \mathbf{X_u}(0;\mathbf{x},\mathbf{v},t) = \mathbf{x} - \dfrac{e^{ct}-1}{c}\mathbf{v} + \displaystyle\sum_{i=0}^{i=N-1}\{\frac{1-e^{-ct_d}}{c}e^{cit_d} - t_d\}\mathbf{u}_i \\ \qquad\qquad\qquad\qquad\qquad\qquad + \{\dfrac{1-e^{-c(t-Nt_d)}}{c}e^{ct} - (t-Nt_d)\}\mathbf{u}_N \\ \mathbf{V_u}(0;\mathbf{x},\mathbf{v},t) = e^{ct}\mathbf{v} - \displaystyle\sum_{i=0}^{i=N-1}(1-e^{-ct_d})e^{cit_d})\mathbf{u}_i - (1-e^{-c(t-Nt_d)})e^{ct}\mathbf{u}_N \end{array}\right. \tag{2.4}$$

Proof. We refer the reader to the article of P.J.O'Rourke for the expression of the direct characteristics (2.3) (see [1]). The end of the proof follows from straightforward calculations. □

Proof of Theorem 2.1. Let $f_{\mathbf{u}}$ be the solution of the model problem (1.7). Since $\Phi_{\mathbf{u}}(t)$ is a bijection, we are allowed to make the change of variables:

$$\tilde{f}_{\mathbf{u}}(\mathbf{x},\mathbf{v},t) = f_{\mathbf{u}}(\Phi_{\mathbf{u}}(t)(\mathbf{x},\mathbf{v}),t) = f_{\mathbf{u}}(\mathbf{X_u}(t;\mathbf{x},\mathbf{v},0),\mathbf{V_u}(t;\mathbf{x},\mathbf{v},0),t) \tag{2.5}$$

where we used the direct characteristics. Derivating $\tilde{f}_{\mathbf{u}}$ with respect to t, we get

$$\begin{array}{l} \partial_t \tilde{f}_{\mathbf{u}}(\mathbf{x},\mathbf{v},t) \\ \quad = \partial_t \mathbf{X_u}\cdot\nabla_{\mathbf{x}} f_{\mathbf{u}}(\mathbf{X_u},\mathbf{V_u},t) + \partial_t\mathbf{V_u}\cdot\nabla_{\mathbf{v}} f_{\mathbf{u}}(\mathbf{X_u},\mathbf{V_u},t) + \partial_t f_{\mathbf{u}}(\mathbf{X_u},\mathbf{V_u},t) \\ \quad = \mathbf{V_u}\cdot\nabla_{\mathbf{x}} f_{\mathbf{u}}(\mathbf{X_u},\mathbf{V_u},t) + \beta(\mathbf{V_u})\cdot\nabla_{\mathbf{v}} f_{\mathbf{u}}(\mathbf{X_u},\mathbf{V_u},t) + \partial_t f_{\mathbf{u}}(\mathbf{X_u},\mathbf{V_u},t). \end{array}$$

Subtracting this expression to the equation (1.2) satisfied by $f_{\mathbf{u}}$, we have

$$\begin{array}{rl} \partial_t \tilde{f}_{\mathbf{u}}(\mathbf{x},\mathbf{v},t) & = -(\nabla_{\mathbf{v}}\cdot\beta)(\mathbf{V_u})f_{\mathbf{u}}(\mathbf{X_u},\mathbf{V_u},t) \\ & = +3cf_{\mathbf{u}}(\mathbf{X_u},\mathbf{V_u},t) \\ & = +3c\tilde{f}_{\mathbf{u}}(\mathbf{x},\mathbf{v},t) \end{array}$$

and the latter equation can be integrated explicitly:

$$\begin{aligned}\tilde{f}_{\mathbf{u}}(\mathbf{x},\mathbf{v},t) &= e^{3ct}\tilde{f}_{\mathbf{u}}(\mathbf{x},\mathbf{v},0)\\ &= e^{3ct}f(\Phi_{\mathbf{u}}(0)(\mathbf{x},\mathbf{v}),0)\\ &= e^{3ct}f_0(\mathbf{x},\mathbf{v}).\end{aligned}$$

In other words, for any $\mathbf{u}$,

$$f_{\mathbf{u}}(\mathbf{x},\mathbf{v},t) = e^{3ct}f_0(\mathbf{X}_{\mathbf{u}}(0;\mathbf{x},\mathbf{v},t),\mathbf{V}_{\mathbf{u}}(0;\mathbf{x},\mathbf{v},t)).$$

Uniqueness follows from the linearity of (1.7) and $f_{\mathbf{u}}$ is in $\mathcal{L}^{1,2}$ because the support of f_0 is compact. □

2.2 Mean solution

Definition 2.3 (mean solution) *Denote by $f_{\mathbf{u}}$ be the solution of the model problem (1.7) for a given $\mathbf{u}(t)$ defined by (1.6). The mean solution $\overline{f}$ of the model problem is the expectation over all the possible distributions $\mathbf{u}$ of $f_{\mathbf{u}}$:*

$$\overline{f}(\mathbf{x},\mathbf{v},t) = \mathbb{E}(f_{\mathbf{u}}) = \int_{\mathbf{u}\in\mathcal{U}(t)} dp(t)(\mathbf{u})f_{\mathbf{u}}(\mathbf{x},\mathbf{v},t). \tag{2.6}$$

The operator $\mathbb{E}$ denotes the expectation operator, $dp(t)(\mathbf{u})$ is the density of probability at time t of $\mathbf{u}$, and $\mathcal{U}(t)$ is the set of all the possible $\mathbf{u}$ defined by (1.6) on the intervall of time $[0,t]$. Note that the expectation operator should be written $\mathbb{E}\equiv\mathbb{E}_{(\kappa,t_d)}$ where (κ,t_d) are the parameters of the modeling.

We prove now that the mean solution satisfies a parabolic equation:

Theorem 2.4 (equation of the mean solution) *The mean solution $\overline{f}$ is the solution in $\mathcal{L}^{1,2}$ of the parabolic system:*

$$\begin{cases}\partial_t\overline{f}+\nabla_{\mathbf{x}}(\overline{f}\mathbf{v})+\nabla_{\mathbf{v}}(\overline{f}\beta_0)-\nabla_{\mathbf{v}}\cdot(D_x\nabla_{\mathbf{x}}\overline{f}+D_v\nabla_{\mathbf{v}}\overline{f})=0\\ f_0\geq 0,\quad f_0\in\mathcal{C}_0^\infty\end{cases} \tag{2.7}$$

where the diffusion coefficients are:

$$\begin{cases}D_x\equiv D_x(t)=c\sigma^2\{(t-Nt_d)-\dfrac{1-e^{-c(t-Nt_d)}}{c}\}\\ D_v\equiv D_v(t)=c\sigma^2(1-e^{-c(t-Nt_d)})\end{cases} \tag{2.8}$$

and $N=[t/t_d]$.

Remark that the mean acceleration appears in the convection term.

Proof. The proof is given in Appendix **A**. It uses the explicit formulas for the characteristics and the law of $\mathbf{u}(t)$. □

Owing to the convexity and monotonicity of the exponential, the difusion coefficients are non negative. Furthermore, the diffusion term $-\nabla_{\mathbf{v}} \cdot (D_x \nabla_{\mathbf{x}} \overline{f} + D_v \nabla_{\mathbf{v}} \overline{f})$ couples the derivatives with respect to $\mathbf{x}$ and $\mathbf{v}$.

Equation (2.7) of Theorem **2.4** is also given in the article of M.W.Reeks (see [2]). The author uses the physical property of Galilean invariance of the equation. In the case of a gaussian random variable $\mathbf{u}(t)$, he proves that the diffusion term $-D_v \nabla_{\mathbf{v}} \overline{f}$ of the Fokker-Planck equation must be replaced by the more general term $-\nabla_{\mathbf{v}} \cdot (D_x \nabla_{\mathbf{x}} \overline{f} + D_v \nabla_{\mathbf{v}} \overline{f})$. In our approach, this result follows from straightforward calculations.

Finally, note that D_x and D_v are some periodic functions of time with period t_d, which may surprise. Actually this comes from our modeling where the fluctuating velocity is piecewise constant. Indeed, the diffusion terms are given by $D_x \nabla_{\mathbf{x}} \overline{f} + D_v \nabla_{\mathbf{v}} \overline{f} = c\mathbb{E}(f_{\mathbf{u}} \mathbf{u})$, which is the correlation between $f_{\mathbf{u}}$ and $\mathbf{u}$. Hence, if $t = it_d$ for some $i \in \mathbb{N}$, the correlation is zero, and then increases as t describes the intervall $[it_d, (i+1)t_d[$.

