Generalized Impedance Boundary Conditions for the Maxwell Equations as Singular Perturbation Problems

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Abstract

In this paper the Maxwell equations in an exterior domain with the generalized impedance boundary conditions of Engquist-Nédélec are considered. The particular form of the assumed boundary conditions can be considered to be a singular perturbation of the Dirichlet boundary conditions. The convergence of the solution of the Maxwell equations with these generalized impedance boundary conditions to that of the corresponding Dirichlet problem is proven. The proof uses a new integral equation method combined with results dealing with singular perturbation problems of a class of pseudo-differential operators.

Key words. Integral Equations, Generalized Impedance Boundary Conditions, Maxwell's Equations, Singular Perturbation Problem, Fredholm system, The Hodge decomposition Theorem.

AMS(MOS) subject classifications. 35C15, 35B25, 78A45, 35S15.

Abbreviated title. Singular Perturbation Problems.

1 Introduction

In recent years, problems involving scattering from thin dielectric layers covering perfectly conducting object have received considerable attention due to the emergence of materials used as coatings on conductors (see e.g. [20], [21], [23], [24], [41], [42], [43], [44], ...). The electromagnetic scattering problem for such obstacles has been rigorously approximated by Engquist and Nédélec [22] and the authors [5] for chiral coatings by introducing new sets of conditions at the surface of the conductor, the generalized impedance boundary conditions which led to problems simpler than the full transmission problem for Maxwell's equations. These generalized impedance boundary conditions relate the tangential electric and magnetic fields, on the exterior boundary of the coated conducting body. The well posedness of the Maxwell system with these boundary conditions was originally treated by the authors in [3]. It was shown by using an integral equation approach that there exists a unique outgoing solution to Maxwell's equations with the generalized impedance boundary condition of Engquist-Nédélec. The analogous boundary conditions in acoustic problems have been treated in [22] and in [9] by Bendali and Lemrabet.

One of the first derivations of approximate boundary conditions in electromagnetics was by Rytov [38] who developed conditions through the second order applicable at the curved surface of a highly conducting object. The simplest and most popular boundary conditions are such that the tangential component of the electric and magnetic fields are related by a constant. They are known as the Leontovich impedance boundary conditions. Impedance boundary conditions are also useful in the study of high frequency diffraction phenomena (see e.g Bouche and Molinet [11] and the references therein).
The generalized impedance boundary conditions are very helpful in simplifying numerical solution of wave problems involving complex structures. They can be used, along with the Silver-Müller radiation condition, to compute the scattered field without requiring detailed modeling of the field quantities inside the dielectric coating [10]. This procedure not only reduces the number of unknowns required to model the diffraction problem, but eliminates the need for Green's function or a variational method for the coating. This second advantage can be significant, particularly if the coating material is complex, such as anisotropic, chiral, or other complex material.

The particular forms of the generalized impedance boundary conditions of Engquist-Nédélec can be considered to be a singular perturbation of the Dirichlet boundary conditions. In this paper, we employ the method of boundary integral equations to discuss the convergence of Maxwell's equations with the generalized impedance boundary conditions as the small parameter tends to zero. By a new boundary integral equation method we obtain that the Maxwell system with the generalized impedance boundary conditions is equivalent to a Fredholm system of pseudo-differential operators. Making use of spectral decompositions of two elliptic self-adjoint pseudo-differential operators, we reduce our singular perturbations to regular perturbations.

The use of integral equations in problems of electromagnetics is a well-known technique. Without attempting to give an exhaustive review of the literature on integral equations in electromagnetics, we mention that various aspects have been treated by Maue [32], Müller [33], Weyl [48], Saunders [39], Calderòn [12], Werner [46], Knauff and Kress [27], Colton and Kress [13] and [14], Gray and Kleinman [16] and Bendali [8]. More recently, the authors [4] have developed a Fredholm system of integral scalar equations for electromagnetic scattering problem by using a Hodge decomposition of the tangential fields on the boundary of the conductor. This method can be used to establish the existence of solutions to Maxwell's equations with more general boundary conditions as in [5].

On the other hand, the theory of singular perturbation problems of partial differential operators and its applications form the subject of a vast literature. The development of this branch of the theory of partial differential equations was greatly stimulated by various problems arising in mechanics, electromagnetics, chemical kinetics, mathematical biology, and modern technology, requiring asymptotic analysis. We refer to the book by Vasil'eva, Butuzov and Kalachev [45] and the references therein for a survey of results on singular problems for partial differential equations.

Many singular perturbation problems in electromagnetics have been studied in the literature. The convergence of the Robin problem for the Helmholtz equation was studied by Kirsch in [26]. The Laplace equation with a Robin boundary condition as a singular perturbation problem was investigated by Ahner and Wiener [1]. Some transmission problems for the Helmholtz equation which can be considered as singular perturbation problems were investigated by Werner [47], Ramm [37] and Wilde [49].

The present paper is organized as follows. Having introduced some standard notations and remarkable identities in section 2 we outline the approach and formulate our main results in section 3. Section 4 is devoted to derive the generalized impedance boundary conditions for thin dielectric layers. In section 5, we reduce the Maxwell equations with these boundary conditions to a Fredholm system of integral equations. Section 7 is devoted to prove our convergence result using a singular perturbation theorem for a class of pseudo-differential operators stated and proved in section 6. We note that our ideas and methods can be extended to more general classes of boundary conditions involving a small parameter, but, for the sake of brevity, we do not consider these more general problems in this paper.

2 Definitions, Notations and Remarkable Identities

Let Ω^e be an exterior smooth domain in $\mathbb{R}^3$ (i.e., an open set such that its boundary is a smooth closed surface of $\mathbb{R}^3$ and the complementary of its closure, denoted by Ω^i, is bounded, connected and has Γ as boundary), and $n(x)$ the outward pointing normal at $x \in \Gamma$. For such a set, there exists a neighborhood of Γ denoted by $\mathcal{O}$ such that, for any $x \in \mathcal{O}$, if x_Γ is the orthogonal projection of x on Γ and $s = (x - x_\Gamma)\,.\,n(x_\Gamma)$, The couple (x_Γ, s) is a local parameterization of Γ. We recall the following usual differential operators on $(\mathcal{D}'(\mathbb{R}^3))^3$ in Cartesian components :

$$\nabla\wedge = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}, \quad \nabla\,. = \nabla^t = (\partial_1, \partial_2, \partial_3),$$

and we introduce the following boundary differential operators defined on tangential fields of Γ : the tangential gradient and divergence respectively denoted by ∇_Γ and div_Γ, the vector and scalar valued tangential rotational operators respectively denoted by $\overrightarrow{\mathrm{curl}}_\Gamma$ and curl_Γ, the Laplace-Beltrami operator on Γ denoted by Δ_Γ and defined by

$$\Delta_\Gamma = \mathrm{div}_\Gamma \, \nabla_\Gamma = -\mathrm{curl}_\Gamma \, \overrightarrow{\mathrm{curl}}_\Gamma .$$

Throughout this paper, for any field u defined on Γ, by $\mathrm{curl}_\Gamma u$ and $\mathrm{div}_\Gamma u$ we mean $\mathrm{curl}_\Gamma u_\Gamma$ and $\mathrm{div}_\Gamma u_\Gamma$ where u_Γ is the tangential component of u on Γ.
We recall the mean curvature $c(x)$ defined as $2c(x) = \nabla \, . \, n(x)$ and the tensor of curvature $R(x)$ defined by $R(x) = \nabla \, n(x)$.
To state our boundary-value problems in suitable mathematical forms, we shall use the following notations for the usual functional spaces :
$\mathcal{D}'(\Omega^e)$ is the space of distributions in Ω^e.
$L^2(\Omega^e)$ is the space of complex square integrable functions defined in Ω^e.
$L^2(\Gamma)$ is the space of complex square integrable functions defined on Γ.
$L^\infty(\mathbb{R}^3)$ is the space of bounded functions defined in $\mathbb{R}^3$.
$H^s(\Gamma)$ the Sobolev space of order $s \in \mathbb{R}$ on Γ. The dual of $H^s(\Gamma)$ is $H^{-s}(\Gamma)$.
$H_0^s(\Gamma) = \Big\{ \varphi \in H^s(\Gamma), \langle \varphi, 1 \rangle_{H^s(\Gamma), H^{-s}(\Gamma)} = 0 \Big\}$.
$H^1(\Omega^e) = \Big\{ \varphi \in L^2(\Omega^e), \nabla \varphi \in (L^2(\Omega^e))^3 \Big\}$.
$H_0^1(\Omega^e) = \Big\{ \varphi \in H^1(\Omega^e), \varphi = 0 \text{ on } \Gamma \Big\}$.
$H(\mathrm{curl}, \Omega^e) = \Big\{ u \in (L^2(\Omega^e))^3, \nabla \wedge u \in (L^2(\Omega^e))^3 \Big\}$.
$H_0(\mathrm{curl}, \Omega^e) = \Big\{ u \in H(\mathrm{curl}, \Omega^e), u \wedge n = 0 \text{ on } \Gamma \Big\}$.
$\overline{H}^{loc}(\mathrm{curl}, \Omega^e)$ is the Fréchet space of fields which are locally in $H(\mathrm{curl}, \Omega^e)$.
$TH^s(\Gamma) = \Big\{ c \in (H^s(\Gamma))^3, c \, . \, n = 0 \Big\}$.
$TH^s(\mathrm{div}, \Gamma) = \Big\{ c \in (H^s(\Gamma))^3, c \, . \, n = 0, \mathrm{div}_\Gamma \, c \in H^s(\Gamma) \Big\}$.
$TH^s(\mathrm{curl}, \Gamma) = \Big\{ c \in (H^s(\Gamma))^3, c \, . \, n = 0, \mathrm{curl}_\Gamma \, c \in H^s(\Gamma) \Big\}$.
$TH^s(\mathrm{curl}, \Gamma)$ and $TH^s(\mathrm{div}, \Gamma)$ are Hilbert spaces. We recall without proof the following duality result

$$TH^{-s-1}(\mathrm{curl}, \Gamma) = \Big(TH^s(\mathrm{div}, \Gamma) \Big)',$$

(see e.g. [36] for a proof) and the following identities. Let u and v be two smooth fields defined in a neighborhood of Γ, w a tangential smooth field on Γ and φ a smooth function defined on Γ. We have

$$\left\{ \begin{array}{l} \mathrm{div}_\Gamma (u \wedge n) = n \, . \Big(\nabla \wedge u \Big)_{|\Gamma} . \\ \mathrm{curl}_\Gamma \, \nabla_\Gamma \, w = 0. \\ \mathrm{div}_\Gamma \, (w \wedge n) = \mathrm{curl}_\Gamma \, w. \end{array} \right. \tag{1}$$

$$\nabla \wedge u = \Big(\mathrm{curl}_\Gamma\, u_\Gamma\Big)\, n + \overrightarrow{\mathrm{curl}}_\Gamma (u\,.\,n) + (R-2c)(u \wedge n) - \frac{\partial}{\partial s}(u \wedge n), \quad (2)$$

where u_Γ is the tangential component of u on Γ : $u_\Gamma = -n \wedge (n \wedge u)$.

$$(R-2c)(u \wedge n) = -R(u_\Gamma). \quad (3)$$

$$(\nabla \wedge u) \wedge n = -\nabla_\Gamma (u\,.\,n) + R(u_\Gamma) + \frac{\partial}{\partial s}(u_\Gamma). \quad (4)$$

$$\nabla\,.\,u = \mathrm{div}_\Gamma\, u_\Gamma + 2c\, u\,.\,n + \frac{\partial}{\partial s}(u\,.\,n). \quad (5)$$

$$\int_\Gamma \mathrm{curl}_\Gamma\, u_\Gamma\, \varphi = \int_\Gamma u_\Gamma\,.\,\overrightarrow{\mathrm{curl}}_\Gamma\, \varphi. \quad (6)$$

3 Setting of the Problem and Main Results

Let E^δ and H^δ be the outgoing solutions of the Maxwell equations :

$$\left\{ \begin{array}{ll} \nabla \wedge E^\delta = i\omega\,\mu\, H^\delta & \text{in } \Omega^e, \\ \nabla \wedge H^\delta = -i\omega\,\varepsilon\, E^\delta & \text{in } \Omega^e, \end{array} \right. \quad (7)$$

with the following boundary conditions on Γ :

$$\begin{array}{l} n \wedge (n \wedge E^\delta) + i\delta\, a^{(1)}\, H^\delta \wedge n + i\delta\, a^{(2)}\, \nabla_\Gamma\, \mathrm{div}_\Gamma\, (H^\delta \wedge n) \\ = g^\delta \in TH^{1/2}(\mathrm{curl}, \Gamma), \end{array} \quad (8)$$

where g^δ converges to g^0 in $TH^{1/2}(\mathrm{curl},\Gamma)$ strongly and $(a^{(j)})_{j=1,2}$ are two strictly positive constants. Throughout the majority of the remainder of this paper $g^0 \in TH^{1/2}(\mathrm{curl},\Gamma)$ is sufficient for our purposes. In certain instances, however, more regularity will be needed and when those situations arise, it will be clearly indicated.

The existence and uniqueness of a solution

$$(E^\delta, H^\delta) \in \overline{H}^{loc}(\mathrm{curl}, \Omega^e) \times \overline{H}^{loc}(\mathrm{curl}, \Omega^e)$$

to the Maxwell system (7) with the boundary conditions (8) satisfying the Silver-Müller radiation condition will be given in this paper. The existence result is established by using an integral approach method originally developped by the authors in [5].

