CONTROL OF PLATE VIBRATIONS BY MEANS OF PIEZOELECTRIC ACTUATORS

MARIUS TUCSNAK

ABSTRACT. We consider initial and boundary value problems modelling the vibrations of a plate with piezoelectric actuator. The simplest model leads to the Bernoulli-Euler plate equation with right hand side given by a distribution concentrated in an interior curve multiplied by a real valued time function representing the voltage applied to the actuator. We prove that, generically with respect to the curve, the plate vibrations can be strongly stabilized and approximatively controlled by means of the voltage applied to the actuator.

1. INTRODUCTION

In this paper we will study control of vibrations for a structure composed of an elastic plate and of a piezoelectric actuator bonded to the plate. The control function is the voltage applied to the actuator. The usual model leads to the Bernoulli-Euler plate equation with right hand side given by a distribution concentrated in an interior curve multiplied by a real valued time function representing the voltage applied to the actuator (cf. [2], [3], [8], [11]).
More precisely let us consider $T > 0$, $u \in L^2(0,T)$, $\Omega \subset \mathbf{R}^2$ an open bounded set, $\Gamma = \partial\Omega$, $Q = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$ and γ a curve included in Ω. We will consider the following initial and boundary value problem

$$w''(x,y,t) + \Delta^2 w(x,y,t) = u(t)\frac{\partial \delta_\gamma}{\partial \nu}, \text{ in } Q, \tag{1.1}$$

$$w(x,y,t) = \frac{\partial w}{\partial \nu}(x,y,t) = 0, \text{ on } \Sigma, \tag{1.2}$$

$$w(x,y,0) = w^0(x,y), \; w'(x,y,0) = w^1(x,y), \text{ in } \Omega, \tag{1.3}$$

where $\frac{\partial \delta_\gamma}{\partial \nu}$ stands for the derivative of the Dirac mass concentrated in γ with respect to the normal to γ. Equation (1.1) was proposed in ([2]), ([11]) as a Bernoulli-Euler model for the vibrations of a clamped plate with piezoelectric actuator. In this model w represents the transverse deflection of the plate, the function $u : [0,T] \to \mathbf{R}$ gives the time variation of the voltage applied to the actuator and γ represents the

1991 *Mathematics Subject Classification.* 35B37, 93C20.
Key words and phrases. plate equation, piezoelectric actuator, strong stabilization, approximate controllability.

boundary of the actuator. The influence of the piezoelectric actuator on the rigidity of the plate is neglected.

For a given voltage $u \in L^2(0,T)$, sharp regularity results for the open loop problem (1.1)-(1.3) were given in [14]. In this paper we shall first study the closed loop problem obtained if the voltage u in (1.1) is given by the feedback law:

$$u(t) = g\left(\int_\gamma \frac{\partial w'}{\partial \nu} d\gamma\right), \tag{1.4}$$

where $g : \mathbf{R} \to \mathbf{R}$ is an increasing function, with $g(0) = 0$. In this case equations (1.1)-(1.4) are dissipative since, at least formally

$$E'(t) = -g\left(\int_\gamma \frac{\partial w'}{\partial \nu} d\gamma\right) \int_\gamma \frac{\partial w'}{\partial \nu} d\gamma \leq 0, \tag{1.5}$$

where $E = E(t)$ is the energy of the plate

$$E(t) = \frac{1}{2}\int_\Omega [|w'(x,t)|^2 + |\Delta w(x,t)|^2] dx.$$

Our main stabilization result asserts that, if the eigenvalues the plate operator are simple[1] and γ satisfies some genericity condition, the energy of the solutions of (1.1)-(1.4) tends to zero if $t \to \infty$, in other words strong stabilization holds. By using the same assumptions as for the stabilization result we show that the system (1.1)-(1.3) can be approximatively controlled. Our result is in fact stronger than approximate controllability as we prove that there exists $T > 0$ such that for all w^0, w^1 finite combinations of eigenfunctions of the plate operator there exists $u \in L^2(0,T)$ such that

$$w(x,T) = w'(x,T) = 0, \ \forall x \in \Omega \tag{1.6}$$

(the result is also generic with respect to γ). The main features of the problems considered in this paper are:

- The control term is supported in an interior curve as in [25]. This fact makes the study of the wellposedness of the closed loop system (1.1)-(1.4) rather delicate (see [25] for the open loop problem). The same feature of our problem implies that strong stabilization or approximate controllability cannot hold if $\int_\gamma \frac{\partial \phi}{\partial \nu} d\gamma = 0$, for some eigenfunction ϕ corresponding to the eigenvalue $-\lambda$ of the bilaplacian. In this case $e^{i\sqrt{\lambda}t}\phi(x)$ is a constant energy solution of (1.1)-(1.4) so we can only show that strong stabilization holds for almost all curves γ. The same remark holds for approximate controllability.
- Unlike in [25] the control u s independent of the space variable, so, in this respect,
(1.1)-(1.4) is similar to a pointwise control problem as in [13], [19], [14], [23]. This is why the assumption of simple eigenvalues is necessary in order to have strong stabilization or approximate controllability.

[1] The assumption of simple eigenvalues excludes some usual domains. However, due to results from [1], this assumption is satisfied generically with respect to Ω see also the example given in section 5.

The plan of this paper is as follows. The second section is devoted to the proof of welposedness results for the closed loop problem (1.1)-(1.4). We obtain more regularity than for the corresponding open loop problem (cf [14]). In the third section we study the asymptotic behavior of solutions for (1.1)-(1.4). The fourth section contains some controllability results. Finally, in the fifth section, we provide some explicit computations in the case of a rectangular simply supported plate.

