REGULARITY OF PLATE EQUATIONS WITH CONTROL CONCENTRATED IN INTERIOR CURVES

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Abstract. We consider initial and boundary value problems modelling the vibration of a plate with piezoelectric actuator. The usual models lead to the Bernoulli-Euler and Kirchhoff plate equations with right hand side given by a distribution concentrated in an interior curve. We obtain regularity results which are stronger than those obtained by simply using the Sobolev regularity of the right hand side. By duality we obtain new trace regularity properties for the solutions of plate equation. Our results provide appropriate function spaces for the control of plates provided with piezoelectric actuators.

1. Introduction

The aim of this paper is to study regularity of solutions for plate equations with control given by the normal derivative of the Dirac mass concentrated in a curve. Our motivation comes from recent work on modelling, analysis and control of elastic structures provided with piezoelectric actuators (see [?], [?], [?], [?]). In [?] and [?] the authors give the well-posedness results implied by the Sobolev regularity of the right hand side. As in the case of wave and plate equations with point control (cf. [?], [?], [?] [?], [?]) we will prove that the regularity obtained by this method is not sharp and we will improve it. We will also show on some examples that our method yields sharp estimates. On the other hand, by a duality argument detailed in Section **??** we obtain new trace regularity properties for plate equations.

More precisely let us consider $T > 0$, $u \in L^2(0,T)$, $\Omega \subset \mathbf{R}^2$ an open bounded set, $\Gamma = \partial\Omega$, $Q = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$ and γ a curve included in Ω. We will

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study the following initial and boundary value problem

$$w''(x,y,t) - \alpha\Delta w''(x,y,t) + \Delta^2 w(x,y,t) = u(t)\frac{\partial\delta_\gamma}{\partial\nu}, \text{ in } Q, \tag{1.1}$$

$$w(x,y,t) = \Delta w(x,y,t) = 0, \text{ on } \Sigma, \tag{1.2}$$

$$w(x,y,0) = 0, \ w'(x,y,0) = 0, \text{ in } \Omega, \tag{1.3}$$

where $\frac{\partial\delta_\gamma}{\partial\nu}$ stands for the derivative of the Dirac mass concentrated in γ with respect to the normal to γ, $\alpha \geq 0$ is a constant and w', w'' denote the time derivatives of w. If $\alpha > 0$ our results also hold if the boundary is clamped, i.e. if we replace (**??**) by

$$w(x,y,t) = \frac{\partial w}{\partial\nu}(x,y,t) = 0, \text{ on } \Sigma, \tag{1.4}$$

Equation (**??**), with $\alpha = 0$, was proposed in ([**?**]), ([**?**]) as a Bernoulli-Euler model for the vibrations of a plate with piezoelectric actuator. In this model w represents the transverse deflection of the plate, the function $u : [0,T] \to \mathbf{R}$ gives the time variation of the voltage applied to the actuator and the effect of rotary inertia of the plate is neglected. If $\alpha > 0$ the rotary inertia of the plate is taken into account by means of the term $\alpha\Delta w''$ in (**??**) (the Kirchhoff model ,cf.[**?**]). In this case equation (**??**) is similar in many respects to a wave equation.

If we suppose that $u \in L^2(0,T)$, by using the Sobolev regularity of the right hand side of (**??**) and classical semigroup methods, (**??**)-(**??**) admit a unique solution satisfying (cf.[**?**], [**?**], [**?**])

$$w \in C([0,T], H^{\frac{3}{2}-\epsilon}(\Omega)) \cap C^1([0,T], H^{\frac{1}{2}-\epsilon}(\Omega)), \text{ if } \alpha > 0, \tag{1.5}$$

$$w \in C([0,T], H^{\frac{1}{2}-\epsilon}(\Omega)) \cap C^1([0,T], H^{-\frac{3}{2}-\epsilon}(\Omega)), \text{ if } \alpha = 0, \tag{1.6}$$

where $\epsilon > 0$ and $H^s(\Omega), s \in \mathbf{R}$ are the usual Sobolev spaces. For the similar one dimensional problem, in the case $\alpha = 0$ (i.e. for the Bernoulli-Euler model), it was proved in [**?**], [**?**] that the solution of (**??**)-(**??**) is by $\frac{1}{2} + \epsilon$ more regular than what (**??**) would yield. The main goal of this paper is to extend this result to the case of two space dimensions and to the Kirchhoff model (i.e. $\alpha > 0$).

Our approach is based on estimates of the Fourier transform of the distribution $\frac{\partial\delta_\gamma}{\partial\nu}$ combined with the Laplace-Fourier transform technique from [**?**], [**?**], [**?**]. For the Bernoulli-Euler and $\Omega = (0,\pi)^2$ we obtain a sharp result by using an estimate on aperiodic Fourier series proved in [**?**] and a number theoretical argument. Similar problems for the wave or plate equation with pointwise control were studied in [**?**], [**?**], [**?**], [**?**], [**?**]. The present work generalizes to the two-dimensional case the regularity results proved in [**?**], [**?**] for the one dimensional analogue of (**??**)-(**??**).

The plan of this paper is as follows. In the second section we state the main results. In the third section we recall a result from [**?**] on trigonometric sums and the relation with trace regularity properties. The fourth section contains some estimates on the Fourier transform of $\frac{\partial\delta_\gamma}{\partial\nu}$. The fifth section is devoted to the proof of the main results. In the last section we show that some of our results are sharp and we give further extensions and comments.

2. Statement of the main results

If $\alpha = 0$ solutions of equation (??) have an infinite speed of propagation, which is not the case if $\alpha > 0$, in which case (??) is similar in many respects to a wave equation. This is why, for $\alpha = 0$, our results will strongly depend on the geometry of the problem.

If $\alpha > 0$ the following result improves by $1/2 + \epsilon$ the regularity given by (??).

Theorem 2.1. Suppose that $\alpha > 0$, $u \in L^2(0,T)$, $\Omega \subset \mathbf{R}^2$ is a regular bounded set and $\gamma \subset \Omega$ is a union of a finite number of curves γ_i, $i = 1, m$, each of them satisfying one of the following conditions:

(H1) There exist $n_i \geq 1$ such that γ_i is C^{n_i+3} and the curvature of γ vanishes at most at one end of γ_i where the tangent line has contact of order n_i.

(H2) γ_i is a segment.

Then the problem (??)-(??) admits a unique solution

$$w \in C([0,T], H^2(\Omega)) \cap C^1([0,T], H^1(\Omega)). \tag{2.1}$$

Moreover there exists a constant $C(T) > 0$ such that

$$||w||_{C([0,T],H^2(\Omega))} + ||w'||_{C([0,T],H^1(\Omega))} \leq C(T)||u||_{L^2(0,T)}, \ \forall u \in L^2(0,T). \tag{2.2}$$

The same results hold for the problem (??), (??), (??).

In the case $\alpha = 0$ we are able to essentially improve (??) only if Ω is a square. For a general Ω we are able to improve (??) only by ϵ. More precisely the folllowing result holds (see [?, p.66] for the definition of $H^{\frac{1}{2}}_{00}(\Omega)$) and $H^{\frac{1}{2}}_{00}(\Omega)$)).

Theorem 2.2. Suppose that $\alpha = 0$, $u \in L^2(0,T)$, $\Omega \subset \mathbf{R}^2$ is a regular bounded set and $\gamma \subset \Omega$ satisfying the assumptions of Theorem ??. Then the problem (??)-(??) admits a unique solution

$$w \in C([0,T], H^{\frac{1}{2}}_{00}(\Omega)) \cap C^1([0,T], [H^{\frac{3}{2}}_{00}(\Omega)]'). \tag{2.3}$$

Moreover there exists a constant $C(T) > 0$ such that

$$||w||_{C([0,T],H^{\frac{1}{2}}_{00}(\Omega))} + ||w'||_{C([0,T],[H^{\frac{3}{2}}_{00}(\Omega)]')} \leq C||u||_{L^2(0,T)}, \ \forall u \in L^2(0,T). \tag{2.4}$$

The same results hold for the problem (??), (??), (??).

In the case $\alpha = 0$ and Ω a square we have the following result, which depends on the geometry of γ.