Our principal motivation in the present work is to study the limiting behavior of the solution of this boundary value problem as the parameter δ tends to zero. In this paper, we shall prove the convergence of (E^δ, H^δ) to (E^0, H^0) defined as the unique outgoing solution of the Maxwell equations with the Dirichlet boundary conditions

$$\left\{\begin{array}{l} \nabla \wedge E^0 = i\omega\,\mu\, H^0 \quad \text{in } \Omega^e, \\ \nabla \wedge H^0 = -i\omega\,\varepsilon\, E^0 \quad \text{in } \Omega^e, \\ n \wedge (n \wedge E^0) = g^0 \quad \text{on } \Gamma, \end{array}\right. \tag{9}$$

and we establish the rate of the convergence and we indicate how the smoothness of g^0 influences this.
Note that if $g^0 \in TH^{1/2}(\text{curl}, \Gamma)$, then the unique solution (E^0, H^0) of the Maxwell system (9) belongs to $\overline{H}^{1,loc}(\Omega^e) \times \overline{H}^{1,loc}(\Omega^e)$. As we shall see later, the convergence of g^δ to g^0 in the Sobolev space $TH^{1/2}(\text{curl}, \Gamma)$ is necessary to prove the convergence of (E^δ, H^δ) to (E^0, H^0) in $H(\text{curl}, \Omega) \times H(\text{curl}, \Omega)$ for any compact Ω contained in Ω^e.

For the convenience of the reader we summarize here the main steps of our method. We begin our analysis by recalling the integral representation formula of the electric field solution of the Maxwell system (7). Taking the tangential trace of the electric field represented by (20) on the boundary Γ, and using the generalized impedance boundary conditions (8), we eliminate the tangential component of the electric field in (20). Using a Hodge decomposition theorem of the Sobolev space $TH^{3/2}(\text{div}, \Gamma)$ whose structure entirely depends on the topology of Γ, we transform the previous vector problem where the unknown is the tangential component of the magnetic field to the Fredholm system of the two integral equations (27) and (28) where the unknowns are scalar functions. This system is shown to be uniquely solvable in $H_0^{5/2}(\Gamma) \times H_0^{7/2}(\Gamma)$. It can be considered as a singular perturbation of the Fredholm system $(45)-(46)$ which is uniquely solvable in $H_0^{3/2}(\Gamma) \times H_0^{5/2}(\Gamma)$. To prove the convergence of the unique solution of $(27) - (28)$ to that of $(45) - (46)$ in $H_0^{3/2}(\Gamma) \times H_0^{5/2}(\Gamma)$ strongly, we rewrite the system $(27) - (28)$ in the form (54). From the apriori-estimates proved in Lemma 6.2, some properties of the pseudo-differential operators under consideration in (54) and by using a Tikhonov regularization argument, we will obtain the claim and we will show the dependence of the rate of convergence on the regularity of g^0.

4 Derivation of the Generalized Impedance Boundary Conditions for Thin Dielectric Layers

This section is devoted to derive the generalized boundary conditions for thin dielectric layers. Let δ be a small strictly positive parameter. Let Ω_1^δ defined as $\Gamma \times]0, \delta[$ be a layer of dielectric material of thickness δ. Let us introduce the exterior domain $\Omega_2^\delta = \mathbb{R}^3 \setminus \overline{\Omega_1^\delta}$, the interface $\Gamma^\delta = \Gamma \times \{s = \delta\}$ and the family of parallel curves Γ_s^δ defined by

$$\Gamma_s^\delta = \Big\{x \in \Omega^e, x = x_\Gamma + sn(x_\Gamma), 0 \leq s \leq \delta\Big\}.$$

Consider now the fields $E_1^\delta, E_2^\delta, H_1^\delta$ and H_2^δ satisfying in a weak sense the Maxwell equations

$$\left\{\begin{array}{l} \nabla \wedge E_1^\delta = i\omega\,\mu_1\,H_1^\delta \quad \text{in } \Omega_1^\delta, \\ \nabla \wedge H_1^\delta = -i\omega\,\varepsilon_1\,E_1^\delta \quad \text{in } \Omega_1^\delta, \\ E_1^\delta \wedge n = 0, \quad \text{on } \Gamma, \end{array}\right. \tag{10}$$

$$\left\{\begin{array}{l} \nabla \wedge E_2^\delta = i\omega\,\mu_2\,H_2^\delta \quad \text{in } \Omega_2^\delta, \\ \nabla \wedge H_2^\delta = -i\omega\,\varepsilon_2\,E_2^\delta \quad \text{in } \Omega_2^\delta, \\ |(E_2^\delta - E^{in})(x) - \sqrt{\dfrac{\mu_2}{\varepsilon_2}}(H_2^\delta - H^{in})(x) \wedge \dfrac{x}{|x|}| \leq \dfrac{C}{|x|^2}, \quad |x| \to +\infty, \end{array}\right. \tag{11}$$

and the transmission conditions

$$E_1^\delta \wedge n = E_2^\delta \wedge n, H_1^\delta \wedge n = H_2^\delta \wedge n, \quad \text{on } \Gamma^\delta. \tag{12}$$

In (10) and (11), the fields (E^{in}, H^{in}) are incident electric and magnetic plane waves and the dielectric coefficients $\varepsilon_1, \varepsilon_2, \mu_1$ and μ_2 are strictly positive constants.

We wish to point out that the Maxwell equations (10) and (11) with the transmission conditions (12) are uniquely solvable in $\overline{H}^{loc}(\text{curl}, \Omega^e) \times \overline{H}^{loc}(\text{curl}, \Omega^e)$. The existence can be shown by using a variational method similar to the one we have used for chiral layers in [5].

From the remarkable identities (4) and (5), we have

$$(\nabla \wedge E_1^\delta) \wedge n = \frac{\partial}{\partial s} E_{\Gamma_s^\delta}^\delta - \nabla_{\Gamma_s^\delta}(E_1^\delta \,.\, n) + R(E_{\Gamma_s^\delta}^\delta). \tag{13}$$

and

$$(\nabla \wedge H_1^\delta) \,.\, n = \text{curl}_{\Gamma_s^\delta} H_{\Gamma_s^\delta}^\delta = \text{div}_{\Gamma_s^\delta}(H_1^\delta \wedge n). \tag{14}$$

Since $E_1^\delta \wedge n = 0$ on Γ, By integrating the identity (13), we obtain

$$E_{\Gamma^\delta}^\delta(x_{\Gamma^\delta}) = \int_0^\delta (\nabla \wedge E_1^\delta) \wedge n \, ds + \int_0^\delta \nabla_{\Gamma_s^\delta}(E_1^\delta \,.\, n) \, ds - \int_0^\delta R(E_{\Gamma_s^\delta}^\delta) \, ds. \tag{15}$$

We use the Maxwell equations (10) to obtain

$$\left\{\begin{array}{l} (\nabla \wedge E_1^\delta) \wedge n = i\omega\mu_1\,H_1^\delta \wedge n, \\ E_1^\delta \,.\, n = \dfrac{i}{\omega\varepsilon_1}\,\text{div}_{\Gamma^\delta}\,(H_1^\delta \wedge n). \end{array}\right. \tag{16}$$

Using the formulae (15) and (16), we can approximate $E^\delta_{\Gamma^\delta}(x_{\Gamma^\delta})$ by

$$\begin{aligned} E^\delta_{\Gamma^\delta}(x_{\Gamma^\delta}) &= i\delta\,\omega\mu_1\, H_1^\delta \wedge n \\ &\quad + \frac{i\delta}{\omega\varepsilon_1}\, \nabla_{\Gamma^\delta}\, \mathrm{div}_\Gamma\, (H_1^\delta \wedge n) + 0(\delta^2) \quad \text{on } \Gamma^\delta, \end{aligned} \tag{17}$$

if we assume that the electric field E_1^δ is smooth in the dielectric layer. Now, from the transmission conditions (12) it follows that

$$-n \wedge (n \wedge E_2^\delta) = i\delta\omega\mu_1\, H_2^\delta \wedge n + \frac{i\delta}{\omega\varepsilon_1} \nabla_{\Gamma^\delta}\, \mathrm{div}_{\Gamma^\delta}\, (H_2^\delta \wedge n) + 0(\delta^2), \tag{18}$$

on Γ^δ. The use of generalized impedance boundary conditions consists in approximating in the exterior domain Ω_2^δ the fields (E_2^δ, H_2^δ) by the fields $(\mathcal{E}^\delta, \mathcal{H}^\delta)$ solutions of the following Maxwell equations

$$\begin{cases} \nabla \wedge \mathcal{E}^\delta = i\omega\,\mu_2\,\mathcal{H}^\delta \quad \text{in } \Omega_2^\delta, \\ \nabla \wedge \mathcal{H}^\delta = -i\omega\,\varepsilon_2\,\mathcal{E}^\delta \quad \text{in } \Omega_2^\delta, \\ -n \wedge (n \wedge \mathcal{E}^\delta) = i\delta\omega\mu_1\,\mathcal{H}^\delta \wedge n + \dfrac{i\delta}{\omega\varepsilon_1} \nabla_{\Gamma^\delta}\, \mathrm{div}_{\Gamma^\delta}\, (\mathcal{H}^\delta \wedge n) \quad \text{on } \Gamma^\delta, \\ |(\mathcal{E}^\delta - E^{in})(x) - \sqrt{\dfrac{\mu_2}{\varepsilon_2}}(\mathcal{H}^\delta - H^{in})(x) \wedge \dfrac{x}{|x|}| \le \dfrac{C}{|x|^2}, \quad |x| \to +\infty. \end{cases} \tag{19}$$

The above generalized impedance boundary conditions was written by Engquist and Nédélec in [22]. Optimal error estimates are established by the authors of the present paper using an integral equation method [5]. It was shown under a regularity assumption on E_1^δ that the estimate

$$||\mathcal{E}^\delta - E_2^\delta||_{L^2(\Omega)} + ||\mathcal{H}^\delta - H_2^\delta||_{L^2(\Omega)} \le C(\Omega)\,\delta^2,$$

holds for any compact region Ω contained in Ω_2^δ where the constant $C(\Omega)$ is independent of δ.

5 Boundary Integral Operators

This section is probably the most important section of this paper because essentially all of what follows is either based on or motivated by the results we are now about to discuss. Our basic aim is to reduce the Maxwell equations (7) with the boundary conditions (8) to a Fredholm system of two integral scalar equations on Γ. This system is used to establish the existence of solutions to the boundary value problem (7) − (8). The uniqueness result follows by using some well-known properties of the Hankel functions. The main advantage of this integral approach lies in the fact that it reduces a vector problem defined

over the unbounded domain Ω^e to a scalar problem defined on the boundary Γ. This fact is, of course, crucial for our singular perturbation analysis.
We begin our analysis by considering classical representation formulae for the electric field E^δ and the magnetic field H^δ. Let $\Phi(k, x-y)$ be the fundamental outgoing solution of the Helmholtz equation :

$$\Phi(k, x-y) \;=\; \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad x \neq y,$$

where $k = \omega\sqrt{\varepsilon\mu}$. Let us first introduce

$$j^\delta \;=\; H^\delta \wedge n, m^\delta \;=\; E^\delta \wedge n, f^\delta \;=\; g^\delta \wedge n, \quad \text{on } \Gamma.$$

Since g^δ is in $TH^{1/2}(\mathrm{curl}, \Gamma)$ we have f^δ in $TH^{1/2}(\mathrm{div}, \Gamma)$.
It is well-known that the electric field E^δ admits the following integral representation :

$$\begin{aligned} E^\delta(x) \;&=\; -\nabla \wedge \int_\Gamma \Phi(k, x-y)\, m^\delta(y)\, d\gamma(y) \\ &\quad -i\omega\mu \int_\Gamma \Phi(k, x-y)\, j^\delta(y)\, d\gamma(y) \\ &\quad -\frac{i}{\omega\varepsilon} \nabla \int_\Gamma \Phi(k, x-y)\, \mathrm{div}_\Gamma\, j^\delta(y)\, d\gamma(y), \quad x \in \Omega^e, \end{aligned} \tag{20}$$

and the magnetic field H^δ is expressed in the form

$$\begin{aligned} H^\delta(x) \;&=\; -\nabla \wedge \int_\Gamma \Phi(k, x-y)\, j^\delta(y)\, d\gamma(y) \\ &\quad +\frac{i}{\omega\mu} \nabla \int_\Gamma \Phi(k, x-y)\, \mathrm{div}_\Gamma m^\delta(y)\, d\gamma(y) \\ &\quad +i\omega\varepsilon \int_\Gamma \Phi(k, x-y)\, m^\delta(y)\, d\gamma(y), \quad x \in \Omega^e. \end{aligned} \tag{21}$$

Passing to the limit $x \in \Gamma$ and taking the tangential component in (20) we find with the help of the identity

$$\begin{aligned} &\lim_{x\in\Omega^e\to\Gamma} n \wedge \int_\Gamma \nabla_x \Phi(k, x-y) \wedge m^\delta(y)\, d\gamma(y) \\ &= -\frac{1}{2} m^\delta(x) + n \wedge \int_\Gamma \nabla_x \Phi(k, x-y) \wedge m^\delta(y)\, d\gamma(y), \end{aligned}$$

proved by Müller in [33] that

$$\begin{aligned}\frac{1}{2}m^\delta(x) &= n \wedge \int_\Gamma \nabla_x \Phi(k, x-y) \wedge m^\delta(y)\, d\gamma(y) \\ &+ i\omega\mu\, n(x) \wedge \int_\Gamma \Phi(k, x-y)\, j^\delta(y)\, d\gamma(y) \\ &- \frac{i}{\omega\varepsilon} \overrightarrow{\mathrm{curl}}_\Gamma \int_\Gamma \Phi(k, x-y)\, \mathrm{div}_\Gamma\, j^\delta(y)\, d\gamma(y).\end{aligned} \tag{22}$$

