Study of a massless electrons approximation model issued from plasma physics and quasineutrality

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Abstract - In this paper, we study the existence and uniqueness in $\mathbb{R}^N$ of Poisson's equation coupled with Boltzmann's relation which gives a particular expression of the electron's density. We consider also the case where the ionic density vanishes. At last, an asymptotic analysis yields a proof of the quasineutrality hypothesis of this model where the Debye length can be neglected in comparison with the size of the system.

Résumé - Cet article étudie l'existence et l'unicité de la solution de l'équation de Poisson dans $\mathbb{R}^N$ couplée à la relation de Boltzmann qui donne une expression particulière de la densité électronique. Le cas d'une densité ionique nulle est examiné. Enfin, une analyse asymptotique permet de prouver la validité de l'hypothèse de quasineutralité de ce modèle où la longueur de Debye est négligeable devant la taille du système.

1 Introduction

We consider a N-dimensional plasma consisting of an electron fluid on a background of fixed ions. We suppose that the electrons are in an isothermal evolution with electronic temperature $T_e \neq 0$, so that the momentum conservation law is given by

$$\partial_t(n_e m_e u_e) + div(n_e m_e u_e \otimes u_e) + \nabla P_e = -e n_e E, \quad (1)$$

and

$$E = -\nabla\varphi \qquad with \qquad -\Delta\varphi = \frac{e}{\epsilon_0}(n_i - n_e), \tag{2}$$

where n_e, m_e, u_e, p_e are the density, mass, velocity and pressure of the electron fluid. The pressure is given by $p_e = n_e K T_e$. The electric field E derives from the potentiel φ and n_i is the fixed ions density. We shall consider the massless electrons approximation, because the electron mass is so small, the momentum equation (1) can be approximated by

$$\frac{\nabla n_e}{n_e} = \frac{e}{KT_e}\nabla\varphi,$$

using the fact that T_e is constant, we obtain the Boltzmann relation

$$n_e = n_0 \; exp(\frac{e\varphi}{KT_e}). \tag{3}$$

Thus the system reduces to

$$-\Delta\varphi + \frac{n_0 e}{\epsilon_0} exp(\frac{e\varphi}{KT_e}) = \frac{e}{\epsilon_0} n_i.$$

We now introduce the following dimensionless variables. Choosing characteristic length L, density n_0, and letting $\psi = \frac{e\varphi}{KT_e}$, $n_i = n_0 n$, we get

$$-\frac{1}{L^2}\Delta\psi + \frac{n_0 e^2}{KT_e\epsilon_0} exp(\psi) = \frac{n_0 e^2}{KT_e\epsilon_0} n,$$

where the Debye length appears, defined by

$$\lambda_D^2 = \frac{\epsilon_0 K T_e}{n_0 e^2}.$$

The equation hence becomes

$$-\varepsilon^2\Delta\psi + \exp(\psi) = n \qquad in \qquad \mathbb{R}^N. \tag{4}$$

where $\varepsilon = \frac{\lambda_D}{L}$ is a small parameter.
The quasineutral Euler model is obtained by taking the limit ε goes to zero so that at least formally $n_e = n_i$. For more physical explanations, one can read [KT] or [C].

In this paper we shall first study the problem

$$-\Delta\varphi + \exp(\varphi) = n \qquad \mathbb{R}^N$$

$$\lim_{|x|\to+\infty} \varphi \quad = \quad 0,$$

and then add the small parameter ε and show that when ε goes to zero, the solution φ_ε of

$$-\varepsilon^2\Delta\varphi_\varepsilon + \exp(\varphi_\varepsilon) = n \qquad \mathbb{R}^N$$

$$\lim_{|x|\to+\infty} \varphi_\varepsilon \quad = \quad 0,$$

goes to $\log(n)$, that is

$$\lim_{\lambda_D\to 0} n_e \;=\; n_i$$

in $L^2(\mathbb{R}^N)$.

2 Existence and uniqueness of the solution of the equation $-\Delta\varphi + \exp(\varphi) = n \quad in \quad \mathbb{R}^N$

We want to solve the problem

$$(P)\begin{cases} -\Delta\varphi + \exp(\varphi) = n \quad in & \mathbb{R}^N \\ \lim_{|x|\to+\infty}\varphi \quad = 0, & \end{cases}$$

with $n \in L^\infty(\mathbb{R}^N)$ and $\exists\, n_1, n_0 > 0$ such that $0 \;<\; n_1 \;\leq n \leq n_0$ a.e.

Theorem 1 *If n belongs to $L^\infty(\mathbb{R}^N)$ and is such that $0 < n_1 \leq n \leq n_0$ a.e., then there exists a function φ in $W^{1,\infty}(\mathbb{R}^N)$ such that*

$$-\Delta\varphi + e^\varphi = n \quad a.e.\ in\ \mathbb{R}^N.$$

Theorem 2 *If $n-1$ belongs to $L^\infty(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$ and is such that $0 < n_1 \leq n \leq n_0$ a.e. then there exists a unique solution φ in $H^1(\mathbb{R}^N)$ of the problem (P). Moreover φ belongs to $W^{1,p}(\mathbb{R}^N)$ for all p such that $2 \leq p \leq +\infty$ and is also in $H^2(\mathbb{R}^N)$.*

In order to prove these theorems, we shall first study the solutions of

$$(PL)\begin{cases} -\Delta\varphi_L + \exp(\varphi_L) = n & in \quad B_L \\ \varphi_L = 0 & in \quad \partial B_L, \end{cases}$$

where B_L is the ball of radius $L > 0$ centered at the origin in $\mathbb{R}^N$.

