GENERIC WELL-POSEDNESS IN A MULTIDIMENSIONAL HYPERBOLIC INVERSE PROBLEM

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Abstract. Let $y(f)$ and $u(p,a,h)$ be the solutions respectively to initial/boundary value problems for hyperbolic equations in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \ge 1)$ with smooth boundary $\partial\Omega$:

$$\left\{\begin{array}{r} y''(x,t) = \Delta y(x,t) - p(x)y(x,t) - f(x)\lambda(x,t) \quad (x \in \Omega,\, t > 0)\\ y(x,0) = y'(x,0) = 0 \qquad (x \in \Omega)\\ y(x,t) = 0 \qquad (x \in \partial\Omega,\, t > 0).\end{array}\right\}$$

$$\left\{\begin{array}{r} u''(x,t) = \Delta u(x,t) - p(x)u(x,t) \qquad (x \in \Omega,\, t > 0)\\ u(x,0) = a(x), \quad u'(x,0) = 0 \qquad (x \in \Omega)\\ u(x,t) = h(x,t) \qquad (x \in \partial\Omega,\, t > 0).\end{array}\right\}$$

Here $u' = \frac{\partial u}{\partial t}$ and $u'' = \frac{\partial^2 u}{\partial t^2}$. For a given $\Gamma \subset \partial\Omega$ and a sufficiently large $T < \infty$, we consider the two inverse problems.
(I) Determine $f(x)$ $(x \in \Omega)$ from $\frac{\partial y(f)}{\partial n}(x,t)$ $(x \in \Gamma,\, 0 < t < T)$, provided that $p(x)$ and $\lambda(x,t)$ are given functions.
(II) Determine $p(x)$ $(x \in \Omega)$ from $\frac{\partial u(p,a,h)}{\partial n}(x,t)$ $(x \in \Gamma,\, 0 < t < T)$, provided that $a(x)$ and $h(x,t)$ are given functions.

For the inverse problem (I), if $\frac{\lambda}{\lambda(\cdot,0)}$ satisfies conditions on regularity, then the observations $\frac{\partial y(f)}{\partial n}$ $(x \in \Gamma,\, 0 < t < T)$ determine f up to additional terms in a finite dimensional subspace of $L^2(\Omega)$. Also an estimate of $\|f\|_{L^2(\Omega)}$ by $\|\frac{\partial y(f)}{\partial n}\|_{H^1(0,T;L^2(\Gamma))}$ is given. Moreover the set of λ guaranteeing unique determination of f from $\frac{\partial y(f)}{\partial n}(x,t)$ $(x \in \Gamma,\, 0 < t < T)$, is dense in an appropriate norm.

For the inverse problem (II), for a given h, we discuss the set $\mathcal{D}_h$ of initial values a guaranteeing L^∞-local well-posedness around a given p, by which we understand that the map $q \longrightarrow \frac{\partial u(q,a,h)}{\partial n}$ from $L^2(\Omega)$ to $H^1(0,T;L^2(\Gamma))$ is injective and has a continuous inverse in an L^∞-neighbourhood of a given p: $\{q \in L^\infty(\Omega); \|q-p\|_{L^\infty(\Omega)} < r\}$. We show that if $p(x) \ge 0$ $(x \in \overline{\Omega})$ and $h(x,t) > 0$ $(x \in \partial\Omega,\, t \ge 0)$ is t-independent, then $\mathcal{D}_h$ is generic in the sense that it contains any points except for a finite number of exceptions in every segment in $C^\infty(\overline{\Omega})$ starting the stationary solution u_0

$$\left\{\begin{array}{r} \Delta u_0(x,t) - p(x)u_0(x,t) = 0 \qquad (x \in \Omega,\, t > 0)\\ u_0(x) = h(x) \qquad (x \in \partial\Omega,\, t > 0).\end{array}\right\}$$

Also openness of $\mathcal{D}_h$ is shown.

Furthermore we discuss a relation between well-posedness of the inverse problems (I) and (II).

The method is based on exact boundary controllability by the Hilbert Uniqueness Method, duality, compactness and analytic perturbations of compact operators.

Keywords. inverse problem, overdetermined boundary data, exact controllability, generic well-posedness, analytic perturbation, compact operator

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§1. Introduction.

In this paper, we are concerned with determination of coefficients of lower order in hyperbolic equations from overdetermined boundary data. That is, we consider an initial/boundary value problem for a hyperbolic equation:

$$
\left\{
\begin{aligned}
u''(x,t) &= \Delta u(x,t) - p(x)u(x,t) && (x \in \Omega,\ 0 < t < T) \\
u(x,t) &= h(x,t) && (x \in \partial\Omega,\ 0 < t < T) \\
u(x,0) &= a(x), \quad u'(x,0) = 0 && (x \in \Omega).
\end{aligned}
\right\} \tag{1.1}
$$

Here, $u' = \frac{\partial u}{\partial t}$, $u'' = \frac{\partial^2 u}{\partial t^2}$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ and $0 < T < \infty$. Throughout this paper, we assume that a and h are given. Henceforth let us denote the solution to (1.1) by $u(p,a,h)(x,t)$.

Our problem can be stated as follows:

Inverse Problem. Let $p, q \in L^\infty(\Omega)$ and let $\Gamma \subset \partial\Omega$ be given.

(1) (Uniqueness) Does

$$
\frac{\partial u(q,a,h)}{\partial n}(x,t) = \frac{\partial u(p,a,h)}{\partial n}(x,t) \qquad (x \in \Gamma,\ 0 < t < T) \tag{1.2}
$$

imply $q(x) = p(x) \quad (x \in \Omega)$?

(2) (Stability) Can we estimate $q - p$ by $\frac{\partial u(q,a,h)}{\partial n} - \frac{\partial u(p,a,h)}{\partial n}$?

Here we set

$$
\frac{\partial u}{\partial n} = \sum_{i=1}^{n} \nu_i(x) \frac{\partial u}{\partial x_i}(x) \quad (x \in \partial\Omega), \tag{1.3}
$$

where $\nu(x) = (\nu_1(x) \ldots, \nu_n(x))$ is the outward unit normal to $\partial\Omega$. Thus we will consider determination of q from overdetermined boundary observation.

This inverse problem is nonlinear in the sense that the correspondence $q \longrightarrow \frac{\partial u(q,a,h)}{\partial n}$ is nonlinear for every fixed a and h. In this paper, for solving the inverse problem, assuming that p is known, we linearize the problem around p with respect to q, so that we get a local result near p. That is, let us consider a similar inverse problem for

$$(1.4)\qquad \left\{\begin{array}{r} y''(x,t) = \Delta y(x,t) - p(x)y(x,t) - f(x)u(p,a,h)(x,t) \\ (x\in\Omega,\, 0<t<T) \\ y(x,t) = 0 \quad (x\in\partial\Omega,\, 0<t<T) \\ y(x,0)=0, \quad y'(x,0)=0 \qquad (x\in\Omega), \end{array}\right\}$$

where $f \in L^2(\Omega)$ and $u(p,a,h)$ is the solution to (1.1). Thus we first consider

Linearized Inverse Problem. Let $\lambda \in H^2(0,T;L^\infty(\Omega))$ be a given function and let us consider

$$(1.5)\qquad \left\{\begin{array}{r} y''(x,t) = \Delta y(x,t) - p(x)y(x,t) - f(x)\lambda(x,t) \qquad (x\in\Omega,\, 0<t<T) \\ y(x,t) = 0 \quad (x\in\partial\Omega,\, 0<t<T) \\ y(x,0)=0, \quad y'(x,0)=0 \qquad (x\in\Omega). \end{array}\right\}$$

Let $y = y(f)$ be the solution to (1.5) for $f \in L^2(\Omega)$. Then we discuss

(1) (Uniqueness) Does

$$(1.6)\qquad \frac{\partial y(f)}{\partial n}(x,t) = 0 \qquad (x\in\Gamma,\, 0<t<T)$$

imply $f(x) = 0 \quad (x\in\Omega)$?

(2) (Stability) Can we estimate f by $\frac{\partial y(f)}{\partial n}$ in appropriate norms ?

We refer to a note [24] by Puel and Yamamoto with respect to our methodology for the linearized inverse problem.

Next we will translate results for the linearized inverse problem into local results for the original nonlinear problem of determining coefficients of terms of lower order.

For the linearized problem, we can roughly state our main results as follows: f is determined up to indeterminate elements in a finite dimensional space which depends on λ and p. More precisely, we have the alternative:

(I) To λ, we can assign $N \ge 1$ and $\phi_k \in L^2(\Omega)\,(1 \le k \le N)$ such that if (1.6) holds, then

$$\lambda(\cdot,0)f \in \mathrm{Span}\{\phi_k\,;\,1 \le k \le N\}.$$

(II) The equality (1.6) implies $f(x) = 0 \quad (x \in \Omega)$.

Moreover we can estimate the norm of $\lambda(\cdot,0)f$ in the quotient space $L^2(\Omega)/\mathrm{Span}\{\phi_k; 1 \le k \le N\}$ by norms of $\frac{\partial y(f)}{\partial n}$ in the case (I) and the $L^2(\Omega)$-norm of $\lambda(\cdot,0)f$ in the case (II).

For the original nonlinear inverse problem, we can roughly state our main results as follows. Let us assume that a and h are given smooth functions with suitable compatibility conditions, and $p(x) \ge 0\,(x \in \overline{\Omega})$, $h(x,t) = h(x)$ is independent of t, $h(x) > 0\,(x \in \partial\Omega)$. Then there exists a dense and open set $\mathcal{D}_h$ in a suitable norm such that for any $a \in \mathcal{D}_h$, the equality (1.3) implies $q(x) = p(x)\,(x \in \Omega)$ and we have estimates

$$\begin{aligned}
&C^{-1}\left\|\frac{\partial u(q,a,h)}{\partial n} - \frac{\partial u(p,a,h)}{\partial n}\right\|_{H^1(0,T;L^2(\Gamma))} \le \|q-p\|_{L^2(\Omega)}\\
\le& C\left\|\frac{\partial u(q,a,h)}{\partial n} - \frac{\partial u(p,a,h)}{\partial n}\right\|_{H^1(0,T;L^2(\Gamma))},
\end{aligned}$$

provided that $\|q-p\|_{L^\infty(\Omega)}$ is sufficiently small.

Our inverse problem is very typical and has been studied extensively. Although we do not know whether $\mathcal{D}_h = \{a \in C^\infty(\overline{\Omega})\,;\, a(x) > 0 \,\text{for}\; x \in \overline{\Omega}\}$, we can conclude that this inverse problem is well-posed generically and locally respectively in the senses that the set $\mathcal{D}_h$ of initial values a's guaranteeing the well-posedness, is dense and sufficiently wide, provided that an unknown coefficient q is restricted to a neighbourhood of a known p. Throughout this paper, we intend to assert that some kind of inverse problems such as our typical one is generically well-posed with reasonable norms, although every inverse problem is indiscriminately considered as ill-posed in the sense that uniqueness does not hold or that we can not establish a stability estimate with suitable norms.

In relation with the linearized inverse problem, we refer to Engl, Scherzer and Yamamoto [6], Yamamoto [32] – [36]. Except for Proposition 5.5 in [6], the determination of f from boundary data is discussed in the case where $\lambda(x,t)$ is x-independent by exact controllability of the dual system.

In relation with our nonlinear inverse problem, we will refer to three types of researches.

(A) Approach for determining coefficients by Carleman's estimate : Bukhgeim and Klibanov [5], Isakov [9], [11] and Klibanov [13].

In Klibanov [13], the following uniqueness result is shown. In (1.1) let us set $\Omega = V_\rho \equiv \{x \in \mathbb{R}^n \,;\, |x|_{\mathbb{R}^n} < \rho\}$ and let us assume that $T > 2\rho$. Then if $a(x) \neq 0\,(x \in V_\rho)$, then

$$\frac{\partial u(q,a,h)}{\partial n}(x,t) = \frac{\partial u(p,a,h)}{\partial n}(x,t) \quad (x \in \partial V_\rho,\, t > 0)$$

implies

$$q(x) = p(x) \quad (x \in V_\rho).$$

Here we note that $p, q \in C^1(\overline{\Omega})$ and the normal derivatives are observed on the whole boundary $\partial\Omega$ and that the method is based on Carleman estimates (e.g. Isakov [9]), so that the argument is within classical solutions.

