LOW FREQUENCY BEHAVIOR OF SOLUTIONS TO ELECTROMAGNETIC SCATTERING PROBLEMS IN CHIRAL MEDIA

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Abstract. This paper deals with the low-frequency scattering problem by a conducting object in a chiral medium. With boundary integral equation methods, we obtain the limit of the scattered field, solution of the Drude-Born-Fedorov equations as frequency tends to zero. Making use of a Hodge decomposition of the tangent fields shows that the asymptotics strongly depend on the topological properties of the domains under consideration. The asymptotic expansion's dependence on the chirality parameter is completely shown.

Key words. Integral Equations, Chiral Media, Drude-Born-Fedorov Equations, Maxwell's Equations, Fredholm System, Low-Frequency, The Hodge Decomposition Theorem.

AMS subject classifications. 35C15, 35B40, 35Q60, 45B05.

1. Introduction. Chiral media are examples of media responding with both magnetic and electric polarization to electric or magnetic excitation. They belong to the general class of bianistropic media. Chiral media can be characterized by a set of constitutive relations in which the electric and magnetic fields are coupled. The coupling strength is given by the magnitude of a quantity known as the chirality admittance. It determines the bulk electromagnetic properties of these materials. Chiral materials are used in many optics applications. Antennas coated with chiral materials have significantly interesting radiation characteristics due to the extra degree of freedom offered by the presence of the chirality admittance. Different expressions exist for the constitutive relations. The Drude-Born-Fedorov constitutive equations are used here. For interesting explanations of these equations and various physical aspects of the propagation inside chiral media, we refer to [17], [16] and the list of references therein which is representative of the work that is presently being conducted in this area. The original investigations of the effect of chirality on the polarization of light, also date back to the nineteenth century. Arago [6], Biot [9], Cauchy [10], Pasteur [22] and Fresnel [11] all examined optical activity in chiral media. The concept of chirality has also played an important role in chemistry with the works of Prelog [24].

In [2], the scattering problem by a non-homogeneous object embedded in a chiral medium was investigated. Representation theorems and the radiation condition for the Drude-Born-Fedorov equations were introduced. The existence and uniqueness of a solution for this problem was proved by using the boundary integral equation method.

The purpose of the present paper is to study the limiting behavior of the Drude-Born-Fedorov equations as frequency tends to zero. The crucial point for our analysis is a new integral approach which was originally developed in [1] to study the impedance boundary conditions for thin curved chiral layers. This method permits to obtain the limiting problem and the corrector terms at any order. We give the full asymptotic expansions of the electric and magnetic fields with respect to the frequency. We prove optimal error estimates and we show by using a Hodge decomposition that the

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asymptotic expansions of the electric and magnetic fields depend on the topological properties of the boundary of the conducting object.

A very close related problem is the time-harmonic boundary value problem for the Maxwell system at low frequencies which has been studied, for example, by Müller-Niemeyer [19], Werner [31], [33], Kress [15] with integral equation methods. Our present method gives in the case of Maxwell's equations a new approach to analyze the behavior of the electric and magnetic fields as frequency tends to zero and to derive the corrector terms.

The present paper is organized as follows. Having introduced some notations in section 2 we analyze the behavior of the electric and magnetic fields as frequency tends to zero in the case of a smooth obstacle with a simply connected boundary. We proceed by reducing the Drude-Born-Fedorov equations to a Fredholm system of two integral equations which is uniquely solvable in the limit case. Full asymptotic expansions of the fields are constructed and optimal error estimations are proved. The final section is devoted to the case of a non simply connected boundary. The dependence of the asymptotic expansions on the chirality parameter is shown.

2. Definitions, Notations and Remarkable Identities. Let Ω be an exterior smooth domain in $I\!R^3$ (i.e. an open set such that its boundary is a smooth closed surface of $I\!R^3$ and the complementary of its closure, denoted by Ω^i, is bounded, connected and has $\partial\Omega$ as boundary), and $n(x)$ the outward pointing normal at $x \in \partial\Omega$. For such a set, there exists a neighborhood of $\partial\Omega$ denoted by $\mathcal{O}$ such that, for any $x \in \mathcal{O}$, if $x_{\partial\Omega}$ is the orthogonal projection of x on $\partial\Omega$ and $s = (x - x_{\partial\Omega}) \,.\, n(x_{\partial\Omega})$, the couple $(x_{\partial\Omega}, s)$ is a local parameterization of $\mathcal{O}$. Then for any smooth function f or any vector field u defined on $\partial\Omega$, we get an extension of each one to $\mathcal{O}$ by setting :

$$\forall\, x \in \mathcal{O}, \tilde{f}(x) \,=\, f(x_{\partial\Omega}), \tilde{u}(x) \,=\, u(x_{\partial\Omega}).$$