Proposition 1 *There exists $\varphi_L \in H_0^1(B_L)$ solution of (PL) and*

$$\forall v \in H_0^1(B_L) \qquad \int_{B_L} \nabla\varphi_L \nabla v + v(\exp(\varphi_L) - 1) = \int_{B_L} (n-1)v. \tag{5}$$

Proof : It is a direct application of the lower- and upper-solutions theory, see [GT] for example.

Proposition 2 *The function φ_L is the unique solution of (PL).*

Proof : Let φ_1 and φ_2 be two solutions of (PL) then (5) yields

$$\forall v \in H_0^1(B_L) \qquad \int_{B_L} (\nabla\varphi_1 - \nabla\varphi_2)\nabla v + v(\exp(\varphi_1) - \exp(\varphi_2)) = 0.$$

Let $v = \varphi_1 - \varphi_2$ then

$$\int_{B_L} |\nabla(\varphi_1 - \varphi_2)|^2 = -\int_{B_L} (\exp(\varphi_1) - \exp(\varphi_2))(\varphi_1 - \varphi_2).$$

By monotony of the exponential function, we have

$$\int_{B_L} (\exp(\varphi_1) - \exp(\varphi_2))(\varphi_1 - \varphi_2) \ \geq \ 0 \qquad \Longrightarrow \|\nabla(\varphi_1 - \varphi_2)\|_{L^2} = 0,$$

so that $\varphi_1 = \varphi_2$ a.e.

Let $\varphi^+ = \max(\varphi, 0)$ and $\varphi^- = -\min(\varphi, 0)$ so that we may write $\varphi = \varphi^+ - \varphi^-$.

Theorem 3 *The function φ_L belongs to L^∞ and for all L*

$$min(0, \log(n_1)) \leq \varphi_L \leq max(0, \log(n_0)) \quad a.e.$$

Let x belong to $\mathbb{R}^N$ and let $B(x,r)$ be the ball centered in x of radius r, in $\mathbb{R}^N$. There exist C and L_0 such that for all $L \geq L_0$

$$\|\nabla\varphi_L\|_{L^\infty(B(x,r))} \leq C,$$

where the constant $C = C(N, n_0, n_1, r)$ does not depend on x.

Proof : Let $K_\alpha = \log(\alpha n_1)$ with $0 < \alpha < \min(1, \frac{1}{n_1})$, then $K_\alpha < 0$ and $v = (\varphi - K_\alpha)^- \in H_0^1(B_L)$ so using (5) we get

$$-\int_{B_L} |\nabla(\varphi_L - K_\alpha)^-|^2 = \int_{B_L} (n - \exp(\varphi_L))(\varphi_L - K_\alpha)^- \leq 0.$$

Let $A = \{x,\ \varphi_L < K_\alpha\}$ then since $e^{\varphi_L} \leq \alpha n_1$, we get

$$(1-\alpha)n_1 \int_A (\varphi_L - K_\alpha)^- \leq 0$$

since $\alpha < 1$, $\varphi_L \geq K_\alpha$ a.e. for all α such that $\alpha < \min(1, \frac{1}{n_1})$. If $n_1 > 1$, when $\alpha \to \frac{1}{n_1}$, we see that $\varphi_L \geq 0$ a.e., if $n_1 \leq 1$, when $\alpha \to 1$, we get $\varphi_L \geq \log(n_1)$ a.e., thus $\varphi_L \geq \min(0, \log(n_1))$ a.e.

Let now $K_\alpha = \log(\alpha n_0)$ with $\alpha > \max(1, \frac{1}{n_0})$ then $K_\alpha > 0$ and $v = (\varphi - K_\alpha)^+ \in H_0^1(B_L)$ so using (5) we get

$$\int_{B_L} |\nabla(\varphi_L - K_\alpha)^+|^2 = \int_{B_L} (n - \exp(\varphi_L))(\varphi_L - K_\alpha)^+ \geq 0.$$

Let $A = \{x,\ \varphi_L > K_\alpha\}$ then using $e^{\varphi_L} \geq \alpha n_0$, we get

$$0 \leq (1-\alpha)n_0 \int_A (\varphi_L - K_\alpha)^+ \leq 0,$$

thus necessarily $\varphi_L \leq \log(\alpha n_0)$ a.e. If $n_0 > 1$, when $\alpha \to 1$, we see that $\varphi_L \leq \log(n_0)$ a.e., if $n_0 \leq 1$, when $\alpha \to \frac{1}{n_0}$, we get $\varphi_L \leq 0$ a.e., thus $\varphi_L \leq \max(0, \log(n_0))$ a.e.
We shall use the following lemma.