2. Wellposedness for the closed loop system

We begin this section by introducing the set of curves which we shall admit as control regions. More precisely we shall denote by $\mathcal{K}$ the set of functions $\gamma : [0,1] \to \Omega \subset R^2$ satisfying to the conditions:

(C1) γ is of class C^1,
(C2) $\gamma(t_1) \neq \gamma(t_2)$, for all $0 \le t_1 < t_2 < 1$.
(C3) $\gamma(0) = \gamma(1)$.
(C4) $\gamma(t) \notin \Gamma$,for all $t \in [0,1]$.

Let us notice that, by Jordan's theorem an admissible curve (i.e. $\gamma \in \mathcal{K}$) divides Ω in two connected components Ω_1 (interior to γ, representing the domain of the actuator) and Ω_2. Concerning the boundary Γ of Ω we shall suppose that it is locally Lipschitz.

Remark 2.1. We notice that (1.1)-(1.4) can be understood as the coupling of two plate equations posed in Ω_1 and Ω_2 respectively. Indeed, by using an integration by parts formula given in [16, p.68], it is easy to check that (1.1) is equivalent to

$$w'' + \Delta^2 w = 0, \text{ in } \Omega_i \times (0,\infty), \ i = 1,2,$$

$$[w] = [\frac{\partial w}{\partial \nu}] = [\frac{\partial \Delta w}{\partial \nu}] = 0, \ [\Delta w] = g(\int_\gamma \frac{\partial w'}{\partial \nu}), \text{ on } \gamma \times (0,\infty),$$

where $[F]$ denotes the jump of the function F across γ.

On $\mathcal{K}$ we introduce the Hausdorff distance between two curves γ_1 and γ_2, denoted by $d(\gamma_1,\gamma_2)$ and defined in the following way (cf. [20, p.31]):

$$\delta(x,\gamma) = \inf_{t\in[0,1]} |x - \gamma(t)|, \ \forall x \in R^2, \ \forall \gamma \in \mathcal{K},$$

$$\rho(\gamma_1,\gamma_2) = \sup_{t\in[0,1]} \delta(\gamma_1(t),\gamma_2),$$

$$d(\gamma_1,\gamma_2) = \max\{\rho(\gamma_1,\gamma_2), \rho(\gamma_2,\gamma_1)\}.$$

It is easy to check that $\mathcal{K}$ endowed with the distance d forms a metric space.

The wellposedness of (1.1)-(1.4) will be studied by the use of the theory of nonlinear monotone operators. We begin by introducing the function space

$$X = H_0^2(\Omega) \times L^2(\Omega),$$

and the operators

$$A, \ B : H_0^2(\Omega) \to H^{-2}(\Omega),$$

defined by

$$Aw = \Delta^2 w, \ \forall w \in H_0^2(\Omega), \tag{2.1}$$

$$Bv = -g\left(\frac{\partial v}{\partial \nu}\right)\frac{\partial \delta_\gamma}{\partial \nu},\ \forall v \in H_0^2(\Omega). \tag{2.2}$$

We suppose that X is endowed with the scalar product

$$\left(\begin{pmatrix} w_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} w_2 \\ v_2 \end{pmatrix}\right)_X = \int_\Omega (\Delta w_1 \Delta w_2 + v_1 v_2)dx,$$

which is obviously equivalent to the usual one. It is well known that A is onto, selfadjoint, strictly positive and that A^{-1} is compact. As a consequence the eigenvalues of A form an increasing sequence $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \ldots$. By using the notations above the system (1.1)-(1.4) can be written in the abstract form

$$w'' + Aw + Bw' = 0, \tag{2.3}$$

$$w(0) = w^0,\ w'(0) = w^1. \tag{2.4}$$

Let us now define

$$\mathcal{D}(\mathcal{A}) = \left\{ U = \begin{pmatrix} w \\ v \end{pmatrix} \in [H_0^2(\Omega)]^2 \text{ s.t. } Aw + Bv \in L^2(\Omega) \right\}, \tag{2.5}$$

and the nonlinear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to X$ defined by

$$\mathcal{A}\begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} -v \\ Aw + Bv \end{pmatrix}. \tag{2.6}$$

Let us first prove the following simple, but important properties of $\mathcal{D}(\mathcal{A})$

Lemma 2.1. If $\gamma \in \mathcal{K}$ and Γ is localy Lipshitz then $\mathcal{D}(\mathcal{A})$ is a dense subspace of X and the imbedding $\mathcal{D}(\mathcal{A}) \subset X$ is compact.

Proof. The density of $\mathcal{D}(\mathcal{A})$ in X is a consequence of the fact that $\mathcal{D}(\mathcal{A})$ contains V, where

$$V = \left\{ \begin{pmatrix} w \\ v \end{pmatrix} \in (H^4(\Omega_1) \cap H^4(\Omega_2) \cap H_0^2(\Omega)) \times H_0^2(\Omega) \text{ s.t. } g(\int_\gamma \frac{\partial v}{\partial \nu})|_\gamma = [\Delta w] \right\},$$

and of the density of V in X.
On the other hand $\begin{pmatrix} w \\ v \end{pmatrix} \in \mathcal{D}(\mathcal{A})$ if and only if there exists $u \in L^2(\Omega)$ such that

$$Aw = u - Bv. \tag{2.7}$$

As, by the trace theorem, $Bv \in H^{-\frac{3}{2}-\epsilon}(\Omega),\ \forall \epsilon > 0,\ \forall v \in [H_0^2(\Omega)]^2$, relation (2.7) implies that $w \in A^{-1}(H^{-\frac{3}{2}-\epsilon}(\Omega)),\ \forall \epsilon > 0$, so $\mathcal{D}(\mathcal{A}) \subset A^{-1}(H^{-\frac{3}{2}-\epsilon}(\Omega)) \times H_0^2(\Omega)$. As the imbeddings $A^{-1}(H^{-\frac{3}{2}-\epsilon}(\Omega)) \subset H_0^2(\Omega)$ and $H_0^2(\Omega) \subset L^2(\Omega)$ are obviously compact, we can conclude that the imbedding $\mathcal{D}(\mathcal{A}) \subset X$ is also compact. □

The welposedness for (1.1)-(1.4) will easily follow from the following result

Lemma 2.2. Suppose that $\gamma \in \mathcal{K}$, Γ is localy Lipschitz and g is increasing, continous, with $g(0) = 0$. Then $\mathcal{A}$ is the generator of a continous semigroup of nonlinear contractions in X.