(B) The case where Ω is one-dimensional : Suzuki [28], Nakagiri [23]. The latter is an extensive survey.

Let $u(p,a,b)$ be the solution to

$$
(1.7) \qquad \left\{
\begin{array}{r}
u''(x,t) = u_{xx}(x,t) - p(x)u(x,t) \quad (0<x<1,\ 0<t<T) \\
\\
u(x,t) = 0 \quad (x=0,1,\ 0<t<T) \\
\\
u(x,0) = a(x), \quad u'(x,0) = b(x) \qquad (0<x<1).
\end{array}
\right\}
$$

In [28], it is proved that if $T > 4$, then

$$
\frac{\partial u(q,c,d)}{\partial x}(x,t) = \frac{\partial u(p,a,b)}{\partial x}(x,t) \quad (x=0,1,\ 0<t<T)
$$

implies

$$
q(x) = p(x),\ c(x) = a(x),\ d(x) = b(x) \quad (0<x<1),
$$

provided that all the components of (a,b) on the eigenfunctions of the first order system in $L^2(\Omega) \times L^2(\Omega)$ associated with (1.7), is non-zero. Then we can estimate $q - p$ by

$$
\left(\frac{\partial u(q,c,d)}{\partial x} - \frac{\partial u(p,a,b)}{\partial x} \right)(x,t) \quad (x=0,1,\ 0<t<T).
$$

This condition of (a,b) can be regarded as generic one, so that we can express that the inverse problem is well-posed "generically". Since the key is the Gel'fand-Levitan theory ([7]), the generalization to the multidimensional case is extremely difficult with the formulation in [28].

(C) Approach by the Dirichlet to Neumann map : Isakov [9]–[11], Rakesch and Symes [25], Sun [27], Sylvester and Uhlmann [29].

¿From the practical point of view, Approach (C) corresponds to repeat of infinite times observations of boundary data, while our formulation is from a single time observation of boundary data. In this sense, our approach is practical, but so far the charaterization of initial values guaranteeing the well-posedness, seems not perfect.

For other types of inverse problems in hyperbolic equations, we should further refer to extensive researches by a Novosibirsk school (Lavrentiev, Reznitskaya and Yakhno [18], Lavrentiev, Romanov and Shis·hatskiĭ[19], Romanov [26]).

By the technique by Reznitskaya (e.g. [18], [26]) and estimates established in Vu and Yamamoto [31], we can translate a similar inverse problem for parabolic equations into (1.1), so that we can reach well-posedness for determination of potential terms in parabolic equations.

Our methodology is based on

(i) duality argument between our inverse problem and exact controllability

(ii) exact controllability by the Hilbert Uniqueness Method (e.g. Komornik [14], Lions [20], [21]).

(iii) a compactness argument

The point (i) is a key in Engl, Scherzer and Yamamoto [5]. These three points are avaiable for other types of equations in mathematical physics (Lagnese [15], [16], Lions [20], [21] for the point (ii), Lions and Magenes [22] for the point (iii), for example). Thus our methodology proposed in this paper is widely effective for showing generic well-posedness of inverse problems of determining coefficients in other types of equations. In this paper, however, we exclusively discuss a wave equation (1.1) in order to show the essence of how to apply our methodology to multidimesnional inverse problems.

In the point of using boundary control systems for inverse spectral problems, we can quote Belishev and Kurylev [3].

This paper is composed of seven sections. In §2 and §3 we state results respectively for the linearized inverse problem and the original nonlinear inverse problem.

§4 is devoted to a relationship between both inverse problems in terms of a linearization. In §5, §6 and §7 we prove the results respectively of §2, §4 and §3.

§2. Main Results for the Linearized Inverse Problem.

Henceforth, we assume that

$$\lambda, \quad \frac{\lambda(\cdot,\cdot)}{\lambda(\cdot,0)} \in H^2(0,T;L^\infty(\Omega)). \tag{2.1}$$

$$p \in L^\infty(\Omega). \tag{2.2}$$

Here for a Banach space X, we set $H^2(0,T;X) = \{u\,;\, u, u', u'' \in L^2(0,T;X)\}$. We recall that $\nu(x)$ is the outward unit normal vector to $\partial\Omega$ at x. For an arbitrarily fixed $x_0 \in \mathbb{R}^n$, we set

$$\Gamma = \{x \in \partial\Omega\,;\, (x - x_0, \nu(x))_{\mathbb{R}^n} > 0\} \tag{2.3}$$

$$R_0 = \sup\{|x - x_0|_{\mathbb{R}^n}\,;\, x \in \partial\Omega\}.$$

This corresponds to the Bardos-Lebeau-Rauch condition ([2]). Our first theorem asserts an alternative like the Fredholm one for the compact operator equation.

Theorem 1. *We assume*

$$T > 2R_0 \tag{2.4}$$

Either one of the following two cases holds.

(1) For given λ and p, there exist $N = N(\lambda,p) \in \mathbb{N}$ and linearly independent $\phi_k = \phi_k(\lambda,p) \in L^2(\Omega)\,(1 \le k \le N)$ such that if

$$\frac{\partial y(f)}{\partial n}(x,t) = 0 \quad (x \in \Gamma,\, 0 < t < T), \tag{2.5}$$

then

$$\lambda(\cdot,0)f \in Span\{\phi_k\,;\, 1 \le k \le N\} \equiv X_0.$$

Moreover there exists a constant $C_1 = C_1(\lambda, p) > 0$ depending only on λ and p such that

$$\|\lambda(\cdot,0)f\|_{L^2(\Omega)/X_0} \le C_1 \left\| \frac{\partial}{\partial t}\frac{\partial y(f)}{\partial n}\right\|_{L^2(\Gamma\times(0,T))} \tag{2.6}$$

for any $f \in L^2(\Omega)$.

(2) For given λ and p, the equality (2.5) implies $f(x) = 0 \quad (x \in \Omega)$. Moreover there exists a constant $C_1 > 0$ such that

$$\|\lambda(\cdot,0)f\|_{L^2(\Omega)} \le C_1 \left\| \frac{\partial}{\partial t}\frac{\partial y(f)}{\partial n}\right\|_{L^2(\Gamma\times(0,T))} \tag{2.7}$$

for any $f \in L^2(\Omega)$.

Corollary 1. *In addition to the assumptions in Theorem 1, we assume*

$$\lambda(x,0) \ge \lambda_0 > 0 \quad (x \in \overline{\Omega}) \tag{2.8}$$

where λ_0 is a constant. Then in the second alternative of Theorem 1, in place of (2.7), we have a lower and upper estimate

$$C_1^{-1}\left\| \frac{\partial}{\partial t}\frac{\partial y(f)}{\partial n}\right\|_{L^2(\Gamma\times(0,T))} \le \|f\|_{L^2(\Omega)} \le C_1 \left\| \frac{\partial}{\partial t}\frac{\partial y(f)}{\partial n}\right\|_{L^2(\Gamma\times(0,T))}$$

for any $f \in L^2(\Omega)$.

Here and henceforth we set

$$\|f\|_{L^2(\Omega)/X_0} = \inf_{f_0\in X_0} \|f - f_0\|_{L^2(\Omega)},$$

the quotient norm in $L^2(\Omega)/X_0$. The second alternative shows that $X_0 = \{0\}$, namely, $N = 0$, in the first alternative and corresponds to well-posedness of our linearized inverse problem. In the first alternative, the uniqueness does not hold, so that the inverse problem is ill-posed. However our boundary observations $\frac{\partial y(f)}{\partial n}(x,t)$ $(x \in \Gamma,\ 0 < t < T)$ determine f up to indeterminate elements in a finite dimensional space $X_0 = X_0(\lambda, p)$, and if we adopt the quotient norm as norm measuring solutions f's, then we can get a stability estmate (2.6).

If we can assume some smallness condition or a special form on λ, we can exclude the ill-posed case (1) in Theorem 1. That is,

Proposition 1. *Let us assume (2.3) and let λ satisfy (2.1).*

(i) If

$$\left\| \frac{\lambda'}{\lambda(\cdot, 0)} \right\|_{L^1(0,T;L^\infty(\Omega))}$$

is sufficiently small, then the first alternative in Theorem 1 cannot occur. In other words, the equality (2.5) implies $f(x) = 0\,(x \in \Omega)$ and we have the estimate (2.6).

(ii) Let (2.4) hold and let λ satisfy (2.1). Moreover if λ has a form

$$\lambda(x,t) = \mu(t)\eta(x) \quad (x \in \Omega,\ t > 0),$$

then the first alternative cannot occur.

The part (i) of this proposition is a slight modification of Proposition 5.5 in Engl, Scherzer and Yamamoto [6], while the part (ii) directly follows from Theorem 1 in Yamamoto [34] or [35].

So far, for a given λ, we do not know any criterion implying the second alternative, the well-posedness. However, in some sense, we "generically" always have the well-posedness. More precisely,

Theorem 2. *Let us put*

$$\Lambda = \{\lambda \in H^2(0,T;L^\infty(\Omega))\,;\, \lambda \quad \textit{satisfies (2.1) and (2.8)}\}.$$

The set

$$\mathcal{D} = \{\lambda \in \Lambda\,;\, \textit{the second alternative in Theorem 1 occurs for} \quad \lambda\} \tag{2.9}$$

is open in Λ in the topology of $H^1(0,T;L^\infty(\Omega))$ and is dense in Λ in the topology of $H^2(0,T;L^\infty(\Omega))$.

§3. Main Results for the Nonlinear Inverse Problem.

In this section, we consider the original inverse problem, the determination of coefficient in a wave equation. Henceforth we fix p and h, while we change a, initial data, in order to attain the well-posedness around p. Let us assume

$$h \in C^\infty(\partial\Omega \times [0,T]) \qquad h(x,t) > 0 \quad (x \in \partial\Omega,\, 0 \le t \le T). \tag{3.1}$$

For a given h, we define an admissible set of initial values a's :

$$\begin{aligned} \mathcal{U}_h =&\{a \in L^\infty(\Omega)\,;\, a(x) > 0\,(x \in \overline{\Omega}) \\ &\qquad \text{such that } u(p,a,h) \in H^2(0,T;L^\infty(\Omega))\}. \end{aligned} \tag{3.2}$$

In (3.2) we are actually required to assume suitable compatibility conditions for a and h on $\partial\Omega \times \{0\}$.

For simplicity of the succeeding statements, we introduce

Definition. *For $a \in \mathcal{U}_h$, the inverse problem*

$$\frac{\partial u(q,a,h)}{\partial n}(x,t)\,(x \in \Gamma,\, 0 < t < T) \implies q(x)\,(x \in \Omega)$$

is called to be L^∞-locally well-posed around p if q and $\widetilde{q}$ are sufficiently close to p in $L^\infty(\Omega)$, then the equality

$$\frac{\partial u(q,a,h)}{\partial n}(x,t) = \frac{\partial u(\widetilde{q},a,h)}{\partial n}(x,t) \quad (x \in \Gamma,\, 0<t<T) \tag{3.3}$$

implies $q(x) = \widetilde{q}(x)$ $(x \in \Omega)$, and there exists a constant $C_2 = C_2(h,a,p)$ such that

$$\|q-\widetilde{q}\|_{L^2(\Omega)} \le C_2 \left\| \frac{\partial u(q,a,h)}{\partial n} - \frac{\partial u(\widetilde{q},a,h)}{\partial n} \right\|_{H^1(0,T;L^2(\Gamma))}. \tag{3.4}$$

Remark. As is seen from Lemma 2 (ii) in §5, we can always prove that

$$C_2^{-1} \left\| \frac{\partial u(q,a,h)}{\partial n} - \frac{\partial u(\widetilde{q},a,h)}{\partial n} \right\|_{H^1(0,T;L^2(\Gamma))} \le \|q-\widetilde{q}\|_{L^2(\Omega)}$$

with some $C_2 = C_2(h,a,p) > 0$. Therefore in order to estimate $\|q-\widetilde{q}\|_{L^2(\Omega)}$, the norm $\|\cdot\|_{H^1(0,T;L^2(\Gamma))}$ for the observation is the best possible.