Recalling the generalized impedance boundary conditions

$$m^\delta = -i\delta a^{(1)} n \wedge j^\delta + i\delta a^{(2)} \overrightarrow{\mathrm{curl}}_\Gamma\, \mathrm{div}_\Gamma\, j^\delta - f^\delta, \tag{23}$$

we eliminate the tangential field m^δ in the integral equation (22). Then, the use of the generalized impedance boundary conditions (8) leads to the following integral equation on the surface Γ

$$\begin{aligned}&\frac{1}{2}\Big(- i\delta a^{(1)} n \wedge j^\delta + i\delta a^{(2)} \overrightarrow{\mathrm{curl}}_\Gamma\, \mathrm{div}_\Gamma\, j^\delta - f^\delta \Big)(x) = \\ &n \wedge \int_\Gamma \nabla_x \Phi(k, x-y) \wedge \Big(- i\delta a^{(1)} n \wedge j^\delta + i\delta a^{(2)} \overrightarrow{\mathrm{curl}}_\Gamma\, \mathrm{div}_\Gamma\, j^\delta - f^\delta \Big)(y)\, d\gamma(y) \\ &+ i\omega\mu\, n(x) \wedge \int_\Gamma \Phi(k, x-y)\, j^\delta(y)\, d\gamma(y) \\ &- \frac{i}{\omega\varepsilon} \overrightarrow{\mathrm{curl}}_\Gamma \int_\Gamma \Phi(k, x-y)\, \mathrm{div}_\Gamma\, j^\delta(y)\, d\gamma(y).\end{aligned} \tag{24}$$

where the tangential field j^δ is the unknown. Note that (20) and (21) solve the Maxwell equations (7) with the boundary conditions (8) provided j^δ is solution of (24). We now try to find $j^\delta \in TH^{3/2}(\mathrm{div}, \Gamma)$ solution of the above integral equation.

Assume now that the boundary Γ is simply connected. Making use of the Hodge decomposition of the Sobolev space $TH^{3/2}(\mathrm{div}, \Gamma)$ we decompose j^δ in the form

$$j^\delta = \overrightarrow{\mathrm{curl}}_\Gamma\, \varphi^\delta + \nabla_\Gamma\, \psi^\delta, \tag{25}$$

where $\varphi^\delta \in H_0^{5/2}(\Gamma)$ and $\psi^\delta \in H_0^{7/2}(\Gamma)$. Inserting the Hodge decomposition (25) into the integral equation (24), we obtain

$$\begin{aligned}
&\frac{1}{2} f^\delta + \frac{1}{2} i\delta a^{(1)} \Big(\nabla_\Gamma \varphi^\delta - \overrightarrow{\mathrm{curl}}_\Gamma\, \psi^\delta \Big) - \frac{i}{2}\, \delta\, a^{(2)}\, \overrightarrow{\mathrm{curl}}_\Gamma\, \Delta_\Gamma\, \psi^\delta \\
&= i\delta a^{(1)}\, n \wedge \int_\Gamma \nabla_x\, \Phi(k, x-y) \wedge \Big(\nabla_\Gamma \varphi^\delta - \overrightarrow{\mathrm{curl}}_\Gamma\, \psi^\delta \Big)(y)\, d\gamma(y) \\
&- i\delta a^{(2)}\, n \wedge \int_\Gamma \nabla_x\, \Phi(k, x-y) \wedge \overrightarrow{\mathrm{curl}}_\Gamma\, \Delta_\Gamma\, \psi^\delta(y)\, d\gamma(y) \\
&+ n \wedge \int_\Gamma \nabla_x\, \Phi(k, x-y) \wedge f^\delta(y)\, d\gamma(y) \qquad\qquad (26) \\
&- i\omega\mu\, n(x) \wedge \int_\Gamma \Phi(k, x-y)\, \overrightarrow{\mathrm{curl}}_\Gamma\, \varphi^\delta(y)\, d\gamma(y) \\
&- i\omega\mu\, n(x) \wedge \int_\Gamma \Phi(k, x-y)\, \nabla_\Gamma\, \psi^\delta(y)\, d\gamma(y) \\
&+ \frac{i}{\omega\varepsilon}\, \overrightarrow{\mathrm{curl}}_\Gamma \int_\Gamma \Phi(k, x-y)\, \Delta_\Gamma\, \psi^\delta(y)\, d\gamma(y).
\end{aligned}$$

The unknowns now become the scalar functions φ^δ and ψ^δ. We will try to find them in the Sobolov spaces $H_0^{5/2}(\Gamma)$ and $H_0^{7/2}(\Gamma)$ respectively.
The use of Hodge decompositions in integral equations was discussed in [2]. This approach reduces the vector problem (24) to a scalar one as we will see later. We must note that the equation (26) derived from a Hodge decomposition depends automatically on the topological properties of the Γ under consideration. If Γ is not simply connected we decompose j^δ as follows

$$j^\delta \;=\; \overrightarrow{\mathrm{curl}}_\Gamma\, \varphi^\delta + \nabla_\Gamma\, \psi^\delta + \alpha^\delta,$$

where α^δ is in $\mathcal{N}(\Gamma) = \Big\{ \alpha, \Delta_\Gamma\, \alpha \,=\, 0 \Big\}$ which is a finite-dimensional vectorial space and its dimension is the first Betti number. In this case, we split the integral equation (26) into three integral equations. In the present paper, we focus our attention only on the case of a simply connected boundary Γ. Let us now rewrite the quantity

$$n \wedge \int_\Gamma \nabla_x\, \Phi(k, x-y) \wedge \overrightarrow{\mathrm{curl}}_\Gamma\, \Delta_\Gamma\, \psi^\delta(y)\, d\gamma(y),$$

as

$$\begin{aligned}
&- \int_\Gamma \partial_{n_x}\, \Phi(k, x-y)\, \overrightarrow{\mathrm{curl}}_\Gamma\, \Delta_\Gamma\, \psi^\delta(y)\, d\gamma(y) \\
&+ \int_\Gamma \nabla_x\, \Phi(k, x-y)\, (n_x - n_y) \,.\, \overrightarrow{\mathrm{curl}}_\Gamma\, \Delta_\Gamma\, \psi^\delta(y)\, d\gamma(y),
\end{aligned}$$

and the quantity

$$n \wedge \int_\Gamma \nabla_x \Phi(k, x-y) \wedge \nabla_\Gamma \varphi^\delta(y)\, d\gamma(y),$$

as

$$\begin{aligned} &-\int_\Gamma \partial_{n_x} \Phi(k, x-y)\, \nabla_\Gamma \varphi^\delta(y)\, d\gamma(y) \\ &+\int_\Gamma \nabla_x \Phi(k, x-y)\, (n_x - n_y) \,.\, \nabla_\Gamma \varphi^\delta(y)\, d\gamma(y). \end{aligned}$$

Under the assumption of a simply connected boundary, we split the integral equation (26) into two distinct integral equations. By taking the tangential divergence of (26) we obtain

$$\begin{aligned} &\frac{1}{2} i\delta a^{(1)}\, \Delta_\Gamma \varphi^\delta - i\omega\mu\, \mathrm{curl}_\Gamma \int_\Gamma \Phi(k, x-y)\, \overrightarrow{\mathrm{curl}}_\Gamma\, \varphi^\delta(y)\ d\gamma(y) \\ &-i\delta a^{(1)}\, \mathrm{div}_\Gamma \int_\Gamma \partial_{n_x} \Phi(k, x-y)\, \nabla_\Gamma\, \varphi^\delta(y)\, d\gamma(y) \\ &+i\delta a^{(1)}\, \mathrm{div}_\Gamma \int_\Gamma \nabla_x \Phi(k, x-y)\, (n_x - n_y) \,.\, \nabla_\Gamma\, \varphi^\delta(y)\, d\gamma(y) \\ &-i\delta a^{(1)}\, \mathrm{curl}_\Gamma \int_\Gamma \nabla_x \Phi(k, x-y) \wedge \overrightarrow{\mathrm{curl}}_\Gamma\, \psi^\delta(y)\ d\gamma(y) \\ &+i\delta a^{(2)} \mathrm{div}_\Gamma \int_\Gamma \partial_{n_x} \Phi(k, x-y)\, \overrightarrow{\mathrm{curl}}_\Gamma\, \Delta_\Gamma\, \psi^\delta(y)\, d\gamma(y) \\ &-i\delta a^{(2)}\, \mathrm{div}_\Gamma \int_\Gamma \nabla_x \Phi(k, x-y)\, (n_x - n_y) \,.\, \overrightarrow{\mathrm{curl}}_\Gamma\, \Delta_\Gamma\, \psi^\delta(y)\, d\gamma(y) \\ &-i\omega\mu\, \mathrm{curl}_\Gamma \int_\Gamma \Phi(k, x-y)\, \nabla_\Gamma\, \psi^\delta(y)\ d\gamma(y) \\ &=\ -i\delta a^{(2)} \mathrm{curl}_\Gamma \int_\Gamma \nabla_x \Phi(k, x-y) \wedge f^\delta(y)\, d\gamma(y) - \frac{1}{2}\, \mathrm{div}_\Gamma\, f^\delta. \end{aligned} \tag{27}$$

From (26), it can also be seen by taking the curl_Γ of (26) that

$$\begin{aligned}
&\frac{i}{2}\,\delta\, a^{(2)}\,\Delta_\Gamma^2\,\psi^\delta + \frac{i}{\omega\varepsilon}\,\Delta_\Gamma \int_\Gamma \Phi(k,x-y)\,\Delta_\Gamma\,\psi^\delta(y)\,d\gamma(y) + \frac{1}{2} i\delta a^{(1)}\,\Delta_\Gamma\,\psi^\delta \\
&+ i\delta a^{(1)}\,\mathrm{curl}_\Gamma \int_\Gamma \partial_{n_x}\Phi(k,x-y)\,\nabla_\Gamma\,\varphi^\delta(y)\,d\gamma(y) \\
&- i\delta a^{(1)}\,\mathrm{curl}_\Gamma \int_\Gamma \nabla_x\,\Phi(k,x-y)\,(n_x-n_y)\,.\,\nabla_\Gamma\,\varphi^\delta(y)\,d\gamma(y) \\
&+ i\delta a^{(1)}\,\mathrm{div}_\Gamma \int_\Gamma \nabla_x\,\Phi(k,x-y)\,\wedge\,\overrightarrow{\mathrm{curl}}_\Gamma\,\psi^\delta(y)\,d\gamma(y) \\
&- i\delta a^{(2)}\mathrm{curl}_\Gamma \int_\Gamma \partial_{n_x}\Phi(k,x-y)\,\overrightarrow{\mathrm{curl}}_\Gamma\,\Delta_\Gamma\,\psi^\delta(y)\,d\gamma(y) \qquad (28) \\
&+ i\delta a^{(2)}\,\mathrm{curl}_\Gamma \int_\Gamma \nabla_x\,\Phi(k,x-y)\,(n_x-n_y)\,.\,\overrightarrow{\mathrm{curl}}_\Gamma\,\Delta_\Gamma\,\psi^\delta(y)\,d\gamma(y) \\
&+ i\omega\mu\,\mathrm{div}_\Gamma \int_\Gamma \Phi(k,x-y)\,\overrightarrow{\mathrm{curl}}_\Gamma\,\varphi^\delta(y)\,d\gamma(y) \\
&+ i\omega\mu\,\mathrm{div}_\Gamma \int_\Gamma \Phi(k,x-y)\,\nabla_\Gamma\,\psi^\delta(y)\,d\gamma(y) \\
&= -i\delta a^{(2)}\,\mathrm{div}_\Gamma \int_\Gamma \nabla_x\,\Phi(k,x-y)\,\wedge\,f^\delta(y)\,d\gamma(y) - \frac{1}{2}\mathrm{curl}_\Gamma\,f^\delta.
\end{aligned}$$

Note that the system $(27)-(28)$ depends on the topological properties of the boundary Γ under consideration. Before showing that the system of the two integral equations (27) and (28) is uniquely solvable, let us first verify that the two quantities

$$\mathrm{curl}_\Gamma \int_\Gamma \nabla_x\,\Phi(k,x-y)\,\wedge\,f^\delta(y)\,d\gamma(y),$$

and

$$\mathrm{div}_\Gamma \int_\Gamma \nabla_x\,\Phi(k,x-y)\,\wedge\,f^\delta(y)\,d\gamma(y),$$

are in $H^{1/2}(\Gamma)$. From the identity

$$\begin{aligned}
&n\,\wedge \int_\Gamma \nabla_x\,\Phi(k,x-y)\,\wedge\,f^\delta(y)\,d\gamma(y) \;=\; -\int_\Gamma \partial_{n_x}\Phi(k,x-y)\,f^\delta(y)\,d\gamma(y) \\
&+ \int_\Gamma \nabla_x\,\Phi(k,x-y)\,(n_x-n_y)\,.\,f^\delta(y)\,d\gamma(y),
\end{aligned}$$

we obtain the claim by recalling that the operator with kernel

$$\partial_{n_x}\Phi(k,x-y) \;=\; \frac{e^{ik|x-y|}}{4\pi|x-y|^2}\Big(ik - \frac{1}{|x-y|}\Big)(x-y,n_x),$$

is of order -1. On the other hand, we have

$$\mathrm{div}_\Gamma f^\delta = \mathrm{curl}_\Gamma g^\delta \in H^{1/2}(\Gamma)$$

and

$$\mathrm{curl}_\Gamma f^\delta = -\mathrm{div}_\Gamma g^\delta \in H^{-1/2}(\Gamma).$$