Lemma 1 *Let Ω be a ball in $\mathbb{R}^N$ and let $u \in H^1(\Omega)$ be a solution of $\Delta u = f$, in Ω. Then*

$$\sup_\Omega d_x |\nabla u(x)| \leq C'(\sup_\Omega |u| + \sup_\Omega {d_x}^2 |f(x)|),$$

where $d_x = d(x, \partial\Omega)$ and C' is a constant depending only on the dimension N.

The proof of Lemma 1 can be found in [GT] for example.

By the above estimates, we know that there exist two constants A and B such that for all L $\|\varphi_L\|_{L^\infty(B_L)} \leq A$ and $\|\exp(\varphi_L) - n\|_{L^\infty(B_L)} \leq B$, where

A and B depend only on n_0 and n_1. Let $\Omega = B(x, r+1)$ and L_0 be such that for all $L \geq L_0$ the ball B(x,r+1) is included in B_L. If $y \in B(x,r)$ then $d_y = d(y, \partial\Omega)$ satisfies $1 \leq d_y \leq r+1$. By Lemma 1 we obtain

$$\|\nabla \varphi_L\|_{L^\infty(B(x,r))} \leq C'(A + (r+1)^2 B).$$

The constant C is equal to C'(A+(r+1)^{2}B) and does not depend on x and L.

Remark 1 *In particular, we have proved that if $x \in \mathbb{R}^N$, there exists L_0 such that $\forall L \geq L_0$,*

$$\|\nabla \varphi_L\|_{L^\infty(B(x,1))} \leq C,$$

where $C = C(N,n_0,n_1)$ does not depend on x and L.

Without loss of generality, we may suppose that $n_0 \geq 1$ and write

$$\varphi_L \leq \log(n_0) \ a.e.$$

Proposition 3 *The following estimates hold*

$$\begin{array}{ll} \|\varphi_L\|_{L^2(B_L)} \leq C\|n-1\|_{L^2(B_L)}, & \\ \|\nabla\varphi_L\|_{L^2(B_L)} \leq \sqrt{C}\,\|n-1\|_{L^2(B_L)}, & \end{array} \qquad \forall L > 0,$$

where the constant C can be taken equal to 2 if $n_1 \geq 1$ and $3+2\frac{n_0}{n_1}$ if $n_1 < 1$.

Proof : In (5), we choose $v = {\varphi_L}^+$ so that we obtain

$$\int_{B_L} |\nabla\varphi_L^+|^2 + \int_{B_L} \varphi_L^+(\exp(\varphi_L) - 1) = \int_{B_L} (n \ - \ 1)\varphi_L^+.$$

Since $\varphi_L^+(e^{\varphi_L} - 1) = \varphi_L^+(e^{\varphi_L^+} - 1)$, after studying the function $x \to x(e^x - 1) - \frac{x^2}{2}$, one can easily show that there exists $\alpha < -1$ such that

$$x \geq \alpha \quad \Rightarrow x(e^x - 1) - \frac{x^2}{2} \geq 0.$$

And $\varphi_L^+ \geq 0$ thus

$$\frac{{\varphi_L^+}^2}{2} \leq \varphi_L^+(e^{\varphi_L^+} - 1),$$

then we get

$$\|\varphi_L^+\|_{L^2(B_L)} \leq 2\,\|n-1\|_{L^2(B_L)}.$$

Now, if we choose $v = \varphi_L$ in (5), we have

$$\int_{B_L} |\nabla \varphi_L|^2 + \int_{B_L} \varphi_L(\exp(\varphi_L) - 1) = \int_{B_L} (n \ - \ 1)\varphi_L$$

and $\forall x, \quad x(e^x - 1) \geq 0$ thus

$$\|\nabla \varphi_L\|^2_{L^2(B_L)} \leq \|\varphi_L\|_{L^2(B_L)} \|n - 1\|_{L^2(B_L)}.$$

If, moreover $n_1 \geq 1$ then Theorem 3 shows that $\varphi_L \geq 0$ a.e. thus $\varphi_L^+ = \varphi_L$ and the above estimates yield

$$\|\nabla \varphi_L\|_{L^2(B_L)} \leq \sqrt{2}\, \|n - 1\|_{L^2(B_L)}.$$

If $n_1 < 1$, we need to estimate the L^2- norm of φ_L^-. By convexity of the exponential function, we have

$$\exp(\varphi_2)(\varphi_1 - \varphi_2) \ \leq \exp(\varphi_1) - \exp(\varphi_2) \ \leq \ \exp(\varphi_1)(\varphi_1 - \varphi_2). \qquad (6)$$

Thus, we obtain

$$\forall \varphi_L \quad e^{\varphi_L} - 1 \leq \varphi_L e^{\varphi_L} = \varphi_L^+ e^{\varphi_L} - \varphi_L^- e^{\varphi_L}$$

thus

$$\varphi_L^- e^{\varphi_L} \leq \varphi_L^+ e^{\varphi_L} + 1 - e^{\varphi_L}.$$