We conclude this Section with the following remark:

Remark 2.5 (formal limit) If we introduce the dependency of $\overline{f} = \overline{f}_{(\kappa,t_d)}$ with respect to κ and t_d, we are allowed to consider, at least formally, the white-noise limit (see next Section) where κ is fixed and t_d tends to 0 for the *mean* solution $\overline{f}_{(\kappa,t_d)}$. Let $< D_x >$ and $< D_v >$ be the mean values of $D_x(t)$ and $D_v(t)$ respectively over a period t_d. It is easy to see that $< D_x >= 0$ and $< D_v >= \frac{c^2\kappa}{2}$ in this limit, which formally leads to the following Fokker-Planck equation for the mean solution $\overline{f}_{\mathbf{u}_{(\kappa,0)}}$:

$$\partial_t \overline{f}_{(\kappa,0)} + \nabla_{\mathbf{x}}(\overline{f}_{(\kappa,0)} \mathbf{v}) + \nabla_{\mathbf{v}}(\overline{f}_{(\kappa,0)} \beta_0) - \nabla_{\mathbf{v}} \cdot (< D_v > \nabla_{\mathbf{v}} \overline{f}_{(\kappa,0)}) = 0. \qquad (2.9)$$

3 Stochastic calculus

3.1 White-noise limit

We now want to model the effect of a gas a turbulent flow field in which the variations of the fluctuating velocity are extremely fast compared to any other characteristic time. Thus, we consider the limit where κ is fixed and t_d tends to zero, that is the diffusion coefficient D_v is a constant and the time frequency of the eddies tends to infinity. In the previous Section, we introduced the idea of calculating a mean solution in this limit, that is identifying $\overline{f}_{(\kappa,0)}$ as the limit $\overline{f}_{(\kappa,0)} = \lim_{t_d \longrightarrow 0} \overline{f}_{(\kappa,t_d)}$.

Instead of considering the limit of the *mean* solution, we prefer to consider again a *particular* solution $f_{\mathbf{u}_{(\kappa,t_d)}}$ of the model problem (1.7), where t_d is positive. We want to define now the limit $f \equiv f_{\mathbf{u}_{(\kappa,0)}}$ as t_d tends to zero of $f_{\mathbf{u}_{(\kappa,t_d)}}$. That is, we want to define a *particular* solution of the model problem in the limiting white-noise. A rigourous definition is given in the following theorem:

Theorem 3.1 (stochastic equation) *Let $f_{\mathbf{u}_{(\kappa,t_d)}}$ be the solution of the model problem (1.7). Then, in the white-noise limit, the solution $f_{\mathbf{u}_{(\kappa,t_d)}}$ converges in* *law in $\mathcal{L}^{1,2}$ to the solution f of the following stochastic problem:*

$$\left\{\begin{array}{l} df + \{\nabla_{\mathbf{X}} \cdot (f\mathbf{v}) + \nabla_{\mathbf{V}} \cdot (f\beta_0) - \nabla_{\mathbf{V}} \cdot (\overline{D_v}\nabla_{\mathbf{V}} f)\}dt + \sqrt{2\overline{D_v}}\nabla_{\mathbf{V}} f \cdot d\mathbf{B}_t = 0 \\ f_0 \geq 0, \quad f_0 \in \mathcal{C}_0^\infty(\mathbb{R}^3_{\mathbf{X}} \times \mathbb{R}^3_{\mathbf{V}}) \end{array}\right. \tag{3.1}$$

where $\overline{D_v} = \dfrac{c^2\sigma^2 t_d}{2} = \dfrac{c^2\kappa}{2}$ *and $\mathbf{B}(t)$ is the brownian motion in $\mathbb{R}^3$.*

Proof. The proof is given in Appendix B. □

From the properties of the brownian motion, we get immediately:

Definition and Corollary 3.2 (mean solution in the white-noise limit)
Let f be the random function defined in theorem **3.1**. *In the white-noise limit, the mean solution $\overline{f}_{(\kappa,0)}$ is:*

$$\overline{f}_{(\kappa,0)} = \mathbb{E}(f) \tag{3.2}$$

where $\mathbb{E}$ is the expectation operator in the white-noise limit.
The mean solution $\overline{f}_{(\kappa,0)}$ is the solution in $\mathcal{L}^{1,2}$ of the parabolic system:

$$\left\{\begin{array}{l} \partial_t\overline{f}_{(\kappa,0)} + \nabla_{\mathbf{X}}(\overline{f}_{(\kappa,0)}\mathbf{v}) + \nabla_{\mathbf{V}}(\overline{f}_{(\kappa,0)}\beta_0) - \nabla_{\mathbf{V}} \cdot (\overline{D_v}\nabla_{\mathbf{V}}\overline{f}_{(\kappa,0)}) = 0 \\ f_0 \geq 0, \quad f_0 \in \mathcal{C}_0^\infty(\mathbb{R}^3_{\mathbf{X}} \times \mathbb{R}^3_{\mathbf{V}}) \end{array}\right. \tag{3.3}$$

where $\overline{D_v}$ is given in Theorem **3.1**.

Notice that this is the equation (2.9) of Remark **2.5**. The first task of our work is now achieved, that is to study the diagram (1.8): we have rigourously shown how the Fokker-Planck equation follows from the white noise limit. We shall now see that the study of the stochastic equation gives us more insight on the droplets behaviour than the Fokker-Planck equation.

3.2 Chaos expansion

We now want to compare the *particular* and *mean* solutions f and $\overline{f}_{(\kappa,0)}$ in the *white-noise limit*. In this Subsection, we prove that the solution f of (3.1) writes in terms of a Wiener Chaos expansion involving the *mean solution* (3.2). Define first the homogeneous chaos spaces for our problem:

Wiener Chaos

Let $t \longrightarrow F(t)$ be a random process with *finite* mean and variance. The Wiener chaos expansion of F is the serie:

$$F(t) = \sum_{n=0}^{+\infty} I_n(t)(F) \tag{3.4}$$

where $I_n(t)(F)$ is a scalar random variable that is in the Wiener chaos space $\mathcal{H}_n$ of order n: let $\mathbf{B}(t)$ denote the Brownian motion in three dimensions. Then, the

Wiener chaos spaces are defined as follows

$$\mathcal{H}_0 = \{h_0 \in L^2[0,T]\}$$

$$\mathcal{H}_1 = \left\{h_1(t) = \int_0^t \mathbf{h_0}(\mathbf{s}) \cdot \mathbf{dB_s}, \quad \mathbf{h_0} \in (\mathcal{H}_\mathbf{0})^\mathbf{3}\right\}$$

$$\vdots$$

$$\mathcal{H}_n = \left\{h_n(t) = \int_0^t \mathbf{h_{n-1}}(\mathbf{s}) \cdot \mathbf{dB_s}, \quad \mathbf{h_{n-1}} \in (\mathcal{H}_\mathbf{n-1})^\mathbf{3}\right\}$$

and the scalar function $I_n(t)(F)$ defined above writes in the form

$$I_n(t)(F) = \sum_{j_1,\ldots,j_n=1}^{3} \int_0^t \cdots \int_0^{t_{n-1}} f^{(j_1,\ldots,j_n)}(t_1,\ldots,t_n)\; dB_{t_1}^{j_1} \ldots dB_{t_n}^{j_n} \tag{3.5}$$

where the scalar functions $f^{(j_1,\ldots,j_n)}(t_1,\ldots,t_n)$ are unknowns, and $d\mathbf{B}_{t_k}^j$ is the j_{th} component of the differential of the Brownian motion $\mathbf{B}(t)$ at time t_k. The space $\mathcal{H}_0$ is the space of the deterministic scalar functions. The space $\mathcal{H}_n$ contains the iterated integrals of order n of the Brownian motion. Consider the scalar product defined with the expectation operator. Then, the Wiener chaos spaces are orthogonal spaces, that is

$$\mathbb{E}(I_n(t)(F)I_m(s)(G))$$
$$= \delta(m,n) \sum_{j_1,\ldots,j_n=1}^{3} \int_0^{\inf(t,s)} \cdots \int_0^{t_{n-1}} f^{(j_1,\ldots,j_n)}(t_1,\ldots,t_n) g^{j_1,\ldots,j_n}(t_1,\ldots,t_n)\; dt_1 \ldots dt_n$$

for any functions F and G.

For the sake of simplicity, we introduce the tensor of order n of the differentials:

$$\left(dB_{t_1}^{j_1} \ldots dB_{t_n}^{j_n}\right)_{j_1,\ldots,j_n} = [d\mathbf{B}_{t_1} \otimes \ldots \otimes d\mathbf{B}_{t_n}]$$

and in the same way we introduce the tensor of rank n: $\mathbf{f}^{(n)} = \left(f^{(j_1,\ldots,j_n)}\right)_{j_1,\ldots,j_n}$. Then, (3.5) simply writes:

$$I_n(t)(F) = \int_0^t \cdots \int_0^{t_{n-1}} \mathbf{f}^{(n)}\; (:)\; [d\mathbf{B}_{t_1} \otimes \ldots \otimes d\mathbf{B}_{t_n}]$$

where (:) is the n-contracting product of tensors.

Remark that if $F(t)$ also depends on the space variable $(\mathbf{x},\mathbf{v})$, then the tensor $\mathbf{f}^{(n)}$ will also depend on $(\mathbf{x},\mathbf{v})$.

Chaos expansion of the particular solution

Since the function f defined by (3.1) is bounded (as a weak limit of the uniformely bounded sequence $f_{\mathbf{u}_{(\kappa,t_d)}}$), it has finite mean and variance. Hence, we are allowed to seek for a chaos expansion of this function.