Theorem 2.3. Suppose that $\epsilon > 0$, $u \in L^2(0,T)$, $\Omega = (0,\pi)^2$ and that $\gamma \subset \Omega$ is a union of a finite number of curves γ_i, $i = 1, m$, each of themstazisfying the assumptions of theorem ??. Then the problem (??)-(??) admits a unique solution

$$w \in C([0,T], H^{\frac{\mu+2}{\mu+1}-\epsilon}(\Omega)) \cap C^1([0,T], H^{-\frac{\mu}{\mu+1}-\epsilon}(\Omega)), \tag{2.5}$$

if the tangent line to γ has contact of order at most μ, $1 \le \mu \le n$ in all points of γ and

$$w \in C([0,T], H^1(\Omega)) \cap C^1([0,T], H^{-1}(\Omega)), \tag{2.6}$$

if γ contains a segment. Moreover there exists a constant $C(T) > 0$ such that

$$||w||_{C([0,T],H^{\beta+2}(\Omega))} + ||w'||_{C([0,T],H^{\beta}(\Omega))} \le C(T)||u||_{L^2(0,T)}, \tag{2.7}$$

$\forall u \in L^2(0,T)$, where $\beta = -\frac{\mu}{\mu+1} - \epsilon$ if the tangent line to γ has contact of order at most $\mu \le n$. and $\beta = -1$ if the curvature of γ vanishes at some point.

Remark 2.1. The greatest regularity of w is obtained if the curvature of γ is bounded away from zero, when $\mu = 1$, so (**??**) becomes

$$w \in C([0,T], H^{\frac{3}{2}-\epsilon}(\Omega)) \cap C^1([0,T], H^{-\frac{1}{2}-\epsilon}(\Omega)), \tag{2.8}$$

This result improves by 1 in the Sobolev scale the property (**??**) obtained by simply using the Sobolev regularity of the right hand side of (**??**). If γ contains the segment the regularity (**??**) is the same as for the one dimensional case studied in [**?**], [**?**].

Let us consider now the following initial and boundary value problem, which is the dual of (**??**)-(**??**).

$$\phi''(x,y,t) - \alpha\Delta\phi''(x,y,t) + \Delta^2\phi(x,y,t) = 0, \text{ in } Q, \tag{2.9}$$

$$\phi(x,y,t) = \Delta\phi(x,y,t) = 0, \text{ on } \Sigma, \tag{2.10}$$

$$\phi(x,y,T) = 0, \ \phi'(x,y,T) = g(x,y). \text{ in } \Omega, \tag{2.11}$$

By a well known duality method (cf. [**?**], [**?**]), which will be briefly recalled in the next section, Theorem **??** (resp. Theorem **??**)) is equivalent to the trace regularity properties given in Proposition **??** (resp. Proposition **??**). More precisely we have:

Proposition 2.4. Suppose that $\alpha > 0$ and that Ω is a regular bounded set, $\gamma \subset \Omega$ is a curve satisfying the assumptions of Theorem **??** and $g \in L^2(\Omega)$. Then the solution of the initial and boundary value problem (**??**)-(**??**) satisfies

$$\int_\gamma \frac{\partial\phi}{\partial\nu} d\gamma \in L^2(0,T). \tag{2.12}$$

Moreover there exists a constant $C > 0$ such that

$$\int_0^T \left(\int_\gamma \frac{\partial\phi}{\partial\nu} d\gamma \right)^2 dt \le C\|g\|^2_{L^2(\Omega)}. \tag{2.13}$$

The same results hold if we replace the boundary condition (**??**) by

$$\phi(x,y,t) = \frac{\partial\phi}{\partial\nu}(x,y,t) = 0, \text{ on } \Sigma. \tag{2.14}$$

Proposition 2.5. Suppose that $\alpha = 0$, Ω is a square and $\gamma \subset \Omega$ is a curve satisfying the assumptions of Theorem **??**. Then we have:

(a) If $\epsilon > 0$ and the tangent line to γ has contact of order ar most μ, $1 \leq \mu \leq n$, the solution ϕ of (**??**)-(**??**) satisfies (**??**) provided that $g \in H^{-\frac{\mu+2}{\mu+1}+\epsilon}(\Omega)$. Moreover

$$\int_0^T \left(\int_\gamma \frac{\partial \phi}{\partial \nu} d\gamma \right)^2 dt \leq C\|g\|^2_{H^{-\frac{\mu+2}{\mu+1}+\epsilon}(\Omega)}. \tag{2.15}$$

(b) If γ contains a segment the solution ϕ of (**??**)-(**??**) satisfies (**??**) provided that $g \in H^{-1}(\Omega)$. Moreover

$$\int_0^T \left(\int_\gamma \frac{\partial \phi}{\partial \nu} d\gamma \right)^2 dt \leq C\|g\|^2_{H^{-1}(\Omega)}. \tag{2.16}$$

Remark 2.2. The trace regularity results above are not consequences of the interior regularity of ϕ. Related "hidden" trace regularity properties for the solutions of wave or plate equations are proved, for example, in [**?**]. However the multiplier technique used in the paper quoted above cannot be used in our case as ϕ doesn't satisfy a homogenous Dircichlet condition on γ.

Remark 2.3. The assumptions on the curve γ essentially exclude some pathological curves which either have a tangent of infinite order (like $y = e^{-\frac{1}{x}}$) or infiniite oscillations (like $y = x^n \sin \frac{1}{x^p}$ with $n >> p \geq 1$).

3. Notations and preliminaries

In order to study the wellposedness for (**??**)-(**??**) we introduce the function spaces $(Y_\beta)_{\beta \in \mathbf{R}}$ defined as follows: if $\beta > 0$ then Y_β is the closure in $H^\beta(\Omega)$ of the set of all functions $\psi \in C^\infty(\bar{\Omega})$ satisfying $\Delta^{2n}\psi = 0$, on Γ for all $n \geq 0$; for negative β Y_β is the dual space of $Y_{-\beta}$ constructed by means of the inner product of $Y_0 = L^2(\Omega)$. Let us notice that if $\Omega = (0, \pi)^2$ and the function g is given by the Fourier series

$$g(x, y) = \sum_{m,n \geq 1} b_{mn} \sin(mx) \sin(ny),$$

the condition $g \in Y_\beta$ is equivalent to

$$||g||_{Y_\beta} = \sum_{m,n \geq 1} (m^2 + n^2)^\beta b^2_{mn} < \infty, \tag{3.1}$$

and that $|| \cdot ||_{Y_\beta} \to \mathbf{R}^2$ is equivalent to the H^β norm.

If $\alpha > 0$ the equivalence of interior regularity for (**??**)-(**??**) and trace regularity for (**??**)-(**??**) is given by the following result.

Lemma 3.1. If $\alpha > 0$ conditions (**??**) and (**??**) hold for all solutions ϕ of (**??**)-(**??**) and any $g \in L^2(\Omega)$ if and only if the solution w of (**??**)-(**??**) satisfies (**??**) and (**??**), for all $u \in L^2(0, T)$. In other words the assertions of Theorem **??** and Proposition **??** are equivalent.

Proof. Assertions similar to the present lemma were proved in [?], [?] ,for the wave equation, as applications of the general transposition method introduced in [?]. However, for the sake of completeness we will sketch the proof. Let us first notice that if we multiply (??) by ϕ and integrate by parts we obtain (at least formally)

$$\int_\Omega w(x,y,\tau)g(x,y)dxdy + \alpha\int_\Omega \nabla w(x,y,\tau)\cdot\nabla g(x,y)dxdy = \tag{3.2}$$

$$= \int_0^\tau u(t)[\int_\gamma \frac{\partial\phi}{\partial\nu}d\gamma]dt.$$

If we suppose now that (??) holds for any $g \in L^2(\Omega)$ we get

$$\int_0^\tau u(t)[\int_\gamma \frac{\partial\phi}{\partial\nu}d\gamma]dt \leq C\|u||_{L^2(0,T)}||g||_{L^2(\Omega)},$$

so, by (??) we obtain $w(\cdot,\cdot,\tau) \in Y_2$ for all $\tau \in [0,T]$. If we replace τ by $\tau + h$ in (??) and take the limit when $h \to 0$ we easily obtain

$$w \in C([0,T], Y_2). \tag{3.3}$$

As w satisfies (??) and $u(t)\dfrac{\partial\delta_\gamma}{\partial\nu} \in L^2((0,T), Y_{-\frac{3}{2}-\epsilon})$ it follows that

$$(I - \alpha\Delta)w'' \in L^2((0,T), Y_{-2}),$$

so

$$w'' \in L^2((0,T), Y_0). \tag{3.4}$$

By applying the intermediate derivative theorem theorem (cf.[?, p.19]) we obtain

$$w' \in L^2((0,T), Y_{-1}). \tag{3.5}$$

Relations (??) and (??) follow now from (??)- (??) and the general lifting result of [?], so we proved the first inplication.