Our first main result for the original inverse problem shows that the local well-posedness is reserved against perturbations of initial data a's in a suitable norm.

Theorem 3. *Let us define an operator A by*

$$\begin{aligned} &(Au)(x) = \Delta u(x) - p(x)u(x) \quad (x\in\Omega) \\ &\mathcal{D}(A) = \{u \in H^2(\Omega); u_{|\partial\Omega} = 0\}. \end{aligned} \tag{3.5}$$

Only in this theorem, in addition to (2.2), we assume that $p \in C^\infty(\overline{\Omega})$. We set

$$l = \left[\frac{n}{4} + \frac{1}{2}\right] + 1, \tag{3.6}$$

where $[\alpha]$ denotes the greatest integer among ones $\le \alpha$. Let us assume that for a, the inverse problem

$$\frac{\partial u(q,a,h)}{\partial n}(x,t)\,(x\in\Gamma,\,0<t<T) \Longrightarrow q(x)\,(x\in\Omega)$$

is L^∞-locally well-posed. Then there exists $\delta(a) > 0$ such that if $b \in L^2(\Omega)$ satisfies

$$\|A^l(b-a)\|_{L^2(\Omega)} + \|b-a\|_{L^2(\Omega)} \le \delta(a), \tag{3.7}$$

then for b, the inverse problem

$$\frac{\partial u(q,b,h)}{\partial n}(x,t)\,(x\in\Gamma,\,0<t<T) \Longrightarrow q(x)\,(x\in\Omega)$$

is L^∞-locally well-posed.

We set

$$\mathcal{D}_h = \{a \in \mathcal{U}_h\,;\text{ the inverse problem is } L^\infty\text{-locally well-posed}$$

$$\text{around } p \text{ for } a\}. \tag{3.8}$$

We do not know whether or not the well-posed set for initial values a's coincides with the admissible set $\mathcal{U}_h$, namely, $\mathcal{D}_h = \mathcal{U}_h$, and possibly $\mathcal{D}_h = \emptyset$ may happen. For determination of coefficients, it is desirable that the set $\mathcal{D}_h$ may be as large as possible. Under an additional condition on h and p, we can show that $\mathcal{D}_h$ is sufficiently general.

Proposition 2. *Let us assume (2.4) and*

$$h(x,t) = h(x) > 0 \quad (x\in\partial\Omega,\ 0\le t\le T), \qquad p(x)\ge 0 \quad (x\in\overline{\Omega}). \tag{3.9}$$

For any $a \in \mathcal{U}_h$ and any $\epsilon > 0$, there exists $a_\epsilon \in \mathcal{D}_h$ such that

$$\|a - a_\epsilon\|_{L^\infty(\Omega)} < \epsilon.$$

In other words, $\mathcal{D}_h$ is dense in $\mathcal{U}_h$ in the topology of $L^\infty(\Omega)$.

Thus we can conclude that our inverse problem is locally well-posed for "generic" choices of a's on an additional assumption (3.9). In fact, we can further prove

Theorem 4 (a characterization of $\mathcal{D}_h$). *Let a_0 be the solution to*

$$\left\{\begin{aligned} \Delta a_0(x) - p(x)a_0(x) = 0 \quad (x \in \Omega) \\ a_0(x) = h(x) \quad (x \in \partial\Omega). \end{aligned}\right\} \tag{3.10}$$

For any $a \in \mathcal{U}_h$, there exists a finite set $E = E(a) \subset [0,1]$ such that

$$\{(1-\xi)a_0 + \xi a\,;\, \xi \in [0,1] \setminus E \cup \{0\}\} \subset \mathcal{D}_h. \tag{3.11}$$

In particular, we note that $a_0 \in \mathcal{D}_h$.

This theorem asserts that the inverse problem is locally well-posed for a's except for a finite number of functions in the segment connecting a_0 and $a \in \mathcal{U}_h$.

§4. Relation between the Nonlinear Inverse Problem and the Linearized Inverse Problem.

For fixed p and h, we assume (2.2) and (3.1), and let $a \in \mathcal{U}_h$. Here we recall that $\mathcal{U}_h$ is defined by (3.2). For studying the relation between the original nonlinear inverse problem and the linearized inverse problem, we define a map $\mathcal{F} : L^\infty(\Omega) \longrightarrow H^1(0,T;L^2(\Gamma))$ by

$$(\mathcal{F}q)(x,t) = \frac{\partial u(q,a,h)}{\partial n}(x,t) \quad (x \in \Gamma,\ 0 < t < T). \tag{4.1}$$

Then $\mathcal{F}$ is well-defined as a map from $L^\infty(\Omega)$ to $H^1(0,T;L^2(\Gamma))$. In fact,

Proposition 3. *If $a \in \mathcal{U}_h$, then $\mathcal{F}q \in H^1(0,T;L^2(\Gamma))$ for any $q \in L^\infty(\Omega)$. Moreover if $\|q-p\|_{L^\infty(\Omega)} \le M_1$: arbitrary positive constant, then there exists a constant $C_3 = C_3(\Omega,T,h,a,p,M_1) > 0$ such that*

$$\|\mathcal{F}q - \mathcal{F}p\|_{H^1(0,T;L^2(\Gamma))} \le C_3\|q-p\|_{L^2(\Omega)}. \tag{4.2}$$

In terms of Corollary 1 in §2, concerning the linearized inverse problem, we have to measure $q-p$ in the topology of $L^2(\Omega)$, not of $L^\infty(\Omega)$ if we adopt $H^1(0,T;L^2(\Gamma))$-norm for $\frac{\partial u(q,a,h)}{\partial n} - \frac{\partial u(p,a,h)}{\partial n}$. However $\mathcal{F}$ may not be Gâteaux differentiable from $L^2(\Omega)$ to $H^1(0,T;L^2(\Gamma))$. More precisely,

Proposition 4. *For fixed p and h, we assume (2.2) and (3.1), and let $a \in \mathcal{U}_h$. Let us consider an initial/boundary value problem for a wave equation*

$$
\left\{
\begin{aligned}
& y''(x,t) = \Delta y(x,t) - p(x)y(x,t) - f(x)u(p,a,h)(x,t) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x \in \Omega,\ 0<t<T) \\
& \qquad\qquad\qquad\qquad y(x,t) = 0 \quad (x \in \partial\Omega,\ 0<t<T) \\
& \qquad y(x,0) = 0, \quad y'(x,0) = 0 \qquad (x \in \Omega).
\end{aligned}
\right\}
\tag{4.3}
$$

Then there exists a unique solution

$$
y = y(p,a,f) \in C^1([0,T];H_0^1(\Omega)) \cap C^2([0,T];L^2(\Omega))
\tag{4.4}
$$

to (4.3) and in addition we have

$$
\frac{\partial y(p,a,f)}{\partial n} \in H^1(0,T;L^2(\partial\Omega)).
\tag{4.5}
$$

Moreover let us fix a constant $M_1 > 0$ arbitrarily and let $\|q\|_{L^\infty(\Omega)} \le M_1$. Then there exists a constant $C_4 = C_4(\Omega,T,M_1,h,a,p) > 0$ such that

$$
\begin{aligned}
&\left\| \left(\frac{\partial u(q,a,h)}{\partial n} - \frac{\partial u(p,a,h)}{\partial n} \right) - \frac{\partial y(p,a,q-p)}{\partial n} \right\|_{H^1(0,T;L^2(\partial\Omega))} \\
\le\, & C_4 \|q-p\|_{L^\infty(\Omega)} \|q-p\|_{L^2(\Omega)}.
\end{aligned}
\tag{4.6}
$$

In other words, from the viewpoint of the Fréchet derivative, we must adopt the $L^\infty(\Omega)$-norm. However we can merely obtain a stability estimate with a weaker

$L^2(\Omega)$-norm in the linearized inverse problem. This discrepancy of norms results in the choice of norms of $q-p$ in Definition in §3.

¿From Proposition 4, taking into consideration this discrepancy, we can show a relation between the nonlinear inverse problem and the linearized inverse problem.

Theorem 5. *For fixed p and h, we assume (2.2) and (3.1), and let $a \in \mathcal{U}_h$. Let us assume that*

$$\frac{\partial y(p,a,f)}{\partial n}(x,t) = 0 \quad (x \in \Gamma,\ 0 < t < T)$$

and $f \in L^2(\Omega)$ imply $f = 0$. Then the nonlinear inverse problem

$$\frac{\partial u(q,a,h)}{\partial n}(x,t)\,(x \in \Gamma,\ 0 < t < T) \implies q(x)\,(x \in \Omega)$$

is L^∞-locally well-posed around p.

Remark. We can prove a converse to Theorem 5. That is, let us assume that the nonlinear inverse problem

$$\frac{\partial u(q,a,h)}{\partial n}(x,t)\,(x \in \Gamma,\ 0 < t < T) \implies q(x)\,(x \in \Omega)$$

is L^∞-locally well-posed around p. Then

$$\frac{\partial y(p,a,f)}{\partial n}(x,t) = 0 \quad (x \in \Gamma,\ 0 < t < T)$$

and $f \in L^\infty(\Omega)$ imply $f = 0$.

§5. Proof of the Results in §2.

Proof of Theorem 1.

We divide the proof into the following three steps.

First Step. In this step, we show a result on exact controllability and apply the duality argument, so that we can reduce our linearized inverse problem to an equation of the second kind. This stage corresponds to the point (i) and (ii) of our methodology mentioned at the end of §1.

We show Lemmata 1 - 4.

Lemma 1 (Komornik [14]**).** *Let $T > 2R_0$ and let $p \in L^\infty(\Omega)$. Then to each $\phi_0 \in L^2(\Omega)$ we can assign $v_0 \in L^2(\Gamma \times (0,T))$ such that the solution $\phi = \phi(v_0)$ to*

$$\begin{cases} \phi''(x,t) = \Delta\phi(x,t) - p(x)\phi(x,t) \qquad (x \in \Omega,\, 0 < t < T) \\ \phi(x,T) = 0, \quad \phi'(x,T) = 0 \qquad (x \in \Omega) \\ \phi(x,t) = 0 \quad (x \in \partial\Omega \setminus \Gamma,\, 0 < t < T) \\ \phi(x,t) = v_0(x,t) \quad (x \in \Gamma,\, 0 < t < T) \end{cases} \tag{5.1}$$

satisfies

$$\phi(x,0) = \phi_0(x), \quad \phi'(x,0) = 0 \qquad (x \in \Omega).$$

In fact, using Komornik [14, Theorem 2.1], we can construct v_0 by the Hilbert Uniqueness Method (e.g. Lions [20], [21]) and there exists a constant $C_5 > 0$ depending only on p (and independent of ϕ_0) such that

$$\|v_0\|_{L^2(\Gamma\times(0,T))} \le C_5 \|\phi_0\|_{L^2(\Omega)}. \tag{5.2}$$

Then we define an operator g by

$$g(\phi_0) = v_0,$$

so that g is a bounded linear operator from $L^2(\Omega)$ to $L^2(\Gamma \times (0,T))$, and the domain of g is $L^2(\Omega)$.

Next we show a result on regularity of the solution $y(f)$ to (1.5).

Lemma 2.