Theorem 3.3 (first expansion) *Let f be the function defined by (3.1). It writes*

$$f(\mathbf{x},\mathbf{v},t)=\overline{f}_{(\kappa,0)}(\mathbf{x},\mathbf{v},t)+\sum_{n=1}^{+\infty}V_n(\mathbf{x},\mathbf{v},t) \tag{3.6}$$

where

$$V_n(\mathbf{x},\mathbf{v},t)=(2\overline{D_v})^{\frac{n}{2}}\int_0^t\ldots\int_0^{t_{n-1}}S(t-t_1)\nabla_{\mathbf{v}}\ldots S(t_{n-1}-t_n)\nabla_{\mathbf{v}}\overline{f}_{(\kappa,0)}(\mathbf{x},\mathbf{v},t_n)$$
$$(:)\ \ [d\mathbf{B}_{t_1}\otimes\ldots\otimes d\mathbf{B}_{t_n}]\,. \tag{3.7}$$

Here $(S(t))_{t>0}$ is the semi-group associated to the evolution equation (3.3) satisfied by the mean function $\overline{f}_{(\kappa,0)}$.

Proof. Let $g=f-\overline{f}_{(\kappa,0)}$. As seen in Appendix B, the functions f and $\overline{f}_{(\kappa,0)}$ are some C^∞ functions of $\mathbf{x}$ and $\mathbf{v}$. From Eqs. (3.1) and (3.3), the function g is the solution of the system:

$$\begin{cases} dg_t=A(g)dt+\sqrt{2\overline{D_v}}\nabla_{\mathbf{v}}f\cdot d\mathbf{B}_t\\ g(\mathbf{x},\mathbf{v},0)=0\end{cases} \tag{3.8}$$

where we have set

$$A(g)=-\nabla_{\mathbf{x}}\cdot(g\mathbf{v})-\nabla_{\mathbf{v}}\cdot(g\beta_0)+\nabla_{\mathbf{v}}\cdot(\overline{D_v}\nabla_{\mathbf{v}}g).$$

The operator A is the infinitesimal generator of the semi-group S. The regularity of f allows us to write

$$g(\mathbf{x},\mathbf{v},t)=\sqrt{2\overline{D_v}}\int_0^t S(t-s)\nabla_{\mathbf{v}}f(\mathbf{x},\mathbf{v},s)\cdot d\mathbf{B}_s.$$

Writting $f=\overline{f}_{(\kappa,0)}+g$ in this relation and iterating the operation, we get

$$g(\mathbf{x},\mathbf{v},t)=\sum_{n=1}^{N}V_n(\mathbf{x},\mathbf{v},t)+R^N(\mathbf{x},\mathbf{v},t) \tag{3.9}$$

where $V_n(t)$ is the function of $\mathcal{H}_n$ defined by (3.7) and $R^N(t)$ is the function of $\oplus_{N+1}^{+\infty}\mathcal{H}_n$ defined by

$$R^N(\mathbf{x},\mathbf{v},t)=(2\overline{D_v})^{\frac{n}{2}}\int_0^t\ldots\int_0^{t_{n-1}}S(t-t_1)\nabla_{\mathbf{v}}\ldots S(t_{n-1}-t_n)\nabla_{\mathbf{v}}g(\mathbf{x},\mathbf{v},t_n)$$
$$(:)\ \ [d\mathbf{B}_{t_1}\otimes\ldots\otimes d\mathbf{B}_{t_n}]\,. \tag{3.10}$$

Owing to the orthogonality of the chaos spaces, the relation (3.9) gives

$$\mathbb{E}(g(\mathbf{x},\mathbf{v},t)^2) = \sum_{n=1}^{N} \mathbb{E}(V_n(\mathbf{x},\mathbf{v},t)^2) + \mathbb{E}(R^N(\mathbf{x},\mathbf{v},t)^2) \geq \sum_{n=1}^{N} \mathbb{E}(V_n(\mathbf{x},\mathbf{v},t)^2).$$

which shows that the serie converges. The linearity of (3.1) shows immediately that the limit $\sum_{n=1}^{+\infty} V_n$ is the unique solution of (3.1) which proves the theorem. □

In order to simplify this expansion, we have to commute the derivation operator ∇_v and the semi-group operator $S_t = S(t)$:

Lemma 3.4 (commutator) *Let $S(t)$ be the semi-group associated to (3.3). We have the operators identities:*

$$\left\{ \begin{array}{l} S_t \nabla_{\mathbf{X}} . = \nabla_{\mathbf{X}} S_t . \\ S_t \nabla_{\mathbf{V}} . = a(t) \nabla_{\mathbf{X}} S_t . + b(t) \nabla_{\mathbf{V}} S_t . \end{array} \right. \quad \textit{where} \quad \left\{ \begin{array}{l} a(t) = \dfrac{1 - e^{-ct}}{c} \\ b(t) = e^{-ct} \end{array} \right. \tag{3.11}$$

Proof. The proof is given in Appendix **A** □

Theorem 3.5 (second expansion) *Let f be the function defined by (3.1). It writes*

$$f(\mathbf{x},\mathbf{v},t) = \overline{f}_{(\kappa,0)}(\mathbf{x},\mathbf{v},t) + \sum_{n=1}^{\infty} (2\overline{D}_v)^{n/2} \sum_{k=0}^{n} \nabla_{\mathbf{X}}{}^{n-k} \nabla_{\mathbf{V}}{}^{k} \overline{f}_{(\kappa,0)}(\mathbf{x},\mathbf{v},t) \ (:) \ \tilde{B}_t^{n,k} \tag{3.12}$$

where $\tilde{B}_t^{n,k}$ is the tensor of rank n defined by

$$\tilde{B}_t^{n,k} = \int_0^t \dots \int_0^{t_{n-1}} \sum_{\Im_n^k} a(t-t_{i_1}) \dots a(t-t_{i_{n-k}}) b(t-t_{i_{n-k+1}}) \dots b(t-t_{i_n})$$
$$(:) \ \ [d\mathbf{B}_{t_1} \otimes \dots \otimes d\mathbf{B}_{t_n}]$$

and $\Im_n^k$ is the set of indices:

$$\Im_n^k = \{(i_1,\dots,i_n); (i_1,\dots,i_{n-k}) \subset (1,\dots,n) \ , \ \textit{and} \ (i_{n-k+1},\dots,i_n) \subset (1,\dots,n) \backslash (i_1,\dots,i_k)\}.$$

Proof. Lemma **3.4** allows us to write

$$\begin{aligned} & S(t-t_1)\nabla_{\mathbf{V}} \otimes \dots \otimes S(t_{n-1}-t_n)\nabla_{\mathbf{V}} \\ & \quad = [a(t-t_1)\nabla_{\mathbf{X}} + b(t-t_1)\nabla_{\mathbf{V}}] \otimes \dots \otimes [a(t-t_n)\nabla_{\mathbf{X}} + b(t-t_n)\nabla_{\mathbf{V}}] S(t-t_n) \\ & \quad = \left(\sum_k \alpha_t^{n,k}(t_1,\dots,t_n) \nabla_{\mathbf{X}}{}^{n-k} \nabla_{\mathbf{V}}{}^{k} \right) S(t-t_n) \end{aligned} \tag{3.13}$$

where $a(t)$ and $b(t)$ are defined in (3.11), and

$$\alpha_t^{n,k}(t_1,\dots,t_n) = \sum_{\Im_n^k} a(t-t_{i_1})\dots a(t-t_{i_{n-k}})b(t-t_{i_{n-k+1}})\dots b(t-t_{i_n})$$

where $\Im_n^k$ is defined in the theorem. Since $S(t-t_n)\overline{f}_{(\kappa,0)}(\mathbf{x},\mathbf{v},t_n) = \overline{f}_{(\kappa,0)}(\mathbf{x},\mathbf{v},t)$, the identity (3.13) enables us to rewrite the general term (3.7) of the serie as shown in Theorem **3.5**. $\square$

Remark 3.6 (generalisation) The model can easily be extended to the case where the turbulent gas speed depends on the space variable (see [9]). In the most general model, $\mathbf{u}(\mathbf{x},t)$ is a stationary random field which satisfies a mixing property in the $t-$direction. The stochastic equation writes as:

$$df + \{\nabla_{\mathbf{X}}.(f\mathbf{v}) + \nabla_{\mathbf{V}}.(f\beta_0) - \nabla_{\mathbf{V}}.(\overline{D}_v\nabla_{\mathbf{V}}f)\}dt + \sqrt{2\overline{D}_v}\nabla_{\mathbf{V}}f.d\mathbf{B}(x,t) = 0$$

where $\mathbf{B}(x,t)$ is a centered gaussian field whose variance is defined by:

$$\mathbb{E}(\mathbf{B}(x,t).\mathbf{B}(y,s)) = \inf(s,t)\mathbb{E}(\mathbf{u}(x-y,0).\mathbf{u}(0,0))$$

Remark that the equation for the mean solution (3.2) and the first chaos expansion (3.6) remains unchanged provided that a slight modification is made in the definition of the homogeneous chaos spaces (it simply consists in replacing $\mathbf{B}(t)$ by $\mathbf{B}(\mathbf{x},t)$ in the definition). However it is not possible to rewrite simply this expansion as we did in (3.12) because the derivation operator $\nabla_{\mathbf{X}}$ now acts on the whole stochastic integral $\int \overline{f}(\mathbf{x},\mathbf{v},t)d\mathbf{B}(\mathbf{x},t)$ and not only on $\overline{f}(\mathbf{x},\mathbf{v},t)$.