For the converse result, if we suppose that (??) and (??) hold for any $u \in L^2(0,T)$, by using again (??) we easily obtain ϕ satisfies (??) and (??) for all $g \in L^2(\Omega)$. □

For the Bernoulli-Euler model, i.e. $\alpha = 0$, the following result is similar to Lemma **??**.

Lemma 3.2. If $\alpha = 0$, the solution ϕ of (??)-(??) satisfies (??), for all $g \in Y_{-\beta-2}$, with

$$\int_0^T \left(\int_\gamma \frac{\partial\phi}{\partial\nu}d\gamma\right)^2 dt \leq C\|g\|^2_{Y_{-\beta-2}}, \tag{3.6}$$

if and only if the solution w of (??)-(??) has the regularity

$$w \in C([0,T], Y_{\beta+2}) \cap C^1([0,T], Y_\beta), \tag{3.7}$$

and satisfies (??) for all $u \in L^2(0,T)$.

As the proof is completely similar to the proof of Lemma **??**, it will be omitted.

We now give a technical estimate on an improper integral which will be needed in the proof of Theorems **??** and **??**. In order to state it let us denote by $\sigma(\alpha)$ the signature of α, i.e. $\sigma(\alpha) = 0$ if $\alpha = 0$ and $\sigma(\alpha) = 1$ if $\alpha > 0$. The estimate is given in the lemma below, which was proved in [?] if $\alpha > 0$ and [?] if $\alpha = 0$.

Lemma 3.3. Let $\lambda > 0$ and $\alpha \geq 0$ be two fixed constants and let $\omega \in \mathbf{R}$ be a parameter. Then

$$I_1(\omega) = \int_0^\infty \frac{\eta^{2+\frac{\sigma(\alpha)}{2}} d\eta}{[\eta^2 + (1+\alpha\eta)(\lambda^2 - \omega^2)]^2 + 4\lambda^2\omega^2(1+\alpha\eta)^2} \leq C < \infty,$$

where C is a constant independent of $\omega \in \mathbf{R}$.

Finally, in the proof of Theorem **??** we will essentially use a theorem on aperiodic Fourier series, due to Meyer (cf. [?]). In order to state this result let us consider a strictly increasing sequence of frequencies $s(j) \in \mathbf{R}$, $s(-j) = s(j)$, $\forall j \in \mathbf{Z}$, $E(j)$ a finite set for all $j \in \mathbf{Z}$ and $b, c : \mathbf{Z} \times \mathbf{Z} \to \mathbf{C}$ such that

$$\sum_{j=-\infty}^{\infty} \sum_{r \in E(j)} |c(j,r)|^2 < \infty. \tag{3.8}$$

We can now state the result proved in [?]

Lemma 3.4. There exist a constant $C > 0$ such that

$$\int_0^T |\sum_{j=-\infty}^{\infty} \sum_{r \in E(j)} b(j,r)c(j,r)e^{its(j)}|^2 dt \leq C \sum_{j=-\infty}^{\infty} \sum_{r \in E(j)} |c(j,r)|^2,$$

if and only if

$$\sum_{k \leq s(j) < k+1} \sum_{r \in E(j)} |b(j,r)|^2 \leq D,$$

where D is a constant not depending on k.

4. Estimates of the Fourier transform of $\dfrac{\partial \delta_\gamma}{\partial \nu}$

If the curvature of γ is bounded away from zero we have an estimate of the Fourier transform of $\dfrac{\partial \delta_\gamma}{\partial \nu}$ which will be essential for the proof of Theorems **??**, **??** and **??**. We will use the notation

$$H(\xi_1, \xi_2) = \int_\gamma \frac{\partial}{\partial \nu}[e^{i(x\xi_1 + y\xi_2)}] d\gamma_{x,y}. \tag{4.1}$$

The following estimate for H can be derived if the curvature of γ is bounded away from zero.

Lemma 4.1. Suppose that $\gamma \subset \Omega$ is piecewise C^4 and its curvature is different from zero at all points. Then there exists a constant $C > 0$ such that

$$|H(\xi)| \leq C|\xi|^{\frac{1}{2}},\ \forall \xi = (\xi_1, \xi_2) \in \mathbf{R}^2. \tag{4.2}$$

Proof. For a fixed $\xi \in R^2$ we can suppose (eventually after dividing γ in smaller curves) that $\nu(x,y)$ is parallel to ξ at most at one extremity of γ. Let us now choose a system of orthogonal coordinates such that $\xi = (0, \xi_2)$. A simple calculation shows that

$$H(\xi) = \xi_2 \int_0^l x'(s) e^{iy(s)\xi_2} ds. \tag{4.3}$$

If ν is never parallel to ξ estimate (**??**) follows simply by integrating by parts in (**??**) (as the derivative of the phase $\Phi(s) = x(s)\xi_1 + y(s)\xi_2$ is different from zero at all points). If we suppose that ξ is orthogonal to γ at the origin of our system of coordinates, and we denote by $(x(s), y(s))$ the arc length parmetrization of γ we obtain, for l small enough, that

$$x'(s) = 1 + s\eta_1(s), \tag{4.4}$$

$$y(s) = As^2 + s^3\eta_2(s), \tag{4.5}$$

$$y'(s) = 2As + s^2\eta_3(s), \tag{4.6}$$

$$y'(s) \geq C's, \text{ with } C' > 0, \ \forall s \in [0, l] \tag{4.7}$$

where C' denotes the minimum of the curvature of γ and $\eta_i : [0, l] \to \mathbf{R}$ are C^1 functions such that $|\eta_i(s)| + |\eta_i'(s)| \leq C$, for all $s \in [0, l]$ and $i = 1, 2, 3$. Relation (**??**) and (**??**) imply

$$H(\xi) = H_1(\xi) + H_2(\xi), \tag{4.8}$$

where

$$H_1(\xi) = \xi_2 \int_0^l s\eta_1(s) e^{iy(s)\xi_2} ds, \ H_2(\xi) = \xi_2 \int_0^l e^{iy(s)\xi_2 ds}.$$

A simple integration by parts gives

$$H_1(\xi) = \left. \frac{s\eta_1(s)}{y'(s)} \right|_0^a - \int_0^a \left[\frac{s\eta_1(s)}{y'(s)} \right]' e^{iy(s)\xi_2} ds,$$

so, since y' is larger that a fixed constant a, H_1 is bounded on R^2. In order to estimate H_2 we write it

$$H_2(\xi) = \xi_2 \int_0^{\frac{1}{\sqrt{\xi_2}}} e^{iy(s)\xi_2 ds} + \xi_2 \int_{\frac{1}{\sqrt{\xi_2}}}^l e^{iy(s)\xi_2 ds}. \tag{4.9}$$

We obviously have

$$|\xi_2 \int_0^{\frac{1}{\sqrt{\xi_2}}} e^{iy(s)\xi_2} ds| \leq \sqrt{\xi_2}. \tag{4.10}$$

For the second term in the right hand side of (**??**), an integration by parts gives

$$\xi_2 \int_{\frac{1}{\sqrt{\xi_2}}}^l e^{iy(s)\xi_2} ds = \left. \frac{1}{y'(s)} e^{iy(s)\xi_2} \right|_{\frac{1}{\sqrt{\xi_2}}}^l - \int_{\frac{1}{\sqrt{\xi_2}}}^l \frac{1}{y'(s)} e^{iy(s)\xi_2} ds. \tag{4.11}$$

By using (??) and (??) we obtain

$$\left|\xi_2 \int_{\frac{1}{\sqrt{\xi_2}}}^{l} e^{iy(s)\xi_2 ds}\right| \leq \frac{C\sqrt{\xi_2}}{C'} = \frac{C\sqrt{|\xi|}}{C'},\ \forall \xi \in R^2, \tag{4.12}$$

where C' was introduced in (??). As our estimates are uniform with respect to ξ from (??) and (??) we obtain the conclusion (??). □

Remark 4.1. If γ is closed (??) reduces to an estimate of the Fourier transform of the indicator function of the interior of γ. In this case the result proved in [?], [?] easily implies (??).