If $n_1 < 1$ then $\log(n_1) \leq \varphi_L \leq \log(n_0)$ and $n_1 \varphi_L^- \leq n_0 \varphi_L^+ + 1 - e^{\varphi_L}$. We have seen that $\varphi_L^+ \in L^2(B_L)$. We still have to prove that $e^{\varphi_L} - 1 \in L^2(B_L)$. We know that $\varphi_L \in H_0^1(B_L) \cap L^\infty(B_L)$ thus $v = e^{\varphi_L} - 1 \in H_0^1(B_L)$. From (5), we get

$$\int_{B_L} |\nabla \varphi_L|^2 e^{\varphi_L} + \int_{B_L} (\exp(\varphi_L) - 1)^2 = \int_{B_L} (n \ - \ 1)(\exp(\varphi_L) - 1),$$

then,

$$\| \exp(\varphi_L) - 1\|_{L^2(B_L)} \leq \|n - 1\|_{L^2(B_L)}.$$

Then the following estimate holds

$$\|\varphi_L^-\|_{L^2(B_L)} \leq (2\frac{n_0}{n_1} + 1)\, \|n - 1\|_{L^2(B_L)}.$$

Thus φ_L satisfies the required estimates.

Proof of Theorem 1 : Let us define

$$\tilde{\varphi}_L = \begin{cases} \varphi_L & in \quad B_L \\ 0 & in \quad \mathbb{R}^N \backslash B_L. \end{cases}$$

As $\varphi_L \in H_0^1(B_L)$, $\tilde{\varphi}_L \in H^1(\mathbb{R}^N)$ and thanks to the properties of φ_L

$$\exists C_1, C_0 \quad such\ that\ \forall L, \quad C_1 \leq \tilde{\varphi}_L \leq C_0 \quad a.e.,$$

where $C_1 = \min(0, \log(n_1))$ and $C_0 = \max(0, \log(n_0))$. Thus, after extracting subsequences, one can also note that

$$\tilde{\varphi}_L \rightharpoonup \varphi \quad in \quad L^\infty(\mathbb{R}^N) \text{ weak-}\star \text{ and in } \mathcal{D}'(\mathbb{R}^N).$$

Furthermore $K = \{\varphi \in L^\infty\ such\ that \quad C_1 \leq \varphi \leq C_0\}$ being a compact set of L^∞weak-$\star$, the limit φ belongs to K. The convergence in $\mathcal{D}'(\mathbb{R}^N)$ implies that $\Delta\tilde{\varphi}_L \rightarrow \Delta\varphi$ in $\mathcal{D}'(\mathbb{R}^N)$. Let K be a compact subset of $\mathbb{R}^N$, then by Theorem 3 there exists L_0 such that for all $L \geq L_0$

$$\|\tilde{\varphi}_L\|_{H^1(K)} \leq C,$$

where the constant C does not depend on L. Thus after extracting subsequences, we get

$$\begin{cases} \tilde{\varphi}_L \rightarrow \varphi & in\ L^2(K)\ strongly, \\ \tilde{\varphi}_L \rightarrow \varphi & a.e.\ in\ K. \end{cases}$$

So that $\exp(\tilde{\varphi}_L) \rightarrow \exp(\varphi)$ a.e. in K and $|\exp(\tilde{\varphi}_L)| \leq n_0 \in \ L^2(K)$. By the Lebesgue dominated convergence theorem, one can see that $\exp(\tilde{\varphi}_L) \rightarrow \exp(\varphi)$ in $L^2(K)$ and this is true for all compact subset of $\mathbb{R}^N$ so that

$$\forall\psi \in \mathcal{D}(\mathbb{R}^N), < -\Delta\tilde{\varphi}_L + \exp(\tilde{\varphi}_L), \psi > \rightarrow < -\Delta\varphi + \exp(\varphi), \psi > .$$

Now for a fixed $\psi \in \mathcal{D}(\mathbb{R}^N)$, $\exists L_0$ such that $supp\psi \subset\subset B_{L_0}$, so we may write

$$< \Delta\tilde{\varphi}_L, \psi > = -\int_{B_L} \nabla\varphi_L \nabla\psi = \int_{B_L} \Delta\varphi_L \psi - \int_{\partial B_L} \frac{\partial\varphi_L}{\partial n} \psi d\sigma.$$

Since the integral on the boundary vanishes $\forall L \geq L_0$ thus

$$< \Delta\tilde{\varphi}_L, \psi > = \int_{B_L} \Delta\varphi_L \psi \quad \forall L \geq L_0.$$

As $\exp(\tilde{\varphi}_L) \in L^1_{loc}(\mathbb{R}^N)$, we obtain

$$< -\Delta\tilde{\varphi}_L + \exp(\tilde{\varphi}_L), \psi >= \int_{B_L} (-\Delta\varphi_L + \exp(\varphi_L))\psi = \int_{B_L} n\psi =< n, \psi >,$$

$\forall L \geq L_0$, so that $-\Delta\varphi + e^{\varphi} = n$ in $\mathcal{D}'(\mathbb{R}^N)$ and a.e.
Let x be fixed in $\mathbb{R}^N$, using Remark 1 and the convergence of $\tilde{\varphi}_L$ to φ in $\mathcal{D}'(\mathbb{R}^N)$, yield the convergence of $\nabla\varphi_L$ to $\nabla\varphi$ in $L^\infty(B(x,1))$weak-$\star$. We may then assert that

$$\|\nabla\varphi\|_{L^\infty(B(x,1))} \leq \liminf \|\nabla\varphi_L\|_{L^\infty(B(x,1))} \leq C,$$

where the constant C = C(N,n_0,n_1) does not depend on x. Now this inequality is satisfied for all x in $\mathbb{R}^N$, so that φ belongs to $W^{1,\infty}(\mathbb{R}^N)$.