Proof. A simple calculation gives:

$$\left(\left(\mathcal{A}\begin{pmatrix} w_1 \\ v_1 \end{pmatrix} - \mathcal{A}\begin{pmatrix} w_2 \\ v_2 \end{pmatrix}\right), \begin{pmatrix} w_1 \\ v_1 \end{pmatrix} - \begin{pmatrix} w_2 \\ v_2 \end{pmatrix}\right)_X =$$

$$= [g(\int_\gamma \frac{\partial v_1}{\partial \nu}) - g(\int_\gamma \frac{\partial v_2}{\partial \nu})) \int_\gamma \frac{\partial}{\partial \nu}(v_1 - v_2) d\gamma \geq 0, \ \forall \begin{pmatrix} w \\ v \end{pmatrix} \in \mathcal{D}(\mathcal{A}),$$

so $\mathcal{A}$ is monotone. In order to prove that $-\mathcal{A}$ generates a semigroup of nonlinear contractions on X it suffices to show that $\mathcal{A}$ is onto. Let $\begin{pmatrix} f \\ h \end{pmatrix} \in X$ and consider the equation

$$\mathcal{A}\begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix}, \tag{2.8}$$

which can be written as

$$v = -f \in H_0^2(\Omega), \tag{2.9}$$

$$Aw + Bv = h \in L^2(\Omega). \tag{2.10}$$

By using (2.9) relation (2.10) becomes

$$Aw = h + Bf \in H^{-2}(\Omega),$$

and, from classical elliptic regularity, we obtain that (2.8) admits a unique solution

$$\begin{pmatrix} w \\ v \end{pmatrix} \in [H_0^2(\Omega)]^2. \tag{2.11}$$

By using (2.10), (2.11) and the definition of $\mathcal{D}(\mathcal{A})$ it is now clear that $\begin{pmatrix} w \\ v \end{pmatrix} \in \mathcal{D}(\mathcal{A})$, so $\mathcal{A}$ is onto. □

As a consequence of Lemma 2.2 and of Theorem 3.1 from [4, p.152] we obtain the following existence and uniqueness result

Theorem 2.3. Suppose that γ and Γ satisfy the assumptions of Lemma 2.2. Then, if $\begin{pmatrix} w^0 \\ w^1 \end{pmatrix} \in \mathcal{D}(\mathcal{A})$, the initial and boundary value problem (1.1)-(1.4) admits a unique strong solution $\begin{pmatrix} w \\ w' \end{pmatrix} \in C([0,\infty), \mathcal{D}(\mathcal{A}))$. Moreover, if $U^0 = \begin{pmatrix} w^0 \\ w^1 \end{pmatrix} \in X$ then the initial and boundary value problem (1.1)-(1.4) admits a unique mild solution (i.e. $\begin{pmatrix} w \\ w' \end{pmatrix} \in C([0,\infty), X)$), which is defined as limit of strong solutions.

3. Asymptotic behavior of solutions for the closed loop system

In this section we shall study the energy decay of solutions of (1.1)-(1.4) when $t \to \infty$. It is by now wellknown that, by the invariance principle of LaSalle the energy decay property is reduced to a unique continuation result. However, for the sake of completeness, we shall sketch the proof of the following result

Lemma 3.1. Suppose that g is continous, strictly increasing, $g(0) = 0$ and that the only function satisfying the conditions

$$u" + \Delta^2 u = 0, \text{ in } \Omega \times (0, \infty) \tag{3.1}$$

$$u = \frac{\partial u}{\partial \nu} = 0, \text{ on } \Gamma \times (0, \infty), \tag{3.2}$$

$$\int_\gamma \frac{\partial u'}{\partial \nu} = 0, \text{ on } (0, \infty), \tag{3.3}$$

is $u \equiv 0$. Then, for any $w^0 \in H_0^2(\Omega)$, $w^1 \in L^2(\Omega)$ the solution w of (1.1)-(1.4) satisfies the energy estimate

$$\lim_{t \to \infty} (\|w(t)\|_{H^2(\Omega)} + \|w'(t)\|_{L^2(\Omega)}) = 0. \tag{3.4}$$