If the curvature of γ has a zero of finite multiplicity an estimate weaker than Lemma **??** holds. More precisely we can prove the following result.

Lemma 4.2. Suppose that γ is of class C^{n+3} with $n \geq 1$, and that the curvature of γ is nonzero at all point of γ, with the possible exception of one point, where the tangent line has contact of order $1 < \mu \leq n$. Then there exists a constant $C > 0$ and a system of orthogonal coordinates such that

$$|H(\xi_1, \xi_2)| \leq C|\xi|^{\frac{\mu}{\mu+1}},\ \text{if}\ \ |\xi_1| \leq |\xi_2|^{\frac{1}{\mu+1}}, \tag{4.13}$$

$$|H(\xi_1, \xi_2)| \leq C\frac{|\xi_2|^{\frac{2\mu-1}{2\mu}}}{|\xi_1|^{\frac{\mu-1}{2\mu}}},\ \text{if}\ \ |\xi_2| \geq |\xi_1| \geq |\xi_2|^{\frac{1}{\mu+1}}, \tag{4.14}$$

$$|H(\xi_1, \xi_2)| \leq C|\xi|^{\frac{1}{2}},\ \text{if}\ \ |\xi_1| \geq |\xi_2|. \tag{4.15}$$

Proof. Let us choose a system of coordinates xOy such that the origin is at the point where the curvature of γ vanishes and Ox is tangent to γ in O. If we denote by $(x(s), y(s)),\ s \in [0, l]$ the arc length parametrization of γ the assumptions of the present lemma imply that for all $s \in [0, l]$ we have

$$x(s) = s + s^2\eta_4(s),\ y(s) = s^{\mu+1} + s^{\mu+2}\eta_5(s),, \tag{4.16}$$

$$y'(s) = (\mu + 1)s^{\mu} + s^{\mu+1}\eta_6(s),\ x'(s) = 1 + s^{2\mu}\eta_7(s), \tag{4.17}$$

(as $[x'(s)]^2 + [y'(s)]^2 \equiv 1$),

$$y''(s) = \alpha(\alpha - 1)s^{\mu-1} + s^{\mu}\eta_8(s),\ y''(s) > 0, \tag{4.18}$$

where $\eta_i : [0, l] \to \mathbf{R}$ are C^1 functions such that $|\eta_i(s)| + |\eta_i'(s)| \leq C$, for all $s \in [0, l]$ and $i = 4, 8$. If $|\xi_2| \leq |\xi_1|$ all stationary phase points of the oscillatory integral defining H are non-degenerate so the method used in the proof of **??** yields (??). This is why we will study only the case

$$|\xi_1| < |\xi_2|.$$

Moreover we will suppose that

$$\xi_2 > 0, \tag{4.19}$$

as the case $\xi_2 < 0$ can be treated in a completely similar manner.

Let us now notice that

$$H(\xi) = \int_0^l [-\xi_1 y'(s) + \xi_2 x'(s)] e^{i[x(s)\xi_1 + y(s)\xi_2]} ds. \tag{4.20}$$

A straightforward integration by parts shows that

$$\left| \int_0^l -\xi_1 y'(s) e^{i[x(s)\xi_1 + y(s)\xi_2]} ds \right| \leq C \frac{|\xi_1|}{|\xi_2|} \leq C_1, \tag{4.21}$$

$$\left| \int_0^l \xi_2 [x'(s) - 1] e^{i[x(s)\xi_1 + y(s)\xi_2]} ds \right| \leq C, \tag{4.22}$$

for all $\xi \in \mathbf{R}^2$, thus we have only to study

$$\widetilde{H}(\xi) = \xi_2 \int_0^l e^{i[x(s)\xi_1 + y(s)\xi_2]} ds.$$

If $\xi_1 > 0$, as we assumed (**??**), the derivative of the phase $\Phi(s) = x(s)\xi_1 + y(s)\xi_2$ doesn't vanish so an estimate stronger than (**??**) holds true. This is why we will suppose that

$$\xi_2 \geq 0, \ \xi_1 \leq 0, \tag{4.23}$$

so the derivative of the phase $\Phi'(s) = x'(s)\xi_1 + y'(s)\xi_2$ vanishes only a point s_0 and (**??**) implies that

$$s_0 = \left(\frac{-\xi_1}{\xi_2} \right)^{\frac{1}{\mu}} + o\left(\left(\frac{-\xi_1}{\xi_2} \right)^{\frac{1}{\mu}} \right), \tag{4.24}$$

We will now distinguish two cases

Case 1. If we suppose that

$$\frac{(-\xi_1)^{\mu+1}}{\xi_2} \leq 1. \tag{4.25}$$

and introduce the notation $v(\xi) = (\xi_2)^{-1/(\mu+1)}$, we obtain

$$\left| \xi_2 \int_0^{v(\xi)} e^{i\Phi(s)} ds \right| \leq |\xi_2|^{\mu/(\mu+1)}. \tag{4.26}$$

On the other hand a simple integration by parts gives

$$\left| \xi_2 \int_{v(\xi)}^l e^{i\Phi(s)} ds \right| \leq C \left. \frac{|\xi_2|}{\Phi'(s)} \right|_{v(\xi)}^l . \tag{4.27}$$

Since l was chosen such that $\Phi'(l)$ is larger than a given constant it suffices now to use (**??**) to obtain

$$\frac{1}{\Phi'(v(\xi))} \sim \xi_2^{\frac{\mu}{\mu+1}},$$

relation (**??**) implies

$$\left|\xi_2 \int_{v(\xi)}^{l} e^{i\Phi(s)} ds\right| \leq C|\xi_2|^{\mu/(\mu+1)}. \tag{4.28}$$

By using (**??**) and (**??**) we obtain now easily the conclusion (**??**).

Case 2. Suppose that

$$|\xi_2| \geq |\xi_1| \geq |\xi_2|^{\frac{1}{\mu+1}}, \tag{4.29}$$

and make the notation

$$u_0 = \frac{1}{\sqrt{\xi_2 s_0^{\mu-1}}}, \tag{4.30}$$

where s_0 was introduced in (**??**). We notice that

$$\left|\xi_2 \int_{s_0-u_0}^{s_0+u_0} e^{i\Phi(s)} ds\right| \leq 2|\xi_2| u_0 \leq \frac{|\xi_2|}{\sqrt{\xi_2 s_0^{\mu-1}}}. \tag{4.31}$$

As for $s \notin [s_0 - u_0, s_0 + u_0]$ the derivative of the phase is not vanishing, it suffices now to estimate $\dfrac{1}{\Phi'(s)}$ for $s \in \{s_0 - u_0, s_0 + u_0\}$. From (**??**) we see that $u_0 = o(s_0)$ so

$$\left|\frac{1}{\Phi'(s_0 \pm u_0)}\right| \leq \frac{C}{|u_0|},$$

which implies

$$\left|\xi_2 \int_{0}^{s_0-u_0} e^{i\Phi(s)} ds + \xi_2 \int_{s_0+u_0} e^{i\Phi(s)} ds\right| \leq C\frac{|\xi_2|}{\sqrt{\xi_2 s_0^{\mu-1}}}. \tag{4.32}$$

From (**??**), (**??**), (**??**) and (**??**) we can now easily deduce the conclusion (**??**). □

Remark 4.2. If γ is closed an estimate similar to (**??**) was proved in ([**?**]).
If we consider curves γ possibly containing a segment we can derive an estimate only on the mean of H with repect to the angle. More precisely we have

Lemma 4.3. Suppose that $\alpha > 0$, $u \in L^2(0,T)$, $\Omega \subset \mathbf{R}^2$ is a regular bounded set and $\gamma \subset \Omega$ is a union of a finite number of curves γ_i, $i = 1, m$, each of them satisfying one of the following conditions:

- (H1) There exist $n_i \geq 2$ such that the curvature of γ vanishes at most at one end of γ_i and at that point the tangent line has contact of order at most n_i at
- (H2) γ_i is a segment.