Proof of Theorem 2 : By Proposition 3, there exists a constant C such that the functions $\tilde{\varphi}_L$ satisfie

$$\|\tilde{\varphi}_L\|_{H^1(\mathbb{R}^N)} \leq C\|n-1\|_{L^2(\mathbb{R}^N)},$$

for all L. Thus, after extracting a subsequence, one can see that $\tilde{\varphi}_L$ converges weakly in $H^1(\mathbb{R}^N)$. As we know yet that $\tilde{\varphi}_L$ converges to φ in $\mathcal{D}'(\mathbb{R}^N)$, we may ensure that φ belongs to $H^1(\mathbb{R}^N)$ and that

$$\tilde{\varphi}_L \rightharpoonup \varphi \quad in \quad H^1(\mathbb{R}^N) \text{ weak.}$$

As $\varphi \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, the function $\exp(\varphi) - 1$ is in $L^2(\mathbb{R}^N)$ and $\Delta\varphi = (\exp(\varphi) - 1) - (n-1)$ belongs to $L^2(\mathbb{R}^N)$, so that φ is in $H^2(\mathbb{R}^N)$.
If φ and ψ are two solutions in $H^1(\mathbb{R}^N)$ of $-\Delta f + \exp(f) = n$ in $\mathbb{R}^N$, then we may write

$$\int_{\mathbb{R}^N} (\nabla\varphi - \nabla\psi)\nabla v + \int_{\mathbb{R}^N} (\exp(\varphi) - \exp(\psi))v = 0, \quad \forall v \in H^1(\mathbb{R}^N).$$

Take v = $\varphi - \psi$ then we obtain

$$\int_{\mathbb{R}^N} |\nabla(\varphi - \psi)|^2 = -\int_{\mathbb{R}^N} (\exp(\varphi) - \exp(\psi))(\varphi - \psi).$$

By monotony of the exponential function, we have $(\exp(\varphi) - \exp(\psi))(\varphi - \psi) \geq 0$ for all φ and ψ, so that $\varphi - \psi$ is a constant a.e. in $\mathbb{R}^N$. Then the equation satisfied by $\varphi - \psi$ becomes $\exp(\varphi) - \exp(\psi) = 0$ a.e. and the

monotony of the exponential function yields $\varphi = \psi$ a.e.
Now we have to see that φ goes to zero when $|x| \to +\infty$. As φ belongs to $H^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$, φ is in all the spaces $W^{1,p}(\mathbb{R}^N)$ for $2 \leq p \leq +\infty$. If $p > N$, we know that the elements of $W^{1,p}(\mathbb{R}^N)$ go to zero when $|x|$ goes to $+\infty$. So that the function φ satisfies the boundary conditions of the problem (P).

Remark 2 *The fact that $n_1 > 0$ is very important to get uniform estimates on φ_L almost for the ${\varphi_L}^-$'s part. So it is natural to ask "what happens if n = 0"?. Of course, in that case, n-1 does not belong to $L^2(\mathbb{R}^N)$ but is still in $L^\infty(\mathbb{R}^N)$ and we shall see that there is no solution of the problem (P) in $H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.*

3 The case where ions density vanishes

We consider the problem

$$(\star) \begin{cases} -\Delta\varphi_L + e^{\varphi_L} = 0 \quad a.e.\ in\ B_L, \\ \varphi_L = 0\ in\ \partial B_L. \end{cases}$$

Proposition 4 *There exists a unique $\varphi_L \in H_0^1(B_L)$ solution of ($\star$) and φ_L satisfies*

$$\int_{B_L} \nabla\varphi_L \nabla v + \int_{B_L} \exp(\varphi_L) v = 0 \quad \forall v \in H_0^1(B_L). \tag{7}$$

Proof : It is exactly the same proof as in Section 2.

Proposition 5 *The function φ_L solution of ($\star$) is negative, belongs to $\mathcal{C}^2(\overline{B_L})$ and is radially symmetric.*

Proof : As $\varphi_L^+ \in H_0^1(B_L)$, using (7) we get $\nabla\varphi_L^+ = 0$ a.e. thus $\varphi_L \leq 0$. As $0 \leq \exp(\varphi_L) \leq 1$, $\exp(\varphi_L) \in L^\infty(B_L)$ and the exponential function being $\mathcal{C}^\infty$, the solution φ_L belongs to $\mathcal{C}^2(\overline{B_L})$ (see [GT] for example). Now, let v $= -\varphi_L$ then v is a positive solution of $\Delta v + f(v) = 0$, $v = 0$ on ∂B_L with $f \in \mathcal{C}^1$, we may then use the results of [GNN] to assert that φ_L is radially symmetric.