We recall the following usual differential operators on $(\mathcal{D}'(I\!R^3))^3$ in Cartesian components, the rotational and the divergence respectively defined by :

$$\nabla\wedge \,=\, \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}, \quad \nabla\,. \,=\, \nabla^t \,=\, (\partial_1, \partial_2, \partial_3).$$

Using the above notations, we introduce the following boundary differential operators :
the tangential gradient of a function f : $\nabla_{\partial\Omega} f \,=\, \nabla \tilde{f}|_{\partial\Omega}$,
the surface divergence of a field u : $\mathrm{div}_{\partial\Omega} u \,=\, \nabla \,.\, \tilde{u}|_{\partial\Omega}$,
the vector rotational of a function : $\overrightarrow{\mathrm{curl}}_{\partial\Omega} f \,=\, \nabla \,\wedge\, (fn)|_{\partial\Omega}$,
the scalar rotational of a vector : $curl_{\partial\Omega} u \,=\, n \,.\, \nabla \,\wedge\, \tilde{u}|_{\partial\Omega}$,
and the Laplace-Beltrami operator on $\partial\Omega$ defined on scalar functions by

$$\Delta_{\partial\Omega} f = \mathrm{div}_{\partial\Omega}\, \nabla_{\partial\Omega} f = -\mathrm{curl}_{\partial\Omega}\, \overrightarrow{\mathrm{curl}}_{\partial\Omega} f.$$

We recall the mean curvature $c(x)$ defined as $2c(x) = \nabla \,.\, n(x)$, the tensor of curvature $R(x) = \nabla\, n(x)$ and the following identities used in the present paper :

$$(1) \qquad \begin{cases} \mathrm{div}_{\partial\Omega}(u \,\wedge\, n) \,=\, n \,.\, \Big(\nabla \,\wedge\, \tilde{u}\Big)|_{\partial\Omega} \,=\, \mathrm{curl}_{\partial\Omega}\, u. \\ \mathrm{div}_{\partial\Omega}\, \overrightarrow{\mathrm{curl}}_{\partial\Omega}\, f \,=\, 0, \mathrm{curl}_{\partial\Omega}\, \nabla_{\partial\Omega} f \,=\, 0. \end{cases}$$

$$\nabla \wedge w = \mathrm{curl}_{\partial\Omega} w_{\partial\Omega}\, n + \overrightarrow{\mathrm{curl}}_{\partial\Omega}(w \,.\, n) + (R - 2c)(w \wedge n) - \frac{\partial}{\partial s}(w \wedge n),$$

where w is a smooth field defined in a neighborhood of $\partial\Omega$ and $w_{\partial\Omega}$ is its tangential component on $\partial\Omega$: $w_{\partial\Omega} = -n \wedge (n \wedge w)$.

$$\begin{aligned}(\nabla\wedge)^2 w \;=\; & (\mathrm{curl}_{\partial\Omega}\, \overrightarrow{\mathrm{curl}}_{\partial\Omega}\,(w \,.\, n))\, n + \mathrm{div}_{\partial\Omega}\,(R\, w_{\partial\Omega})\, n + \mathrm{div}_{\partial\Omega}\,(\frac{\partial}{\partial s}\, w_{\partial\Omega}) \\ & + \overrightarrow{\mathrm{curl}}_{\partial\Omega}\, \mathrm{curl}_{\partial\Omega}\, w_{\partial\Omega} + (R - 2c - \frac{\partial}{\partial s})\,(-\nabla_{\partial\Omega}(w \,.\, n) + R\, w_{\partial\Omega} + \frac{\partial}{\partial s}\, w_{\partial\Omega}).\end{aligned}$$