(i) Let $q \in L^\infty(\Omega)$ *and* $\|q\|_{L^\infty(\Omega)} \le M_1$ *for some constant* $M_1 > 0$. *Let* $\theta = \theta(x,t)$ *be the solution to*

$$\left\{ \begin{array}{r} \theta''(x,t) = \Delta\theta(x,t) - q(x)\theta(x,t) + G(x,t) \quad (x \in \Omega,\, 0<t<T) \\ \\ \theta(x,t) = 0 \quad (x \in \partial\Omega,\, 0<t<T) \\ \\ \theta(x,0) = \theta_0(x), \quad \theta'(x,0) = \theta_1(x) \quad (x \in \Omega), \end{array} \right\} \tag{5.3}$$

where $\theta_0 \in H_0^1(\Omega)$, $\theta_1 \in L^2(\Omega)$ *and* $G \in L^1(0,T;L^2(\Omega))$. *Then there exists a unique solution* $\theta \in C^0([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ *to (5.3) such that*

$$\begin{aligned} &\|\theta\|_{L^\infty(0,T;H_0^1(\Omega))} + \|\theta'\|_{L^\infty(0,T;L^2(\Omega))} \\ \le & C_6' \left(\|\nabla\theta_0\|_{L^2(\Omega)} + \|\theta_1\|_{L^2(\Omega)} + \|G\|_{L^1(0,T;L^2(\Omega))} \right). \end{aligned} \tag{5.4}$$

In addition, $\frac{\partial\theta}{\partial n} \in L^2(\partial\Omega \times (0,T))$ *and we have*

$$\left\| \frac{\partial\theta}{\partial n} \right\|_{L^2(\partial\Omega\times(0,T))} \le C_6' \left(\|\nabla\theta_0\|_{L^2(\Omega)} + \|\theta_1\|_{L^2(\Omega)} + \|G\|_{L^1(0,T;L^2(\Omega))} \right). \tag{5.5}$$

Here a constant $C_6' = C_6'(\Omega, T, M_1) > 0$ *is independent of* θ_0, θ_1 *and* G.

(ii) Moreover if $\theta_0 = \theta_1 = 0$ *and* $G' \in L^1(0,T;L^2(\Omega))$, *then*

$$\begin{aligned} &\theta \in C^1([0,T];H_0^1(\Omega)) \cap C^2([0,T];L^2(\Omega)) \\ &\left\| \frac{\partial\theta}{\partial n} \right\|_{H^1(0,T;L^2(\partial\Omega))} \le C_6' \left(\|G\|_{L^1(0,T;L^2(\Omega))} + \|G'\|_{L^1(0,T;L^2(\Omega))} \right). \end{aligned} \tag{5.6}$$

(iii) We assume that λ *satisfies (2.1). For any* $f \in L^2(\Omega)$, *we have*

$$y(f) \in C^1([0,T];H_0^1(\Omega)) \cap C^2([0,T];L^2(\Omega)) \tag{5.7}$$

and

$$\left\| \frac{\partial y(f)}{\partial n} \right\|_{H^1(0,T;L^2(\partial\Omega))} \le C_6 \|f\|_{L^2(\Omega)}. \tag{5.8}$$

Here a constant $C_6 = C_6(\lambda, M_1) > 0$ *is independent of* f.

Finally we show

Lemma 3 (duality equality). *We assume* $\lambda \in H^1(0,T;L^\infty(\Omega))$. *For any* $v_0 \in L^2(\Gamma\times(0,T))$, *we have*

$$\int_\Omega f(x)\left(\phi(v_0)(x,0)\lambda(x,0)+\int_0^T \lambda'(x,t)\phi(v_0)(x,t)dt\right)dx \tag{5.9}$$
$$=\int_0^T\int_\Gamma \frac{\partial}{\partial t}\frac{\partial y(f)}{\partial n}(x,t)v_0(x,t)d\sigma dt.$$

Proof of Lemma 2. (i) The existence of θ and the estimate of (5.4) are proved by the eigenfunction expansion of θ (e.g. Lemme 3.6 (p.39) in Lions [21]). Let us prove the estimate (5.5). By Théorème 4.1 (p.44) in [21] (or Lasiecka, Lions and Triggiani [17]), we see

$$\left\|\frac{\partial\theta}{\partial n}\right\|_{L^2(\partial\Omega\times(0,T))}\le C_6'\Bigg(\|\nabla\theta_0\|_{L^2(\Omega)}+\|\theta_1\|_{L^2(\Omega)}$$
$$+\|q\theta\|_{L^1(0,T;L^2(\Omega))}+\|G\|_{L^1(0,T;L^2(\Omega))}\Bigg).$$

On the other hand, we get

$$\|q\theta\|_{L^1(0,T;L^2(\Omega))}\le\|q\|_{L^\infty(\Omega)}\|\theta\|_{L^1(0,T;L^2(\Omega))}$$
$$\le M_1C_6'\left(\|\nabla\theta_0\|_{L^2(\Omega)}+\|\theta_1\|_{L^2(\Omega)}+\|G\|_{L^1(0,T;L^2(\Omega))}\right)$$

by (5.4). Therefore we see (5.5).

(ii) Let us consider

$$\left\{\begin{aligned} &\mu''(x,t)=\Delta\mu(x,t)-q(x)\mu(x,t)+G'(x,t)\quad (x\in\Omega,\ 0<t<T)\\ &\mu(x,t)=0\quad (x\in\partial\Omega,\ 0<t<T)\\ &\mu(x,0)=0,\quad \mu'(x,0)=G(x,0)\qquad (x\in\Omega).\end{aligned}\right\}$$

By the part (i) of this lemma, we see that $\mu\in C^0([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))$ and

$$\left\|\frac{\partial\mu}{\partial n}\right\|_{L^2(\partial\Omega\times(0,T))}\le C_6'\left(\|G'\|_{L^1(0,T;L^2(\Omega))}+\|G(\cdot,0)\|_{L^2(\Omega)}\right).$$

Therefore setting

$$\widetilde{\theta}(x,t) = \int_0^t \mu(x,s)ds \qquad (x \in \Omega,\ 0 \le t \le T),$$

we can verify that this $\widetilde{\theta}$ satisfies (5.3) with $\theta_0 = \theta_1 = 0$. By uniqueness of weak solutions, we get $\theta \in C^1([0,T]; H_0^1(\Omega)) \cap C^2([0,T]; L^2(\Omega))$ and

$$\left\| \frac{\partial}{\partial t}\frac{\partial \theta}{\partial n} \right\|_{L^2(\partial\Omega\times(0,T))} \le C_6' \left(\|G'\|_{L^1(0,T;L^2(\Omega))} + \|G(\cdot,0)\|_{L^2(\Omega)} \right),$$

with which we combine (5.5) with $\theta_0 = \theta_1 = 0$, so that we can reach (5.6), the conclusion of (ii).

(iii) Since λ satisfies (2.1), from (ii) we see (5.7) and

$$\begin{aligned}
&\left\| \frac{\partial y(f)}{\partial n} \right\|_{H^1(0,T;L^2(\partial\Omega))} \\
\le & C_6' \left(\|f\lambda'\|_{L^1(0,T;L^2(\Omega))} + \|f\lambda\|_{L^1(0,T;L^2(\Omega))} \right) \\
\le & C_6' \|f\|_{L^2(\Omega)} \left(\|\lambda'\|_{L^1(0,T;L^\infty(\Omega))} + \|\lambda\|_{L^1(0,T;L^\infty(\Omega))} \right),
\end{aligned}$$

which is (5.8), the conclusion of (iii). Thus the proof of Lemma 2 is complete.

Proof of Lemma 3. First, assuming that $v_0 \in C_0^\infty(\Gamma \times (0,T))$, we see that $\phi = \phi(v_0)$ is sufficiently smooth on $\overline{\Omega} \times [0,T]$. Moreover $z = y(f)'$ satisfies

$$\text{(5.10)} \quad \left\{ \begin{aligned}
&z''(x,t) = \Delta z(x,t) - p(x)z(x,t) - f(x)\lambda'(x,t) \quad (x \in \Omega,\ 0 < t < T) \\
&z(x,t) = 0 \quad (x \in \partial\Omega,\ 0 < t < T) \\
&z(x,0) = 0, \quad z'(x,0) = -f(x)\lambda(x,0) \qquad (x \in \Omega),
\end{aligned} \right.$$

so that by Lemma 2, according to the definition of the solution $\phi(v_0)$ by the transposition method (e.g. Lions and Magenes [22]), we can calculate

$$\int_\Omega \left(\int_0^T y(f)'''(x,t)\phi(v_0)(x,t)dt \right) dx$$

as follows.

$$\int_\Omega \left(\int_0^T z''(x,t)\phi(v_0)(x,t)dt \right) dx$$
$$= \int_\Omega \left([z'(x,t)\phi(v_0)(x,t)]_{t=0}^{t=T} - \int_0^T z'(x,t)\phi(v_0)'(x,t)dt \right) dx$$

(by integration by parts)

$$= - \int_\Omega \left(\int_0^T z'(x,t)\phi(v_0)'(x,t)dt \right) dx + \int_\Omega f(x)\lambda(x,0)\phi(v_0)(x,0)dx$$

(by the initial condition of z and the final condition of $\phi(v_0)$)

$$= \int_\Omega \left(- [z(x,t)\phi(v_0)'(x,t)]_{t=0}^{t=T} + \int_0^T z(x,t)\phi(v_0)''(x,t)dt \right) dx$$
$$+ \int_\Omega f(x)\lambda(x,0)\phi(v_0)(x,0)dx$$

(by integration by parts)

$$= \int_\Omega \left(\int_0^T z(x,t)\phi(v_0)''(x,t)dt \right) dx + \int_\Omega f(x)\lambda(x,0)\phi(v_0)(x,0)dx$$

(by the initial condition of z and the final condition of $\phi(v_0)$).

Therefore using the first equations in (5.10) and (5.1), we have

$$\int_0^T \left(\int_\Omega \phi(v_0)(x,t)\Delta z(x,t) - z(x,t)\Delta\phi(v_0)(x,t)dx \right) dt$$
$$= \int_\Omega f(x) \left(\phi(v_0)(x,0)\lambda(x,0) + \int_0^T \lambda'(x,t)\phi(v_0)(x,t)dt \right) dx.$$

Applying the Green formula, taking into consideration the boundary conditions of z and $\phi(v_0)$, we reach (5.9) for any $v_0 \in C_0^\infty(\Gamma \times (0,T))$.

Finally let $v_0 \in L^2(\Gamma \times (0,T))$. Since $C_0^\infty(\Gamma \times (0,T))$ is dense in $L^2(\Gamma \times (0,T))$, there exist $v_n \in C_0^\infty(\Gamma \times (0,T)) \quad (n \ge 1)$ such that

$$\|v_n - v_0\|_{L^2(\Gamma\times(0,T))} \longrightarrow 0$$

as $n \longrightarrow \infty$. Similarly to Théorème 4.2 (pp.46-47) in Lions [21] or [17], we have

$$\|\phi(v_n) - \phi(v_0)\|_{L^\infty(0,T;L^2(\Omega))} \longrightarrow 0$$

as $n \longrightarrow \infty$. Therefore by Lemma 2 and $\lambda' \in L^2(0,T;L^\infty(\Omega))$, we can make n tend to ∞ in (5.9) with v_n, so that we can get (5.9) for any $v_0 \in L^2(\Gamma \times (0,T))$. Thus the proof of Lemma 3 is complete.

Combining Lemma 3 with Lemma 1, we have a key lemma.

Lemma 4. *On the assumptions (2.1), for any $\phi_0 \in L^2(\Omega)$, we have*

$$\begin{aligned}&\int_\Omega \lambda(x,0)f(x)(\phi_0(x) + (S\phi_0)(x))dx\\ &= \int_0^T \int_\Gamma \frac{\partial}{\partial t}\frac{\partial y(f)}{\partial n}(x,t)g(\phi_0)(x,t)d\sigma dt,\end{aligned}$$

where we define $S = S(\lambda, p) : L^2(\Omega) \longrightarrow L^2(\Omega)$ by

$$(S\phi_0)(x) = \int_0^T \frac{\lambda'(x,t)}{\lambda(x,0)}\phi(g(\phi_0))(x,t)dt \quad (x \in \Omega).$$

Here we recall that $\frac{\lambda'}{\lambda(\cdot,0)} \in H^1(0,T;L^\infty(\Omega))$ by the assumption (2.1).