Then there exists a constant $C > 0$ such that

$$\int_0^{2\pi} |H(\rho\cos\theta, \rho\sin\theta)|^2 d\theta \leq C\rho, \ \forall \rho > 0. \tag{4.33}$$

Proof. It is clear that it suffices to prove (**??**) in each of the following situations:

1) the curvature of γ is bounded away from zero;
2) the curvature of γ vanishes at only one point s_0 and the tangent line has contact of order at most n at s;
3) γ is a segment.

Case 1. This is the simplest case as (**??**) follows directly from Lemma **??**.
Case 2. A simple calculation shows that

$$|\xi|^{\frac{\mu}{\mu+1}} \leq \frac{|\xi_2|^{\frac{2\mu-1}{2\mu}}}{|\xi_1|^{\frac{\mu-1}{2\mu}}}, \text{ if } |\xi_2|^{\frac{1}{\mu+1}} \geq |\xi_1|, \tag{4.34}$$

Inequality (**??**) combined with Lemma **??** implies that there exists $C > 0$ such that

$$|H(\rho\cos\theta, \rho\sin\theta)| \leq C\frac{|\xi_2|^{\frac{2\mu-1}{2\mu}}}{|\xi_1|^{\frac{\mu-1}{2\mu}}}, \ \forall \xi \in \mathbf{R}^2, \tag{4.35}$$

Using polar coordinates ($\xi_1 = \rho\cos\theta,\ \xi_2 = \rho\sin\theta$) the inequality above implies

$$\int_0^{2\pi} |H(\rho\cos\theta, \rho\sin\theta)|^2 d\theta \leq C\rho \int_0^{\frac{\pi}{2}} \frac{(\sin\theta)^{\frac{2\mu-1}{\mu}}}{(\cos\theta)^{\frac{\mu-1}{\mu}}} \leq C\rho \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{(\cos\theta)^{\frac{\mu-1}{\mu}}}. \tag{4.36}$$

If make the change of variables $\eta = \cos\theta$ in (**??**), as $0 < \frac{\mu-1}{\mu} < 1$, we obviously obtain the conclusion (**??**).
Case 3. If γ is a segment we can choose the system of coordinates such that

$$\gamma = \{(x, 0) \in \mathbf{R}^2 | 0 \leq x \leq l\},$$

with $l > 0$. A simple calculation shows that in this case we have

$$|H(\xi)| = \frac{|\xi_2|}{|\xi_1|} |\sin(\frac{l\xi_1}{2})|,$$

and by passing to polar coordinates we obtain

$$\int_0^{2\pi} |H(\rho\cos\theta, \rho\sin\theta)|^2 d\theta = 4J(\rho), \tag{4.37}$$

where

$$J(\rho) = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta \sin^2(\frac{l\rho\cos\theta}{2})}{\cos^2\theta} d\theta. \tag{4.38}$$

In order to estimate $J(\rho)$ let us first notice that

$$J(\rho) \leq \int_0^{\frac{\pi}{2}} \frac{\sin\theta \sin^2(\frac{l\rho\cos\theta}{2})}{\cos^2\theta} d\theta,$$

so, by the change of variable $k = \frac{l\rho}{2}\sin\theta$ we easily obtain that

$$J(\rho) \leq \frac{K_0 l\rho}{2}, \tag{4.39}$$

with $K_0 = \displaystyle\int_0^\infty \frac{\sin^2(k)}{k^2} dk$. Relation (??) and (??) obviously imply (??). $\square$

5. Proof of the main results

The natural idea for proving regularity results for (??)-(??) is to adapt one of the methods used for the wave or plate equations with point control. The methods of Lions and Nirenberg (cf.[?]) are based on explicit solutions for the corresponding free space problem, so it seems difficult to adapt them in our situation as no explicit solutions of (??) are avaiable (even if we replace Ω by $\mathbf{R}^2$).

This is why we will use techniques inspired from [?], [?], [?] (for the proof of theorems ?? and ??) and [?] (for the proof of Theorem ??).

The method inspired by [?], [?], [?] is based on Laplace-Fourier transforms for the corresponding free space problem followed by changes of variables leading to a standard problem in Ω. This method gives sharp results in the case $\alpha > 0$ where the regularity of the Cauchy problem in $\mathbf{R}^2$ is preserved for the initial and boundary value problem in Ω. The same technique applied to the case $\alpha = 0$ gives results which are not sharp, as suggested by the one dimensional case studied in [?], [?]. This is why, for $\alpha = 0$ and Ω a square, we will use the dual problem (??)-(??) combined with estimates on aperiodic Fourier series and a number theoretical result.

In order to prove Theorem ?? and Theorem ?? let us first consider the Cauchy problem

$$\psi''(x,y,t) - \alpha\Delta\psi''(x,y,t) + \Delta^2\psi(x,y,t) = u(t)\frac{\partial\delta_\gamma}{\partial\nu}, \text{ in } \mathbf{R}^2 \times (0,\infty), \tag{5.1}$$

$$\psi(x,y,0) = 0,\ \psi'(x,y,0) = 0. \text{ in } \mathbf{R}^2, \tag{5.2}$$

Regularity for (??)-(??) is given by the following result.

Lemma 5.1. If $\alpha \geq 0,\ u \in L^2(0,T)$ and γ is a C^2 curve, the initial and boundary value problem (??)-(??) has a unique solution ψ and

$$\psi \in C([0,T], H^{\frac{3+\sigma(\alpha)}{2}}(\mathbf{R}^2)) \cap C^1([0,T], H^{\frac{3\sigma(\alpha)-1}{2}}(\mathbf{R}^2)). \tag{5.3}$$

Proof. As noticed in [?],[?] for the plate equation with point control it suffices to prove that

$$\psi \in L^2((0,T), H^{\frac{3+\sigma(\alpha)}{2}}(\mathbf{R}^2)), \tag{5.4}$$

and then apply (??) combined with the general lifting result from [?].

Let us denote by $\widehat{\psi}(\xi,\zeta),\ \zeta = \lambda + i\omega,\ \lambda > 0,\ \omega \in \mathbf{R},\ \xi \in R^2$ the Laplace (in t) Fourier (in (x,y)) transform of ψ given by

$$\widehat{\psi}(\xi,\zeta) = (2\pi)^{-1}\int_0^\infty e^{-\zeta t}\int_{\mathbf{R}^2} e^{-i(x\xi_1+y\xi_2)}\psi(x,y,t)dxdydt.$$

From (??) we get

$$\widehat{\psi}(\xi,\zeta) = -(2\pi)^{-1}\frac{\widehat{u}(\zeta)\int_\gamma \frac{\partial}{\partial\nu}[e^{-i(x\xi_1+y\xi_2)}]d\gamma_{x,y}}{\zeta^2(1+\alpha|\xi|^2)+|\xi|^4}. \tag{5.5}$$

On the other hand ψ satisfies (**??**) if and only if

$$e^{-\lambda t}\psi(x,y,t)\in L^2((0,\infty),H^2(\mathbf{R}^2)),$$

which, by Parseval identity (cf.[**?**, p.212]), is equivalent to

$$|\xi|^{\frac{3+\sigma(\alpha)}{2}}\widehat{\psi}(\xi,\lambda+i\omega)\in L^2(\mathbf{R}^2_\xi\times\mathbf{R}^1_\omega). \tag{5.6}$$

From (**??**) we have

$$|||\xi|^{\frac{3+\sigma(\alpha)}{2}}\widehat{\psi}||^2_{L^2(\mathbf{R}^2_\xi\times\mathbf{R}^1_\omega)}=$$

$$=\frac{1}{4\pi^2}\int_{\mathbf{R}}|\widehat{u}(\lambda+i\omega)|^2\int_{\mathbf{R}^2}\frac{|\int_\gamma\frac{\partial}{\partial\nu}[e^{-i(x\xi_1+y\xi_2)}]d\gamma_{x,y}|^2|\xi|^{3+\sigma(\alpha)}}{|\zeta^2(1+\alpha|\xi|^2)+|\xi|^4|^2}d\xi d\omega,$$

so in order to obtain (**??**) it suffices to prove the estimate

$$\int_{\mathbf{R}^2}\frac{|\int_\gamma\frac{\partial}{\partial\nu}[e^{-i(x\xi_1+y\xi_2)}]d\gamma_{x,y}|^2|\xi|^{3+\sigma(\alpha)}}{|\zeta^2(1+\alpha|\xi|^2)+|\xi|^4|^2}d\xi= \tag{5.7}$$

$$=\int_{\mathbf{R}^2}\frac{|\int_\gamma\frac{\partial}{\partial\nu}[e^{-i(x\xi_1+y\xi_2)}]d\gamma_{x,y}|^2|\xi|^{3+\sigma(\alpha)}}{[(\lambda^2-\omega^2)(1+\alpha|\xi|^2)+|\xi|^4]^2+4\lambda^2\omega^2(1+\alpha|\xi|^2)^2}d\xi\le C<\infty,$$

uniformely in $\omega\in\mathbf{R}$.