In order to show that the system $(27)-(28)$ is uniquely solvable, let us now recall some well-known results concerning the boundary integral operators used in (27) and (28). It is known, from Nédélec [35], that the integral operator

$$\varphi \longmapsto \mathrm{curl}_\Gamma \int_\Gamma \Phi(k, x-y)\, \overrightarrow{\mathrm{curl}}_\Gamma\, \varphi(y)\; ds(y), \tag{29}$$

is an isomorphism from $H^s_0(\Gamma)$ onto $H^{s-1}_0(\Gamma)$. The operator

$$\psi \longmapsto \Delta_\Gamma \int_\Gamma \Phi(i, x-y)\, \Delta_\Gamma \psi(y)\; ds(y), \tag{30}$$

is an isomorphism from $H^s_0(\Gamma)$ onto $H^{s-3}(\Gamma)$. The operator

$$\psi \longmapsto \Delta_\Gamma \int_\Gamma \Big(\Phi(i, x-y) - \Phi(k, x-y)\Big)\, \Delta_\Gamma \psi(y)\; ds(y), \tag{31}$$

is a compact operator from $H^s(\Gamma)$ into $H^{s-3}(\Gamma)$. The operators

$$\psi \longmapsto \mathrm{curl}_\Gamma \int_\Gamma \partial_{n_x}\, \Phi(k, x-y)\, \overrightarrow{\mathrm{curl}}_\Gamma\, \Delta_\Gamma\, \psi(y)\, d\gamma(y), \tag{32}$$

$$\psi \longmapsto \mathrm{curl}_\Gamma \int_\Gamma \nabla_x\, \Phi(k, x-y)\; (n_x - n_y)\, .\, \overrightarrow{\mathrm{curl}}_\Gamma\, \Delta_\Gamma\, \psi(y)\, d\gamma(y), \tag{33}$$

$$\psi \longmapsto \mathrm{div}_\Gamma \int_\Gamma \partial_{n_x}\, \Phi(k, x-y)\, \overrightarrow{\mathrm{curl}}_\Gamma\, \Delta_\Gamma\, \psi(y)\, d\gamma(y), \tag{34}$$

$$\psi \longmapsto \mathrm{div}_\Gamma \int_\Gamma \nabla_x\, \Phi(k, x-y)\; (n_x - n_y)\, .\, \overrightarrow{\mathrm{curl}}_\Gamma\, \Delta_\Gamma\, \psi(y)\, d\gamma(y), \tag{35}$$

are compact from $H^s(\Gamma)$ into $H^{s-4}(\Gamma)$ and they are bounded operators from $H^s(\Gamma)$ into $H^{s-3}(\Gamma)$. The operator

$$\varphi \longmapsto \mathrm{div}_\Gamma \int_\Gamma \Phi(k, x-y)\, \overrightarrow{\mathrm{curl}}_\Gamma\, \varphi(y)\; ds(y), \tag{36}$$

is compact from $H^s(\Gamma)$ into $H^{s-2}(\Gamma)$. The operators

$$\psi \longmapsto \mathrm{div}_\Gamma \int_\Gamma \Phi(k, x-y)\, \nabla_\Gamma\, \psi(y)\; ds(y), \tag{37}$$

and

$$\psi \longmapsto \mathrm{curl}_\Gamma \int_\Gamma \Phi(k, x-y)\, \nabla_\Gamma\, \psi(y)\; ds(y). \tag{38}$$

are compact from $H^s(\Gamma)$ into $H^{s-2}(\Gamma)$. The operators

$$\varphi \longmapsto \mathrm{curl}_\Gamma \int_\Gamma \nabla_x\, \Phi(k, x-y)\; (n_x - n_y)\, .\, \nabla_\Gamma\, \varphi(y)\, d\gamma(y), \tag{39}$$

$$\varphi \longmapsto \mathrm{div}_\Gamma \int_\Gamma \nabla_x\, \Phi(k, x-y)\; (n_x - n_y)\, .\, \nabla_\Gamma\, \varphi(y)\, d\gamma(y), \tag{40}$$

$$\varphi \longmapsto \mathrm{curl}_\Gamma \int_\Gamma \partial_{n_x}\, \Phi(k, x-y)\, \nabla_\Gamma\, \varphi(y)\, d\gamma(y), \tag{41}$$

and

$$\varphi \longmapsto \mathrm{div}_\Gamma \int_\Gamma \partial_{n_x}\, \Phi(k, x-y)\, \nabla_\Gamma\, \varphi(y)\, d\gamma(y), \tag{42}$$

are compact from $H^s(\Gamma)$ into $H^{s-2}(\Gamma)$.
The above results follow from the fact that the operators with kernels

$$\Phi(k, x-y), \partial_{n_x}\, \Phi(k, x-y), \nabla_x\, \Phi(k, x-y)\; (n_x - n_y)\, .$$

are of order -1.

We have now the following lemma.

Lemma 5.1 *The Fredholm system* $(27)-(28)$ *is uniquely solvable in* $H_0^{5/2}(\Gamma) \times H_0^{7/2}(\Gamma)$.

Proof. Since the system $(27)-(28)$ is a compact perturbation of the invertible one from $H_0^{5/2}(\Gamma) \times H_0^{7/2}(\Gamma)$ onto $H_0^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ given by

$$\begin{pmatrix} \frac{i}{2}\delta a^{(1)} \Delta_\Gamma\, \varphi^\delta & 0 \\ 0 & \frac{i}{2}\delta a^{(2)} \Delta_\Gamma^2\, \psi^\delta \end{pmatrix}$$

it suffices to show that it has at most one solution in $H_0^{5/2}(\Gamma) \times H_0^{7/2}(\Gamma)$. Let $(\mathcal{E}^\delta, \mathcal{H}^\delta)$ be an outgoing solution of the Maxwell equations

$$\begin{cases} \nabla \wedge \mathcal{E}^\delta \;=\; i\omega\mu\mathcal{H}^\delta, \\ \nabla \wedge \mathcal{H}^\delta \;=\; -i\omega\varepsilon\mathcal{E}^\delta, \end{cases} \tag{43}$$

with the boundary conditions

$$n \wedge (n \wedge \mathcal{E}^\delta) + i\delta\, a^{(1)}\; \mathcal{H}^\delta \wedge n + i\delta\, a^{(2)}\, \nabla_\Gamma\, \mathrm{div}_\Gamma\, (\mathcal{H}^\delta \wedge n) \;=0, \quad \text{on } \Gamma. \tag{44}$$

If $(\varphi^\delta, \psi^\delta)$ is a non trivial solution of the system (27) – (28) then the Maxwell system (43) with the boundary conditions (44) has a non trivial solution. It is given by the integral representation formulae

$$\left\{\begin{array}{rcl}\mathcal{E}^\delta &=& -\nabla \wedge \int_\Gamma \Phi(k, x-y)\, m^\delta(y)\, d\gamma(y) \\ && -i\omega\mu \int_\Gamma \Phi(k, x-y)\, j^\delta(y)\, d\gamma(y) \\ && -\dfrac{i}{\omega\varepsilon} \nabla \int_\Gamma \Phi(k, x-y)\, \mathrm{div}_\Gamma\, j^\delta(y)\, d\gamma(y) \\ \mathcal{H}^\delta &=& \dfrac{1}{i\omega\mu} \nabla \wedge \mathcal{E}^\delta,\end{array}\right.$$

where $j^\delta = \overrightarrow{\mathrm{curl}}_\Gamma \varphi^\delta + \nabla_\Gamma \psi^\delta$ and

$$m^\delta \;=\; -i\delta a^{(1)} n \wedge j^\delta + i\delta a^{(2)} \overrightarrow{\mathrm{curl}}_\Gamma\, \mathrm{div}_\Gamma\, j^\delta.$$

Integrating by parts the Maxwell equations (43) over $\Omega_R^e = \Big\{ |x| < R \Big\} \cap \Omega^e$ yields

$$\int_{\Omega_R^e} |\nabla \wedge \mathcal{E}^\delta|^2 - \omega^2 \varepsilon\mu \int_{\Omega_R^e} |\mathcal{E}^\delta|^2 + \delta\omega\mu a^{(1)} \int_\Gamma |\mathcal{H}^\delta \wedge n|^2$$
$$-\delta\omega\mu a^{(2)} \int_\Gamma |\mathrm{div}_\Gamma (\mathcal{H}^\delta \wedge n)|^2 + i\omega\mu \int_{|x|=R} (\mathcal{H}^\delta \wedge n)\,.\,\overline{\mathcal{E}^\delta} \;=\; 0.$$

By taking the imaginary part in the above expression, we obtain

$$\Re e \left(\int_{|x|=R} (\mathcal{H}^\delta \wedge n)\,.\,\overline{\mathcal{E}^\delta} \right) \;=\; 0.$$

Classical arguments, based on the uniqueness theorem of Cauchy-Kowaleska and the well known properties of the electromagnetic operator on $|x| = R$ insure that $\mathcal{E}^\delta = \mathcal{H}^\delta = 0$ in Ω^e (see e.g. [5]) and the proof of the uniqueness is complete. □

Let us consider now the system of the two integral equations on Γ given by

$$\begin{array}{l} -i\omega\mu\, \mathrm{curl}_\Gamma \int_\Gamma \Phi(k, x-y)\, \overrightarrow{\mathrm{curl}}_\Gamma\, \varphi^0(y)\, d\gamma(y) \\ -i\omega\mu\, \mathrm{curl}_\Gamma \int_\Gamma \Phi(k, x-y)\, \nabla_\Gamma\, \psi^0(y)\, d\gamma(y) \\ = -\dfrac{1}{2}\, \mathrm{div}_\Gamma\, f^0, \end{array} \qquad (45)$$

and

$$
\begin{aligned}
&-\frac{i}{\omega\varepsilon}\,\Delta_\Gamma \int_\Gamma \Phi(k,x-y)\,\Delta_\Gamma\,\psi^0(y)\;d\gamma(y)\\
&+i\omega\mu\,\mathrm{div}_\Gamma \int_\Gamma \Phi(k,x-y)\,\overrightarrow{\mathrm{curl}}_\Gamma\,\varphi^0(y)\;d\gamma(y)\\
&+i\omega\mu\,\mathrm{div}_\Gamma \int_\Gamma \Phi(k,x-y)\,\nabla_\Gamma\,\psi^0(y)\;d\gamma(y)\\
&=\;-\frac{1}{2}\mathrm{curl}_\Gamma\, f^0,
\end{aligned}
\tag{46}
$$

where $f^0 = g^0 \wedge n$.
Having proved that the system $(45)-(46)$ is uniquely solvable, we prove in the next section the convergence of $(\varphi^\delta,\psi^\delta)$ to (φ^0,ψ^0) in $H^{3/2}(\Gamma)\times H^{5/2}(\Gamma)$, strongly. The unique solvability of $(45)-(46)$ in $H_0^{3/2}(\Gamma)\times H_0^{5/2}(\Gamma)$ is discussed in the following lemma.

Lemma 5.2 *The Fredholm system* $(45)-(46)$ *is uniquely solvable in* $H_0^{3/2}(\Gamma)\times H_0^{5/2}(\Gamma)$.

Proof. It is clear that the system $(45)-(46)$ is of Fredholm type. As in Lemma 5.2, it suffices to prove that the system

$$
\left\{
\begin{aligned}
&-i\omega\mu\,\mathrm{curl}_\Gamma \int_\Gamma \Phi(k,x-y)\,\overrightarrow{\mathrm{curl}}_\Gamma\,\varphi^0(y)\;d\gamma(y)\\
&-i\omega\mu\,\mathrm{curl}_\Gamma \int_\Gamma \Phi(k,x-y)\,\nabla_\Gamma\,\psi^0(y)\;d\gamma(y)\;=\;0,\\
&-\frac{i}{\omega\varepsilon}\,\Delta_\Gamma \int_\Gamma \Phi(k,x-y)\,\Delta_\Gamma\,\psi^0(y)\;d\gamma(y)\\
&+i\omega\mu\,\mathrm{div}_\Gamma \int_\Gamma \Phi(k,x-y)\,\overrightarrow{\mathrm{curl}}_\Gamma\,\varphi^0(y)\;d\gamma(y)\\
&+i\omega\mu\,\mathrm{div}_\Gamma \int_\Gamma \Phi(k,x-y)\,\nabla_\Gamma\,\psi^0(y)\;d\gamma(y)\;=\;0.
\end{aligned}
\right.
\tag{47}
$$

has only the trivial solution in $H_0^{3/2}(\Gamma)\times H_0^{5/2}(\Gamma)$. To do this, let us consider $(\varphi^0,\psi^0)\in H_0^{3/2}(\Gamma)\times H_0^{5/2}(\Gamma)$ a solution of the system (47) and introduce the fields E^0 and H^0 for $x\in\Omega^e$ as

$$
\left\{
\begin{aligned}
E^0(x) \;&=\; -i\omega\mu \int_\Gamma \Phi(k,x-y)\,\Big(\overrightarrow{\mathrm{curl}}_\Gamma\,\varphi^0 + \nabla_\Gamma\psi^0\Big)(y)\;d\gamma(y)\\
&\quad\; -\frac{i}{\omega\varepsilon}\nabla \int_\Gamma \Phi(k,x-y)\,\Delta_\Gamma\,\psi^0(y),\\
H^0(x) \;&=\; -\nabla\,\wedge \int_\Gamma \Phi(k,x-y)\,\Big(\overrightarrow{\mathrm{curl}}_\Gamma\,\varphi^0 + \nabla_\Gamma\psi^0\Big)(y)\;d\gamma(y).
\end{aligned}
\right.
\tag{48}
$$