Second Step. In this step, we are concerned with the third point (iii) of our methodology mentioned at the end of §1. That is, we prove

Lemma 5. *On the assumptions (2.4) and (2.1) - (2.2), the operator $S = S(\lambda, p)$ is a compact operator from $L^2(\Omega)$ to $L^2(\Omega)$.*

Proof of Lemma 5. First we show

Lemma 6. *Let θ be a solution to*

$$(5.11) \qquad \left\{\begin{aligned} \theta''(x,t) = \Delta\theta(x,t) - p(x)\theta(x,t) \qquad (x \in \Omega,\, 0 < t < T)\\ \theta(x,T) = 0, \quad \theta'(x,T) = 0 \qquad (x \in \Omega)\\ \theta(x,t) = 0 \quad (x \in \partial\Omega \setminus \Gamma,\, 0 < t < T)\\ \theta(x,t) = u_0(x,t) \quad (x \in \Gamma,\, 0 < t < T).\end{aligned}\right.$$

If $u_0 \in {}^0H^1(0,T;L^2(\Gamma)) \equiv \{u \in H^1(0,T;L^2(\Gamma))\,;\, u(\cdot,T)=0\}$, *then* $\theta \in C^0([0,T];H^s(\Omega))$ *for any* $0 \le s < \frac{1}{2}$ *and the map :* $u_0 \longrightarrow \theta$ *from* ${}^0H^1(0,T;L^2(\Gamma))$ *to* $C^0([0,T];H^s(\Omega))$ *is continuous.*

Proof of Lemma 6. Similarly to Théorème 4.2 (pp.46-47) in [21] or [17], we have

$$\theta \in C^0([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega)). \tag{5.12}$$

By the change of variables $t \longrightarrow T-t$, setting $\widetilde{\theta}(x,t) = \theta(x,T-t)$ and $\widetilde{u_0}(x,t) = u_0(x,T-t)$, we may consider (5.13) in place of (5.11):

$$\left\{\begin{array}{r} \widetilde{\theta}''(x,t) = \Delta\widetilde{\theta}(x,t) - p(x)\widetilde{\theta}(x,t) \qquad (x \in \Omega,\ 0<t<T) \\ \widetilde{\theta}(x,0)=0, \quad \widetilde{\theta}'(x,0)=0 \qquad (x\in\Omega) \\ \widetilde{\theta}(x,t)=0 \quad (x \in \partial\Omega\setminus\Gamma,\ 0<t<T) \\ \widetilde{\theta}(x,t) = \widetilde{u_0}(x,t) \quad (x\in\Gamma,\ 0<t<T). \end{array}\right\} \tag{5.13}$$

Henceforth we will denote $\widetilde{\theta}$ and $\widetilde{u_0}$ by respectively θ and u_0, for simplicity. Thus it is sufficient to prove that if $u_0 \in {}_0H^1(0,T;L^2(\Gamma)) \equiv \{u \in H^1(0,T;L^2(\Gamma))\,;\, u(\cdot,0) = 0\}$, then the solution θ to (5.13) belongs to $C^0([0,T];H^s(\Omega))$ for any $0 \le s < \frac{1}{2}$ and the map : $u_0 \longrightarrow \theta$ from ${}_0H^1(0,T;L^2(\Gamma))$ to $C^0([0,T];H^s(\Omega))$ is continuous.

Now by (5.12), we see $\theta \in C^0([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$. Since $u_0 \in {}_0H^1(0,T;L^2(\Gamma))$, setting

$$u(x,t) = \begin{cases} u_0(x,t), & x \in \Gamma \\ 0, & x \in \Omega\setminus\Gamma, \end{cases}$$

we see that

$$u \in {}_0H^1(0,T;L^2(\partial\Omega)). \tag{5.14}$$

Let us set

$$
\begin{aligned}
\psi(x,t) &= \int_0^t \theta(x,s)ds \quad (x \in \Omega,\ 0 < t < T) \\
w(x,t) &= \int_0^t u(x,s)ds \quad (x \in \partial\Omega,\ 0 < t < T).
\end{aligned} \tag{5.15}
$$

Then

$$
\psi \in C^1([0,T]; L^2(\Omega)) \cap C^2([0,T]; H^{-1}(\Omega)) \tag{5.16}
$$

and

$$
w \in {}_0H^2(0,T; L^2(\partial\Omega)) \equiv \{u \in H^2(0,T; L^2(\partial\Omega))\,;\, u(\cdot,0) = u'(\cdot,0) = 0\}. \tag{5.17}
$$

Moreover ψ satisfies

$$
\left\{
\begin{aligned}
\psi''(x,t) = \Delta\psi(x,t) - p(x)\psi(x,t) \qquad &(x \in \Omega,\ 0 < t < T) \\
\psi(x,0) = 0, \quad \psi'(x,0) = 0 \qquad &(x \in \Omega) \\
\psi(x,t) = w(x,t) \quad &(x \in \partial\Omega,\ 0 < t < T).
\end{aligned}
\right\} \tag{5.18}
$$

Now we consider the boundary value problem for an elliptic equation.

$$
-\Delta\eta(x) = 0 \quad (x \in \Omega) \qquad \eta(x) = \mu(x) \quad (x \in \partial\Omega).
$$

Then by Theorem 7.4 (p.188) in Volume I of Lions and Magenes [22], if $\mu \in L^2(\partial\Omega)$, then there exists a solution $\eta = \eta(\mu) \in H^s(\Omega)$ for $0 \le s < \frac{1}{2}$, and

$$
\begin{gathered}
\text{the map } \mu \longmapsto \eta \text{ is continuous from} \\
L^2(\partial\Omega) \text{ to } H^s(\Omega)\,(0 \le s < \frac{1}{2}).
\end{gathered} \tag{5.19}
$$

We define $\eta(x,t)$ by

$$
\left\{
\begin{aligned}
-\Delta\eta(x,t) = 0 \qquad &(x \in \Omega,\ 0 < t < T) \\
\eta(x,t) = w(x,t) \qquad &(x \in \partial\Omega,\ 0 < t < T),
\end{aligned}
\right\} \tag{5.20}
$$

where w is given by (5.15). By (5.17) and (5.19) we see that

$$\eta \in H^2(0,T;H^s(\Omega)) \quad (0 \le s < \frac{1}{2}) \tag{5.21}$$

and

$$\eta(x,0) = \eta'(x,0) = 0 \qquad (x \in \Omega). \tag{5.22}$$

Setting $\xi(x,t) = \psi(x,t) - \eta(x,t)\,(x \in \Omega,\ 0 < t < T)$, we see from (5.16) and (5.21) that

$$\xi \in C^1([0,T];L^2(\Omega)) \cap H^2(0,T;H^{-1}(\Omega)) \tag{5.23}$$

and from (5.18), (5.20) and (5.22) that

$$\left\{\begin{aligned} &\xi''(x,t) = \Delta\xi(x,t) - p(x)\xi(x,t) + f_1(x,t) + f_2(x,t) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x \in \Omega,\ 0 < t < T) \\ &\qquad\qquad \xi(x,0) = 0, \quad \xi'(x,0) = 0 \qquad (x \in \Omega) \\ &\qquad\qquad \xi(x,t) = 0 \quad (x \in \partial\Omega,\ 0 < t < T), \end{aligned}\right\} \tag{5.24}$$

where

$$f_1(x,t) = -\eta''(x,t), \quad f_2(x,t) = -p(x)\eta(x,t) \qquad (x \in \Omega,\ 0 < t < T). \tag{5.25}$$

By (5.21) and $p \in L^\infty(\Omega)$, it follows that

$$f_1 \in L^2(0,T;H^s(\Omega)), \quad f_2 \in H^2(0,T;L^2(\Omega)). \tag{5.26}$$

Let ξ_i $(i = 1,2)$ satisfy

$$\left\{\begin{aligned} &\xi_i''(x,t) = \Delta\xi_i(x,t) - p(x)\xi_i(x,t) + f_i(x,t) \qquad (x \in \Omega,\ 0 < t < T) \\ &\qquad\qquad \xi_i(x,0) = 0, \quad \xi_i'(x,0) = 0 \qquad (x \in \Omega) \\ &\qquad\qquad \xi_i(x,t) = 0 \quad (x \in \partial\Omega,\ 0 < t < T). \end{aligned}\right\} \tag{5.27}$$

Then by uniqueness of solutions, we obtain $\xi = \xi_1 + \xi_2$. Moreover we can prove

$$\xi_1 \in C^1([0,T]; H^s(\Omega)) \tag{5.28}$$

and

$$\xi_2 \in C^1([0,T]; H_0^1(\Omega)). \tag{5.29}$$

In fact, since $f_2 \in H^2(0,T; L^2(\Omega))$, the relation (5.29) follows from Lemma 2 (ii). As for (5.28), we can proceed as follows. If $f_1 \in L^2(0,T; L^2(\Omega))$, then $\xi_1' \in C^0([0,T]; L^2(\Omega))$, and if $f_1 \in L^2(0,T; H_0^1(\Omega))$, then $\xi_1' \in C^0([0,T]; H^1(\Omega))$ (e.g. Theorem 8.2 (p.275, Vol.I) in [22]). Consequently, by the interpolation theorem (Chapter 1, §5, §9 and Theorem 14.2 (p.95, Vol.I) in [22]), we reach (5.28) for $f_1 \in L^2(0,T; H^s(\Omega))$. Thus (5.28) and (5.29) are seen.

The relations (5.28) and (5.29) imply

$$\xi' \in C^0([0,T]; H^s(\Omega)) \tag{5.30}$$

for any $s < \frac{1}{2}$. On the other hand, from (5.21) and

$$H^1(0,T; H^s(\Omega)) \subset C^0([0,T]; H^s(\Omega))$$

(e.g. Theorem 3.1 (p.19, Vol.I) in [22]), it follows that

$$\eta' \in C^0([0,T]; H^s(\Omega)).$$

Combining this with (5.30) we obtain $\psi' = \xi' + \eta' = \theta \in C^0([0,T]; H^s(\Omega))$ for $0 \le s < \frac{1}{2}$. Thus we see: if $u_0 \in {}_0H^1(0,T; L^2(\Gamma))$, then $\theta \in C^0([0,T]; H^s(\Omega))$. Also the continuity of the map: $u_0 \longmapsto \theta$ from ${}_0H^1(0,T : L^2(\Gamma))$ to $C^0([0,T]; H^s(\Omega))$ is established by the above argument, so that the proof of Lemma 6 is complete.

We proceed to the proof of Lemma 5. We set

$$\theta(x,t) = \int_t^T \phi(v_0)(x,s)ds \quad (x \in \Omega,\, 0 < t < T) \tag{5.31}$$

for the solution $\phi(v_0)$ to (5.1) with $v_0 = g(\phi_0) \in L^2(\Gamma \times (0,T))$. Then

$$\left\{\begin{aligned} \theta''(x,t) = \Delta\theta(x,t) - p(x)\theta(x,t) \qquad (x \in \Omega,\, 0 < t < T)\\ \theta(x,T) = 0, \quad \theta'(x,T) = 0 \qquad (x \in \Omega)\\ \theta(x,t) = 0 \quad (x \in \partial\Omega \setminus \Gamma,\, 0 < t < T)\\ \theta(x,t) = \int_t^T g(\phi_0)(x,s)ds \quad (x \in \Gamma,\, 0 < t < T). \end{aligned}\right\} \tag{5.32}$$

Let us denote $\theta(x,t)$ by $\theta(\phi_0)(x,t)\,(x \in \Omega,\, 0 < t < T)$. Since $\int_t^T g(\phi_0)(x,s)ds \in {}^0H^1(0,T;L^2(\Gamma))$, Lemma 6 implies that $\theta(\phi_0) \in C^0([0,T];H^s(\Omega))$ and that $\phi_0 \mapsto (\theta(\phi_0)(\cdot,0), \theta(\phi_0))$ is a bounded map from $L^2(\Omega)$ to $H^s(\Omega) \times C^0([0,T];H^s(\Omega))$ for $0 \le s < \frac{1}{2}$. By $\theta \in C^1([0,T];L^2(\Omega)) \cap C^2([0,T];H^{-1}(\Omega))$ and (2.1), we can carry out integration by parts in $\lambda(\cdot,0)S(\phi_0)(x)$, and by (5.31) we have

$$\begin{aligned} &\lambda(x,0)(S\phi_0)(x)\\ =&\lambda'(x,0)\theta(\phi_0)(x,0) + \int_0^T \lambda''(x,t)\theta(\phi_0)(x,t)dt \qquad (x \in \Omega). \end{aligned}$$

Consequently by (2.1), we obtain

$$(S\phi_0)(x) = \frac{\lambda'(x,0)}{\lambda(x,0)}\theta(\phi_0)(x,0) + \int_0^T \frac{\lambda''(x,t)}{\lambda(x,0)}\theta(\phi_0)(x,t)dt.$$

Therefore it follows that S is a bounded linear operator from $L^2(\Omega)$ to $H^s(\Omega)$. Since the embedding $H^s(\Omega) \longrightarrow L^2(\Omega)$ is compact (e.g. [22]), the proof of Lemma 5 is complete.