3.3 Application

The expansion (3.1) is useful for the comparison of f (or any given functional of f) and its mean. We shall give an example for the simplest functional, the density of particles (we restrict ourselves to the 1-d case).

Theorem 3.7 (error estimate) *Let σ_x and σ_v be two positive numbers. We take as initial data the following gaussian function:*

$$f_0(x,v) = \frac{1}{\sqrt{2\pi}\sigma_x}exp\left(-\frac{x^2}{2{\sigma_x}^2}\right)\frac{1}{\sqrt{2\pi}\sigma_v}exp\left(-\frac{v^2}{2{\sigma_v}^2}\right) \tag{3.14}$$

and we consider the densities $\rho_0(x,t) = \int_v f_0 dv$, $\rho(x,t) = \int_v f dv$ and $\overline{\rho}(x,t) = \int_v \overline{f}_{(\kappa,0)}dv$. We can write the error

$$\rho(x,t) - \overline{\rho}(x,t) = \sum_{n=1}^{\infty}(2\overline{D}_v)^{n/2}{\nabla_{\mathbf{X}}}^n\overline{\rho}(x,t)\tilde{B}_t^{n,n} \tag{3.15}$$

and we have

$$\forall t, \quad \frac{\mathbb{E}\left(\|\rho(x,t)-\overline{\rho}(x,t)\|^2_{\mathbb{L}^2(dx)}\right)}{\|\rho_0(x)\|^2_{\mathbb{L}^2(dx)}} \leq \inf\left(\frac{\overline{D}_v}{c^2\sigma_x^2}t, \frac{\sigma_x}{\left(\sigma_x^2+\frac{\sigma_v^2}{c^2}\right)^{\frac{1}{2}}}\right). \tag{3.16}$$

Note that this is a global result in the sense that the norms $\|g\|_{\mathbb{L}^2(dx)}$ in the estimate are calculated thanks to all the values of g in the physical space.

Proof. The expansion (3.15) is an obvious consequence of the chaos expansion (3.12), because $\nabla_{\mathbf{V}}\rho = 0$. In order to calculate the norm in $\mathbb{L}^2(dx)$, we calculate the norm in the Fourier space $\mathbb{L}^2(d\xi)$. Thus, introduce the Fourier transforms $\hat{\overline{\rho}}(\xi,t)$ and $\hat{\rho}(\xi,t)$ of $\overline{\rho}(x,t)$ and $\rho(x,t)$ respectively. The expansion (3.15) writes in the Fourier space:

$$\hat{\rho}(\xi,t)-\hat{\overline{\rho}}(\xi,t) = \sum_{n=1}^{\infty}(2D_v)^{n/2}(\imath\xi)^n\hat{\overline{\rho}}(\xi,t)\tilde{B}_t^{n,n} \tag{3.17}$$

where

$$\tilde{B}_t^{n,n} = \int_0^t \cdots \int_0^{t_{n-1}} a(t-t_1)\dots a(t-t_n)dB_{t_1}\dots dB_{t_n} \tag{3.18}$$

thanks to Theorem **3.5**. Hence,

$$\mathbb{E}((\tilde{B}_t^{n,n})^2) = \int_0^t \cdots \int_0^{t_{n-1}} a(t-t_1)^2\dots a(t-t_n)^2 dt_1\dots dt_n = \frac{1}{n!}\left(\int_0^t a(s)^2 ds\right)^n.$$

Calculate now the expectation of the error (3.16):

$$\mathbb{E}\left(\|\rho(x,t)-\overline{\rho}(x,t)\|^2_{\mathbb{L}^2(dx)}\right) = \int_\xi \mathbb{E}(\hat{\rho}(\xi,t)-\hat{\overline{\rho}}(\xi,t))^2 d\xi.$$

Owing to the orthogonality of the chaos spaces, we have

$$\begin{aligned}\mathbb{E}(\hat{\rho}(\xi,t)-\hat{\overline{\rho}}(\xi,t))^2 &= \sum_{n=1}^{\infty}(2\overline{D}_v)^n\frac{\left(\int_0^t a(s)^2ds\right)^n}{n!}|\xi|^{2n}|\hat{\overline{\rho}}(\xi,t)|^2 \\ &= \left[\exp\left(2\overline{D}_v\left(\int_0^t a(s)^2ds\right)|\xi|^2\right)-1\right]|\hat{\overline{\rho}}(\xi,t)|^2.\end{aligned}$$

For the sake of brevity, we shall not give all the details of the calculation (see [10]): since f_0 is gaussian, we can compute explicitly $\hat{\overline{\rho}}(\xi,t)$ thanks to the formula given in Appendix **A**. Finally, we have for all t the equality:

$$\frac{\mathbb{E}\left(\|\rho(x,t)-\overline{\rho}(x,t)\|^2_{\mathbb{L}^2(dx)}\right)}{\|\rho_0(x)\|^2_{\mathbb{L}^2(dx)}} = \left(\frac{1}{\sqrt{\phi_1(t)}}-\frac{1}{\sqrt{\phi_1(t)+\phi_2(t)}}\right)\sigma_x$$

where

$$\begin{cases} \phi_1(t) = \sigma_x^2 + \left(\frac{1-e^{-ct}}{c}\right)^2 \sigma_v^2 \\ \phi_2(t) = 2\overline{D}_v \left(\int_0^t a(s)^2 ds\right) = \frac{\overline{D}_v}{c^3}(2ct - 3 + 4e^{-ct} - e^{-2ct}) \end{cases}$$

and the result follows easily. □

Note that for $t \ll 1$, the error increases slowly since it is equivalent to $\frac{\overline{D}_v}{3\sigma_v^3} t^3$. The same method may be used for other linear quantities of f and $f - \overline{f}_{(\kappa,0)}$.

4 Conclusion

We insisted on the difference between a particular solution and a mean solution. When we want to compute the solution, we may use one of these two functions. In the direct simulation, we would compute the path of the droplets for a particular occurence of the turbulent gas velocity field. The solution obtained corresponds to the function $f_{\mathbf{u}_{(\kappa,t_d)}}$ of Theorem **2.1**. On the other hand, one may compute the mean solution by solving the parabolic equations (2.7) or (3.3). Hence, this study contributes to understanding and comparing these two approaches.

Furthermore, in the white-noise limit, the mean solution obeys the Fokker-Planck equation. If we consider a steady, homogeneous gas flow, we may look for some equilibrium solutions of the spray. The explicit expression of the mean solution proves that the velocity distribution of the droplets tends to a maxwellian distribution. An other approach is possible. By using an entropy function for the spray, L.Sainsaulieu considers a steady, non necessarily homogeneous, turbulent gas flow and writes the equilibrium distributions for the spray (see [11]).

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A Proofs of Theorems and Lemmas.

Proof of Theorem 2.4:

For the sake of simplicity, we omit the index (κ, t_d) in the proof below. Taking the expectation of (1.7), we obtain

$$\partial_t \overline{f} + \nabla_{\mathbf{x}}(\overline{f}\mathbf{v}) + \nabla_{\mathbf{v}}(\overline{f}\beta_0) + \nabla_{\mathbf{v}}(\mathbb{E}(cf_{\mathbf{u}}\mathbf{u})) = 0 \tag{A.1}$$

so that we just have to compute $\mathbb{E}(cf_{\mathbf{u}}\mathbf{u})$. Since $\mathbf{u}(t) \equiv \mathbf{u}_N$ in our notations, we write

$$\begin{aligned}\mathbb{E}(cf_{\mathbf{u}}\mathbf{u}) &= c\int_{\mathcal{U}(t)} dp(t)(\mathbf{u})f_{\mathbf{u}}\mathbf{u}(t) \\ &= c\int_{\mathcal{U}(t-t_d)} dp(t-t_d)\int_{\mathbb{R}^3} dp(\mathbf{u}_N)f_{\mathbf{u}}\mathbf{u}_N\end{aligned} \tag{A.2}$$

where we have $\mathcal{U}(t) = \mathbb{R}^{3([t/t_d]+1)}$, and

$$dp(t)(\mathbf{u}) = \prod_{i=0}^{[t/t_d]}\prod_{j=1}^{3} dp(u_i^j) = \prod_{i=0}^{[t/t_d]}\prod_{j=1}^{3} G(u_i^j)du_i^j \tag{A.3}$$

where the function $G_\sigma(u)$ is the gaussian function defined by (1.5). Hence,we have to compute

$$\int_{\mathbb{R}^3} dp(\mathbf{u}_N)f_{\mathbf{u}}\mathbf{u}_N = \begin{bmatrix} \displaystyle\int_{\mathbb{R}^3} dp(\mathbf{u}_N)f_{\mathbf{u}}u_N^1 \\ \displaystyle\int_{\mathbb{R}^3} dp(\mathbf{u}_N)f_{\mathbf{u}}u_N^2 \\ \displaystyle\int_{\mathbb{R}^3} dp(\mathbf{u}_N)f_{\mathbf{u}}u_N^3 \end{bmatrix} \tag{A.4}$$

where (u_N^1, u_N^2, u_N^3) are the three components of $\mathbf{u}_N$. Write for example the first component. Thanks to the definition of $\mathbf{u}(t)$ and the expression (2.2) of the solution $f_{\mathbf{u}}$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} dp(\mathbf{u}_N)f_{\mathbf{u}}u_N^1 &= \int_{\mathbb{R}^2} dp(u_N^1)dp(u_N^2)\int_{\mathbb{R}} G(u_N^1)f_{\mathbf{u}}(\mathbf{x},\mathbf{v},t)u_N^1du_N^1 \\ &= \int_{\mathbb{R}^2} dp(u_N^1)dp(u_N^2) \\ &\qquad e^{3ct}\int_{\mathbb{R}} du_N^1u_N^1G(u_N^1)f_0(\mathbf{X}_{\mathbf{u}}(0;\mathbf{x},\mathbf{v},t),\mathbf{V}_{\mathbf{u}}(0;\mathbf{x},\mathbf{v},t)).\end{aligned}$$