It is clear that

$$\nabla \wedge E^0 \;=\; i\omega\mu H^0, \quad \text{in } \Omega^e. \tag{49}$$

We shall verify that

$$\nabla \wedge H^0 \;=\; -i\omega\varepsilon E^0, \quad \text{in } \Omega^e. \tag{50}$$

We have

$$\nabla \wedge H^0 \;=\; -(\nabla \wedge)^2 \int_\Gamma \Phi(k, x-y) \left(\overrightarrow{\text{curl}}_\Gamma\, \varphi^0 + \nabla_\Gamma \psi^0\right)(y)\; d\gamma(y), \quad x \in \Omega^e.$$

By using the identity

$$-(\nabla\,)^2 = \Delta - \nabla\,\text{div},$$

together with

$$\Delta\,\Phi(k, x-y) \;=\; -\omega^2\varepsilon\mu\,\Phi(k, x-y), \quad \text{for } x \in \Omega^e \text{ and } y \in \Gamma,$$

we obtain that

$$\begin{aligned} \nabla \wedge H^0 \;&=\; \omega^2\varepsilon\mu \int_\Gamma \Phi(k, x-y) \left(\overrightarrow{\text{curl}}_\Gamma\, \varphi^0 + \nabla_\Gamma \psi^0\right)(y)\; d\gamma(y) \\ &\quad -\nabla \int_\Gamma \nabla_x \Phi(k, x-y)\,.\left(\overrightarrow{\text{curl}}_\Gamma\, \varphi^0 + \nabla_\Gamma \psi^0\right)(y)\; d\gamma(y). \end{aligned} \tag{51}$$

Now, from $\nabla_x \Phi(k, x-y) = -\nabla_y \Phi(k, x-y)$, it follows by integrating by parts that

$$\nabla \int_\Gamma \nabla_x\, \Phi(k, x-y)\,.\left(\overrightarrow{\text{curl}}_\Gamma\, \varphi^0 + \nabla_\Gamma \psi^0\right)(y)\; d\gamma(y) \;=\; \int_\Gamma \Phi(k, x-y)\,.\,\Delta_\Gamma \psi^0(y)\; d\gamma(y). \tag{52}$$

Combining (52) and (51) gives (50). Hence, the fields E^0 and H^0 satisfy the Maxwell system (49) – (50) in the exterior domain Ω^e. From the system (47), we obtain

$$\text{curl}_\Gamma \left(n \wedge E^0\right) \;=\; \text{div}_\Gamma \left(n \wedge E^0\right) \;=\; 0, \quad \text{on } \Gamma. \tag{53}$$

Since Γ is simply connected, the above identities give $E^0 \wedge n = 0$ on Γ and from the unique solvability of the exterior Maxwell equations, it follows that $E^0 = H^0 = 0$ in Ω^e, and so, we have from the integral representations of E^0 and H^0 (see for example Colton and Kress [13])

$$\overrightarrow{\text{curl}}_\Gamma\, \varphi^0 + \nabla_\Gamma \psi^0 \;=\; 0,$$

which yields to $\varphi^0 = \psi^0 = 0$. □

6 Singular Perturbation Problems

Our aim in this section is to state and prove a singular problem for a Fredholm system of pseudo-differential operators. To keep the presentation short, we do not introduce an abstract framework. However, we shall give the study in the form which can be adapted to other problems of the same type. The results obtained in this section will be used in the next one when we discuss the convergence of the solution of the Maxwell equations with the generalized impedance boundary conditions to that of the corresponding Dirichlet problem. Let us consider the system given by

$$\left\{\begin{array}{rcl} \delta\,\mathcal{L}_1\,\psi^\delta + \left(\mathcal{M}_1 + \delta\,\mathcal{N}_1^1\right)\psi^\delta + \left(\mathcal{N}_1^2 + \delta\,\mathcal{N}_1^3\right)\varphi^\delta & = & h_1^\delta, \\ \delta\,\mathcal{L}_2\,\varphi^\delta + \left(\mathcal{M}_2 + \delta\,\mathcal{N}_2^1\right)\varphi^\delta + \left(\mathcal{N}_2^2 + \delta\,\mathcal{N}_2^3\right)\psi^\delta & = & h_2^\delta, \end{array}\right. \tag{54}$$

where $\mathcal{L}_1$ is an isomorphism from $H_0^{7/2}(\Gamma)$ onto $H^{-1/2}(\Gamma)$, and is positive definite operator. We mean by that $(\mathcal{L}_1\psi^\delta, \psi^\delta)_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} > 0$, if $\psi^\delta \neq 0$.
$\mathcal{L}_2$ is an isomorphism from $H_0^{5/2}(\Gamma)$ onto $H_0^{1/2}(\Gamma)$, and is positive definite operator : $(\mathcal{L}_2\varphi^\delta, \varphi^\delta)_{L^2(\Gamma),L^2(\Gamma)} > 0$, if $\varphi^\delta \neq 0$.
$\mathcal{M}_1$ is an isomorphism from $H_0^{5/2}(\Gamma)$ onto $H^{-1/2}(\Gamma)$, and is positive definite operator : $(\mathcal{M}_1\psi^\delta, \psi^\delta)_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} > 0$, if $\psi^\delta \neq 0$.
$\mathcal{M}_2$ is an isomorphism from $H_0^{3/2}(\Gamma)$ onto $H_0^{1/2}(\Gamma)$, and is positive definite operator : $(\mathcal{M}_2\varphi^\delta, \varphi^\delta)_{L^2(\Gamma),L^2(\Gamma)} > 0$, if $\varphi^\delta \neq 0$.
$\mathcal{N}_1^1 : H_0^{7/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is compact, and it is a bounded operator from $H_0^{5/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$.
$\mathcal{N}_2^1 : H_0^{5/2}(\Gamma) \to H^{1/2}(\Gamma)$ is compact, and it is a bounded operator from $H_0^{5/2}(\Gamma)$ into $H^{3/2}(\Gamma)$.
$\mathcal{N}_1^2 : H_0^{5/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is compact.
$\mathcal{N}_2^2 : H_0^{3/2}(\Gamma) \to H^{1/2}(\Gamma)$ is compact.
$\mathcal{N}_1^3 : H_0^{3/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is compact.
$\mathcal{N}_2^3 : H_0^{7/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is compact, and it is a bounded operator from $H_0^{5/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$.
$h_1^\delta \in H^{-1/2}(\Gamma)$ and converges to h_1^0 as δ tends to zero.
$h_2^\delta \in H^{1/2}(\Gamma)$ and converges to h_2^0, $H^{1/2}(\Gamma)$ strongly as δ tends to zero.
We assume that the system

$$\left\{\begin{array}{rcl} \mathcal{M}_1\,\psi + \mathcal{N}_1^2\,\varphi & = & 0, \\ \mathcal{M}_2\,\varphi + \mathcal{N}_2^2\,\psi & = & 0, \end{array}\right. \tag{55}$$

has only the trivial solution in $H_0^{5/2}(\Gamma) \times H_0^{3/2}(\Gamma)$.
Let us introduce the operators

$$\mathcal{K}_j \;=\; \mathcal{L}_j^{-1/2}\,\mathcal{M}_j\,\mathcal{L}_j^{-1/2}, \quad \text{for } j = 1, 2.$$

The operators $\mathcal{K}_1 : H_0^{1/2}(\Gamma) \to H_0^{1/2}(\Gamma)$ and $\mathcal{K}_2 : H_0^{1/2}(\Gamma) \to H_0^{1/2}(\Gamma)$ are compact. Indeed, $\mathcal{K}_1(H_0^{1/2}(\Gamma)) = H_0^{3/2}(\Gamma)$ and $\mathcal{K}_2(H_0^{1/2}(\Gamma)) = H_0^{3/2}(\Gamma)$. Both $\mathcal{K}_1$ and $\mathcal{K}_2$ are self-adjoint operators, $\mathcal{K}_1(H_0^{3/2}(\Gamma)) = H_0^{5/2}(\Gamma)$ and $\mathcal{K}_2(H_0^{3/2}(\Gamma)) = H_0^{5/2}(\Gamma)$.
Hence, both $\mathcal{K}_1 : H_0^{5/2}(\Gamma) \to H_0^{5/2}(\Gamma)$ and $\mathcal{K}_2 : H_0^{3/2}(\Gamma) \to H_0^{3/2}(\Gamma)$ are compact, self-adjoint, positive definite operators.
Let the countable set of positive eigenvalues of $\mathcal{K}_1$ be denoted by $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \ldots$ with associated orthonormal eigenfunctions $\psi_1, \psi_2, \ldots, \psi_n, \ldots$. From the spectral theorem

$$\mathcal{K}_1 \psi \;=\; \sum_{n\geq 0} \lambda_n \, (\psi, \psi_n) \, \psi_n,$$

and if $\mathcal{F}$ is a continuous function defined on $[0, \lambda_1]$ then

$$\mathcal{F}(\mathcal{K}_1) \, \psi \;=\; \sum_{n\geq 0} \mathcal{F}(\lambda_n) \, (\psi, \psi_n) \, \psi_n.$$

Analogously, from the spectral theorem, we get

$$\mathcal{K}_2 \varphi \;=\; \sum_{n\geq 0} \mu_n \, (\varphi, \varphi_n) \, \varphi_n,$$

for each $\varphi \in H_0^{1/2}(\Gamma)$, where $0 < \mu_1 \leq \mu_2 \leq \ldots \leq \ldots$ is the set of eigenvalues of the operator $\mathcal{K}_2$ with associated orthonormal eigenfunctions $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$.
We shall now recall several well-known results which are special cases of results established by Groetsch [19]. Similar results were used by Ahner and Wierner to prove the convergence of an exterior problem for Laplace's equation with a Robin boundary condition to the corresponding Dirichlet problem.
By assumption, the equation

$$\mathcal{K}_1 \psi = h, \tag{56}$$

is uniquely solvable in $H_0^{1/2}(\Gamma)$ if $h \in H_0^{3/2}(\Gamma)$. The inverse is given by

$$\mathcal{K}_1^{-1} \, h \;=\; \sum_{n\geq 0} \frac{1}{\lambda_n} \, (h, \psi_n) \, \psi_n, \quad \forall \, h \in H_0^{3/2}(\Gamma). \tag{57}$$

Since the equation (56) is uniquely solvable, the Picard's criterion (see Theorem 1.2.6 [19]) is then satisfied and the sum $\sum_{n\geq 0} \frac{1}{\lambda_n} \, (h, \psi_n) \psi_n$ converges in $H_0^{1/2}(\Gamma)$.
It is to be noted that the convergence of the series $\sum_{n\geq 0} \frac{1}{\lambda_n} \, (h, \psi_n) \psi_n$ is equivalent to the unique solvability of the equation (56).
Now, by a Tikhonov regularization argument (see e.g Groetsch [19]), we get that

$$\Big(\delta I + \mathcal{K}_1\Big)^{-1} \, h$$

converges strongly in $H_0^{1/2}(\Gamma)$ to $\mathcal{K}_1^{-1}h$ as δ tends to zero for each $h \in H_0^{3/2}(\Gamma)$. The convergence follows from the dominated convergence theorem : the function $\dfrac{1}{\delta+t}$ is continuous on $[0,\infty[$ for all $\delta > 0$, it converges to $\dfrac{1}{t}$ for each $t \in]0,\infty[$ as δ tends to zero and $\dfrac{t}{\delta+t}$ is uniformly bounded by 1 on $[0,\infty[$.
Now, it is easily seen that

$$\Big(\delta I + \mathcal{K}_1\Big)^{-1} h - \mathcal{K}_1^{-1}\, h \;=\; \sum_{n\geq 0}\Big(\frac{1}{\delta+\lambda_n} - \frac{1}{\lambda_n}\Big)\,(h,\psi_n)\,\psi_n, \quad \forall\, h \in H_0^{3/2}(\Gamma).$$

Therefore, if h is such that

$$\sum_{n\geq 0}\frac{1}{\lambda_n^4}\,|(h,\psi_n)|^2 \;<\; \infty, \tag{58}$$

then the rate of convergence of $\Big(\delta I + \mathcal{K}_1\Big)^{-1} h$ to $\mathcal{K}_1^{-1}h$ is $0(\delta)$.
Once again, by Picard's criterion, the convergence of the series (58) is equivalent to the unique solvability of the equation $\mathcal{K}_1^2\,\psi = h$. It follows that if $h \in H_0^{5/2}(\Gamma)$ then the rate of convergence of $\Big(\delta I + \mathcal{K}_1\Big)^{-1} h$ to $\mathcal{K}_1^{-1}h$ in $H^{1/2}(\Gamma)$ is $0(\delta)$. By rate of convergence we mean here that there exists a strictly positive constant C independent of δ such that the estimate

$$||\Big(\delta I + \mathcal{K}_1\Big)^{-1} h - \mathcal{K}_1^{-1}h||_{H^{1/2}(\Gamma)} \;\leq\; C\,\delta,$$

holds uniformly for δ small enough. The constant C depends, of course, on h.
Finally, let us state and prove the following lemma which is a slight generalization of the above convergence result.