Third Step. ¿From Lemma 4 we can derive

$$(1 + S^*)(\lambda(\cdot,0)f) = g^*\left(\frac{\partial}{\partial t}\frac{\partial y(f)}{\partial n}\right) \tag{5.33}$$

for any $f \in L^2(\Omega)$. Here and henceforth $g^* : L^2(\Gamma \times (0,T)) \longrightarrow L^2(\Omega)$ and $S^* : L^2(\Omega) \longrightarrow L^2(\Omega)$ are the adjoint operators of $g : L^2(\Omega) \longrightarrow L^2(\Gamma \times (0,T))$ and $S : L^2(\Omega) \longrightarrow L^2(\Omega)$ by identifying the Hilbert spaces with themselves. By Lemma 5, S^* is also compact (e.g. Brezis [4]). Therefore by the Fredholm alternative, we have either

(1) -1 is an eigenvalue of S^*.

(2) -1 is not an eigenvalue of S^*.

Case (1). We prove that this case corresponds to the first alternative in Theorem 1. By compactness of S^*, there exist $N \ge 1$ and linearly independent $\phi_k \in L^2(\Omega)\,(1 \le k \le N)$ such that $(1+S^*)\phi_k = 0\,(1 \le k \le N)$ (e.g. [4]). Assuming that $\frac{\partial y(f)}{\partial n}(x,t) = 0\,(x \in \Gamma,\, 0 < t < T)$, we have to prove that $\lambda(\cdot,0)f = \sum_{k=1}^N \alpha_k\phi_k$ with some $\alpha_k \in \mathbb{R}$, that is, $\lambda(\cdot,0)f \in \mathrm{Ker}(1+S^*)$. This is straightforward from (5.33). Moreover since $1+S^*$ is injective from $L^2(\Omega)/X_0$ to $L^2(\Omega)$, again by the Fredholm alternative, the estimate (2.6) is seen.

Case (2). By the Fredholm alternative, we see that -1 is in the resolvent set of S^*, so that the equality (2.5) implies $f(x) = 0\,(x \in \Omega)$ and

$$\|\lambda(\cdot,0)f\|_{L^2(\Omega)} \le C_7 \left\| g^*\left(\frac{\partial}{\partial t}\frac{\partial y(f)}{\partial n}\right)\right\|_{L^2(\Omega)} \le C_7C_5 \left\|\frac{\partial}{\partial t}\frac{\partial y(f)}{\partial n}\right\|_{L^2(\partial\Omega\times(0,T))}$$

by (5.2). This completes the proof of Theorem 1.

We conclude this section with proofs of Corollary 1, Proposition 1 and Theorem 2.

Proof of Corollary 1. It is sufficient to prove

$$\left\|\frac{\partial}{\partial t}\frac{\partial y(f)}{\partial n}\right\|_{L^2(\partial\Omega\times(0,T))} \le C_8\|f\|_{L^2(\Omega)}$$

with some constant $C_8 = C_8(\lambda, p) > 0$. This inequality is directly derived from Lemma 2, since λ satisfies (2.1).

Proof of Proposition 1. The part (ii) is shown in Yamamoto [34], [35]. We proceed to the part (i). By Theorem 1, it is sufficient to prove that -1 is not an eigenvalue of S^*, namely, -1 is not an eigenvalue of S. Let $(1+S)h_0 = 0$. Then, setting $C_9 = \int_0^T \left\| \frac{\lambda'(\cdot,t)}{\lambda(\cdot,0)} \right\|_{L^\infty(\Omega)} dt$, we have

$$
\begin{aligned}
\|h_0\|_{L^2(\Omega)} &= \left\| \int_0^T \frac{\lambda'(\cdot,t)}{\lambda(\cdot,0)} \phi(g(h_0))(\cdot,t)dt \right\|_{L^2(\Omega)} \\
&\le \int_0^T \left\| \frac{\lambda'(\cdot,t)}{\lambda(\cdot,0)} \right\|_{L^\infty(\Omega)} \|\phi(g(h_0))(\cdot,t)\|_{L^2(\Omega)} dt \\
&\le C_9 \sup_{0\le t\le T} \|\phi(g(h_0))(\cdot,t)\|_{L^2(\Omega)} \\
&\le C_9 \|g(h_0)\|_{L^2(\Gamma\times(0,T))} \qquad \text{(by Théorème 4.2 (pp.46-47) in [21])} \\
&\le C_9 \|h_0\|_{L^2(\Omega)} \qquad \text{(by (5.2))}.
\end{aligned}
$$

Since C_9 is sufficiently small, this inequality implies $h_0 = 0$. Thus we see that -1 is not an eigenvalue of S.

Proof of Theorem 2. We set

$$
(S_\lambda \phi_0)(x) = \int_0^T \frac{\lambda'(x,t)}{\lambda(x,0)} \phi(g(\phi_0))(x,t)dt \quad (x \in \Omega).
$$

First we will prove the openness of $\mathcal{D}$. Let $\lambda \in \Lambda$ and let $\mu \in \Lambda$ satisfy (2.1) and $\|\lambda - \mu\|_{H^1(0,T;L^\infty(\Omega))} \le \epsilon$: sufficiently small. Then we have $\lambda(x,0), \mu(x,0) \ge$

$C_{10}(\epsilon) \equiv C_{10} > 0$ $(x \in \overline{\Omega})$. By direct calculations, we get

$$
\begin{aligned}
&\|(S_\lambda - S_\mu)(\phi_0)\|_{L^2(\Omega)} \\
\leq &\int_0^T \left(\left\| \frac{\lambda'(\cdot,t)(\mu(\cdot,0)-\lambda(\cdot,0))}{\lambda(\cdot,0)\mu(\cdot,0)} \right\|_{L^\infty(\Omega)} \right. \\
+ &\left. \left\| \frac{\lambda'(\cdot,t)-\mu'(\cdot,t)}{\mu(\cdot,0)} \right\|_{L^\infty(\Omega)} \right) \|\phi(g(\phi_0))(\cdot,t)\|_{L^2(\Omega)} dt \\
\leq &\left(\frac{T}{C_{10}^2} \sup_{0\leq t\leq T} \|\lambda'(\cdot,t)\|_{L^\infty(\Omega)} \times \|\mu(\cdot,0)-\lambda(\cdot,0)\|_{L^\infty(\Omega)} \right. \\
+ &\left. \frac{1}{C_{10}} \|\lambda'-\mu'\|_{L^2(0,T;L^\infty(\Omega))} \right) \times \sup_{0\leq t\leq T} \|\phi(g(\phi_0))(\cdot,t)\|_{L^2(\Omega)} \\
\leq &\left(C_{11}\|\lambda\|_{H^2(0,T;L^\infty(\Omega))} + C_{11} \right) \times \epsilon \times \|g(\phi_0)\|_{L^2(\Gamma\times(0,T))} \\
&\text{(by Théorème 4.2 (pp.46-47) in [21])} \\
\leq &C_{11}\epsilon\|\phi_0\|_{L^2(\Omega)} \left(\|\lambda\|_{H^2(0,T;L^\infty(\Omega))} + 1 \right) \\
\equiv &\epsilon C_{11}(T,\lambda)\|\phi_0\|_{L^2(\Omega)}.
\end{aligned}
$$

Therefore we obtain

$$
\lim_{\epsilon\to 0} \|S_\lambda - S_\mu\|_{L^2(\Omega)\to L^2(\Omega)} = 0. \tag{5.34}
$$

By Theorem 1, $\lambda \in \mathcal{D}$ is equivalent to that $-1 \in \rho(S_\lambda)$: the resolvent set of S_λ. Therefore, by the stability of the resolvent set (e.g. Theorem 1.16 (p.196) in Kato [12]) and (5.34), if $\|\lambda - \mu\|_{H^1(0,T;L^\infty(\Omega))}$ is sufficiently small, then $-1 \in \rho(S_\mu)$ follows. This means that $\mathcal{D}$ is an open set in Λ in the topology of $H^1(0,T;L^\infty(\Omega))$.

Finally we will prove the density of $\mathcal{D}$. Let $\lambda \in \Lambda$. We have to show that for any $\epsilon > 0$ there exists $\lambda_\epsilon \in \mathcal{D}$ such that $\|\lambda_\epsilon - \lambda\|_{H^2(0,T;L^\infty(\Omega))} \leq \epsilon$. We take $\lambda_0 \in L^\infty(\Omega)$ such that $\lambda_0(x) > 0$ $(x \in \overline{\Omega})$. Then $\lambda_0 \in \Lambda$ and for $0 \leq \theta \leq 1$, we have

$$
S_{(1-\theta)\lambda+\theta\lambda_0}\phi_0 = \int_0^T \frac{(1-\theta)\lambda'(\cdot,t)}{(1-\theta)\lambda(\cdot,0)+\theta\lambda_0(\cdot,0)} \phi(g(\phi_0))(\cdot,t)dt.
$$

Henceforth $\mathcal{L}(L^2(\Omega))$ is the Banach space of all bounded linear operators from $L^2(\Omega)$ to $L^2(\Omega)$. Thus $S_{(1-\theta)\lambda+\theta\lambda_0}$ is analytic in $\theta \in (0,1)$ with values in $\mathcal{L}(L^2(\Omega))$ and by Lemma 5, $S_{(1-\theta)\lambda+\theta\lambda_0} : L^2(\Omega) \longrightarrow L^2(\Omega)$ is compact for $\theta \in [0,1]$. Moreover we have $S_{\lambda_0} = 0$ by $\lambda_0' = 0$, that is, $-1 \in \rho(S_{\lambda_0})$. Therefore by a theorem on analytic perturbation (e.g. Theorem 1.9 (p.370) in Kato [12]), there exists a finite set $\Theta \subset [0,1]$ such that for any $\theta \in [0,1] \setminus \Theta$ it follows that $-1 \in \rho(S_{(1-\theta)\lambda+\theta\lambda_0})$, namely, $(1-\theta)\lambda + \theta\lambda_0 \in \mathcal{D}$ for any $\theta \in [0,1] \setminus \Theta$. Therefore for any $\epsilon > 0$, there exists

$$\delta < \frac{\epsilon}{\|\lambda - \lambda_0\|_{H^2(0,T;L^\infty(\Omega))}}$$

such that $\lambda_\epsilon \equiv (1-\delta)\lambda + \delta\lambda_0 \in \mathcal{D}$, which implies the existence of $\lambda_\epsilon \in \mathcal{D}$ such that

$$\|\lambda_\epsilon - \lambda\|_{H^2(0,T;L^\infty(\Omega))} < \epsilon.$$

Thus the proof of Theorem 2 is complete.

§6. Proof of the Results in §4.