Proof. We follow the outline of [9]. Let us denote by $S(t)$ the semigroup of contractions generated by the operator $\mathcal{A}$ defined in (2.6). By density it is enough to prove that $S(t) \begin{pmatrix} w^0 \\ w^1 \end{pmatrix} \to 0$ for all $\begin{pmatrix} w^0 \\ w^1 \end{pmatrix} \in \mathcal{D}(\mathcal{A})$. By Lemma 2.1 $\mathcal{D}(\mathcal{A}) \subset X$ is compact, so, by using Theorem 3 from [10], we obtain that the orbit $orb \begin{pmatrix} w^0 \\ w^1 \end{pmatrix} = \bigcup_{t \geq 0} S(t) \begin{pmatrix} w^0 \\ w^1 \end{pmatrix}$ is precompact in X for all $\begin{pmatrix} w^0 \\ w^1 \end{pmatrix} \in \mathcal{D}(\mathcal{A})$. It follows that the ω-limit set $\omega \begin{pmatrix} w^0 \\ w^1 \end{pmatrix} = \{y \in X | y = \lim_{n \to \infty} S(t_n) \begin{pmatrix} w^0 \\ w^1 \end{pmatrix}, \text{ with } t_n \to \infty\}$ is nonvoid for all $\begin{pmatrix} w^0 \\ w^1 \end{pmatrix} \in \mathcal{D}(\mathcal{A})$. On the other hand, by LaSalle's invariance principle (see [9],[10]), if $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \omega \begin{pmatrix} w^0 \\ w^1 \end{pmatrix}$ then

$$S(t) \begin{pmatrix} u^0 \\ u^1 \end{pmatrix} = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix} \in \omega \begin{pmatrix} w^0 \\ w^1 \end{pmatrix}, \ \forall t \geq 0,$$

$$\left\| \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix} \right\|_X = \left\| \begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \right\|_X, \ \forall t \geq 0.$$

Relations above and (1.5) imply that u satisfies (3.1)-(3.3) so $u \equiv 0$ which implies that $\omega \begin{pmatrix} w^0 \\ w^1 \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ and the conclusion of the present lemma. $\square$

For particular geometries of Ω it is easy to construct examples of curves γ such that the assumption of Lemma 3.1 doesn't hold true (it suffices to have $\int_\gamma \frac{\partial\phi}{\partial\nu} d\gamma = 0$ for some eigenfunction of A). However we shall prove that generically the assumption of Lemma 3.1 holds true. In order to precise the notion of genericity let us recall a definition. If X is a topological space, the subset $F \subset X$ is called residual in X if F is a countable intersection of open, dense sets in X. If F is residual then F is dense in X. Moreover, the complement of F is a set of first category in X.

The following theorem will be essential for the proof of the stabilization result

Theorem 3.2. If A has simple eigenvalues then there exists a set $\mathcal{N}$, residual in $\mathcal{K}$, such that, for all $\gamma \in \mathcal{N}$, the only solution of (3.1)-(3.3) is $u \equiv 0$.

As a consequence of Lemma 3.1 and Theorem 3.2 we obviously obtain our main stabilization result

Theorem 3.3. If A has simple eigenvalues, g is continous, strictly increasing, with $g(0) = 0$ and $\gamma \in \mathcal{N}$ the solution w of (1.1)-(1.4) satisfies

$$\lim_{t\to\infty} (||w(t)||_{H^1(\Omega)} + ||w'(t)||_{L^2(\Omega)}) = 0. \tag{3.5}$$

In order to define the set $\mathcal{N}$ in Theorem 3.2, let us denote by $\lambda_i, i \geq 1$ the eigenvalues of A, supposed to be simple, i.e. $0 < \lambda_1 < \lambda_2 < \ldots \lambda_n < \ldots$. The following definition generalizes the notion of strategic point, wellknown for pointwise control problems.

Definition 3.1 *We say that a curve $\gamma \in \mathcal{K}$ is strategic and we write $\gamma \in \mathcal{N}$ if $\int_\gamma \frac{\partial\phi}{\partial\nu} d\gamma \neq 0$, for all eigenvector ϕ of A.*

The rest of this section is devoted to the proof of Theorem 3.2. We begin by stating a lemma which will easily imply Theorem 3.2 (the proof of the lemma will be given at the end of this section).

Lemma 3.4. The set $\mathcal{N}$ is residual in $\mathcal{K}$.

Proof of Theorem 3.2. Consider $\gamma \in \mathcal{N}$ and u satisfying (3.1)-(3.3). If we denote u' by ψ we see that ψ satisfies the conditions

$$\psi" + \Delta^2\psi = 0, \text{ in } \Omega \times (0, \infty) \tag{3.6}$$

$$\psi = \frac{\partial\psi}{\partial\nu} = 0, \text{ on } \Gamma \times (0, \infty), \tag{3.7}$$

$$\int_\gamma \frac{\partial\psi}{\partial\nu} = 0, \ \forall t \in (0, \infty). \tag{3.8}$$

Suppose that $\psi(0)$ (repectively $\psi'(0)$) admits the Fourier expansion

$$\psi(x, 0) = \sum_{n\geq 1} a_n w_n(x) (\text{ respectively } \psi'(x, 0) = \sum_{n\geq 1} b_n w_n(x)),$$

where $(w_n)_{n\geq 1}$ is a basis of eigenfunctions of A. A simple calculation shows that $\int_\gamma \frac{\partial\psi}{\partial\nu} d\gamma$ will be given by

$$\int_\gamma \frac{\partial\psi}{\partial\nu} = \sum_{n\geq 1}[a_n \cos(\sqrt{\lambda_n}t) + \frac{b_n}{\sqrt{\lambda_n}} \sin(\sqrt{\lambda_n}t)] \int_\gamma \frac{\partial w_n}{\partial\nu} d\gamma, \ \forall t \geq 0. \tag{3.9}$$

On the other hand (cf. [15])

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T e^{i\sqrt{\lambda_m}t} e^{i\sqrt{\lambda_n}t} = 0, \quad \text{if } m \neq n. \tag{3.10}$$

From (3.9) and (3.10) it follows that $\int_\gamma \frac{\partial\psi}{\partial\nu} d\gamma = 0$ for all $t \geq 0$ if and only if

$$a_n \int_\gamma \frac{\partial w_n}{\partial\nu} d\gamma = b_n \int_\gamma \frac{\partial w_n}{\partial\nu} d\gamma = 0, \ \forall n \geq 1. \tag{3.11}$$

As $\gamma \in \mathcal{N}$ from (3.11) we obtain that $a_n = b_n = 0$, $\forall n \geq 1$, so $\psi \equiv 0$ in $\Omega \times (0, T)$. By using the relation $\psi = u'$ and (3.1) it follows that $\Delta^2 u = 0$, which combined with (3.2) gives $u \equiv 0$ in $\Omega \times (0, T)$. □

The last part of this section is devoted to the proof of Lemma 3.4. Similar ideas were used in [1] for proving genericity of simple eigenvalues for the laplacian.