Lemma 6.1 *If h^δ converges to h in $H_0^{3/2}(\Gamma)$ strongly, then $\Big(\delta I + \mathcal{K}_1\Big)^{-1} h^\delta$ converges to $\mathcal{K}_1^{-1}h$ in $H_0^{1/2}(\Gamma)$ strongly as δ tends to zero. Furthermore, if $h \in H_0^{5/2}(\Gamma)$, then the rate of convergence is $0(\delta + ||h^\delta - h||_{H^{3/2}(\Gamma)})$.*

Proof. We first show that $\Big(\delta I + \mathcal{K}_1\Big)^{-1} h^\delta \to \mathcal{K}_1^{-1}h$ in $H_0^{1/2}(\Gamma)$ strongly. To do this, we observe that

$$\begin{aligned}\Big(\delta I + \mathcal{K}_1\Big)^{-1} h^\delta - \mathcal{K}_1^{-1}\, h \;&=\; \sum_{n\geq 0}\Big(\frac{1}{\delta+\lambda_n} - \frac{1}{\lambda_n}\Big)\,(h^\delta - h,\psi_n)\,\psi_n \\ &\quad + \sum_{n\geq 0}\Big(\frac{1}{\delta+\lambda_n} - \frac{1}{\lambda_n}\Big)\,(h,\psi_n)\,\psi_n \\ &\quad + \sum_{n\geq 0}\frac{1}{\lambda_n}\,(h^\delta - h,\psi_n)\,\psi_n.\end{aligned} \tag{59}$$

The convergence is then obvious. Now, if h is in $H_0^{5/2}(\Gamma)$, we can deduce from

$$\left\{\begin{array}{l} ||\displaystyle\sum_{n\geq 0}\Big(\frac{1}{\delta+\lambda_n}-\frac{1}{\lambda_n}\Big)\,(h,\psi_n)\,\psi_n||_{H^{1/2}(\Gamma)}\;\leq\; C\,\delta\,||h||_{H^{5/2}(\Gamma)},\\ ||\displaystyle\sum_{n\geq 0}\Big(\frac{1}{\delta+\lambda_n}-\frac{1}{\lambda_n}\Big)\,(h^\delta-h,\psi_n)\,\psi_n||_{H^{1/2}(\Gamma)}\;\leq\; C\;||h^\delta-h||_{H^{3/2}(\Gamma)},\\ ||\displaystyle\sum_{n\geq 0}\frac{1}{\lambda_n}\,(h^\delta-h,\psi_n)\,\psi_n||_{H^{1/2}(\Gamma)}\;\leq\; C\,||h^\delta-h||_{H^{3/2}(\Gamma)}, \end{array}\right.$$

where C is a constant independent of δ that the rate of the convergence is

$$0(\delta+||h^\delta-h||_{H^{3/2}(\Gamma)}).$$

□

Note that similar results are valid for the operator $\mathcal{K}_2$.
Now, it can be easily seen that

$$\mathcal{L}_2^{-1/2}\mathcal{N}_2^1\,\varphi^\delta\in H_0^{3/2}(\Gamma),\mathcal{L}_1^{-1/2}\Big(\mathcal{N}_1^2+\delta\mathcal{N}_1^3\Big)\,\varphi^\delta\in H_0^{3/2}(\Gamma),\quad\forall\,\varphi^\delta\in H_0^{5/2}(\Gamma),$$

$$\mathcal{L}_1^{-1/2}\mathcal{N}_1^1\,\psi^\delta\in H_0^{3/2}(\Gamma),\mathcal{L}_2^{-1/2}\Big(\mathcal{N}_2^2+\delta\mathcal{N}_2^3\Big)\,\psi^\delta\in H_0^{3/2}(\Gamma),\quad\forall\,\psi^\delta\in H_0^{7/2}(\Gamma).$$

and

$$\mathcal{L}_1^{-1/2}\,h_1^\delta\in H_0^{3/2}(\Gamma),\mathcal{L}_2^{-1/2}\,h_2^\delta\in H_0^{3/2}(\Gamma),\quad\forall\,h_1^\delta\in H^{-1/2}(\Gamma),\forall\,h_2^\delta\in H^{1/2}(\Gamma).$$

It follows that we can rewrite the system (54) as

$$\left\{\begin{array}{l} \tilde{\psi^\delta}+\delta\,\Big(\delta\,I+\mathcal{K}_1\Big)^{-1}\mathcal{L}_1^{-1/2}\mathcal{N}_1^1\mathcal{L}_1^{-1/2}\,\tilde{\psi^\delta}\\ +\Big(\delta\,I+\mathcal{K}_1\Big)^{-1}\,\mathcal{L}_1^{-1/2}\Big(\mathcal{N}_1^2+\delta\,\mathcal{N}_1^3\Big)\,\mathcal{L}_2^{-1/2}\tilde{\varphi^\delta}\;=\Big(\delta\,I+\mathcal{K}_1\Big)^{-1}\,\mathcal{L}_1^{-1/2}\,h_1^\delta,\\ \tilde{\varphi^\delta}+\delta\,\Big(\delta\,I+\mathcal{K}_2\Big)^{-1}\mathcal{L}_2^{-1/2}\mathcal{N}_2^1\mathcal{L}_2^{-1/2}\,\tilde{\varphi^\delta}\\ +\Big(\delta\,I+\mathcal{K}_2\Big)^{-1}\mathcal{L}_2^{-1/2}\Big(\mathcal{N}_2^2+\delta\,\mathcal{N}_2^3\Big)\mathcal{L}_1^{-1/2}\,\tilde{\psi^\delta}\;=\Big(\delta\,I+\mathcal{K}_2\Big)^{-1}\,\mathcal{L}_2^{-1/2}\,h_2^\delta, \end{array}\right. \tag{60}$$

where

$$\left\{\begin{array}{lcl} \tilde{\psi^\delta} &=& \mathcal{L}_1^{1/2}\,\psi^\delta,\\ \tilde{\varphi^\delta} &=& \mathcal{L}_2^{1/2}\,\varphi^\delta. \end{array}\right.$$

Our purpose now is to prove the convergence of the unique solution of the above system to that of

$$\left\{\begin{array}{lcl} \tilde{\psi^0}+\mathcal{K}_1^{-1}\,\mathcal{L}_1^{-1/2}\mathcal{N}_1^2\mathcal{L}_2^{-1/2}\,\tilde{\varphi^0} &=& \mathcal{K}_1^{-1}\,\mathcal{L}_1^{-1/2}\,h_1^0,\\ \tilde{\varphi^0}+\mathcal{K}_2^{-1}\mathcal{L}_2^{-1/2}\mathcal{N}_2^2\mathcal{L}_1^{-1/2}\,\tilde{\psi^0} &=& \mathcal{K}_2^{-1}\,\mathcal{L}_2^{-1/2}\,h_2^0. \end{array}\right. \tag{61}$$

The convergence of $(\psi^\delta, \varphi^\delta)$ to the unique solution of the following system

$$\left\{ \begin{array}{lcl} \psi^0 + \mathcal{M}_1^{-1}\, \mathcal{N}_1^2\, \varphi^0 & = & \mathcal{M}_1^{-1}\, h_1^0, \\ \varphi^0 + \mathcal{M}_2^{-1} \mathcal{N}_2^2\, \psi^0 & = & \mathcal{M}_2^{-1}\, h_2^0, \end{array} \right. \tag{62}$$

will be obvious. Note that the unique solvability of (61) and (62) follows from our assumption on the unique solvability of (55). We begin by proving the following a priori-estimates.

Lemma 6.2 *There exists a strictly positive constant C independent of δ such that*

$$||\psi^\delta||_{H^{5/2}(\Gamma)} + ||\varphi^\delta||_{H^{3/2}(\Gamma)} \leq C. \tag{63}$$

Proof. Let us prove by contradiction that the quantity $\alpha^\delta = ||\psi^\delta||_{H^{5/2}(\Gamma)} + ||\varphi^\delta||_{H^{3/2}(\Gamma)}$ is uniformly bounded with respect to the parameter δ. Suppose that we have a subsequence, still denoted by α^δ such that it converges to $+\infty$ as δ tends to zero. Let us define u^δ and v^δ by

$$u^\delta = \frac{\psi^\delta}{\alpha^\delta}, v^\delta = \frac{\varphi^\delta}{\alpha^\delta}.$$

It is easily seen that

$$\left\{ \begin{array}{lcl} \delta\, \mathcal{L}_1\, u^\delta + \Big(\mathcal{M}_1 + \delta\, \mathcal{N}_1^1\Big)\, u^\delta + \Big(\mathcal{N}_1^2 + \delta\, \mathcal{N}_1^3\Big)\, v^\delta & = & \dfrac{h_1^\delta}{\alpha^\delta}, \\ \delta\, \mathcal{L}_2\, v^\delta + \Big(\mathcal{M}_2 + \delta\, \mathcal{N}_2^1\Big)\, v^\delta + \Big(\mathcal{N}_2^2 + \delta\, \mathcal{N}_2^3\Big)\, u^\delta & = & \dfrac{h_2^\delta}{\alpha^\delta}. \end{array} \right. \tag{64}$$

It follows that (u^δ, v^δ) converges to $(u, v), H^{5/2}(\Gamma) \times H^{3/2}(\Gamma)$ weakly where (u, v) is such that

$$\left\{ \begin{array}{lcl} \mathcal{M}_1\, u + \mathcal{N}_1^2\, v & = & 0, \\ \mathcal{M}_2\, v + \mathcal{N}_2^2\, u & = & 0. \end{array} \right. \tag{65}$$

By assumption, it follows that $u = v = 0$ and so, $u^\delta \to 0, H^{5/2}(\Gamma)$ weakly and $v^\delta \to 0, H^{3/2}(\Gamma)$ weakly. Now, multiplying the first equation in the system (64) by u^δ and the second by v^δ and integrating on Γ gives

$$\left\{ \begin{array}{lcl} \delta\, (\mathcal{L}_1\, u^\delta, u^\delta) + (\Big(\mathcal{M}_1 + \delta\, \mathcal{N}_1^1\Big)\, u^\delta, u^\delta) + (\Big(\mathcal{N}_1^2 + \delta\, \mathcal{N}_1^3\Big)\, v^\delta, u^\delta) & = & (\dfrac{h_1^\delta}{\alpha^\delta}, u^\delta) \\ \delta\, (\mathcal{L}_2\, v^\delta, v^\delta) + (\Big(\mathcal{M}_2 + \delta\, \mathcal{N}_2^1\Big)\, v^\delta, v^\delta) + (\Big(\mathcal{N}_2^2 + \delta\, \mathcal{N}_2^3\Big)\, u^\delta, v^\delta) & = & (\dfrac{h_2^\delta}{\alpha^\delta}, v^\delta). \end{array} \right. \tag{66}$$

Since the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are positive definite and u^δ and v^δ converge to zero weakly, it follows from (66) that $(\mathcal{M}_1 u^\delta, u^\delta)$ and $(\mathcal{M}_2 v^\delta, v^\delta)$ converge to zero as δ tends to zero. Therefore, $u^\delta \to 0, H^{5/2}(\Gamma)$ strongly and $v^\delta \to 0, H^{3/2}(\Gamma)$ strongly. This leads to a contradiction since

$$||u^\delta||_{H^{5/2}(\Gamma)} + ||v^\delta||_{H^{3/2}(\Gamma)} = 1.$$

□

We also need the following lemma.

Lemma 6.3 *There exists a strictly positive constant C independent of δ such that*

$$\begin{aligned} &||\Big(\delta\, I + \mathcal{K}_1\Big)^{-1} \mathcal{L}_1^{-1/2} \mathcal{N}_1^1\, \psi^\delta||_{H^{1/2}(\Gamma)} + ||\Big(\delta\, I + \mathcal{K}_1\Big)^{-1} \mathcal{L}_1^{-1/2} \mathcal{N}_1^3\, \varphi^\delta||_{H^{1/2}(\Gamma)} \\ &+||\Big(\delta\, I + \mathcal{K}_2\Big)^{-1} \mathcal{L}_2^{-1/2} \mathcal{N}_2^1\, \varphi^\delta||_{H^{1/2}(\Gamma)} \\ &+||\Big(\delta\, I + \mathcal{K}_2\Big)^{-1} \mathcal{L}_2^{-1/2} \mathcal{N}_2^3\, \psi^\delta||_{H^{1/2}(\Gamma)} \le C. \end{aligned} \tag{67}$$

Proof. We first recall that the linear operator $\mathcal{K}_1$ is compact from $H_0^{1/2}(\Gamma)$ into $H_0^{1/2}(\Gamma)$ and it is an isomorphism from $H_0^{1/2}(\Gamma)$ onto $H_0^{3/2}(\Gamma)$. To prove that the quantity

$$||\Big(\delta\, I + \mathcal{K}_1\Big)^{-1} \mathcal{L}_1^{-1/2} \mathcal{N}_1^1\, \psi^\delta||_{H^{1/2}(\Gamma)}$$

is uniformly bounded with respect to the parameter δ, it suffices to show that

$$||\mathcal{L}_1^{-1/2} \mathcal{N}_1^1\, \psi^\delta||_{H^{3/2}(\Gamma)}$$

is uniformly bounded. This holds from Lemma 6.2. The estimates of the remainder quantities follow from exactly the same arguments. □

After these technical preliminaries, we are now in position to prove the main result of this section.