Let $u_L(r) = \varphi_L(x_1, ..., x_N)$ where r is the Euclidean norm of the point $(x_1, ..., x_N)$ of $\mathbb{R}^N$. The above properties of φ_L show that u_L is a solution in $\mathcal{C}^2([0,L])$ of the following equation

$$-\,(u_L{}'' + \frac{N-1}{r}u_L{}') + e^{u_L} = 0, \tag{8}$$

with $u_L(L) = 0$ and $u_L{}'(0) = 0$ if $N > 1$. If $N = 1$, φ_L is solution of

$$-\varphi_L{}'' + e^{\varphi_L} = 0,$$

with $\varphi_L(-L) = \varphi_L(L) = 0$.

Proposition 6 *The function u_L increases monotically on [0,L] and has an absolute minimum on $r = 0$.*

Proof : If $N = 1$, let $\psi = \varphi_L(-x)$ then ψ satisfies the problem $(\star)$ and by the uniqueness of the solution $\psi = \varphi_L$ and φ_L is an even function. Thus $\varphi_L'(0) = 0$. As $\varphi_L'' \geq 0$, 0 is the only extremum and actually a minimum. If $N > 1$, then u_L is solution of

$$-(r^{N-1}u_L{}'' + (N-1)r^{N-2}u_L{}') + r^{N-1}e^{u_L} = 0,$$

that we can write

$$(r^{N-1}u_L{}')' = r^{N-1}e^{u_L}.$$

We integrate it between 0 and $x > 0$ to get

$$x^{N-1}u_L{}'(x) = \int_0^x r^{N-1}e^{u_L}dr.$$

Thus $u_L{}'(x) > 0$ on]0,L] and $u_L{}'(0) = 0$ by continuity so that 0 is an absolute minimum. Let $\alpha_L = u_L(0)$ then the following results hold.

Theorem 4 *There exists L_0 such that for all $L \geq L_0$,*

$$\alpha_L \leq -\frac{1}{2}\log(\frac{L^2-1}{4(1+(L-1)(N-1))}).$$

So that the sequence α_L satisfies

$$\lim_{L\to+\infty}\alpha_L = -\infty \textit{ that is } \lim_{L\to+\infty}\varphi_L(0) = -\infty.$$

Proof : Equation (8) implies that

$$r{u_L}'' + (N-1){u_L}' \geq re^{\alpha_L}. \tag{9}$$

Let $A \leq L$, then $\int_0^A r{u_L}'' = (A-1){u_L}'(A)$. After integrating (9) between 0 and A, we obtain

$$(A-1){u_L}'(A) + (N-1)(u_L(A) - u_L(0)) \geq \frac{A^2}{2}e^{\alpha_L}.$$

So that

$$(A-1){u_L}'(A) \geq \frac{A^2}{2}e^{\alpha_L} + (N-1)\alpha_L.$$

If $A \geq 1$, then we get

$${u_L}'(A) \geq \frac{A}{2}e^{\alpha_L} + (N-1)\alpha_L. \tag{10}$$

Now we integrate (10) between 1 and L and we obtain

$$-u_L(1) \geq \frac{(L^2-1)}{4}e^{\alpha_L} + (N-1)(L-1)\alpha_L,$$

and $u_L(1) \geq \alpha_L$, so that $-\alpha_L \geq \lambda e^{\alpha_L}$ where

$$\lambda = \frac{L^2-1}{4(1+(L-1)(N-1))}.$$

Thus there exists $L_0 > 1$ such that for all $L \geq L_0$, $\lambda > 1$. We may then assert that $-\alpha_L \geq \frac{1}{2}\log\lambda$ and thus

$$u_L(0) \leq -\frac{1}{2}\log\lambda \longrightarrow -\infty \; when \; L \to +\infty.$$

In fact, if $-\alpha_L < \frac{1}{2}\log\lambda$, then

$$-\alpha_L \geq \lambda e^{\alpha_L} \geq \sqrt{\lambda} \geq \frac{1}{2}\log\lambda > -\alpha_L,$$

which is impossible. In the case of N = 1, ${u_L}'' = e^{u_L} \geq e^{\alpha_L}$, and ${u_L}'(A) \geq Ae^{\alpha_L}$, give the same conclusion. We have thus proved that

$$\lim_{L\to+\infty} \varphi_L(0) = -\infty.$$

Proposition 7 *The equation* $-\Delta\varphi+\exp(\varphi)=0$ *has no solution in* $H^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$.