We shall need the following result

Lemma 3.5. Consider an orthonormal basis of $L^2(\Omega)$ formed by eigenvectors of A denoted by $(w_i)_{i=1,\infty}$. Then the set Z of points $(x, y) \in \Omega$ such that $\Delta w_i(x, y) \neq 0$, $\forall i \geq 1$ is dense in Ω.

Proof. According to [6] the functions w_i are $C^2(\Omega)$, so the set of points $(x, y) \in \Omega$ such that $\Delta w_i(x, y) \neq 0$, is open in Ω for all $i \geq 1$. On the other hand if $\Delta w_i \equiv 0$ in an open set $\omega_0 \subset \Omega$ then $w_i(x, y) = \frac{1}{\lambda_i}\Delta^2 w_i(x, y) = 0$ in ω_0 and by Holmgren's uniqueness theorem we obtain the contradiction $w_i \equiv 0$ in Ω. This is why the interior of the set of points $(x, y) \in \Omega$ such that $\Delta w_i(x, y) = 0$, is void, for all $i \geq 1$. By applying now Baire's lemma we obtain that Z is dense. □

Proof of Lemma 3.4. . Let us define the sets

$$A_n = \{\gamma \in \mathcal{K} \text{ s.t. } \int_\gamma \frac{\partial w_i}{\partial\nu} d\gamma \neq 0, \ \forall i = 1, \ldots, n\}.$$

The conclusion of Lemma 3.4 can be obtained as a consequence of the following two assertions:

(1) The set A_n is open in $\mathcal{K}, \forall n \geq 0$.
(2) The set A_{n+1} is dense in A_n, $\forall n \geq 0$.

In order to prove assertion (1) we shall use induction after n. Let us suppose that A_n is open for $n = 1, \ldots, k$ Let us first notice that

$$A_{k+1} = A_k \cap \{\gamma \in \mathcal{K} \text{ s.t. } \int_\gamma \frac{\partial w_{k+1}}{\partial\nu} d\gamma \neq 0\}.$$

On the other hand it is simple to prove that

$$\{\gamma \in \mathcal{K} \text{ s.t. } \int_\gamma \frac{\partial w_{k+1}}{\partial \nu} d\gamma \neq 0\},$$

is open in $\mathcal{K}$ so A_{k+1} is also open.

Let us now check property (2). For $\gamma \in A_n - A_{n+1}$ and an arbitrary $\epsilon > 0$ we shall construct a curve $\gamma'' \in K$ such that

$$\gamma'' \in A_{n+1},\ d(\gamma, \gamma'') < \epsilon \tag{3.12}$$

By Lemma 3.5 we can deform γ into a simple curve curve $\gamma' \in A_n$ such that $d(\gamma, \gamma') < \frac{\epsilon}{2}$ and $\Delta w_i(x_0, y_0) > 0$ for some $(x_0, y_0) \in \gamma'$. If $\int_{\gamma'} \frac{\partial w_{n+1}}{\partial \nu} d\gamma \neq 0$ there is nothing to prove. This is why we can suppose

$$\int_{\gamma'} \frac{\partial w_{n+1}}{\partial \nu} d\gamma' = 0. \tag{3.13}$$

Due to the C^2 regularity of w_i we can suppose that there exists $\eta \in (0, \frac{\epsilon}{2})$ such that

$$\Delta w(x, y) > 0, \forall (x, y) \in B(x_0, y_0; \eta), \tag{3.14}$$

where by $B(x_0, y_0; \eta)$ we donoted the ball of center (x_0, y_0) and of radius η. Let us now define γ'' as the boundary of $\Omega' \cup B(x_0, y_0; \eta)$, where Ω' is the open set contained at the interior of γ'. It is clear that

$$d(\gamma, \gamma'') < \epsilon. \tag{3.15}$$

On the other hand we have

$$\int_{\gamma''} \frac{\partial w_{n+1}}{\partial \nu} d\gamma'' = \int_{\Omega'} \Delta w_{n+1}(x, y) dxdy + \int_{\Omega''} \Delta w_{n+1}(x, y) dxdy, \tag{3.16}$$

where $\Omega'' = B(x_0, y_0; \eta) - \Omega'$. Relations (3.13), (3.14) and (3.16) imply that $\int_{\gamma''} \frac{\partial w_{n+1}}{\partial \nu} d\gamma'' \neq 0$ so $\gamma'' \in A_{n+1}$. By using now (3.15) we conclude that A_{n+1} is dense in A_n. $\square$

Remark 3.1. All results in this section hold if we replace the boundary condition (1.2) by

$$w(x, y, t) = \Delta w(x, y, t) = 0, \quad \text{on } \Gamma \times (0, T). \tag{3.17}$$

Moreover our method can be easily addapted for the stabilization of a Kirhhoff plate equation, obtained if we replace equation (1.1) by

$$w''(x, y, t) - \alpha \Delta w'' + \Delta^2 w(x, y, t) = u(t) \frac{\partial \delta_\gamma}{\partial \nu}, \text{ in } Q, \tag{3.18}$$

with $\alpha > 0$.