Theorem 6.4 *The unique solution $(\psi^\delta, \varphi^\delta)$ of the system* (54) *converges $H^{5/2}(\Gamma) \times H^{3/2}(\Gamma)$ strongly to the unique solution (ψ^0, φ^0) of the system* (62). *Furthermore, if $h_1^0 \in H^{1/2}(\Gamma)$ and $h_2^0 \in H^{3/2}(\Gamma)$, then the rate of the convergence is*

$$0(\delta + ||h_1^\delta - h_1^0||_{H^{-1/2}(\Gamma)} + ||h_2^\delta - h_2^0||_{H^{1/2}(\Gamma)}).$$

Proof. Since φ^δ converges to $\varphi, H^{3/2}(\Gamma)$ weakly and ψ^δ converges to $\psi, H^{5/2}(\Gamma)$ weakly, it follows that $\varphi^\delta \to \varphi, H^{1/2}(\Gamma)$ strongly and $\psi^\delta \to \psi, H^{3/2}(\Gamma)$, strongly. Now, from the properties of the operators $\mathcal{N}_1^2$ and $\mathcal{N}_2^2$ that

$$\mathcal{L}_1^{-1/2} \mathcal{N}_1^2 \varphi^\delta \to \mathcal{L}_1^{-1/2} \mathcal{N}_1^2 \varphi, H^{3/2}(\Gamma), \text{ strongly},$$

$$\mathcal{L}_2^{-1/2}\mathcal{N}_2^2\psi^\delta \to \mathcal{L}_2^{-1/2}\mathcal{N}_2^2\psi, H^{3/2}(\Gamma), \text{ strongly},$$

$$\Big(\delta\, I+\mathcal{K}_1\Big)^{-1}\mathcal{L}_1^{-1/2}\mathcal{N}_1^2\,\varphi^\delta \to \mathcal{K}_1^{-1}\,\mathcal{L}_1^{-1/2}\mathcal{N}_1^2\,\varphi, H^{1/2}(\Gamma), \text{ strongly},$$

and

$$\Big(\delta\, I+\mathcal{K}_2\Big)^{-1}\mathcal{L}_2^{-1/2}\mathcal{N}_2^2\,\psi^\delta \to \mathcal{K}_2^{-1}\mathcal{L}_2^{-1/2}\mathcal{N}_2^2\,\psi, H^{1/2}(\Gamma), \text{ strongly}.$$

On the other hand, from Lemma 6.3, we obtain

$$\delta\Big(\delta\, I+\mathcal{K}_1\Big)^{-1}\mathcal{L}_1^{-1/2}\mathcal{N}_1^1\,\psi^\delta \to 0, H^{1/2}(\Gamma), \text{ strongly},$$

$$\delta\Big(\delta\, I+\mathcal{K}_1\Big)^{-1}\mathcal{L}_1^{-1/2}\mathcal{N}_1^3\,\varphi^\delta \to 0, H^{1/2}(\Gamma), \text{ strongly},$$

$$\delta\Big(\delta\, I+\mathcal{K}_2\Big)^{-1}\mathcal{L}_2^{-1/2}\mathcal{N}_2^1\,\varphi^\delta \to 0, H^{1/2}(\Gamma), \text{ strongly},$$

and

$$\delta\Big(\delta\, I+\mathcal{K}_2\Big)^{-1}\mathcal{L}_2^{-1/2}\,\mathcal{N}_2^3\,\psi^\delta \to 0, H^{1/2}(\Gamma), \text{ strongly}.$$

From (60) and (61), it is easily shown that

$$\begin{aligned}
&\tilde{\psi^\delta}-\tilde{\psi^0}+\delta\left(\delta\, I+\mathcal{K}_1\right)^{-1}\mathcal{L}_1^{-1/2}\mathcal{N}_1^1\mathcal{L}_1^{-1/2}\,\tilde{\psi^\delta}\\
&+\Big(\delta\, I+\mathcal{K}_1\Big)^{-1}\mathcal{L}_1^{-1/2}\mathcal{N}_1^2\mathcal{L}_2^{-1/2}\tilde{\varphi^\delta}-\mathcal{K}_1^{-1}\,\mathcal{L}_1^{-1/2}\mathcal{N}_1^2\mathcal{L}_2^{-1/2}\tilde{\varphi^0}\\
&+\delta\left(\delta\, I+\mathcal{K}_1\right)^{-1}\mathcal{L}_1^{-1/2}\mathcal{N}_1^3\,\mathcal{L}_2^{-1/2}\tilde{\varphi^\delta}\\
&=\Big(\delta\, I+\mathcal{K}_1\Big)^{-1}\mathcal{L}_1^{-1/2}\,h_1^\delta-\mathcal{K}_1^{-1}\mathcal{L}_1^{-1/2}\,h_1^0,
\end{aligned}\tag{68}$$

and

$$\begin{aligned}
&\tilde{\varphi^\delta}-\tilde{\varphi^0}+\delta\left(\delta\, I+\mathcal{K}_2\right)^{-1}\mathcal{L}_2^{-1/2}\mathcal{N}_2^1\,\mathcal{L}_2^{-1/2}\tilde{\varphi^\delta}\\
&+\Big(\delta\, I+\mathcal{K}_2\Big)^{-1}\mathcal{L}_2^{-1/2}\mathcal{N}_2^2\mathcal{L}_1^{-1/2}\tilde{\psi^\delta}-\mathcal{K}_2^{-1}\mathcal{L}_2^{-1/2}\mathcal{N}_2^2\mathcal{L}_1^{-1/2}\tilde{\psi^0}\\
&+\delta\left(\delta\, I+\mathcal{K}_2\right)^{-1}\mathcal{L}_2^{-1/2}\mathcal{N}_2^3\,\mathcal{L}_1^{-1/2}\tilde{\psi^\delta}\\
&=\Big(\delta\, I+\mathcal{K}_2\Big)^{-1}\mathcal{L}_2^{-1/2}\,h_2^\delta-\mathcal{K}_2^{-1}\mathcal{L}_2^{-1/2}\,h_2^0.
\end{aligned}\tag{69}$$

From the above two equations, straightforward combinations of the above convergence results shows that $(\tilde{\psi^\delta},\tilde{\varphi^\delta})\to(\tilde{\psi^0},\tilde{\varphi^0})$, in $H^{1/2}(\Gamma)\times H^{1/2}(\Gamma)$, strongly and then, $(\psi^\delta,\varphi^\delta)\to(\psi^0,\varphi^0)$, in $H^{5/2}(\Gamma)\times H^{3/2}(\Gamma)$, strongly.

Now, if $h_1^0 \in H^{1/2}(\Gamma)$ and $h_2^0 \in H^{3/2}(\Gamma)$, then $\mathcal{L}_1^{-1} h_1^0 \in H_0^{5/2}(\Gamma)$ and $\mathcal{L}_2^{-1} h_2^0 \in H_0^{5/2}(\Gamma)$. From Lemma 6.1, it follows that the rate of the convergence of

$$\Big(\delta\, I + \mathcal{K}_1\Big)^{-1} \mathcal{L}_1^{-1/2}\, h_1^\delta \to \mathcal{K}_1^{-1} \mathcal{L}_1^{-1/2}\, h_1^0 \;=\; \mathcal{L}_1^{1/2} \mathcal{M}_1^{-1}\, h_1^0$$

is $0(\delta + ||h_1^\delta - h_1^0||_{H^{-1/2}(\Gamma)})$ and

$$\Big(\delta\, I + \mathcal{K}_2\Big)^{-1} \mathcal{L}_2^{-1/2}\, h_2^\delta \longrightarrow \mathcal{K}_2^{-1} \mathcal{L}_2^{-1/2}\, h_2^0 \;=\; \mathcal{L}_2^{1/2} \mathcal{M}_2^{-1}\, h_2^0$$

as $0(\delta + ||h_2^\delta - h_2^0||_{H^{1/2}(\Gamma)})$. Now, let us observe that

$$\Big(\delta\, I + \mathcal{K}_1\Big)^{-1} \mathcal{L}_1^{-1/2}\, \mathcal{N}_1^2 \varphi^0 \to \mathcal{K}_1^{-1} \mathcal{L}_1^{-1/2}\, \mathcal{N}_1^2 \varphi^0,$$

and

$$\Big(\delta\, I + \mathcal{K}_2\Big)^{-1} \mathcal{L}_2^{-1/2}\, \mathcal{N}_2^2 \psi^0 \to \mathcal{K}_2^{-1} \mathcal{L}_2^{-1/2}\, \mathcal{N}_2^2 \psi^0$$

as $0(\delta)$.
It is now easy to see that we can rewrite (68) in the form

$$\psi^\delta - \psi^0 + \mathcal{Q}_1(\varphi^\delta - \varphi) \;=\; 0(\delta + ||h_1^\delta - h_1^0||_{H^{-1/2}(\Gamma)} + ||h_2^\delta - h_2^0||_{H^{1/2}(\Gamma)}), \quad (70)$$

where by $0(\delta + ||h_1^\delta - h_1^0||_{H^{-1/2}(\Gamma)} + ||h_2^\delta - h_2^0||_{H^{1/2}(\Gamma)})$ we mean a quantity that converges in $H^{5/2}(\Gamma)$, strongly with the rate

$$0(\delta + ||h_1^\delta - h_1^0||_{H^{-1/2}(\Gamma)} + ||h_2^\delta - h_2^0||_{H^{1/2}(\Gamma)})$$

and $\mathcal{Q}_1 \; : H^{3/2}(\Gamma) \to H^{5/2}(\Gamma)$ is a compact operator. Analogously, we can deduce from (69) that

$$\varphi^\delta - \varphi^0 + \mathcal{Q}_2(\psi^\delta - \psi^0) \;=\; 0(\delta + ||h_1^\delta - h_1^0||_{H^{-1/2}(\Gamma)} + ||h_2^\delta - h_2^0||_{H^{1/2}(\Gamma)}), \quad (71)$$

where here by $0(\delta + ||h_1^\delta - h_1^0||_{H^{-1/2}(\Gamma)} + ||h_2^\delta - h_2^0||_{H^{1/2}(\Gamma)})$ we mean a quantity that converges in $H^{3/2}(\Gamma)$, strongly with the rate $0(\delta + ||h_1^\delta - h_1^0||_{H^{-1/2}(\Gamma)} + ||h_2^\delta - h_2^0||_{H^{1/2}(\Gamma)})$ and $\mathcal{Q}_2 \; : H^{5/2}(\Gamma) \to H^{3/2}(\Gamma)$ is a compact operator. The claim is obtained from (70) and (71) by contradiction as in Lemma 6.2. □

7 Convergence of the Solution

In this section we prove the convergence of the electric and magnetic fields as δ tends to zero and we establish its rate. If $\Re e(k) = 0$, then the system of pseudo-differential equations (27) – (28) can be written as (54), where

$$\mathcal{L}_1 \;=\; \frac{1}{2} a^{(2)}\, \Delta_\Gamma^2,$$

$$\mathcal{M}_1 = \frac{1}{\omega\varepsilon}\,\Delta_\Gamma \int_\Gamma \Phi(k, x-y)\,\Delta_\Gamma - \omega\mu\,\mathrm{div}_\Gamma \int_\Gamma \Phi(k, x-y)\,\nabla_\Gamma,$$

$$\mathcal{N}_1^1 = \frac{1}{2}a^{(1)}\,\Delta_\Gamma + a^{(1)}\mathrm{div}_\Gamma \int_\Gamma \nabla_x\Phi \wedge \overrightarrow{\mathrm{curl}}_\Gamma - a^{(2)}\,\mathrm{curl}_\Gamma \int_\Gamma \partial_{n_x}\Phi\,\overrightarrow{\mathrm{curl}}_\Gamma\,\Delta_\Gamma$$
$$+a^{(2)}\,\mathrm{curl}_\Gamma \int_\Gamma \nabla_x\Phi\,(n_x - n_y)\,.\,\overrightarrow{\mathrm{curl}}_\Gamma\,\Delta_\Gamma,$$

$$\mathcal{N}_1^2 = \omega\mu\,\mathrm{div}_\Gamma \int_\Gamma \Phi\,\overrightarrow{\mathrm{curl}}_\Gamma,$$

$$\mathcal{N}_1^3 = a^{(1)}\,\mathrm{curl}_\Gamma \int_\Gamma \partial_{n_x}\Phi\,\nabla_\Gamma - a^{(1)}\,\mathrm{curl}_\Gamma \int_\Gamma \nabla_x\Phi\,(n_x - n_y)\,.\,\nabla_\Gamma,$$

$$h_1^\delta = -a^{(2)}\,\mathrm{div}_\Gamma \int_\Gamma \nabla_x\Phi \wedge f^\delta + \frac{i}{2}\mathrm{curl}_\Gamma f^\delta,$$

$$\mathcal{L}_2 = -\frac{1}{2}a^{(1)}\,\Delta_\Gamma,$$

$$\mathcal{M}_2 = \omega\mu\,\mathrm{curl}_\Gamma \int_\Gamma \Phi(k, x-y)\,\mathrm{curl}_\Gamma,$$

$$\mathcal{N}_2^1 = a^{(1)}\,\mathrm{div}_\Gamma \int_\Gamma \partial_{n_x}\Phi\,\nabla_\Gamma - a^{(1)}\,\mathrm{div}_\Gamma \int_\Gamma \nabla_x\Phi\,(n_x - n_y)\,.\,\nabla_\Gamma,$$

$$\mathcal{N}_2^2 = \omega\mu\,\mathrm{curl}_\Gamma \int_\Gamma \Phi\,\nabla_\Gamma,$$

$$\mathcal{N}_2^3 = a^{(1)}\,\mathrm{curl}_\Gamma \int_\Gamma \overrightarrow{\mathrm{curl}}_\Gamma \nabla_x\Phi \wedge - a^{(2)}\,\mathrm{div}_\Gamma \int_\Gamma \partial_{n_x}\Phi\,\overrightarrow{\mathrm{curl}}_\Gamma\,\Delta_\Gamma$$
$$+a^{(2)}\,\mathrm{div}_\Gamma \int_\Gamma \nabla_x\Phi\,(n_x - n_y)\,.\,\overrightarrow{\mathrm{curl}}_\Gamma\,\Delta_\Gamma,$$

and

$$h_2^\delta = a^{(2)}\,\mathrm{curl}_\Gamma \int_\Gamma \nabla_x\Phi \wedge f^\delta - \frac{i}{2}\mathrm{div}_\Gamma f^\delta.$$