Proof of Poposition 3. Let us set

$$w(x,t) = u(q,a,h)(x,t) - u(p,a,h)(x,t) \quad (x \in \Omega,\ 0 < t < T). \tag{6.1}$$

Then

$$\left\{\begin{array}{r} w''(x,t) = \Delta w(x,t) - q(x)w(x,t) + (p-q)(x)u(p,a,h)(x,t) \\ (x \in \Omega,\ 0 < t < T) \\ w(x,0) = w'(x,0) = 0 \qquad (x \in \Omega) \\ w(x,t) = 0 \qquad (x \in \partial\Omega,\ 0 < t < T). \end{array}\right\} \tag{6.2}$$

Since $u(p,a)$ satisfies (2.1), by $a \in \mathcal{U}_h$, we readily see from Lemma 2 (ii) in §5 that

$$\left\| \frac{\partial w}{\partial n} \right\|_{H^1(0,T;L^2(\partial\Omega))} \le C_3 \|q - p\|_{L^2(\Omega)},$$

which is (4.2), the conclusion of Proposition 3.

Proof of Proposition 4. Similarly to Lemme 3.6 (p.39) in [21], we get

$$\begin{aligned}
&\|y(p,a,q-p)\|_{L^\infty(0,T;H_0^1(\Omega))} + \|y'(p,a,q-p)\|_{L^\infty(0,T;L^2(\Omega))} \\
\le& C_{14}\|(q-p)u(p,a,h)\|_{L^1(0,T;L^2(\Omega))} \\
\le& C_{14}\|u(p,a,h)\|_{L^1(0,T;L^\infty(\Omega))}\|q-p\|_{L^2(\Omega)}. \qquad (6.3)
\end{aligned}$$

For the estimate of $\frac{\partial}{\partial n}(u(q,a,h) - u(p,a,h) - y(p,a,q-p))$, we put

$$d(x,t) = (u(q,a,h) - u(p,a,h) - y(p,a,q-p))(x,t) \quad (x \in \Omega,\ 0<t<T).$$

Then d satisfies

$$\left\{\begin{aligned}
d''(x,t) = \Delta d(x,t) - q(x)d(x,t) + (p(x)-q(x))y(p,a,q-p)(x,t) \\
(x\in\Omega,\ 0<t<T) \\
d(x,0) = d'(x,0) = 0 \qquad (x\in\Omega) \\
d(x,t) = 0 \qquad (x\in\partial\Omega,\ 0<t<T).
\end{aligned}\right.$$

By Lemma 2 (ii) in §5, $\|q\|_{L^\infty(\Omega)} \le M_1$, we obtain

$$\begin{aligned}
&\left\|\frac{\partial d}{\partial n}\right\|_{H^1(0,T;L^2(\partial\Omega))} \\
\le& C_6'\left(\|(q-p)y'(p,a,q-p)\|_{L^1(0,T;L^2(\Omega))} + \|(q-p)y(p,a,q-p)\|_{L^1(0,T;L^2(\Omega))}\right).
\end{aligned}$$

Consequently we have

$$\begin{aligned}
&\left\|\frac{\partial d}{\partial n}\right\|_{H^1(0,T;L^2(\partial\Omega))} \\
\le& C_6'\|q-p\|_{L^\infty(\Omega)}\left(\|y'(p,a,q-p)\|_{L^1(0,T;L^2(\Omega))} + \|y(p,a,q-p)\|_{L^1(0,T;L^2(\Omega))}\right) \\
\le& C_6'\|q-p\|_{L^\infty(\Omega)} \times C_{12}\|u(p,a,h)\|_{L^1(0,T;L^\infty(\Omega))}\|q-p\|_{L^2(\Omega)}
\end{aligned}$$

by (6.3). Thus we reach the conclusion of Proposition 4.

Proof of Theorem 5.

(I) Let us assume that in (1.5) with $\lambda = u(p, a, h)$, the condition $\frac{\partial y(p,a,f)}{\partial n}(x, t) = 0\,(x \in \Gamma,\, 0 < t < T)$ and $f \in L^2(\Omega)$ imply $f = 0$. Since $u(p, a, h)(x, 0) = a(x) > 0\,(x \in \overline{\Omega})$, the condition (2.8) holds. Therefore we can apply Corollary 1 to obtain

$$C_1 \left\| \frac{\partial y(p, a, f)}{\partial n} \right\|_{H^1(0,T;L^2(\Gamma))} \ge \|f\|_{L^2(\Omega)}$$

for any $f \in L^2(\Omega)$. Let $q \in L^\infty(\Omega)$ and $\|q - p\|_{L^\infty(\Omega)}$ be sufficiently small. Then we can take a constant $M_1 > 0$ such that $\|q\|_{L^\infty(\Omega)} \le M_1$. Consequently by Proposition 4, we have

$$\begin{aligned}
&\left\| \frac{\partial u(q, a, h)}{\partial n} - \frac{\partial u(p, a, h)}{\partial n} \right\|_{H^1(0,T;L^2(\Gamma))} \\
=&\left\| \frac{\partial y(p, a, q-p)}{\partial n} - \left(\frac{\partial y(p, a, q-p)}{\partial n} - \frac{\partial (u(q, a, h) - u(p, a, h))}{\partial n} \right) \right\|_{H^1(0,T;L^2(\Gamma))} \\
\ge& \left\| \frac{\partial y(p, a, q-p)}{\partial n} \right\|_{H^1(0,T;L^2(\partial\Omega))} \\
-& \left\| \frac{\partial (u(q, a, h) - u(p, a, h))}{\partial n} - \frac{\partial y(p, a, q-p)}{\partial n} \right\|_{H^1(0,T;L^2(\Gamma))} \\
\ge& (C_1^{-1} - C_4 \|q - p\|_{L^\infty(\Omega)}) \|q - p\|_{L^2(\Omega)}.
\end{aligned}$$

Therefore for $q \in L^\infty(\Omega)$ where $\|q-p\|_{L^\infty(\Omega)}$ is so small that $C_1^{-1} - C_4\|q-p\|_{L^\infty(\Omega)} > 0$, we can reach the L^∞-local well-posedness around p of the nonlinear inverse problem.

§7. Proof of the Results in §3.

Proof of Theorem 3.

First Step. In this step, we show

Lemma 7. *Let us denote the solution to an initial value problem*

$$
\left\{
\begin{aligned}
&u''(x,t) = \Delta u(x,t) - p(x)u(x,t) \qquad (x \in \Omega,\ 0 < t < T)\\
&u(x,0) = \eta(x), \quad u'(x,0) = 0 \qquad (x \in \Omega)\\
&u(x,t) = 0 \qquad (x \in \partial\Omega,\ 0 < t < T)
\end{aligned}
\right\}
\tag{7.1}
$$

by $u(\eta)$. *For* $k \in \mathbb{N}$, *if* $\eta \in \mathcal{D}(A^k)$, *then there exists a constant* $C_{13} = C_{13}(k,p,\Omega,T) > 0$ *independent of* η *such that*

$$
\begin{aligned}
&\|u(\eta)\|_{L^2(0,T;H^{2k}(\Omega))} + \|u(\eta)'\|_{L^2(0,T;H^{2k-1}(\Omega))}\\
\le& C_{13}(\|A^k\eta\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)}).
\end{aligned}
\tag{7.2}
$$

Proof of Lemma 7. We can rewrite (7.1) as an evolution equation in $H_0^1(\Omega) \times L^2(\Omega)$:

$$
\left\{
\begin{aligned}
&U'(t) = \mathcal{A}U(t) \qquad (0 < t < T)\\
&U(0) = \begin{pmatrix} \eta \\ 0 \end{pmatrix}
\end{aligned}
\right\}
\tag{7.3}
$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix}$$

and $U(t) = \begin{pmatrix} u(\eta)(t) \\ u(\eta)'(t) \end{pmatrix}$. Then $\mathcal{A}$ generates a C_0-semigroup in $H_0^1(\Omega) \times L^2(\Omega)$ (e.g. [4]). Therefore we get

$$U(t) = \begin{pmatrix} u(\eta)(t) \\ u(\eta)'(t) \end{pmatrix} = e^{\mathcal{A}t}\begin{pmatrix} \eta \\ 0 \end{pmatrix} \quad (0 < t < T).$$

On the other hand, by direct computations, for $\eta \in \mathcal{D}(A^k)$ and $k \in \mathbb{N}$, we have $\mathcal{A}^{2k-1}\begin{pmatrix} \eta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ A^k\eta \end{pmatrix}$, so that $\eta \in \mathcal{D}(A^k)$ implies $\begin{pmatrix} \eta \\ 0 \end{pmatrix} \in \mathcal{D}(\mathcal{A}^{2k-1})$. Consequently by Théorème VII.5 (p.111) in [4], we see $U \in C^0([0,T];\mathcal{D}(\mathcal{A}^{2k-1}))$.

By $\mathcal{A}^{2k-1}\begin{pmatrix} u(\eta)(t) \\ u(\eta)'(t) \end{pmatrix} = \begin{pmatrix} A^{k-1}u(\eta)'(t) \\ A^k u(\eta)(t) \end{pmatrix}$, it follows that $A^{k-1}u(\eta)' \in C^0([0,T]; H_0^1(\Omega))$, $A^k u(\eta) \in C^0([0,T]; L^2(\Omega))$ and

$$\|A^k u(\eta)\|_{L^\infty(0,T;L^2(\Omega))} + \|A^{k-1}u(\eta)'\|_{L^\infty(0,T;H_0^1(\Omega))} \le C'_{13}\|A^k\eta\|_{L^2(\Omega)}. \tag{7.4}$$

We can take a constant $C_{14} > 0$ such that

$$\|v\|_{H^{2k}(\Omega)} \le C_{14}\|(-A + C_{14})^k v\|_{L^2(\Omega)} \quad (v \in \mathcal{D}((-A + C_{14})^k)$$

(e.g. [22]), so that by the interpolation theorem (e.g. Adams [1]) we have $\|D_x^\alpha v\|_{L^2(\Omega)} \le \epsilon\|A^k v\|_{L^2(\Omega)} + C(\epsilon,\alpha)\|v\|_{L^2(\Omega)}$ for $|\alpha| \le 2k-1$. Therefore we see

$$\|(-A + C_{14})^k v\|_{L^2(\Omega)} \le C_{14}\left(\|A^k v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\right).$$

Here we set $\alpha = (\alpha_1, ..., \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $D_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$ and $|\alpha| = \alpha_1 + ... + \alpha_n$. Thus

$$\|v\|_{H^{2k}(\Omega)} \le C_{14}\left(\|A^k v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\right) \qquad (v \in \mathcal{D}(A^k)). \tag{7.5}$$

Moreover since

$$\begin{aligned} C_{15}^{-1}\|(-A + C_{14})^{\frac{1}{2}} v\|_{L^2(\Omega)} &\le \|v\|_{H_0^1(\Omega)} \\ &\le C_{15}\|(-A + C_{14})^{\frac{1}{2}} v\|_{L^2(\Omega)}, \end{aligned}$$

we have

$$\begin{aligned} &\|v\|_{H^{2k-1}(\Omega)} \le C_{14}\|(-A + C_{14})^{k-\frac{1}{2}} v\|_{L^2(\Omega)} \\ =&C_{14}\|(-A + C_{14})^{k-1}(-A + C_{14})^{\frac{1}{2}} v\|_{L^2(\Omega)} \\ \le&C_{14}\left(\|A^{k-1}(-A + C_{14})^{\frac{1}{2}} v\|_{L^2(\Omega)} + \|(-A + C_{14})^{\frac{1}{2}} v\|_{L^2(\Omega)}\right) \\ &\text{(by the interpolation theorem)} \\ =&C_{14}(\|A^{k-1}v\|_{H_0^1(\Omega)} + \|v\|_{H_0^1(\Omega)}) \end{aligned}$$

for $v \in \mathcal{D}((-A+C_{14})^{k-\frac{1}{2}})$. That is, if $A^{k-1}v \in H_0^1(\Omega)$ and $v \in H_0^1(\Omega)$, then $v \in \mathcal{D}((-A+C_{14})^{k-\frac{1}{2}})$ and

$$\|v\|_{H^{2k-1}(\Omega)} \leq C_{14}\left(\|A^{k-1}v\|_{H_0^1(\Omega)} + \|v\|_{H_0^1(\Omega)}\right). \tag{7.6}$$