4. Controllability results

In this section we shall study exact and approximate controllability for the open loop problem (1.1)-(1.3). As we already noticed in the introduction the problem presents some similarities with the pointwise control of plate equations. This is why we shall use ideas and methods from [13]. By using standard semigroup methods it is easy to verify that (1.1)-(1.3) is wellposed in $L^2(\Omega)\times H^{-2}(\Omega)$, for any given $u\in L^2(0,T)$. In order to state the controllability results, let us give a definition.

Definition 5.1. *An initial data $(w^0,w^1)\in L^2(\Omega)\times H^{-2}(\Omega)$ is called exactly L^2-controllable in γ at time T if there exists $u\in L^2(0,T)$ such that the solution w of (1.1)-(1.3) satisfies (1.6). If the the exactly L^2-controllable initial data form a dense subset of $L^2(\Omega)\times H^{-2}(\Omega)$ we say that approximate controllability holds in $L^2(\Omega)\times H^{-2}(\Omega)$.*

It is by now well-known that the possibility of solving the controllability problem is related to the amount of information revealed by the restriction to $(0,T)$ of the application $t\to\int_\gamma \frac{\partial\phi}{\partial\nu}d\gamma$ where ϕ satisfies

$$\phi''+A\phi=0,\ \text{ in } (0,T), \tag{4.1}$$

$$\phi(0)=\phi^0\in H_0^2(\Omega),\ \phi'(0)=\phi^1\in L^2(\Omega), \tag{4.2}$$

and the operator A was introduced by (2.1). More precisely, if we denote by $<\cdot,\cdot>$ the H^s-H^{-s} duality, and we apply the Hilbert Uniqueness Method (H.U.M.) (cf. [18]) we obtain

Proposition 4.1. A given state $(w^0,w^1)\in L^2(\Omega)\times H^{-2}(\Omega)$ is exactly L^2-controllable in γ at time T if and only if there exists $C>0$ (depending on (w^0,w^1)) such that

$$|<w^0,\phi^1>-<w^1,\phi^0>|\le C\left[\int_0^T(\int_\gamma\frac{\partial\phi}{\partial\nu}d\gamma)^2dt\right]^{\frac12}, \tag{4.3}$$

for all $(\phi^0,\phi^1)\in H_0^2(\Omega)\times L^2(\Omega)$ and ϕ satisfying (4.1)-(4.2).

Let us denote by D the vector space of all finite linear combinations of eigenvectors of A. From Proposition 4.1 it easily follows that the conditions

- A has simple eigenvalues
- γ is a strategic curve, i.e. $\gamma\in\mathcal{N}$, where the set $\mathcal{N}$ was defined in section 3,

are necessary in order to have exact controllability of all states in $D\times D$. The main result of this section asserts that the converse holds true if and only if T is large enough. In order to give a precise statement of our results we shall need the counting function

$$n(t)=\ \text{card}\ \{\mu\in C,\ \mu\le t\},$$

where C is the increasing sequence of square roots of A and card E is the cardinal number of the set E. If we denote by S the two dimensional Lebesgue measure of

Ω, according to an extension of the classical Weyl formula (see [13]), the function n satisfies the asymptotic estimate

$$n(t) = dt + O(t^{\frac{1}{2}}), \tag{4.4}$$

where $d = \frac{1}{4\pi}S$. We shall give now our main controllability result

Theorem 4.2. Let Ω be a bounded smooth domain or a rectangle in $\mathbf{R}^2$ for which all the eigenvalues of A are simple and γ a strategic curve, i.e. $\gamma \in \mathcal{N}$. Then

- For every $T > \frac{S}{2}$ all initial data in $D \times D$ are exactly L^2-controllable in γ at time T.
- If $T < \frac{S}{2}$ there exist initial data in $D \times D$ which are not exactly L^2-controllable in γ at time T.

In order to prove Theorem 4.2 we will need a result of harmonic analysis, proved in [13]. In order to state it consider $\Lambda = (\mu_n)_{n\in\mathbf{Z}}$ a sequence of distinct real numbers and a fuction f of the form

$$f(t) = \sum_{k\in J} f_k e^{i\mu_k t}, \tag{4.5}$$

J beeing a finite subset of $\mathbf{Z}$. By slightly simplifying Corollary 2.3.5 from [13] we have the following result

Lemma 4.3. Let $\Lambda = (\mu_n)_{n\in\mathbf{Z}}$ a sequence of distinct real numbers. Assume that, for some $d \geq 0$ and $0 \leq \alpha < 1$, we have

$$\operatorname{card}\{\mu \in \Lambda,\ 0 \leq \mu \leq t\} = dt + O(t^\alpha), \tag{4.6}$$

$$\operatorname{card}\{\mu \in \Lambda,\ -t \leq \mu \leq 0\} = dt + O(t^\alpha).$$

Than the following assertions hold true

(A) For each $T > 2\pi d$ and for each $n \in \mathbf{Z}$ there exists a constant $C_n > 0$ such that
$$|f_n| \leq C_n \left(\int_0^T |f(t)|^2 dt\right)^{\frac{1}{2}},$$
for each function f of the form (4.5) with $n \in J$.

(B) On the other hand for each $T < 2\pi d$ and for each finite sequence $(\alpha_n)_{n\in F}$ of complex numbers having a non zero term, there exists no constant $C > 0$ such that
$$|\sum_{n\in F} \alpha_n f_n| \leq C \left(\int_0^T |f(t)|^2 dt\right)^{\frac{1}{2}},$$
for each function f of the form (4.5) with $F \subset J$.