By using polar coordinates $(\xi_1=\rho\cos\theta,\ \xi_2=\rho\sin\theta)$, we obtain

$$\int_{\mathbf{R}^2}\frac{|\int_\gamma\frac{\partial}{\partial\nu}[e^{-i(x\xi_1+y\xi_2)}]d\gamma_{x,y}|^2|\xi|^{3+\sigma(\alpha)}d\xi}{[(\lambda^2-\omega^2)(1+\alpha|\xi|^2)+|\xi|^4]^2+4\lambda^2\omega^2(1+\alpha|\xi|^2)^2}=$$

$$=\int_0^\infty\frac{\rho^{4+\sigma(\alpha)}}{[(\lambda^2-\omega^2)(1+\alpha\rho^2)+\rho^4]^2+4\lambda^2\omega^2(1+\alpha\rho^2)^2}\int_0^{2\pi}|H(\rho\cos\theta,\rho\sin\theta)|^2d\theta d\rho,$$

where H was introduced by (**??**). From the relation above and Lemma **??** we get

$$\int_{\mathbf{R}^2}\frac{|\int_\gamma\frac{\partial}{\partial\nu}[e^{-i(x\xi_1+y\xi_2)}]d\gamma_{x,y}|^2|\xi|^{3+\sigma(\alpha)}d\xi}{[(\lambda^2-\omega^2)(1+\alpha|\xi|^2)+|\xi|^4]^2+4\lambda^2\omega^2(1+\alpha|\xi|^2)^2}\le$$

$$\le\frac{1}{2\pi}\int_0^\infty\frac{\rho^{5+\sigma(\alpha)}}{[(\lambda^2-\omega^2)(1+\alpha\rho^2)+\rho^4]^2+4\lambda^2\omega^2(1+\alpha\rho^2)^2}.$$

Finally, if we put $\rho^2=\eta$ and we use Lemma **??** we obtain (**??**). $\square$

In order to prove Theorems **??** and **??** we will compare the regularity of the solution w of (**??**)-(**??**) with the regularity of the solution ψ of (**??**)-(**??**), which is given by Lemma **??**. Our method yields the same regularity for w and ψ if $\alpha>0$ but w is less regular than ψ if $\alpha=0$.

The same method was used in [**?**], [**?**] for plate equations with point control so we will only sketch the proofs and refer to [**?**], [**?**] for the details.

Proof of Theorem **??**. If $\alpha > 0$, by Lemma **??** the solution ψ of (**??**)-(**??**) satisfies

$$\psi \in C([0,T], H^2(\mathbf{R}^2)) \cap C^1([0,T], H^1(\mathbf{R}^2)). \tag{5.8}$$

As ψ satisfies (**??**) we also have

$$\psi \in C^2([0,T], L^2(\mathbf{R}^2)). \tag{5.9}$$

Let us consider $v \in C_0^\infty(\Omega)$ such that $v \equiv 1$ in a neighbourhood of γ and define

$$\psi_c(x,t) = v(x)\psi(x,t), \; h(x,t) = \psi_c(x,t) - w(x,t), \tag{5.10}$$

where w is the solution of (**??**)-(**??**). After some calculations completely similar to those detailed in [**?**] we obtain that h is a solution of the initial and boundary value problem

$$h''(x,y,t) - \alpha\Delta h''(x,y,t) + \Delta^2 h(x,y,t) = F(x,y,t), \text{ in } Q, \tag{5.11}$$

$$h(x,y,t) = \Delta h(x,y,t) = 0, \text{ on } \Sigma, \tag{5.12}$$

$$h(x,y,0) = 0, \; h'(x,y,0) = 0, \text{ in } \Omega, \tag{5.13}$$

with

$$F(x,y,t) = 4\nabla v \cdot \nabla(\Delta\psi) + 2(\Delta v)\Delta\psi + 4\nabla(\Delta v)\cdot\nabla\psi +$$
$$+4\left[\nabla\left(\frac{\partial v}{\partial x}\right)\cdot\left(\frac{\partial \psi}{\partial x}\right) + \nabla\left(\frac{\partial v}{\partial y}\right)\cdot\nabla\left(\frac{\partial \psi}{\partial y}\right)\right] + (\Delta^2 v)\psi - \alpha(\Delta v)\psi'' - \alpha\nabla v\nabla\phi''.$$

From (**??**) and (**??**) we easily see that $F \in C([0,T], Y_{-1})$, so, by applying Lemma 2.5 from [**?**] we obtain $w \in C([0,T], Y_2) \cap C^1([0,T], Y_1)$, so, by (**??**), w has the regularity (**??**).

If the boundary condition (**??**) is replaced by (**??**) the function h defined in (**??**) satisfies (**??**), (**??**) and the boundary condition

$$h(x,h,t) = \frac{\partial h}{\partial \nu}(x,y,t) = 0, \quad \text{on } \Sigma,$$

and by applying Proposition 3.4 from [**?**] we obtain that

$$h \in C([0,T], H_0^2(\Omega)) \cap C^1([0,T], H_0^1(\Omega)),$$

so w satisfies (**??**). ☐

The same method applied to the case $\alpha = 0$ yields a regularity for w which is by $\frac{1}{2}$ smaller (in the H^s scale) than the regularity of the solution ψ of the free space problem.

Proof of Theorem **??**. If $\alpha = 0$ Lemma **??** shows that the solution ψ of (**??**)-(**??**) satisfies

$$\psi \in C([0,T], H^{\frac{3}{2}}(\mathbf{R}^2)) \cap C^1([0,T], H^{-\frac{1}{2}}(\mathbf{R}^2)) \cap C^2([0,T], H^{-\frac{5}{2}}(\mathbf{R}^2)), \tag{5.14}$$

As in the proof of Theorem **??** if we consider $\psi_c(x,t)$ and $h(x,t)$ defined by (**??**) and by simple calculations we find that h is a solution of the initial and boundary value problem

$$h''(x,y,t) + \Delta^2 h(x,y,t) = G(x,y,t), \text{ in } Q, \tag{5.15}$$

$$h(x,y,t) = \Delta h(x,y,t) = 0, \text{ on } \Sigma, \tag{5.16}$$

$$h(x,y,0) = 0, \; h'(x,y,0) = 0, \text{ in } \Omega, \tag{5.17}$$

with

$$G(x,y,t) = 4\nabla v \cdot \nabla(\Delta\psi) + 2(\Delta v)\Delta\psi +$$
$$+4\nabla(\Delta v)\cdot\nabla\psi + 4\left[\nabla\left(\frac{\partial v}{\partial x}\right)\cdot\left(\frac{\partial \psi}{\partial x}\right) + \nabla\left(\frac{\partial v}{\partial y}\right)\cdot\nabla\left(\frac{\partial \psi}{\partial y}\right)\right] + (\Delta^2 v)\psi.$$

From (**??**) we see that $G \in C([0,T],[H_{00}^{\frac{3}{2}}(\Omega)]')$ so, by standard results for plate equations (see [**?**]), we obtain $h \in C([0,T],H_{00}^{\frac{1}{2}}(\Omega)) \cap C^1([0,T],[H_{00}^{\frac{3}{2}}(\Omega)]')$, so w satisfies (**??**). □

We will now show that the result proved in Theorem **??** can be improved, at least if Ω is a square. In this particular case the conclusion of Theorem **??** is sharp and it shows that the regularity of w almost coincides with the regularity of the free space problem.

Proof of Theorem **??**. We will in fact prove Proposition **??** which, by Lemma **??**, is equivalent to Theorem **??**.