Under the assumption $\Re e(k) = 0$, the operators $\mathcal{L}_1^{-1/2}\mathcal{M}_1\mathcal{L}_1^{-1/2}$ and $\mathcal{L}_2^{-1/2}\mathcal{M}_2\mathcal{L}_1^{-1/2}$ are self-adjoint and the assumptions needed in the beginning of the above section are satisfied. The convergence of the system $(27)-(28)$ as δ tends to zero follows then from Theorem 6.1.
The extension to the general case $\Re e(k) \neq 0$ will follow from the observation that the difference between the operator

$$\Delta_\Gamma \int_\Gamma \Phi(i, x-y)\,\Delta_\Gamma,$$

and

$$\Delta_\Gamma \int_\Gamma \Phi(k, x-y)\,\Delta_\Gamma,$$

is compact from $H^{5/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$ and the difference between

$$\operatorname{curl}_\Gamma \int_\Gamma \Phi(i, x-y)\, \overrightarrow{\operatorname{curl}}_\Gamma,$$

and

$$\operatorname{curl}_\Gamma \int_\Gamma \Phi(k, x-y)\, \overrightarrow{\operatorname{curl}}_\Gamma,$$

is compact from $H^{3/2}(\Gamma)$ into $H^{1/2}(\Gamma)$. To see this, we first rewrite the system (27) – (28) as

$$\left\{\begin{array}{rcl} \delta\,\mathcal{L}_1\,\psi^\delta + \Big(\mathcal{M}_1 + \mathcal{R}_1 + \delta\,\mathcal{N}_1^1\Big)\,\psi^\delta + \Big(\mathcal{N}_1^2 + \delta\,\mathcal{N}_1^3\Big)\,\varphi^\delta &=& h_1^\delta, \\ \delta\,\mathcal{L}_2\,\varphi^\delta + \Big(\mathcal{M}_2 + \mathcal{R}_2 + \delta\,\mathcal{N}_2^1\Big)\,\varphi^\delta + \Big(\mathcal{N}_2^2 + \delta\,\mathcal{N}_2^3\Big)\,\psi^\delta &=& h_2^\delta, \end{array}\right. \tag{72}$$

where

$$\begin{aligned} \mathcal{M}_1 \;&=\; -\frac{1}{\omega\varepsilon}\,\Delta_\Gamma \int_\Gamma \Phi(i, x-y)\,\Delta_\Gamma + \omega\mu\,\operatorname{div}_\Gamma \int_\Gamma \Phi(k=i, x-y)\,\nabla_\Gamma, \\ \mathcal{R}_1 \;&=\; -\frac{1}{\omega\varepsilon}\,\Delta_\Gamma \int_\Gamma \Big(\Phi(k, x-y) - \Phi(i, x-y)\Big)\,\Delta_\Gamma \\ &\quad +\omega\mu\,\operatorname{div}_\Gamma \int_\Gamma \Big(\Phi(k, x-y) - \Phi(k=i, x-y)\Big)\,\nabla_\Gamma, \\ \mathcal{M}_2 \;&=\; \omega\mu\,\operatorname{curl}_\Gamma \int_\Gamma \Phi(i, x-y)\,\overrightarrow{\operatorname{curl}}_\Gamma, \end{aligned}$$

and

$$\mathcal{R}_2 \;=\; \omega\mu\,\operatorname{curl}_\Gamma \int_\Gamma \Big(\Phi(k, x-y) - \Phi(i, x-y)\Big)\,\overrightarrow{\operatorname{curl}}_\Gamma.$$

The operators $\mathcal{L}_1^{-1/2}\mathcal{M}_1\mathcal{L}_1^{-1/2}$ and $\mathcal{L}_2^{-1/2}\mathcal{M}_2\mathcal{L}_2^{-1/2}$ are self-adjoint and the operators $\mathcal{R}_1$ and $\mathcal{R}_2$ are compact from $H^{5/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$ and from $H^{3/2}(\Gamma)$ into $H^{1/2}(\Gamma)$, respectively.

Analogous to (60), we rewrite the system (72) in the form

$$\left\{\begin{array}{l} \tilde{\psi}^\delta + \delta\,\Big(\delta\,I + \mathcal{K}_1\Big)^{-1}\mathcal{L}_1^{-1/2}\mathcal{N}_1^1\mathcal{L}_1^{-1/2}\,\tilde{\psi}^\delta + \Big(\delta\,I + \mathcal{K}_1\Big)^{-1}\mathcal{L}_1^{-1/2}\mathcal{R}_1\mathcal{L}_1^{-1/2}\,\tilde{\psi}^\delta \\ +\Big(\delta\,I + \mathcal{K}_1\Big)^{-1}\mathcal{L}_1^{-1/2}\Big(\mathcal{N}_1^2 + \delta\,\mathcal{N}_1^3\Big)\,\mathcal{L}_2^{-1/2}\tilde{\varphi}^\delta = \Big(\delta\,I + \mathcal{K}_1\Big)^{-1}\mathcal{L}_1^{-1/2}\,h_1^\delta, \\ \tilde{\varphi}^\delta + \delta\,\Big(\delta\,I + \mathcal{K}_2\Big)^{-1}\mathcal{L}_2^{-1/2}\mathcal{N}_2^1\,\mathcal{L}_2^{-1/2}\tilde{\varphi}^\delta + \Big(\delta\,I + \mathcal{K}_2\Big)^{-1}\mathcal{L}_2^{-1/2}\mathcal{R}_2\,\mathcal{L}_2^{-1/2}\tilde{\varphi}^\delta \\ +\Big(\delta\,I + \mathcal{K}_2\Big)^{-1}\mathcal{L}_2^{-1/2}\Big(\mathcal{N}_2^2 + \delta\,\mathcal{N}_2^3\Big)\,\mathcal{L}_1^{-1/2}\tilde{\psi}^\delta = \Big(\delta\,I + \mathcal{K}_2\Big)^{-1}\mathcal{L}_2^{-1/2}\,h_2^\delta. \end{array}\right. \tag{73}$$

Now, we want to show the convergence of the system (72) to

$$\left\{\begin{array}{lcl}\psi^0 + \mathcal{M}_1^{-1}\,\mathcal{R}_1\,\psi^0 + \mathcal{M}_1^{-1}\,\mathcal{N}_1^2\,\varphi^0 & = & \mathcal{M}_1^{-1}\,h_1^0,\\ \varphi^0 + \mathcal{M}_2^{-1}\,\mathcal{R}_2\,\varphi^0 + \mathcal{M}_2^{-1}\mathcal{N}_2^2\,\psi^0 & = & \mathcal{M}_2^{-1}\,h_2^0.\end{array}\right. \tag{74}$$

Analogous to Theorem 6.1, we have the following theorem.

Theorem 7.1 *The unique solution* $(\psi^\delta, \varphi^\delta)$ *of the system* (72) *converges* $H^{5/2}(\Gamma)\times H^{3/2}(\Gamma)$ *strongly to the unique solution* (ψ^0, φ^0) *of the system* (74)*. Furthermore, if* $h_1^0 \in H^{1/2}(\Gamma)$ *and* $h_2^0 \in H_0^{3/2}(\Gamma)$*, then the rate of the convergence is*

$$0(\delta + ||h_1^\delta - h_1^0||_{H^{-1/2}(\Gamma)} + ||h_2^\delta - h_2^0||_{H^{1/2}(\Gamma)}).$$

Proof. Proceeding as in Theorem 6.1, we see that the proof is established if we can show that $\Big(\delta\, I + \mathcal{K}_1\Big)^{-1}\mathcal{L}_1^{-1/2}\mathcal{R}_1\,\psi^\delta$ converges to $\mathcal{K}_1^{-1}\mathcal{L}_1^{-1/2}\mathcal{R}_1\,\psi^0$ in $H^{1/2}(\Gamma)$ strongly and $\Big(\delta\, I + \mathcal{K}_2\Big)^{-1}\mathcal{L}_2^{-1/2}\mathcal{R}_2\,\varphi^\delta$ converges to $\mathcal{K}_2^{-1}\mathcal{L}_2^{-1/2}\mathcal{R}_2\,\varphi^0$ in $H^{1/2}(\Gamma)$ strongly. To establish this it suffices to observe that from the properties of the operator $\mathcal{R}_1$ and $\mathcal{R}_2$, it follows that $\mathcal{L}_1^{-1/2}\mathcal{R}_1\,\psi^\delta$ converges to $\mathcal{L}_1^{-1/2}\mathcal{R}_1\,\psi^0$ in $H^{3/2}(\Gamma)$ strongly and $\mathcal{L}_2^{-1/2}\mathcal{R}_2\,\varphi^\delta$ converges to $\mathcal{L}_2^{-1/2}\mathcal{R}_2\,\varphi^0$ in $H^{3/2}(\Gamma)$ strongly. □

Finally, we state and prove the main result of this paper.

Theorem 7.2 *The unique solution of the Maxwell system* (7) *with the boundary conditions* (8) *converges to the unique solution of the Maxwell equations* (9) *in* $\overline{H}^{loc}(\mathrm{curl}, \Omega^e)\times \overline{H}^{loc}(\mathrm{curl}, \Omega^e)$*, strongly. Furthermore, if* g^0 *is in* $TH^{3/2}(\mathrm{curl}, \Gamma)$ *then the rate of the convergence of the electric and magnetic fields is* $0(\delta + ||g^\delta - g^0||_{TH^{1/2}(\mathrm{curl},\Gamma)})$.

Proof. From the integral representations for E^δ, H^δ, E^0, and H^0, we have

$$\begin{array}{rcl}||E^\delta - E^0||_{L^2(\Omega)} + ||H^\delta - H^0||_{L^2(\Omega)} & \leq & C(\Omega)\,\Big(||m^\delta - f^0||_{TH^{-1/2}(\mathrm{div},\Gamma)}\\ & & \qquad + ||j^\delta - j^0||_{TH^{-1/2}(\mathrm{div},\Gamma)}\Big),\end{array} \tag{75}$$

for any compact region Ω contained in Ω^e where the tangential field

$$j^0 = \nabla_\Gamma \psi^0 + \overrightarrow{\mathrm{curl}}_\Gamma\, \varphi^0,$$

and the constant $C(\Omega)$ depends only on Ω; however, it is finite for every compact set Ω contained in Ω^e. From the generalized impedance boundary conditions (23), it is clear that

$$||m^\delta - f^0||_{TH^{-1/2}(\mathrm{div},\Gamma)} \leq \delta\, C\Big(||\varphi^\delta||_{H^{3/2}(\Gamma)} + ||\psi^\delta||_{H^{5/2}(\Gamma)}\Big) + ||g^\delta - g^0||_{TH^{-1/2}(\mathrm{curl},\Gamma)}.$$

From Lemma 6.2, it follows that $||m^\delta - m^0||_{TH^{-1/2}(\mathrm{div},\Gamma)} \to 0$. On the other hand, we also have,

$$||j^\delta - j^0||_{TH^{-1/2}(\mathrm{div},\Gamma)} \leq C\left(||\varphi^\delta - \varphi^0||_{H^{1/2}(\Gamma)} + ||\psi^\delta - \psi^0||_{H^{3/2}(\Gamma)}\right).$$

Theorem 7.1 shows that $||j^\delta - j^0||_{TH^{-1/2}(\mathrm{div},\Gamma)} \to 0$ as $\delta \to 0$. Since both

$$||m^\delta - m^0||_{TH^{-1/2}(\mathrm{div},\Gamma)} \to 0 \text{ and } ||j^\delta - j^0||_{TH^{-1/2}(\mathrm{div},\Gamma)} \to 0$$

as $\delta \to 0$, it follows that $(E^\delta, H^\delta) \to (E^0, H^0)$ where by convergence here we mean that for any compact region Ω in Ω^e, the expression on the left-side of (75) vanishes as δ tends to zero. Now, by our assumption on the regularity of g^0, it follows from Theorem 7.1 that the expression on the right-side of (75) converges to zero as $0(\delta + ||g^\delta - g^0||_{TH^{1/2}(\mathrm{curl},\Gamma)})$.
To conclude this paper, we wish to point out that from the estimate

$$||j^\delta - j^0||_{TH^{1/2}(\mathrm{div},\Gamma)} \leq C\left(||\varphi^\delta - \varphi^0||_{H^{3/2}(\Gamma)} + ||\psi^\delta - \psi^0||_{H^{5/2}(\Gamma)}\right),$$

and Theorem 7.1 it follows that $||j^\delta - j^0||_{TH^{1/2}(\mathrm{div},\Gamma)} \to 0$ as $\delta \to 0$ with the rate $0(\delta + ||g^\delta - g^0||_{TH^{1/2}(\mathrm{curl},\Gamma)})$. It is not possible to obtain a similar convergence result for m^δ since it does not in general belong to the Sobolev space $TH^{1/2}(\mathrm{div},\Gamma)$. □

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