Proof of Theorem 4.2. Let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \ldots$ be the eigenvalues of A and $w_1, w_2, \ldots, w_k, \ldots$ the associated basis of eigenvectors of A. Suppose that ϕ^0, ϕ^1 in (4.2) can be written as

$$\phi^0(x,y) = \sum_{k\geq 1} a_k w_k(x,y),\ \phi^1(x,y) = \sum_{k\geq 1} b_k w_k(x,y).$$

A simple calculation shows that the solution ϕ of (4.1)-(4.2) is given by

$$\phi(x,y,t)=\sum_{k\geq 1}[a_k\cos(\sqrt{\lambda_k}t)+\frac{b_k}{\sqrt{\lambda_k}}\sin(\sqrt{\lambda_k}t)]w_k(x,y). \tag{4.7}$$

As equation (1.1) is linear and we study only the controllability of finite combinations of eigenvectors of A it suffices to apply Proposition 4.1 only in the cases:

1 $w^0=0,\ w^1=w_n,\ n\geq 1,$
2 $w^0=w_m,\ m\geq 1,\ w^1=0.$

We shall only consider the first case, as the second can be treated in a completely similar manner. From Proposition 4.1 and relation (4.7) it folllows that the initial state exactly L^2-controllable in γ at time T if and only if there exists a constant $C_n>0$ such that

$$|a_n|\leq \tag{4.8}$$

$$\leq C_n\left\{\int_0^T\left[\sum_{k\geq 1}a_k\int_\gamma\frac{\partial w_k}{\partial\nu}d\gamma\cos(\sqrt{\lambda_k}t)+\frac{b_k}{\sqrt{\lambda_k}}\int_\gamma\frac{\partial w_k}{\partial\nu}d\gamma\sin(\sqrt{\lambda_k}t)\right]^2dt\right\}^{\frac{1}{2}}.$$

If we write the trigonometric sum from the right hand side of (4.8) as a sum of complex exponentials, relation (4.4) and assertion (A) of Lemma 4.3imply that, for all $n\geq 1$ there exists $D_n>0$ such that

(4.9)
$$\left\{\int_0^T\left[\sum_{k\geq 1}a_k\int_\gamma\frac{\partial w_k}{\partial\nu}d\gamma\cos(\sqrt{\lambda_k}t)+\frac{b_k}{\sqrt{\lambda_k}}\int_\gamma\frac{\partial w_k}{\partial\nu}d\gamma\sin(\sqrt{\lambda_k}t)\right]^2dt\right\}^{\frac{1}{2}}\geq$$

$$\geq D_n\left(|a_n+\frac{b_n}{\sqrt{\lambda_n}}|+|a_n-\frac{b_n}{\sqrt{\lambda_n}}|\right)|\int_\gamma\frac{\partial w_n}{\partial\nu}d\gamma|,\ \forall T>2\pi d=\frac{S}{2}.$$

As we supposed that $\gamma\in\mathcal{N}$, i.e. $\displaystyle\int_\gamma\frac{\partial w_k}{\partial\nu}d\gamma\neq 0,\ \forall k\geq 1$ relation (4.9) implies that (4.8) holds true provided that $T>\frac{S}{2}$. By Theorem 4.1 we obtain that initial states in $D\times D$ are exactly L^2-controllable in γ at time T, for all $T>\frac{S}{2}$.

On the other hand, if $T<\frac{S}{2}$ assertion (B) of Lemma 4.3 implies that (4.8) cannot hold for all $(\phi^0,\phi^1)\in H_0^2(\Omega)\times L^2(\Omega)$ so Theorem 4.1 implies that the initial state $(0,w_n)$ is not exactly L^2-controllable in γ at time T, for any $T<\frac{S}{2}$. $\square$

An imediate consequence of Theorem 4.2 is

Corollary 4.4. Let Ω be a bounded smooth domain or a rectangle in $\mathbf{R}^2$ for which all the eigenvalues of A are simple, γ a strategic curve and $T>\frac{S}{2}$. Then all initial states in $L^2(\Omega)\times H^{-2}(\Omega)$ are approximatively L^2-controllable in γ at time T.

Remark 4.1. All results in this section hold if we replace the boundary condition (1.2) by (3.17). In the one dimensional case, i.e. if we take $\Omega = (0, \pi)$ and $\gamma = \{\xi, \eta\} \subset (0, \pi)$ we notice that d in (4.4) is equal to zero. This implies that, generically with respect to ξ, η the initial states in $D \times D$ are controllable in arbitrarily small time. Moreover if we replace (1.2) by (3.17) it was proved in [23] and [24] that there exist ξ, $\eta \in (0, \pi)$ such that all initial states in $[H^3(\Omega) \cap H_0^1(\Omega)] \times [H^2(\Omega) \cap H_0^1(\Omega)]$ are exactly L^2-controllable in γ at time T.

Remark 4.2. The results in this section don't hold for the Kirchhoff model, i.e. if we replace (1.1) by (3.18). In this case we have to replace the operator A in (2.3), (4.1) by $\widetilde{A} = (I - \alpha\Delta)^{-1}A$. This fact implies that the asymptotic behavior of the eigenvalues changes in a radical way. More precisely, if we denote by $E = \{\mu_1, \mu_2, \ldots, \mu_n \ldots\}$ the increasing sequence of square roots of eigenvalues of $\widetilde{A}$, and define $n_1(t) = \mathrm{card}\{\mu \in E,\ \mu \leq t\}$, we have the Weyl type formula

$$n_1(t) = O(t^2). \tag{4.10}$$

Relation (4.10) implies that the Beurling-Malliavin density of the sequence (μ_n) is infinity, so there exists no time $T > 0$ such that all initial data in $D \times D$ are exactly L^2-controllable in γ at time T (see [5] for the definition of the Beurling-Malliavin density and [13] for the applications to control theory). This implies that, in the same case, we cannot obtain uniform stabilization in the energy space (cf.[18], [21]). Let us also notice that the lack of exact controllability in the usual Sobolev spaces for the Kirchhoff model also follows from a result proved in [22].