For the sake of simplicity we will replace t by $T - t$ in (**??**)-(**??**), so condition (**??**) becomes

$$\phi'(x,y,0) = g(x,y), \quad \text{in } \Omega. \tag{5.18}$$

If we suppose that g in (**??**) can be developed in the Fourier series

$$g(x,y) = \sum_{m,n\geq 1} (m^2+n^2) a_{mn} \sin(mx)\sin(ny), \tag{5.19}$$

a simple calculation shows that the solution ϕ of (**??**), (**??**), (**??**) is given by

$$\phi(x,y,t) = \sum_{i=1}^{\infty} \sum_{m^2+n^2=i} a_{mn} \sin(mx)\sin(ny)\sin[(m^2+n^2)t],$$

which implies that

$$\int_\gamma \frac{\partial\phi}{\partial\nu} = \sum_{i=1}^{\infty}\sum_{m^2+n^2=i} a_{mn}\int_\gamma \{\frac{\partial}{\partial\nu}[\sin(mx)\sin(ny)]\}d\gamma \sin[(m^2+n^2)t]. \tag{5.20}$$

In order to apply Lemma **??** we have to estimate the quantity

$$S_k = \sum_{m^2+n^2=k} |\int_\gamma \{\frac{\partial}{\partial\nu}[\sin(mx)\sin(ny)]\}d\gamma|^2 \tag{5.21}$$

We will distinguish three cases:

First case. Suppose that the curvature of γ never vanishes. By applying Lemma **??** we obtain the existence of a constant $C > 0$ such that

$$|\int_\gamma \{\frac{\partial}{\partial\nu}[\sin(mx)\sin(ny)]\}d\gamma|^2 \leq C\sqrt{m^2+n^2}, \; \forall m,n \geq 1. \tag{5.22}$$

By using relations (**??**) and (**??**) we obtain

$$S_k \leq \sqrt{k} r(k), \tag{5.23}$$

where $r(k)$ is the number of representations of k in the form $k = m^2 + n^2$ with $m, n \in \mathbf{Z}$. Let us fix $\epsilon > 0$. By Theorem 338 from [**?**, p.270] there exists a constant $C > 0$ such that

$$r(k) \leq Ck^{\epsilon},$$

uniformely with respect to k, so (**??**) implies that

$$S_k \leq Ck^{\frac{1}{2}+\epsilon},\ \forall k \geq 1.$$

Relation above can also be written as

$$\sum_{m^2+n^2=k} \left| \frac{\int_\gamma \{\frac{\partial}{\partial \nu}[\sin(mx)\sin(ny)]\} d\gamma}{(m^2+n^2)^{\frac{1}{4}+\frac{\epsilon}{2}}} \right|^2 \leq C, \tag{5.24}$$

By Lemma **??** relations (**??**) and (**??**) imply the existence of a constant $C > 0$ such that

$$\int_0^T \left(\int_\gamma \frac{\partial \phi}{\partial \nu} d\gamma \right)^2 dt \leq C \sum_{m,n \geq 1} (m^2+n^2)^{\frac{1}{2}+\epsilon} a_{mn}^2. \tag{5.25}$$

As from (**??**) and (**??**) we obviously have

$$\sum_{m,n \geq 1} (m^2+n^2)^{\frac{1}{2}+\epsilon} = ||g||^2_{Y_{-\frac{3}{2}+\epsilon}},$$

relation (**??**) implies

$$\int_0^T \left(\int_\gamma \frac{\partial \phi}{\partial \nu} d\gamma \right)^2 dt \leq C ||g||^2_{Y_{-\frac{3}{2}+\epsilon}},$$

which shows that (**??**) holds true at least if $\mu = 1$.

Second case. Supose that the curvature of γ vanishes at some points where the tangent line to γ has contact of order at most μ, with $2 \leq \mu$. In order to estimate S_k defined by (**??**) we will first use the decomposiion

$$S_k = S_k^1 + S_k^2 + S_k^3, \tag{5.26}$$

where

$$S_k^1 = \sum_{(m,n)\in A_1} |\int_\gamma \{\frac{\partial}{\partial \nu}[\sin(mx)\sin(ny)]\} d\gamma|^2,$$

$$A_1 = \{(m,n) \in \mathbf{Z} \times \mathbf{Z} | m^2+n^2 = k,\ m \leq n^{\frac{1}{\mu+1}}\},$$

$$S_k^2 = \sum_{(m,n)\in A_2} |\int_\gamma \{\frac{\partial}{\partial \nu}[\sin(mx)\sin(ny)]\} d\gamma|^2,$$

$$A_2 = \{(m,n) \in \mathbf{Z} \times \mathbf{Z} | m^2+n^2 = k,\ n^{\frac{1}{\mu+1}} \leq m \leq n\},$$

$$S_k^3 = \sum_{(m,n)\in A_1} |\int_\gamma \{\frac{\partial}{\partial \nu}[\sin(mx)\sin(ny)]\} d\gamma|^2,$$

$$A_3 = \{(m,n) \in \mathbf{Z} \times \mathbf{Z} | m^2 + n^2 = k,\ m \geq n\}.$$

We can now apply Lemma **??**. By using (**??**) we obviously have

$$S_k^1 \leq C k^{\frac{\mu}{2(\mu+1)}} \mathrm{card}(A_1), \tag{5.27}$$

where by $\mathrm{card}(A_1)$ we denoted the cardinal number of A_1. A simple calculation shows that $\mathrm{card}(A_1) \leq k^{\frac{1}{2(\mu+1)}}$ so from (**??**) we obtain

$$S_k^1 \leq C k^{\frac{1}{2}}. \tag{5.28}$$

In order to estimate S_k^2 we use the definition of A_2, relation (**??**) and the asymptotic behavior of the fonction $r(k)$ (introduced in the study of the first case) to obtain

$$S_k^2 \leq C k^{\epsilon} \frac{n^{\frac{2\mu-1}{2\mu}}}{m^{\frac{\mu-1}{\mu}}} \leq C k^{\frac{\mu}{\mu+1}+\epsilon}. \tag{5.29}$$

As for S_k^3 the estimate is completely similar to those given in the first case, so we have

$$S_k^3 \leq C k^{\frac{1}{2}+\epsilon}. \tag{5.30}$$

From (**??**)-(**??**) it follows that

$$S_k \leq C k^{\frac{\mu}{\mu+1}+\epsilon}.$$

Relation above can also be written as

$$\sum_{m^2+n^2=k} \left| \frac{\int_\gamma \{\frac{\partial}{\partial \nu}[\sin(mx)\sin(ny)]\} d\gamma}{(m^2+n^2)|^{\frac{\mu}{2(\mu+1)}+\frac{\epsilon}{2}}} \right|^2 \leq C, \tag{5.31}$$

uniformely with respect to k. By Lemma **??** relations (**??**) and (**??**) imply the existence of a constant $C > 0$ such that

$$\int_0^T \left(\int_\gamma \frac{\partial \phi}{\partial \nu} d\gamma \right)^2 dt \leq C \sum_{m,n \geq 1} (m^2+n^2)^{\frac{\mu}{\mu+1}+\epsilon} a_{mn}^2. \tag{5.32}$$

As we obviously have

$$\sum_{m,n \geq 1} (m^2+n^2)^{\frac{\mu}{\mu+1}+\epsilon} = ||g||^2_{Y_{-\frac{\mu+2}{\mu+1}+\epsilon}},$$

from (**??**) we obtain

$$\int_0^T \left(\int_\gamma \frac{\partial \phi}{\partial \nu} d\gamma \right)^2 dt \leq C ||g||^2_{Y_{-\frac{\mu+2}{\mu+1}+\epsilon}},$$

which shows that (**??**) also holds if $\mu \geq 2$.

Third case. Let us now suppose that γ is a part of a straight line. For the sake of simplicity consider that

$$\gamma = \{(x, y_0) \in (0,\pi)^2 | a \leq x \leq a + l\},$$

where $a, y_0, a+l \in (0,\pi)$ with $l > 0$. A simple calculation shows that, in this case,

$$\int_\gamma \{\frac{\partial}{\partial \nu}[\sin(mx)\sin(ny)]\}d\gamma = 2\frac{n}{m}\sin\left[\frac{ml}{2}\right]\sin\left[\frac{m(2a+l)}{2}\right]\cos(ny_0),$$

and from (??) we have

$$S_k \leq 4 \sum_{m^2+n^2=k} \frac{n^2}{m^2}. \tag{5.33}$$

On the other hand

$$\sum_{m^2+n^2=k} \frac{n^2}{m^2} \leq k \sum_{m=1}^{\infty} \frac{1}{m^2} - r(k) \leq Ck,$$

uniformely with respect to k. Relation above and (??) imply that

$$S_k \leq Ck, \ \forall k \geq 1,$$

or, equivalently, we have

$$\sum_{m^2+n^2=k} \frac{n^2}{m^2(m^2+n^2)} \leq C, \ \forall k \geq 1. \tag{5.34}$$

From (??), (??), (??) and Lemma ?? we deduce that

$$\int_0^T \left(\int_\gamma \frac{\partial \phi}{\partial \nu} d\gamma\right)^2 dt \leq C \sum_{m,n\geq 1} (m^2+n^2)a_{mn}^2 = ||g||_{Y_{-1}}^2,$$

which is exactly the conclusion (??) of Proposition ??. □

6. Further extensions and comments

In this section we will show that the results proved in this paper are sharp, at least if γ is flat and we will discuss some possible extensions.