To state our boundary-value problem in a suitable mathematical form, we shall use the following notations for the usual functional spaces :
$\mathcal{D}'(\Omega)$ is the space of distributions in Ω.
$L^2(\Omega)$ is the space of complex square integrable functions defined in Ω.
$L^\infty(R^3)$ is the space of bounded functions defined in R^3.
$H_0^s(\partial\Omega) = \Big\{ \varphi \in H^s(\partial\Omega), \langle \varphi, 1\rangle_{H^s(\partial\Omega), H^{-s}(\partial\Omega)} = 0 \Big\}$.
$H^1(\Omega) = \Big\{ \varphi \in L^2(\Omega), \nabla\varphi \in (L^2(\Omega))^3 \Big\}$.
$H_0^1(\Omega) = \Big\{ \varphi \in H^1(\Omega), \varphi = 0 \,\mathrm{on}\, \partial\Omega \Big\}$.
$H(curl,\Omega) = \Big\{ u \in (L^2(\Omega))^3, \nabla \wedge u \in (L^2(\Omega))^3 \Big\}$.
$H_0(curl,\Omega) = \Big\{ u \in H(curl,\Omega), u \wedge n = 0 \,\mathrm{on}\, \partial\Omega \Big\}$.
$\overline{H}^{loc}(curl,\Omega)$ is the Fréchet space of fields which are locally in $H(curl,\Omega)$.
$TH^{-1/2}(curl,\partial\Omega) = \Big\{ c \in (H^{-1/2}(\partial\Omega))^3, c \,.\, n = 0, curl_{\partial\Omega}\, c \in H^{-1/2}(\partial\Omega) \Big\}$.
$TH^{-1/2}(\mathrm{div},\partial\Omega) = \Big\{ c \in (H^{-1/2}(\partial\Omega))^3, c \,.\, n = 0, \mathrm{div}_{\partial\Omega}\, c \in H^{-1/2}(\partial\Omega) \Big\}$.
$TH^{-1/2}(curl,\partial\Omega)$ and $TH^{-1/2}(\mathrm{div},\partial\Omega)$ are Hilbert spaces. We recall without proof the following duality result (see e.g. [21] for a proof)

$$TH^{-1/2}(curl,\partial\Omega) = \Big(TH^{-1/2}(\mathrm{div},\partial\Omega)\Big)'.$$

3. The Drude-Born-Fedorov Equations. Electromagnetic fields in non-homogeneous chiral media satisfy the Drude-Born-Fedorov constitutive relations which may be written in the following form [17]

$$\left\{ \begin{array}{l} D^{(\omega)} = \varepsilon \Big(E^{(\omega)} + \beta \nabla \wedge E^{(\omega)} \Big), \\ B^{(\omega)} = \mu \Big(H^{(\omega)} + \beta \nabla \wedge H^{(\omega)} \Big), \end{array} \right. \tag{2}$$

where the permittivity ε and the permeability μ are two strictly positive constants and the chirality admittance β is assumed to be frequency-dependent. We assume that it is a smooth function of the frequency ω and we denote by

$$\beta_0 = \beta(\omega = 0) \text{ and } \beta_1 = \frac{d\beta}{d\omega}(\omega = 0).$$

A direct substitution of the constitutive relations (2) into Maxwell's equations

$$\left\{ \begin{array}{l} \nabla \wedge E^{(\omega)} = i\omega\, B^{(\omega)}, \\ \nabla \wedge H^{(\omega)} = -i\omega\, D^{(\omega)}, \end{array} \right. \tag{3}$$

gives the monochromatic constitutive relations

$$(4)\qquad \begin{cases} \left(1-\omega^2\beta^2\varepsilon\mu\right) D^{(\omega)} &= \varepsilon E^{(\omega)} + i\omega\beta\varepsilon\mu\, H^{(\omega)} \quad \text{in } \Omega, \\ \left(1-\omega^2\beta^2\varepsilon\mu\right) B^{(\omega)} &= \mu H^{(\omega)} - i\omega\beta\varepsilon\mu\, E^{(\omega)} \quad \text{in } \Omega. \end{cases}$$

Use of the first two monochromatic Maxwell's equations (3), $\nabla \,.\, B^{(\omega)} = 0$ and $\nabla \,.\, D^{(\omega)} = 0$, in conjunction with the above constitutive relations (4) easily yields the fact that $E^{(\omega)}$ and $H^{(\omega)}$ are also divergence-less.
Furthermore, combining (2) to (4) yields to the Drude-Born-Fedorov system

$$(5)\qquad \begin{cases} \nabla \wedge E^{(\omega)} &= \omega^2 \dfrac{\beta\varepsilon\mu}{1-\omega^2\beta^2\varepsilon\mu} E^{(\omega)} + i\omega \dfrac{\mu}{1-\omega^2\beta^2\varepsilon\mu} H^{(\omega)} \quad \text{in } \Omega, \\ \nabla \wedge H^{(\omega)} &= \omega^2 \dfrac{\beta\varepsilon\mu}{1-\omega^2\beta^2\varepsilon\mu} H^{(\omega)} - i\omega \dfrac{\varepsilon}{1-\omega^2\beta^2\varepsilon\mu} E^{(\omega)} \quad \text{in } \Omega. \end{cases}$$

We denote

$$\begin{cases} \gamma_1 = \dfrac{\sqrt{\varepsilon\mu}}{1-\omega\sqrt{\varepsilon\mu}\beta}, \quad \omega_1 = \dfrac{\omega}{1-\omega\sqrt{\varepsilon\mu}\beta}, \\ \gamma_2 = \dfrac{\sqrt{\varepsilon\mu}}{1+\omega\sqrt{\varepsilon\mu}\beta}, \quad \omega_2 = \dfrac{\omega}{1+\omega\sqrt{\varepsilon\mu}\beta}. \end{cases}$$