5. An example

Suppose that Ω is a rectangle of the form $(0, \pi) \times (0, L)$. In order to do some explicit computations we assume that the plate is simply supported, i.e. we consider the initial and boundary value problem (1.1), (3.17), (1.3). In this case the eigenfunctions of A are

$$\Phi_{mn}(x, y) = \sin(mx) \sin(\frac{n\pi y}{L}),\ \forall m, n \geq 1.$$

It is clear that the eigenvalues of A are simple if and only if $\left(\frac{L}{\pi}\right)^2$ is not a rational number.

Moreover suppose that the actuator occupies the rectangle $\omega = [\xi_1, \eta_1] \times [\xi_2, \eta_2]$, with

$$0 < \xi_1 < \eta_1 < \pi,\ 0 < \xi_2 < \eta_2 < L.$$

If we denote by γ the boundary of ω a simple calculation gives

$$\int_\gamma \frac{\partial \phi_{mn}}{\partial \nu} d\gamma = -\left(m^2 + \frac{n^2\pi^2}{L^2}\right) \int_\omega \Phi_{mn}(x, y) dx dy. \tag{5.1}$$

Relation (5.1) implies that γ is strategic if and only if

$$\frac{\eta_1 \pm \xi_1}{\pi} \notin Q,\ \frac{\eta_2 \pm \xi_2}{L} \notin Q.$$

Conditions above show that, at least in this case, the set of strategic curves $\mathcal{N}$ is not stable with respect to small perturbations.

References

1. M. H. Albert, *Genericity of simple eigenvalues for elliptic PDE's*, Proc. Amer. Math. Soc., 48(1975), 413-418.
2. H. T. Banks and R. C. Smith, *The modelling of piezoceramic patch interactions with shells, plates and beams*, ICASE Report No. 92-66.
3. H. T. Banks, W. Fang, R. J. Silcox and R. C. Smith, *Approximation methods for control of tye acoustic/sructure interaction with piezoceramic actuators*, preprint
4. V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, Ed. Academiei, Bucharest, 1976.
5. A. Beurling and P. Malliavin, *On the closure of characters and the zeros of entire functions*, Acta Math., 118(1967), p.79-83
6. H. Brezis, *Analyse fonctionnelle*, Masson, Paris, 1988
7. F. Browder, *Problèmes non linéaires*, Université de Montréal, Montréal, 1966.
8. E. F. Crawley and E. H. Anderson, *Detailed models for piezoceramic actuation of beams*, J. of Intell. Mater; Syst., 1(1990), p.4-25.
9. C. Dafermos, *Asymptotic behavoir of solutions of evolution equations* in 'Nonlinear evolution equations (M.G. Crandall, Ed.), 103-123, Academic Press, New york, 1978.
10. C. Dafermos and M. Slemrod, *Asymptotic behavior of nonlinear contraction semigroups*, J. Diff. Eq., 13(1973), 97-106
11. Ph. Destuynder, I. Legrain, L. Castel, N. Richard, *Theoretical, numerical and experimental discussion of the use of piezoelectric devices for control-structure interaction* Eur. J. Mech., A/Solids, 11(1992), pp. 181–213.
12. A. Haraux, *Stabilization of trajectories for some weakly damped hyperbolic equations*, J. Diff. Eq., 59(1985), 145-154.
13. A. Haraux and S. Jaffard, *Pointwise and spectral control of plate vibrations*, Rev. Mat. Iberoamericana, 7(1991), p.1-23
14. S. Jaffard and M. Tucsnak, *Regularity of plate equations with control concentrated in interior curves*, preprint
15. Y. Katznelson, *An introduction to harmonic analysis*, Dover, New York, 1968.
16. J. E. Lagnese, *Boundary stabilization of thin plates*, SIAM, Philadelphia, 1989
17. J. E. Lagnese and J. L. Lions, *Modelling, analysis and control of thin elastic plates*, Masson, Paris, 1988
18. J. L. Lions, *Contrôlabilité exacte des systèmes distribués*, Masson, Paris, 1988.
19. J. L. Lions, *Pointwise control for distributed systems*, in Control and estimation in distributed parameter systems, H, T. Banks ed., SIAM, Philadelphia, 1992, pp.1-41.
20. O. Pironneau, Optimal shape design for elliptic systems, Springer, Berlin, 1984
21. D. L. Russell, *Controllability and stabilizability theory for linear partial differential equations*, SIAM Review, 20(1978), 639-679.
22. D. Tataru, *Exact controllability and lack of exact controllability with finitely many unbounded controls*, private communication
23. M. Tucsnak, *Contrôle d'une poutre avec actionneur piézoélectrique*, C.R. Acad. Sci., Série 1, 319(1994), p.697-702
24. M. Tucsnak, *Regularity and exact controllability for a beam with piezoelectric actuator*, to appear in SIAM J. on Control
25. M. Tucsnak and E. Zuazua, *Asymptotic behavior of solutions for the wave equation with control concentrated in interior curves*, preprint

M. Tucsnak, Centre de Mathématiques Appliqués (CMAP), Ecole Polytechnique, 91128 Palaiseau and Université de Versailles, France

E-mail address: tucsnak@cmapx.polytechnique.fr