Furthermore setting $k=1$ in (7.4), we obtain

$$\|u(\eta)'\|_{L^\infty(0,T;H_0^1(\Omega))} \leq C_{13}\|A\eta\|_{L^2(\Omega)}. \tag{7.7}$$

It follows from $\eta \in \mathcal{D}(A^k)$ $(k \geq 1)$ that $\eta \in H_0^1(\Omega)$ and similarly to Lemme 3.6 (p.39) in [21], we can see that

$$\|u(\eta)\|_{L^\infty(0,T;H_0^1(\Omega))} \leq C_{13}\|\eta\|_{H_0^1(\Omega)}. \tag{7.8}$$

Therefore we have

$$\begin{aligned}
&\|u(\eta)\|_{L^\infty(0,T;H^{2k}(\Omega))} + \|u(\eta)'\|_{L^\infty(0,T;H^{2k-1}(\Omega))}\\
\leq& C_{14}\left(\|A^k u(\eta)\|_{L^\infty(0,T;L^2(\Omega))} + \|u(\eta)\|_{L^\infty(0,T;L^2(\Omega))}\right)\\
+&C_{14}\left(\|A^{k-1}u(\eta)'\|_{L^\infty(0,T;H_0^1(\Omega))} + \|u(\eta)'\|_{L^\infty(0,T;H_0^1(\Omega))}\right)\\
&\text{(by (7.5) and (7.6))}\\
\leq& C_{14}\left(\|A^k u(\eta)\|_{L^\infty(0,T;L^2(\Omega))} + \|A^{k-1}u(\eta)'\|_{L^\infty(0,T;H_0^1(\Omega))}\right)\\
+&C_{14}\|\eta\|_{H_0^1(\Omega)} + C_{14}\|A\eta\|_{L^2(\Omega)}\\
&\text{(by (7.7) and (7.8))}\\
\leq& C_{13}\left(\|A^k\eta\|_{L^2(\Omega)} + \|A\eta\|_{L^2(\Omega)} + \|\eta\|_{H_0^1(\Omega)}\right) \quad \text{(by (7.4))}.
\end{aligned} \tag{7.9}$$

Again applying the interpolation inequality, we have

$$\|A\eta\|_{L^2(\Omega)} \leq C_{14}\|A^k\eta\|_{L^2(\Omega)} + C_{14}\|\eta\|_{L^2(\Omega)}$$

and

$$\|\eta\|_{H_0^1(\Omega)} \le C_{14}\|A^k\eta\|_{L^2(\Omega)} + C_{14}\|\eta\|_{L^2(\Omega)},$$

so that (7.9) implies (7.2), the conclusion. Thus the proof of Lemma 7 is complete.

Second Step. Let $b \in L^2(\Omega)$ satisfy $b - a \in \mathcal{D}(A^l)$ with $l \ge 1$ defined by (3.6). Then by the embedding theorem (e.g. [1]), we have $\|v\|_{L^\infty(\Omega)} \le C'_{15}\|v\|_{H^{2l-1}(\Omega)}$. Therefore (7.2) implies

$$\begin{aligned} &\|u(b-a)\|_{L^2(0,T;L^\infty(\Omega))} + \|u(b-a)'\|_{L^2(0,T;L^\infty(\Omega))} \\ \le& C_{15}\left(\|A^l(b-a)\|_{L^2(\Omega)} + \|b-a\|_{L^2(\Omega)}\right). \end{aligned} \tag{7.10}$$

We will estimate

$$\left\|\frac{\partial}{\partial n}(u(q,b,h) - u(q,a,h)) - \frac{\partial}{\partial n}(u(p,b,h) - u(p,a,h))\right\|_{H^1(0,T;L^2(\Gamma))}.$$

For simplicity, we set

$$w_1(x,t) = u(q,b,h)(x,t) - u(q,a,h)(x,t), \quad w_2(x,t) = u(p,b,h)(x,t) - u(p,a,h)(x,t).$$

Then $d = w_1 - w_2$ satisfies

$$\left\{\begin{aligned} d''(x,t) = \Delta d(x,t) - q(x)d(x,t) + (p(x) - q(x))w_2(x,t) \\ (x \in \Omega,\ 0 < t < T) \\ d(x,0) = d'(x,0) = 0 \qquad (x \in \Omega) \\ d(x,t) = 0 \qquad (x \in \partial\Omega,\ 0 < t < T). \end{aligned}\right\}$$

Noting that $w_2 = u(b-a)$ by the notation in Lemma 7, and applying Lemma 2 (ii), we get

$$
\begin{aligned}
&\left\| \frac{\partial d}{\partial n} \right\|_{H^1(0,T;L^2(\Gamma))} \\
\le& C_6 \left(\|(p-q)w_2'\|_{L^1(0,T;L^2(\Omega))} + \|(p-q)w_2\|_{L^1(0,T;L^2(\Omega))} \right) \\
\le& C_6 \|q-p\|_{L^2(\Omega)} \left(\|u(b-a)'\|_{L^1(0,T;L^\infty(\Omega))} + \|u(b-a)\|_{L^1(0,T;L^\infty(\Omega))} \right) \\
\le& C_{15} \|q-p\|_{L^2(\Omega)} \left(\|A^l(b-a)\|_{L^2(\Omega)} + \|(b-a)\|_{L^2(\Omega)} \right) \\
\le& C_{15}\delta(a)\|q-p\|_{L^2(\Omega)}
\end{aligned}
\tag{7.11}
$$

by (3.7).

Finally we will derive the L^∞-local well-posedness for b. Let $\|q-p\|_{L^\infty(\Omega)}$ be sufficiently small. Then we have

$$
\begin{aligned}
&\left\| \frac{\partial u(q,b,h)}{\partial n} - \frac{\partial u(p,b,h)}{\partial n} \right\|_{H^1(0,T;L^2(\Gamma))} \\
\ge& \left\| \frac{\partial u(q,a,h)}{\partial n} - \frac{\partial u(p,a,h)}{\partial n} \right\|_{H^1(0,T;L^2(\Gamma))} \\
-& \left\| \frac{\partial}{\partial n}(u(q,b,h) - u(q,a,h)) - \frac{\partial}{\partial n}(u(p,b,h) - u(p,a,h)) \right\|_{H^1(0,T;L^2(\Gamma))}
\end{aligned}
$$

(by the triangle inequality)

$$
\ge C_2^{-1}\|q-p\|_{L^2(\Omega)} - C_{15}\delta(a)\|q-p\|_{L^2(\Omega)}
$$

(by the L^∞-local well-posedness for a and (7.11))

$$
= (C_2^{-1} - C_{15}\delta(a))\|q-p\|_{L^2(\Omega)}.
$$

If we take sufficiently small $\delta(a) > 0$ such that $C_2^{-1} - C_{15}\delta(a) > 0$, then this inequality implies the L^∞-local well-posedness for b. Thus the proof of Theorem 3 is complete.

Proof of Theorem 4. By the assumption (3.9), there exists a unique solution $a_0 \in H^2(\Omega)$ to (3.10) and by the maximum principle (e.g. Gilbarg and Trudinger

[8]), we obtain

$$0 < C_{16}^{-1} \le a_0(x) \le C_{16} \qquad (x \in \overline{\Omega}) \tag{7.12}$$

with some constant $C_{16} > 0$. In particular, we have

$$u(p, a_0, h)(x,t) = a_0(x) \quad (x \in \Omega,\ 0 < t < T), \qquad a_0 \in \mathcal{U}_h. \tag{7.13}$$

We choose a sufficiently small $\epsilon > 0$, so that $(1-\xi)a_0(x) + \xi a(x) > 0$ for any $\xi \in [-\epsilon, \epsilon+1]$ and all $x \in \overline{\Omega}$. This is possible by (7.12) and $a(x) > 0$ $(x \in \overline{\Omega})$. Let us set

$$a(\xi) = (1-\xi)a_0 + \xi a \qquad (\xi \in [-\epsilon, \epsilon+1]).$$

Then by the linearity of $y(p,a,f)$ and $u(p,a,h)$ with respect to a, we have

$$u(p, a(\xi), h) = (1-\xi)a_0 + \xi u(p,a,h),$$

$$y(p, a(\xi), f) = (1-\xi)y(p, a_0, f) + \xi y(p,a,f),$$

and

$$\begin{aligned} S(\xi)(\phi_0)(x) &\equiv S(u(p, a(\xi), h), p)(\phi_0)(x) \\ &= \int_0^T \frac{\xi u(p,a,h)'(x,t)}{(1-\xi)a_0(x) + \xi a(x)} \phi(g(\phi_0))(x,t)dt \quad (\phi_0 \in L^2(\Omega)) \end{aligned} \tag{7.14}$$

by (7.13). By (7.14), the operator $S(\xi)$ is well-defined for $r \in D_0$ which is some domain in $\mathbb{C}$ containing the segment $[-\epsilon, 1+\epsilon]$ for a sufficiently small $\epsilon > 0$. Moreover by $a \in \mathcal{U}_h$, we see

$$u(p, a(\xi), h) = (1-\xi)a_0 + \xi u(p,a,h) \in H^2(0,T; L^\infty(\Omega)). \tag{7.15}$$

Therefore by Lemma 5,

(i) $S(\xi)$ is a compact operator for each $\xi \in D_0$.

Moreover

(ii) $S(\xi) : D_0 \ni \xi \longmapsto S(\xi) \in \mathcal{L}(L^2(\Omega))$ is analytic in $\xi \in D_0$.

By (i) and (ii), we can apply a theorem on analytic perturbation (e.g. Theorem 1.9 (p.370) in Kato [12]), so that we have either

(1) -1 is an eigenvalue of $S(\xi)$ for any $\xi \in [-\epsilon, 1+\epsilon]$.

or

(2) there exists a finite set $E = E(a) \subset (0, 1]$ such that $-1 \in \rho(S(\xi))$ for any $\xi \in [0, 1] \setminus E$.

The first case (1) can not happen by $-1 \in \rho(S(0))$. In fact, by (7.14) we obtain $S(0)(\phi_0)(x) = 0$, and $-1 \in \rho(S(0)) = \rho(0)$.

Therefore for any $\xi \in [0, 1] \setminus E \cup \{0\}$, we get $-1 \in \rho(S(u(p, a(\xi), h), p))$, that is, if $f \in L^2(\Omega)$ satisfies

$$\frac{\partial y(p, a(\xi), f)}{\partial n}(x, t) = 0 \quad (x \in \Gamma,\, 0 < t < T),$$

then $f(x) = 0\,(x \in \Omega)$ for any $\xi \in [0, 1] \setminus E \cup \{0\}$. By Theorem 5, we reach (3.11), the conclusion of Theorem 4.

Proof of Proposition 2. Let $a \in \mathcal{U}_h$ be arbitrary, and let $a_0 \in L^\infty(\Omega)$ be the solution to (3.10). For sufficiently small $\epsilon > 0$, we consider parametrized initial values $a(\xi) = (1-\xi)a_0 + \xi a$ for $\xi \in [-\epsilon, 1+\epsilon]$ and the compact operator $S(u(p, a(\xi), h), p)$. Repeating the argument in the proof of Theorem 4, we conclude that $-1 \in \rho(S(u(p, a(\xi), h), p))$ except for a finite number of $\xi \in [0, 1]$, which implies that we can choose a sequence $\xi_n\,(n \geq 1)$ such that $\lim_{n\to\infty} \xi_n = 1$ and the inverse problem

$$\frac{\partial u(q, a(\xi_n), h)}{\partial n}(x,t) \quad (x \in \Gamma,\, 0 < t < T) \Longrightarrow q(x)\, (x \in \Omega)$$

is L^∞-locally well-posed for $n \geq 1$. Since $\|a - a(\xi_n)\|_{L^\infty(\Omega)} = |1 - \xi_n| \|a - a_0\|_{L^\infty(\Omega)}$ for any $\epsilon > 0$, there exists $a_\epsilon \in \mathcal{D}_h$ such that $\|a - a_\epsilon\|_{L^\infty(\Omega)} \leq \epsilon$. Thus the proof of Proposition 2 is complete.

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