Proof : Let us suppose that there exists φ in $H^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ such that $-\Delta\varphi+\exp(\varphi)=0$. Thus for all $v\in\mathcal{D}(\mathbb{R}^N)$ such that $v\geq 0$, the following inequality holds

$$-<\Delta\varphi, v>+<\exp(\varphi)-1, v>\leq 0.$$

Since $\varphi\in H^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, $\exp(\varphi)-1\in L^2(\mathbb{R}^N)$ and we may write

$$\int_{\mathbb{R}^N}\nabla\varphi\nabla v+\int_{\mathbb{R}^N}(\exp(\varphi)-1)v\leq 0,$$

for all $v\in H^1(\mathbb{R}^N)$ with $v\geq 0$ a.e. Let $v=\varphi^+$, since $x(e^x-1)\geq\frac{x^2}{2}$ for all $x\geq 0$, we obtain $\|\varphi^+\|_{H^1(\mathbb{R}^N)}\leq 0$, that is $\varphi^+=0$ a.e. So that φ is a lower solution of the following problem

$$(\star)\left\{\begin{array}{l}-\Delta\varphi_L+e^{\varphi_L}=0\quad a.e.\ in\ B_L,\\ \varphi_L=0\ in\ \partial B_L.\end{array}\right.$$

Since $\exp(\varphi)\in L^2(B_L)$, for all $v\in H_0^1(B_L)$ we have

$$\int_{B_L}\nabla(\varphi-\varphi_L)\nabla v+\int_{B_L}(\exp(\varphi)-\exp(\varphi_L))v=0.$$

Let $v=(\varphi-\varphi_L)^+$ then $v\in H_0^1(B_L)$ and $(\exp(\varphi)-\exp(\varphi_L))(\varphi-\varphi_L)^+\geq 0$. So that $(\varphi-\varphi_L)^+=0$ a.e. and for all L, $\varphi\leq\varphi_L$ a.e. in B_L. Now,

$$\|\varphi\|_{L^\infty(\mathbb{R}^N)}\geq\|\varphi\|_{L^\infty(B_L)}\geq\|\varphi_L\|_{L^\infty(B_L)}=|\varphi_L(0)|\longrightarrow+\infty$$

when L goes to $+\infty$, hence φ is not in $L^\infty(\mathbb{R}^N)$. This contradicts our hypothesis.

4 Asymptotic analysis : validity of quasi-neutrality

Let $\varepsilon\in[0,1]$, we consider the problem

$$(P_\varepsilon)\quad\left\{\begin{array}{ll}-\varepsilon^2\Delta\varphi_\varepsilon+\exp(\varphi_\varepsilon)=n & in\ \mathbb{R}^N,\ \ N\leq 3\\ \lim_{|x|\to+\infty}\varphi_\varepsilon=0, & \end{array}\right.$$

with n $\in L^\infty(\mathbb{R}^N),\ 0 < n_1 \leq n \leq n_0,\ n-1 \in L^1(\mathbb{R}^N)$ or $L^2(\mathbb{R}^N)$.

By Sections 2, we already know that (P_ε) has a unique solution $\varphi_\varepsilon \in L^\infty \cap H^1(\mathbb{R}^N)$. Furthermore,

$$\exists C_1, C_0,\ \forall\varepsilon,\ C_1 \leq \varphi_\varepsilon \leq C_0\ a.e.,$$

$$\exists C_3,\ \forall\varepsilon,\ \|\varphi_\varepsilon\|_{L^2(\mathbb{R}^N)} \leq C_3,$$

$$and\ \varphi_\varepsilon \in H^2(\mathbb{R}^N).$$

F.Golse and R.Sentis have proved in [GS] that if Ω is a bounded domain of $\mathbb{R}^3$ with specific geometric conditions, $\|\varphi_\varepsilon - \log n\|_{L^2(\Omega)} = O(\sqrt{\varepsilon})$. We shall prove here that φ_ε converges to $\log n$ in $L^2(\mathbb{R}^N)$ and more precisely show that $\|\varphi_\varepsilon - \log n\|_{L^2(\mathbb{R}^N)} = O(\varepsilon)$.

Theorem 5 *Let g = log(n), if g belongs to $H^1(\mathbb{R}^N)$, then $\varphi_\varepsilon \longrightarrow \log(n)$ in $L^2(\mathbb{R}^N)$ strongly, when ε goes to 0. More precisely ,*

$$\|\varphi_\varepsilon - \log n\|_{L^2(\mathbb{R}^N)} = O(\varepsilon).$$

Proof : In a natural way, let $\psi_\varepsilon = \varphi_\varepsilon - \log(n)$ then ψ_ε is solution of

$$-\varepsilon^2\Delta\psi_\varepsilon + n(\exp(\psi_\varepsilon) - 1) = \varepsilon^2\Delta g \quad in\ \mathcal{D}'(\mathbb{R}^N).$$