In order to show that the result in Theorem ?? is sharp, by Lemma ?? it suffices to prove the following result.

Proposition 6.1. For all α, $\epsilon > 0$ there exist $\Omega \subset \mathbf{R}^2$, a curve $\gamma \subset \Omega$ and $g \in Y_{-\epsilon}$ such that the solution ϕ of (??)-(??) does not satisfy (??).

Proof. Let us take $\Omega = (0,\pi)^2$ and $\gamma = \{(x, \frac{\pi}{2}) | \frac{\pi}{2} \leq x \leq \frac{3\pi}{4}\}$ and consider the sequence of functions

$$g_n(x,y) = (\sin x) \sum_{k=1}^{n} \frac{(2k+1)^2+1}{\sqrt{1+\alpha[(2k+1)^2+1]}} a_k \sin[(2k+1)y],$$

with $a_k = k^{-\frac{3}{2}}$, for all $k \geq 1$. A simple calculation shows that

$$\lim_{n\to\infty} ||g_n||_{L^2(\Omega)} = \infty, \tag{6.1}$$

and

$$g_n \to g, \quad \text{in } Y_{-\epsilon}, \tag{6.2}$$

where

$$g(x,y) = (\sin x)\sum_{k=1}^{\infty}\frac{(2k+1)^2+1}{\sqrt{1+\alpha[(2k+1)^2+1]}}a_k \sin[(2k+1)y]. \tag{6.3}$$

On the other hand the solution of the initial and boundary value problem (??)-(??) with initial data g_n is given by

$$\phi_n(x,y,t) = (\sin x)\sum_{k=1}^{n} a_k \sin[(2k+1)y]\sin\left\{\frac{(2k+1)^2+1}{\sqrt{1+\alpha[(2k+1)^2+1]}}t\right\},$$

which implies that

$$\int_\gamma \frac{\partial\phi_n}{\partial\nu} = \sqrt{2}\sum_{k=1}^{n}(-1)^k(2k+1)a_k \sin\left\{\frac{(2k+1)^2+1}{\sqrt{1+\alpha[(2k+1)^2+1]}}t\right\}. \tag{6.4}$$

As the sequence $\lambda_k = \frac{(2k+1)^2+1}{\sqrt{1+\alpha[(2k+1)^2+1]}}$ has an asymptotic gap, by applying the Ball-Slemrod generalization of the Ingham inequality (cf.[?]) from (??) we obtain that, for T large enough,

$$\int_0^T (\int_\gamma \frac{\partial\phi_n}{\partial\nu})^2 dt \geq K\sum_{k=1}^{n}(2k+1)^2 a_k^2 \geq C||\phi_n||^2_{L^2(\Omega)}, \tag{6.5}$$

where the constants K and C are independent of n. If we denote by ϕ the solution of (??)-(??) with g given by (??), relation (??), combined with (??) and (??) implies that

$$\int_\gamma \frac{\partial\phi}{\partial\nu} \notin L^2(0,T),$$

so the result in Theorem ?? is sharp. □

By a completely similar method we can prove that the result in Proposition ??, and equivalently in Theorem ?? is sharp. More precisely we have

Proposition 6.2. If $\alpha = 0$, $\epsilon > 0$ there exist $\Omega \subset \mathbf{R}^2$, a curve $\gamma \subset \Omega$ and $g \in Y_{-1-\epsilon}$ such that the solution ϕ of (??)-(??) does not satisfy (??).

An interesting open problem is the generalization of Theorem ?? for more general domains. The methods in this paper can still be applied if Ω is a rectangle or a disk but for a general Ω a different method should be used.

Finally let us notice that the results in Theorem ?? and Theorem ?? can be easily generalized to the case of n space dimensions, with $n \geq 3$, provided that $\gamma \subset \Omega$ is a closed hypersurface with nonvanishing principal curvatures. More precisely we obtain this generalization by applying the Laplace-Fourier transform and a result from [?] for estimating the Fourier transform of $\frac{\partial\delta_\gamma}{\partial\nu}$. If some principal curvatures are vanishing the generalization to the n space dimensions of the results in this paper seems an open question.

References

1. J. M. Ball and M. Slemrod, *Nonharmonic Fourier series and the stabilization pf semilinear control systems*, Comm. pure and appl. math., 32(1979), 555-587.
2. H. T. Banks and R. C. Smith, *The modelling of piezoceramic patch interactions with shells, plates and beams*, ICASE Report No. 92-66.
3. H. T. Banks, W. Fang, R. J. Silcox and R. C. Smith, *Approximation methods for control of tye acoustic/sructure interaction with piezoceramic actuators*, preprint
4. E. F. Crawley and E. H. Anderson, *Detailed models for piezoceramic actuation of beams*, J. of Intell. Mater; Syst., 1(1990), p.4-25.
5. Ph. Destuynder, I. Legrain, L. Castel, N. Richard, *Theoretical, numerical and experimental discussion of the use of piezoelectric devices for control-structure interaction* Eur. J. Mech., A/Solids, 11(1992), pp. 181–213.
6. G. Doetsch *Introduction to the theory and applications of the Laplace Transformation*, Springer Verlag, New York/Berlin, 1974
7. Hardy and Wright, *An introduction to the theory of numbers*, Clarendon Press, Oxford, 1975.
8. C. S. Herz, *Fourier transforms related to convex sets*, Ann.of Math., 75(1962); p.81-92
9. D. G. Kendall,*On the number of lattice-points inside a random oval*, QuartJ.Math.oxford Ser.19(1948), p.1-26
10. J. E. Lagnese, *Boundary stabilization of thin plates*, SIAM, Philadelphia, 1989
11. J. E. Lagnese and J. L. Lions, *Modelling, analysis and control of thin elastic plates*, Masson, Paris, 1988
12. I. Lasiecka and R. Triggiani, *A lifting theorem for the time regularity of solutions to abstract equations with unbounded operators and applications to hyperbolic equations*, Proc. Amer. Math. Soc., 10(1988), pp. 745–755.
13. J. L. Lions, *Contrôlabilité exacte des systèmes distribués*, Masson, Paris, 1988.
14. J. L. Lions, *Pointwise control for distributed systems*, in Control and estimation in distributed parameter systems, H, T. Banks ed., SIAM, Philadelphia, 1992, pp.1-41.
15. J. L. Lions and E. Magenes, *Nonhomogenous boundary value problems*, Springer, Berlin, 1972.
16. Y. Meyer, *Etude d'un modele mathématique issu du contrôle des structures spatiales déformables*, in "Nonlinear Partial Differential Equations and their Applications", vol. 2, Research notes in Mathematics, Pitman, Boston 1985, p.234-342.
17. B. Randol, *On the Fourier transform of the indicator function of a planar set*, Trans. A.M.S.,139(1969), p.271-278
18. R. Triggiani, *Regularity with point control. Part 2. Kirchhoff equations*, J. Diff.Eq., 103(1993), p.394– 420.
19. R. Triggiani, *Regularity with point control. Part 1. Wave equations and Bernoulli-Euler equations*, Springer Verlag LNCIS
20. R. Triggiani, *Interior and boundary regularity of the wave equation with interior point control*, Differential and Integral Equations, 6 (1993), pp.111– 129.
21. M. Tucsnak, *Contrôle d'une poutre avec actionneur piézoélectrique*, C.R. Acad. Sci., Série 1, 319(1994), p.697-702
22. M. Tucsnak, *Regularity and exact controllability for a beam with piezoelectric actuator*, to appear in SIAM J. on Control