We note $(\mathbf{X}^-,\mathbf{V}^-) \equiv (\mathbf{X}_{\mathbf{u}}(0;\mathbf{x},\mathbf{v},t),\mathbf{V}_{\mathbf{u}}(0;\mathbf{x},\mathbf{v},t))$ the reverse characteristics defined in (2.4). They may be written in the form:

$$\begin{cases}\mathbf{X}^- = \mathbf{X}_0^- + a(t)\mathbf{u}_N \\ \mathbf{V}^- = \mathbf{V}_0^- + b(t)\mathbf{u}_N\end{cases}$$

where $(\mathbf{X}_0^-,\mathbf{V}_0^-)$ are independent of $\mathbf{u}_N$, $a(t) = \{\frac{1-e^{-c(t-Nt_d)}}{c}e^{ct} - (t-Nt_d)\}$ and $b(t) = -(1-e^{-c(t-Nt_d)})e^{ct}$. In other words, introducing the derivatives of the reverse characteristics with respect to $\mathbf{u}(t)$,we have $(\partial\mathbf{X}^-/\partial\mathbf{u}) = a(t)\mathbf{Id}_{\mathbb{R}}^3$ and $(\partial\mathbf{V}^-/\partial\mathbf{u}) = b(t)\mathbf{Id}_{\mathbb{R}}^3$. We now have to compute the 1-d integral:

$$e^{3ct}\left(\int_{-\infty}^0 + \int_0^{+\infty}\right) du_N^1u_N^1G(u_N^1)f_0(\mathbf{X}_0^- + a(t)\mathbf{u}_N,\mathbf{V}_0^- + b(t)\mathbf{u}_N).$$

Integrate each term thanks to the primitives

$$H^+(u) = -\int_u^{+\infty} G_\sigma(u)udu \quad if\ u \geq 0$$

$$H^-(u) = \int_{-\infty}^{u} G_\sigma(u)udu \qquad if\ u \leq 0.$$

Since $G_\sigma(u)$ is a gaussian function, an easy calculation allows us to write

$$H^\pm(u) = \pm\sigma^2 G_\sigma(u)$$

and an integration by part with respect to u_N^1 gives

$$\begin{aligned}
& e^{3ct}\int_{\mathbb{R}} du_N^1 u_N^1 G(u_N^1) f_0(\mathbf{X}^-, \mathbf{V}^-) \\
&= e^{3ct}[H^-(0) + H^+(0)] f_0(\mathbf{X}_0^-, \mathbf{V}_0^-) \\
&\quad + e^{3ct}\sigma^2 \int_{\mathbb{R}} dp(u_N^1)[a(t)(\partial_{x^1} f_0)(\mathbf{X}^-, \mathbf{V}^-) + b(t)(\partial_{v^1} f_0)(\mathbf{X}^-, \mathbf{V}^-)]
\end{aligned}$$

where ∂_{x^1} and ∂_{v^1} are the partial derivatives with respect to the first coordinates of the position and velocity variables. Note that $[H^-(0) + H^+(0)] = 0$. Repeating the calculation for the other components of the integral (A.4) we can finally write the expectation (A.2):

$$\mathbb{E}(cf_{\mathbf{u}}\mathbf{u})(\mathbf{x}, \mathbf{v}, t) = c\sigma^2 \int_{\mathcal{U}(t)} dp(t)(\mathbf{u}) e^{3ct}[a(t)\nabla_{\mathbf{X}} f_0(\mathbf{X}^-, \mathbf{V}^-) + b(t)\nabla_{\mathbf{V}} f_0(\mathbf{X}^-, \mathbf{V}^-)]. \tag{A.5}$$

In the sequel, we write $(\mathbf{X}^+, \mathbf{V}^+) \equiv (\mathbf{X_u}(t; \mathbf{x}, \mathbf{v}, 0), \mathbf{V_u}(t; \mathbf{x}, \mathbf{v}, 0))$ the direct characteristics defined by (2.3). The derivatives of the direct characteristics with respect to $\mathbf{u}(t)$ are the matrices $(\partial\mathbf{X}^+/\partial\mathbf{u}) = \{(t - Nt_d) - \dfrac{1 - e^{-c(t-Nt_d)}}{c}\}\mathbf{Id}_{\mathbb{R}}^3$, $(\partial\mathbf{V}^+/\partial\mathbf{u}) = (1 - e^{-c(t-Nt_d)})\mathbf{Id}_{\mathbb{R}}^3$. To achieve the proof of the theorem we just have to rewrite (A.5) by using only the derivatives of $\overline{f}$ rather than those of f_0.

Lemma A.1 *We have the equality:*

$$\begin{aligned}
& c\sigma^2 \int_{\mathcal{U}(t)} dp(t)(\mathbf{u}) e^{3ct}[(\partial\mathbf{X}^-/\partial\mathbf{u})\nabla_{\mathbf{X}} f_0(\mathbf{X}^-, \mathbf{V}^-) + (\partial\mathbf{V}^-/\partial\mathbf{u})\nabla_{\mathbf{V}} f_0(\mathbf{X}^-, \mathbf{V}^-)] \\
&= -c\sigma^2[(\partial\mathbf{X}^+/\partial\mathbf{u})\nabla_{\mathbf{X}}\overline{f} + (\partial\mathbf{V}^+/\partial\mathbf{u})\nabla_{\mathbf{V}}\overline{f}].
\end{aligned} \tag{A.6}$$

Recall Eq. (A.1)

$$\partial_t \overline{f} + \nabla_{\mathbf{X}}(\overline{f}\mathbf{v}) + \nabla_{\mathbf{V}}(\overline{f}\beta_0) + \nabla_{\mathbf{V}}(\mathbb{E}(cf_{\mathbf{u}}\mathbf{u})) = 0$$

where $\mathbb{E}(cf_{\mathbf{u}}\mathbf{u})$ is now given by the second term of Eq. (A.6). Thanks to the expression of $(\partial \mathbf{X}^+/\partial \mathbf{u})$ and $(\partial \mathbf{V}^+/\partial \mathbf{u})$ given above, we have $\mathbb{E}(cf_{\mathbf{u}}\mathbf{u}) = -(D_x \nabla_{\mathbf{x}} \overline{f} + D_v \nabla_{\mathbf{v}} \overline{f})$ where D_x and D_v are the diffusion coefficients defined by (2.8). This concludes the proof of Theorem **2.4**. □

Proof of Lemma A.1. According to the explicit expression (2.2) of $f_{\mathbf{u}}$ and the Definition **2.3** of $\overline{f}$, we have:

$$\overline{f}(\mathbf{x}, \mathbf{v}, t) = \int_{\mathbf{u}\in\mathcal{U}(t)} dp(t)(\mathbf{u}) e^{3ct} f_0(\mathbf{X}^-, \mathbf{V}^-).$$

Let $(\partial \mathbf{X}^-/\partial \mathbf{x})$, $(\partial \mathbf{V}^-/\partial \mathbf{x})$, $(\partial \mathbf{X}^-/\partial \mathbf{v})$ and $(\partial \mathbf{V}^-/\partial \mathbf{v})$ be the derivatives of the reverse characteristics with respect to the space variables. Setting

$$\mathbf{M} = \begin{bmatrix} (\partial \mathbf{X}^-/\partial \mathbf{x}) & (\partial \mathbf{V}^-/\partial \mathbf{x}) \\ (\partial \mathbf{X}^-/\partial \mathbf{v}) & (\partial \mathbf{V}^-/\partial \mathbf{v}) \end{bmatrix} \tag{A.7}$$

we write the derivatives of $\overline{f}$ with respect to the space variables:

$$\begin{bmatrix} \nabla_{\mathbf{x}} \overline{f} \\ \nabla_{\mathbf{v}} \overline{f} \end{bmatrix} (\mathbf{x}, \mathbf{v}, t) = \int_{\mathbf{u}\in\mathcal{U}(t)} dp(t)(\mathbf{u}) e^{3ct} \mathbf{M} \begin{bmatrix} \nabla_{\mathbf{x}} f_0 \\ \nabla_{\mathbf{v}} f_0 \end{bmatrix} (\mathbf{X}^-, \mathbf{V}^-). \tag{A.8}$$