And $\psi_\varepsilon \in H^1(\mathbb{R}^N)$, thus $\forall\varphi \in \mathcal{D}(\mathbb{R}^N)$,

$$\varepsilon^2 < \nabla\psi_\varepsilon, \nabla\varphi > + < n(\exp(\psi_\varepsilon) - 1), \varphi >= -\varepsilon^2 < \nabla g, \nabla\varphi >,$$

where $\nabla\psi_\varepsilon$, $n(\exp(\psi_\varepsilon) - 1)$, $\nabla g \in L^2(\mathbb{R}^N)$, then by density, we get

$$\varepsilon^2 \int_{\mathbb{R}^N} \nabla\psi_\varepsilon \nabla v + \int_{\mathbb{R}^N} n(\exp(\psi_\varepsilon) - 1)v = -\varepsilon^2 \int_{\mathbb{R}^N} \nabla g \nabla v, \qquad (11)$$

for all v $\in H^1(\mathbb{R}^N)$. Let $v = \psi_\varepsilon^+$, then using (11)

$$\begin{aligned} \varepsilon^2 \textstyle\int_{\mathbb{R}^N} |\nabla\psi_\varepsilon^+|^2 + \int_{\mathbb{R}^N} n(\exp(\psi_\varepsilon) - 1)\psi_\varepsilon^+ &= -\varepsilon^2 \textstyle\int_{\mathbb{R}^N} \nabla g \nabla \psi_\varepsilon^+, \\ &\leq \varepsilon^2 (\|\nabla\psi_\varepsilon^+\|_{L^2}^2 + \tfrac{1}{4}\|\nabla g\|_{L^2}^2). \end{aligned}$$

So that,

$$\int_{\mathbb{R}^N} n(\exp(\psi_\varepsilon) - 1)\psi_\varepsilon^+ \leq \frac{\varepsilon^2}{4}\|\nabla g\|_{L^2}^2.$$

Since

$$\frac{\psi_\varepsilon^{+2}}{2} \leq (\exp(\psi_\varepsilon) - 1)\psi_\varepsilon^+,$$

see Section 2 for example, thus

$$\|\psi_\varepsilon^+\|_{L^2}^2 \leq \frac{\varepsilon^2}{2n_1}\|\nabla g\|_{L^2}^2.$$

So that,

$$\psi_\varepsilon^+ \longrightarrow 0 \; in \; L^2 \; strongly.$$

On the other side, using convexity of the exponential function (6), we obtain

$$\psi_\varepsilon^- \exp(\psi_\varepsilon) \leq \psi_\varepsilon^+ \exp(\psi_\varepsilon) + 1 - \exp(\psi_\varepsilon).$$

As $\psi_\varepsilon \in L^\infty$ and is uniformly bounded with respect to ε, ∃C, D such that $\forall\varepsilon$,

$$C\psi_\varepsilon^- \leq D\psi_\varepsilon^+ + 1 - \exp(\psi_\varepsilon).$$

If we show that $\exp(\psi_\varepsilon) - 1 \in L^2$ and goes to 0 in L^2 strongly then $\psi_\varepsilon^- \longrightarrow 0$ in L^2 strongly, and ψ_ε too, so that $\varphi_\varepsilon \longrightarrow \log n$ in L^2 strongly when ε goes to 0.

Lemma 2 *The following results hold*

$$\exp(\psi_\varepsilon) - 1 \in L^2(\mathbb{R}^N) \; and \; \|\exp(\psi_\varepsilon) - 1\|_{L^2} = O(\varepsilon).$$

Proof :

$$|\exp(\psi_\varepsilon) - 1| \leq \frac{1}{n_1}|\exp(\varphi_\varepsilon) - n| \leq \frac{1}{n_1}(|\exp(\varphi_\varepsilon) - 1| + |n - 1|),$$

and each term of the right side of the above inequality belongs to L^2, thus $\exp(\psi_\varepsilon) - 1$ belongs to L^2. Moreover,

$$|\nabla(\exp(\psi_\varepsilon) - 1)| \leq D|\nabla\psi_\varepsilon| \in L^2(\mathbb{R}^N),$$

thus $\exp(\psi_\varepsilon) - 1$ belongs to $H^1(\mathbb{R}^N)$. We choose v $= \exp(\psi_\varepsilon) - 1$ in (11), then we obtain

$$\varepsilon^2\|\nabla\psi_\varepsilon\|_{L^2}^2 + \int_{\mathbb{R}^N} n(\exp(\psi_\varepsilon) - 1)^2 \leq \varepsilon^2 D\|\nabla g\|_{L^2}\|\nabla\psi_\varepsilon\|_{L^2}.$$

Thus,

$$\|\exp(\psi_\varepsilon) - 1\|_{L^2}^2 \leq \frac{\varepsilon^2}{n_1}\frac{D}{4C}\|\nabla g\|_{L^2}^2 \longrightarrow 0,$$

when ε goes to 0.

Remark 3 *If g belongs to $H^2(\mathbb{R}^N)$, then the following estimates hold*

$$\|\psi_\varepsilon^+\|_{L^2}^2 \leq \frac{4\varepsilon^4}{{n_1}^2}\|\Delta g\|_{L^2}^2,$$

$$\|\exp(\psi_\varepsilon) - 1\|_{L^2}^2 \leq \frac{\varepsilon^4}{{n_1}^2}\frac{D}{4C}\|\nabla g\|_{L^2}^2,$$

and

$$\|\varphi_\varepsilon - logn\|_{L^2(\mathbb{R}^N)} = O(\varepsilon^2).$$

Thus, with more regularity, the convergence is fastened.

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