From the definition of the reverse and direct characteristics, we have for any t:

$$\forall(\mathbf{x}, \mathbf{v}),\ \forall \mathbf{u},\ \begin{cases} \mathbf{X}^-(\mathbf{X}^+(\mathbf{x}, \mathbf{v}, \mathbf{u}), \mathbf{V}^+(\mathbf{x}, \mathbf{v}, \mathbf{u}), \mathbf{u}) = \mathbf{x} \\ \mathbf{V}^-(\mathbf{X}^+(\mathbf{x}, \mathbf{v}, \mathbf{u}), \mathbf{V}^+(\mathbf{x}, \mathbf{v}, \mathbf{u}), \mathbf{u}) = \mathbf{v} \end{cases}$$

Derivating this equality with respect to $\mathbf{u}$ gives

$$-\begin{bmatrix} (\partial \mathbf{X}^-/\partial \mathbf{u}) \\ (\partial \mathbf{V}^-/\partial \mathbf{u}) \end{bmatrix} = \mathbf{M}^t \begin{bmatrix} (\partial \mathbf{X}^+/\partial \mathbf{u}) \\ (\partial \mathbf{V}^+/\partial \mathbf{u}) \end{bmatrix}$$

where $\mathbf{M}^t$ is the transpose of the matrix $\mathbf{M}$ defined in (A.7). Hence, the first term of (A.6) is rewritten as

$$-\int_{\mathcal{U}(t)} dp(t)(\mathbf{u}) e^{3ct} [(\partial \mathbf{X}^+/\partial \mathbf{u}), (\partial \mathbf{V}^+/\partial \mathbf{u})] \cdot \mathbf{M} \begin{bmatrix} \nabla_{\mathbf{x}} f_0 \\ \nabla_{\mathbf{v}} f_0 \end{bmatrix} (\mathbf{X}^-, \mathbf{V}^-)$$

that is, since the derivatives $(\partial \mathbf{X}^+/\partial \mathbf{u})$ and $(\partial \mathbf{V}^+/\partial \mathbf{u})$ are independent of $\mathbf{u}$

$$-[(\partial \mathbf{X}^+/\partial \mathbf{u}), (\partial \mathbf{V}^+/\partial \mathbf{u})] \int_{\mathcal{U}(t)} dp(t)(\mathbf{u}) e^{3ct} \cdot \mathbf{M} \begin{bmatrix} \nabla_{\mathbf{x}} f_0 \\ \nabla_{\mathbf{v}} f_0 \end{bmatrix} (\mathbf{X}^-, \mathbf{V}^-)$$

and from (A.8), it also writes

$$-[(\partial\mathbf{X}^+/\partial\mathbf{u}),(\partial\mathbf{V}^+/\partial\mathbf{u})]\cdot\left[\begin{array}{c}\nabla_{\mathbf{X}}\overline{f}\\ \\ \nabla_{\mathbf{V}}\overline{f}\end{array}\right](\mathbf{x},\mathbf{v},t).$$

This concludes the proof of Lemma **A.1** □

Proof of Lemma 3.4:

Since $\nabla_{\mathbf{X}}$ commutates with the generator A, it also commutates with S and the first part of the lemma is proved.
We write $(\mathbf{X}_0^-,\mathbf{V}_0^-)$ the reverse characteristics (2.4) with no turbulent velocity $\mathbf{u}$, that is:

$$\left\{\begin{array}{l}\mathbf{X}_0^-(\mathbf{x},\mathbf{v},t)=\mathbf{x}-\dfrac{e^{ct}-1}{c}\mathbf{v}\\ \mathbf{V}_0^-(\mathbf{x},\mathbf{v},t)=e^{ct}\mathbf{v}\end{array}\right.$$

and introduce now a new function $\tilde{f}$:

$$\overline{f}_{(\kappa,0)}(\mathbf{x},\mathbf{v},t)=e^{3ct}\tilde{f}(\mathbf{X}_0^-(\mathbf{x},\mathbf{v},t),\mathbf{V}_0^-(\mathbf{x},\mathbf{v},t),t).$$

We prove in Appendix C that $\tilde{f}$ writes in terms of a semi-group operator: $\tilde{f}=\tilde{S}_t f_0$ where $\tilde{S}$ is a convolution semi-group whose $\tilde{s}$ is the kernel, that is:

$$\tilde{f}(\mathbf{x},\mathbf{v},t)=\tilde{S}_t f_0=\int_{\mathbb{R}^3_{\mathbf{X}}}\int_{\mathbb{R}^3_{\mathbf{V}}}\tilde{s}_t(\mathbf{x}-\mathbf{x}',\mathbf{v}-\mathbf{v}')f_0(\mathbf{x}',\mathbf{v}')d\mathbf{x}'d\mathbf{v}'.$$

Consequently, the operator S satisfies for any function Φ:

$$S_t\Phi(\mathbf{x},\mathbf{v})=\int_{\mathbb{R}^3_{\mathbf{X}}}\int_{\mathbb{R}^3_{\mathbf{V}}}\tilde{s}_t(\mathbf{X}_0^-(\mathbf{x},\mathbf{v},t)-\mathbf{x}',\mathbf{V}_0^-(\mathbf{x},\mathbf{v},t)-\mathbf{v}')\Phi(\mathbf{x}',\mathbf{v}')d\mathbf{x}'d\mathbf{v}'. \quad \text{(A.9)}$$

Note that S_t is not a convolution operator. Let $(\partial\mathbf{X}_0^-/\partial\mathbf{x})$, $(\partial\mathbf{V}_0^-/\partial\mathbf{x})$, $(\partial\mathbf{X}_0^-/\partial\mathbf{v})$ and $(\partial\mathbf{V}_0^-/\partial\mathbf{v})$ be the derivatives of the reverse characteristics with respect to the space variables. Introducing the matrix

$$\mathcal{M}=\left[\begin{array}{cc}(\partial\mathbf{X}_0^-/\partial\mathbf{x}) & (\partial\mathbf{V}_0^-/\partial\mathbf{x})\\ \\ (\partial\mathbf{X}_0^-/\partial\mathbf{v}) & (\partial\mathbf{V}_0^-/\partial\mathbf{v})\end{array}\right]$$

we compute the derivative of (A.9) with respect to the space variables and get

$$\left[\begin{array}{c}\nabla_{\mathbf{X}}\\ \\ \nabla_{\mathbf{V}}\end{array}\right](S_t\Phi)=\mathcal{M}S_t\left(\left[\begin{array}{c}\nabla_{\mathbf{X}}\\ \\ \nabla_{\mathbf{V}}\end{array}\right]\Phi\right) \quad \text{(A.10)}$$

because the derivatives of the characteristics are independent of $\mathbf{x}$ and $\mathbf{v}$. We write now $(\mathbf{X}_0^+, \mathbf{V}_0^+)$ the direct characteristics (2.3) with no turbulent velocity $\mathbf{u}$, that is:

$$\begin{cases} \mathbf{X}_0^+(\mathbf{x}, \mathbf{v}, t) = \mathbf{x} + \dfrac{1 - e^{-ct}}{c}\mathbf{v} \\ \mathbf{V}_0^+(\mathbf{x}, \mathbf{v}, t) = e^{-ct}\mathbf{v} \end{cases}$$

and calculate the inverse of M:

$$\mathcal{M}^{-1} = \begin{bmatrix} (\partial \mathbf{X}_0^+/\partial \mathbf{x}) & (\partial \mathbf{V}_0^+/\partial \mathbf{x}) \\ (\partial \mathbf{X}_0^+/\partial \mathbf{v}) & (\partial \mathbf{V}_0^+/\partial \mathbf{v}). \end{bmatrix}$$

This allows us to rewrite (A.10):

$$S_t\left(\begin{bmatrix} \nabla_{\mathbf{X}} \\ \nabla_{\mathbf{V}} \end{bmatrix} \Phi\right) = \mathcal{M}^{-1} \begin{bmatrix} \nabla_{\mathbf{X}} \\ \nabla_{\mathbf{V}} \end{bmatrix} (S_t\Phi)$$

and the lemma is proved □

B Stochastic Calculus.

The problem involves limit theorems for the solution of an evolution equation with random wide-band noise and has already been studied by several authors: Bouc and Pardoux first studied the case of a parabolic equation where the noise is driven by an ergodic finite dimensional diffusion (see [12]). Then Kushner and Huang generalized these results when the noise satisfies only a mixing property (see [13]). Both of these works use strongly the hilbertian framework associated to the parabolic equation. Using the technique of stochastic flows of diffeomorphisms, Kunita extended previous theorems to transport equations but the functional framework is then a space of differentiable functions (see [9]).

The turbulent accelaration u does not fulfill all Kunita's hypotheses: it satisfies the required ergodicty condition but is not bounded when t_d is fixed because it is gaussian. Nevertheless, as in the finite-dimensional case, the diffusion-approximation result that we need can be extended to the gaussian case: we will not give any details and we refer to Carmona and Fouque [14] who did it for a more complicate problem.

Applied to Eq. (1.7), Kunita's result is the following:

Proposition B.1 (Kunita's result) *Let:*

$$\overline{D}_v = \frac{c^2}{2} \lim_{td \to 0} \int_0^\infty \mathbb{E}\left(u^j(0)u^j(t)\right) dt = \frac{c^2\kappa}{2}.$$

Let $(\mathbf{B}(t))_{0 \le t \le T}$ *be a three dimensional standard brownian motion.*
Then $f_{\mathbf{u}_{(\kappa, t_d)}}$ *converges in distribution in the space* $C^\infty(\mathbb{R}^3_{\mathbf{X}} \times \mathbb{R}^3_{\mathbf{V}}) \times C[0, T]$ *to the unique solution of the following stochastic partial differential equation:*

$$f = f_0 - \int_0^t (\nabla_{\mathbf{X}} \cdot (f\mathbf{v}) + \nabla_{\mathbf{V}} \cdot (f\beta_0))\, ds + \sqrt{2\overline{D}_v} \int_0^t \nabla_{\mathbf{V}} f \circ d\mathbf{B}_s.$$

In that equation, the last integral is a Stratonovich integral. This equation writes also in the Itô sense:

$$f = f_0 - \int_0^t \left(\nabla_{\mathbf{x}} \cdot (f\mathbf{v}) - c\nabla_{\mathbf{v}} \cdot (f\beta_0) \right) ds + \overline{D}_v \int_0^t \Delta_{\mathbf{v}} f ds + \sqrt{2\overline{D}_v} \int_0^t \nabla_{\mathbf{v}} f \cdot d\mathbf{B}_s$$

which is nothing but (3.1). This convergence means that for each continuous function F on $C^\infty(\mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_{\mathbf{v}}) \times C[0,T]$, one has:

$$\mathbb{E}F(f_{\mathbf{u}}) \longrightarrow_{t_d \to 0} \mathbb{E}F(f).$$

However we are not fully satisfied with this result because functions which are physically interesting for the solution of equation (1.7) are the moments:

$$F(f_{\mathbf{u}}) = \int_{\mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_{\mathbf{v}}} f_{\mathbf{u}}(\mathbf{x}, \mathbf{v}, t) \|\mathbf{x}\|^p \|\mathbf{v}\|^q d\mathbf{x} d\mathbf{v}.$$

For this purpose it is enough to extend the convergence result to the space $L^1(\mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_{\mathbf{v}}, \Phi(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v}) \times C[0,T]$ where $\Phi(\mathbf{x}, \mathbf{v})$ is smooth and slowly increasing at infinity. Thanks to previous convergence it is enough to show that the sequence $(f_{\mathbf{u}})$ is tight in this space because finite-dimensional laws already converge. The tightness criterium is given by the Dunford-Pettis theorem and we have to prove the two following assumptions:

(i) $\sup_{t_d, t \leq T} \mathbb{E}\|f_{\mathbf{u}}\|_{L^1(\mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_{\mathbf{v}}, \Phi(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v})} < \infty$

(ii) $\forall \epsilon,\ \exists \eta,\ \forall A \subset \mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_{\mathbf{v}},\ |A| < \eta \Rightarrow \sup_{t_d, t \leq T} \mathbb{E} \int_A f_{\mathbf{u}}(\mathbf{x}, \mathbf{v}, t) \Phi(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} \leq \epsilon$

From a physical point of view, (i) means that the droplets do not go too fast to infinity and (ii) means that there is no accumulation of droplets in a given region of the phase space.
To prove (i), we consider the ball B_L of center 0 and radius L in $\mathbb{R}^3$. Integrating (1.7) over $B_L \times B_L$, we get:

$$\begin{aligned} &\int_{B_L} d\mathbf{v} \int_{B_L} d\mathbf{x} (f_{\mathbf{u}} - f_0) \Phi \\ &\quad + \int_0^t ds \int_{B_L} d\mathbf{v} \int_{B_L} d\mathbf{x} \left(\nabla_{\mathbf{x}} \cdot (f_{\mathbf{u}} \mathbf{v}) + \nabla_{\mathbf{v}} \cdot (f_{\mathbf{u}} \beta_0) + c \nabla_{\mathbf{v}} f_{\mathbf{u}} \cdot \mathbf{u} \right) \Phi = 0. \end{aligned} \tag{B.1}$$

And we prove below that the second integral goes to zero in the mean as L goes to infinity which proves (i) when $\|f_0\|_{L^1(\mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^3_{\mathbf{v}}, \Phi(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v})} < \infty$: let us take care of the first term $T_1(L, \mathbf{u}) = \int_0^t ds \int_{B_L} d\mathbf{v} \int_{B_L} d\mathbf{x}\ \nabla_{\mathbf{x}} \cdot (f_{\mathbf{u}} \mathbf{v}) \Phi$. Using Stokes's formula, we obtain:

$$T_1(L, \mathbf{u}) = \int_0^t ds \int_{B_L} d\mathbf{v} \oint_{\partial B_L} f_{\mathbf{u}} \mathbf{v} \cdot \nabla_{\mathbf{x}} \Phi dS_{\mathbf{x}}$$

taking the expectation and denoting by Ω_L the set $B_L \times \partial B_L$, we get:

$$\mathbb{E}\int_0^t ds \int_{\Omega_L} f_{\mathbf{u}}\mathbf{v}\cdot\nabla_{\mathbf{x}}\Phi dS_{\mathbf{x}} d\mathbf{v} \leq |\Omega_L| T e^{3cT} \sup_{\Omega_L} \|\nabla_{\mathbf{x}}\Phi\| \sup_{t,t_d,\Omega_L} \mathbb{E} f_0(\mathbf{X}_{\mathbf{u}}^{-1}, \mathbf{V}_{\mathbf{u}}^{-1}).$$

Owing to the properties of Φ, there exists an integer p such that $\sup_{\Omega_L} \|\nabla_{\mathbf{x}}\Phi\| \leq L^p$ and, since f_0 has compact support, we have:

$$\sup_{t,t_d,\Omega_L} \mathbb{E} f_0(\mathbf{X}_{\mathbf{u}}^{-1}, \mathbf{V}_{\mathbf{u}}^{-1}) \leq \|f_0\|_\infty \sup_{t,t_d,\Omega_L} \mathbb{P}((\mathbf{X}_{\mathbf{u}}^{-1}(\mathbf{x},\mathbf{v},t), \mathbf{V}_{\mathbf{u}}^{-1}(\mathbf{v},t)) \in \mathrm{supp}(f_0)).$$

We shall prove that this last quantity goes exponentially fast to zero as L goes to infinity which proves that $T_1(L,\mathbf{u})$ goes to zero in the mean as L goes to infinity. We come back to the expression (2.2) of $f_{\mathbf{u}}$ in terms of the characteristics:

$$f_{\mathbf{u}}(\mathbf{x},\mathbf{v},t) = e^{3ct} f_0(\mathbf{X}_{\mathbf{u}}^{-1}(\mathbf{x},\mathbf{v},t), \mathbf{V}_{\mathbf{u}}^{-1}(\mathbf{v},t)).$$

As t_d goes to zero, it is possible to prove that $\mathbf{V}_{\mathbf{u}}(\mathbf{v},t)$ converges in the law sense in the space $C(\mathbb{R}^3_{\mathbf{V}} \times [0,T])$ to the Ornstein-Uhlenbeck process:

$$\tilde{\mathbf{V}}(\mathbf{v},t) = \mathbf{v}e^{-ct} + \sqrt{2\overline{D}_v}\int_0^t e^{-c(t-s)} d\mathbf{W}_s$$

where $\mathbf{W}$ is a brownian motion in $\mathbb{R}^3$. Since $\mathbf{V}_{\mathbf{u}}$ and $\tilde{\mathbf{V}}$ have exponential moments, this implies that:

$$\mathbb{E}\exp(|\mathbf{V}_{\mathbf{u}}(\mathbf{v},t)|) \longrightarrow_{t_d\to 0} \mathbb{E}\exp(|\tilde{\mathbf{V}}(\mathbf{v},t)|).$$

Let K be a compact set of $\mathbb{R}^3_{\mathbf{V}}$, for a constant $C(T,K)$, we have:

$$\sup_{t,t_d,\mathbf{V}\in K} \mathbb{E}\exp(|\mathbf{V}_{\mathbf{u}}(\mathbf{v},t)|) \leq C(T,K)$$

thus:

$$\sup_{t,t_d,\mathbf{V}\in K} \mathbb{P}(|\mathbf{V}_{\mathbf{u}}(\mathbf{v},t)| > L) \leq e^{-L} C(T,K).$$

We can prove a similar property for the flow $\mathbf{X}_{\mathbf{u}}(\mathbf{x},\mathbf{v},t)$ and, in terms of $\mathbf{V}_{\mathbf{u}}^{-1}$ and $\mathbf{X}_{\mathbf{u}}^{-1}$, this tells us that, for each compact set K of $\mathbb{R}^3_{\mathbf{X}} \times \mathbb{R}^3_{\mathbf{V}}$, there exists a constant $\tilde{C}(T,K)$ such that:

$$\sup_{t,t_d,\|\mathbf{V}\|+\|\mathbf{X}\|>L} \mathbb{P}((\mathbf{X}_{\mathbf{u}}^{-1}(\mathbf{x},\mathbf{v},t), \mathbf{V}_{\mathbf{u}}^{-1}(\mathbf{v},t)) \in K) \leq e^{-L}\tilde{C}(T,K) \tag{B.2}$$

which is exactly what we needed to conclude.

The second term is of the same kind but the last one, $\int_0^t ds \int_{B_L} d\mathbf{v} \int_{B_L} d\mathbf{x}\mathbf{u}\cdot\nabla_{\mathbf{V}} f_{\mathbf{u}}\Phi$, is more complicate because $\mathbf{u}$ is not bounded. We do not want to go into the details of its estimation and we only mention that, in order to take care of that term, we have to carry out the perturbed test function method in equality (B.1) by introducing a small perturbation of the form:

$$\int_0^t \int_s^T \mathbb{E}\mathbf{u}(r)/\mathcal{F}_r ds \int_{B_L} d\mathbf{v} \int_{B_L} d\mathbf{x}\ \nabla_{\mathbf{V}} f_{\mathbf{u}}\Phi$$

$(\mathcal{F}_r)$ being the filtration generated by the turbulent speed before time r; we refer to Carmona and Fouque [14] where this is rigourosly done in the gaussian case.
The second part of the tightness criterium is an easy consequence of the first one because $f_{\mathbf{u}}\Phi$ is locally bounded. Let $\epsilon > 0$. Then, coming back to the proof of (i), we can choose L such that:

$$\sup_{t_d} \mathbb{E} \int_{\mathbb{R}^6/(B_L)^2} f_{\mathbf{u}} \Phi d\mathbf{x} d\mathbf{v} < \epsilon/2.$$

We choose next η such that:

$$\|f_0\|_\infty \left(\sup_{(B_L)^2} \Phi \right) \eta < \epsilon/2.$$

Thus, for each subset A with $|A| < \eta$, one has:

$$\mathbb{E} \int_A f_{\mathbf{u}} \Phi d\mathbf{x} d\mathbf{v} \leq \sup_{t_d} \mathbb{E} \int_{A \cap (B_L)^2} f_{\mathbf{u}} \Phi d\mathbf{x} d\mathbf{v} + \sup_{t_d} \mathbb{E} \int_{\mathbb{R}^6/(B_L)^2} f_{\mathbf{u}} \Phi d\mathbf{x} d\mathbf{v} \leq \epsilon$$

which concludes the proof of Theorem **3.1**. □

C Solution of the Fokker-Planck equation

We write the explicit formula for the solution of the Fokker-Planck equation (3.3).

Theorem C.1 (solution of the Fokker-Planck equation) *Let $\overline{f}$ be the solution of the parabolic system:*

$$\left\{ \begin{array}{l} \partial_t \overline{f} + \nabla_{\mathbf{X}}(\overline{f}\mathbf{v}) + \nabla_{\mathbf{V}}(\overline{f}\beta_0) - \nabla_{\mathbf{V}} \cdot (D \nabla_{\mathbf{V}} \overline{f}) = 0 \\ f_0 \geq 0, \quad f_0 \in \mathcal{C}_0^\infty(\mathbb{R}^3_{\mathbf{X}} \times \mathbb{R}^3_{\mathbf{V}}) \end{array} \right. \tag{C.1}$$

where $\beta_0(\mathbf{v}) = -c\mathbf{v}$. Then, the function $\overline{f}$ writes:

$$\overline{f}(\mathbf{x}, \mathbf{v}, t) = e^{3ct} \int_{\mathbb{R}^3_{\mathbf{X}}} \int_{\mathbb{R}^3_{\mathbf{V}}} G(\mathbf{X}_0^-(\mathbf{x}, \mathbf{v}, t) - \mathbf{x}', \mathbf{V}_0^-(\mathbf{x}, \mathbf{v}, t) - \mathbf{v}', t) f_0(\mathbf{x}', \mathbf{v}') d\mathbf{x}' d\mathbf{v}'$$

where $(\mathbf{X}_0^-, \mathbf{V}_0^-)$ are the reverse characteristics (2.4) with no turbulent velocity, that is:

$$\left\{ \begin{array}{l} \mathbf{X}_0^-(\mathbf{x}, \mathbf{v}, t) = \mathbf{x} - \dfrac{e^{ct} - 1}{c} \mathbf{v} \\ \mathbf{V}_0^-(\mathbf{x}, \mathbf{v}, t) = e^{ct} \mathbf{v} \end{array} \right.$$

and G is defined by

$$G(\mathbf{x}, \mathbf{v}, t) = \frac{1}{(2\Pi)^3 \Delta^{\frac{3}{2}}} exp \left[- \left(a(t)\mathbf{x}^2 + 2h(t)\mathbf{x} \cdot \mathbf{v} + b(t)\mathbf{v}^2 \right) / 2\Delta(t) \right]$$

where

$$\begin{cases} a(t) = 2D\displaystyle\int_0^t (e^{cs})^2 ds & , \quad b(t) = 2D\displaystyle\int_0^t \left(\frac{e^{cs}-1}{c}\right)^2 ds \\ h(t) = -2D\displaystyle\int_0^t e^{cs}\left(\frac{1-e^{cs}}{c}\right) ds & , \quad \Delta(t) = a(t)b(t) - h^2(t) \end{cases}$$

One checks that $\overline{f}$ is smooth with respect to $(\mathbf{x},\mathbf{v})$ and is in $\mathcal{L}^{\infty,\in}$.

Proof. Make the change of variables:

$$\overline{f}(\mathbf{x},\mathbf{v},t) = e^{3ct}\tilde{f}(\mathbf{X}_0^-(\mathbf{x},\mathbf{v},t),\mathbf{V}_0^-(\mathbf{x},\mathbf{v},t),t). \tag{C.2}$$

It is equivalent to write

$$\tilde{f}(\mathbf{x},\mathbf{v},t) = e^{-3ct}\overline{f}(\mathbf{X}_0^+(\mathbf{x},\mathbf{v},t),\mathbf{V}_0^+(\mathbf{x},\mathbf{v},t),t)$$

where $(\mathbf{X}_0^+,\mathbf{V}_0^+)$ are the direct characteristics (2.3) with no turbulent velocity, that is:

$$\begin{cases} \mathbf{X}_0^+(\mathbf{x},\mathbf{v},t) = \mathbf{x} + \dfrac{1-e^{-ct}}{c}\mathbf{v} \\ \mathbf{V}_0^+(\mathbf{x},\mathbf{v},t) = e^{-ct}\mathbf{v} \end{cases}$$

One easily writes the system for $\tilde{f}$:

$$\begin{cases} \partial_t\tilde{f} - D\left[\left(\dfrac{e^{ct}-1}{c}\right)^2\Delta_{\mathbf{X}}\tilde{f} - 2\dfrac{(e^{ct}-1)e^{ct}}{c}(\nabla_{\mathbf{X}}\cdot\nabla_{\mathbf{V}})\tilde{f} + e^{2ct}\Delta_{\mathbf{V}}\tilde{f}\right] = 0 \\ \tilde{f}(\mathbf{x},\mathbf{v},0) = f_0(\mathbf{x},\mathbf{v}) \end{cases} \tag{C.3}$$

Recall the lemma (see [3]):

Lemma C.2 *Let $\Phi(t)$ and $\Psi(t)$ be two arbitrary functions of time. Consider the system*

$$\begin{cases} \partial_t G = \Phi^2(t)\ \Delta_{\mathbf{X}}G + 2\Phi(t)\Psi(t)\ (\nabla_{\mathbf{X}}\cdot\nabla_{\mathbf{V}})G + \Psi^2(t)\ \Delta_{\mathbf{V}}G \\ G(\mathbf{x},\mathbf{v},t=0) = \delta_{(\mathbf{X})}\otimes\delta_{(\mathbf{V})} \end{cases} \tag{C.4}$$

where δ denotes the delta function. The function G writes

$$G(\mathbf{x},\mathbf{v},t) = \frac{1}{(2\Pi)^3\Delta^{\frac{3}{2}}}exp\left[-\left(a(t)\mathbf{x}^2 + 2h(t)\mathbf{x}\cdot\mathbf{v} + b(t)\mathbf{v}^2\right)/2\Delta(t)\right] \tag{C.5}$$

where

$$\begin{cases} a(t) = 2\displaystyle\int_0^t \Psi(s)^2 ds & , \quad b(t) = 2\displaystyle\int_0^t \Phi(s)^2 ds \\ h(t) = -2\displaystyle\int_0^t \Phi(s)\Psi(s) ds & , \quad \Delta(t) = a(t)b(t) - h^2(t) \end{cases}$$

Proof. The proof is given in Chapter II of the article of S.Chandrasekhar (see [3]). $\square$

Let us denote by G the kernel defined in (C.5) with $\Phi(t) = -\sqrt{D}\dfrac{e^{ct}-1}{c}$ and $\Psi(t) = \sqrt{D}e^{ct}$. The solution $\tilde{f}$ of (C.3) writes $\tilde{f} = G \star f_0$, which concludes the proof of Theorem **C.1**. $\square$

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