We are interested in the scattering problem for the Drude-Born-Fedorov equations for an incident plane wave defined as

$$(6)\qquad \begin{cases} E^{in}_{\omega}(x) = q_1\, e^{i\omega\gamma_1 p_1 \,.\, x} + q_2\, e^{i\omega\gamma_2 p_2 \,.\, x}, \\ H^{in}_{\omega}(x) = -i\sqrt{\dfrac{\varepsilon}{\mu}} \left(q_1\, e^{i\omega\gamma_1 p_1 \,.\, x} - q_2\, e^{i\omega\gamma_2 p_2 \,.\, x} \right), \end{cases}$$

where the complex vectors $p_{1,2}$ and $q_{1,2}$ satisfy the following relations :

$$p_1 \,.\, q_1 = 0, p_1 \wedge q_1 = -iq_1, p_2 \,.\, q_2 = 0, \text{ and } p_2 \wedge q_2 = iq_2.$$

The vector field E^{in}_{ω} is a combination of left-circularly polarized plane wave and right-circularly polarized one and it satisfies the isotropic Drude-Born-Fedorov equations in R^3.
We introduce the scalar three-dimensional outgoing fundamental solutions of the Helmholtz equations corresponding to γ_j for $j = 1, 2$ as

$$g_j(\omega, x-y) = \frac{e^{i\omega\gamma_j|x-y|}}{4\pi|x-y|}, \quad \text{for } x \neq y.$$

4. The limiting Behavior of the Drude-Born-Fedorov Equations.

4.1. Representation Theorems and Boundary Integral Equations. The purpose of this paper is to study the limiting behavior of the solution $(E^{(\omega)}, H^{(\omega)})$ of

the Drude-Born-Fedorov system

$$
(7)\qquad \begin{cases} \nabla \wedge E^{(\omega)} = \dfrac{\omega^2\varepsilon\mu\beta}{1-\omega^2\beta^2\varepsilon\mu}\,E^{(\omega)} + \dfrac{i\omega\mu}{1-\omega^2\beta^2\varepsilon\mu}\,H^{(\omega)} \quad \text{in } \Omega, \\ \nabla \wedge H^{(\omega)} = \dfrac{\omega^2\varepsilon\mu\beta}{1-\omega^2\beta^2\varepsilon\mu}\,H^{(\omega)} - \dfrac{i\omega\varepsilon}{1-\omega^2\beta^2\varepsilon\mu}\,E^{(\omega)} \quad \text{in } \Omega, \\ n \wedge E^{(\omega)} = 0 \quad \text{on } \partial\Omega, \\ |\Big(E^{(\omega)} - E^{in}_{\omega}\Big) - \sqrt{\dfrac{\mu}{\varepsilon}}\,\Big(H^{(\omega)} - H^{in}_{\omega}\Big) \wedge \dfrac{x}{|x|}| \leq \dfrac{C}{|x|^2}, \quad |x| \to +\infty, \end{cases}
$$

as frequency ω tends to zero. The existence and uniqueness of a solution $(E^{(\omega)}, H^{(\omega)})$ to (7) in $\overline{H}^{loc}(curl,\Omega) \times \overline{H}^{loc}(curl,\Omega)$ satisfying the classical Silver-Müller radiation condition was proved in [4] and [2]. The uniqueness follows from the Bohren decomposition of the electric and magnetic fields and the existence is established by using an integral equation approach.

Let us first describe the essential steps of our method.
We begin our analysis by recalling the representation formulae (8) and (9) of the electric and magnetic fields solutions of the Drude-Born-Fedorov system (7). In these representation formulae, the unknown is the tangential component of the magnetic field $H^{(\omega)}$ on the boundary $\partial\Omega$.
Taking the tangential trace of the electric field represented by (8) on $\partial\Omega$, multiplying by a test field in $TH^{-1/2}(\mathrm{div},\partial\Omega)$ and integrating over $\partial\Omega$, we reduce the previous problem to a variational problem (10) where the tangential component of the magnetic field is the unknown.
Using the Hodge decomposition of the space $TH^{-1/2}(\mathrm{div},\partial\Omega)$ whose structure entirely depends on the topology of $\partial\Omega$, we transform the previous vector problem (10) to the Fredholm system of two integral equations (12) where the two unknowns are scalar functions in $H_0^{1/2}(\partial\Omega)$ and $H_0^{3/2}(\partial\Omega)$.
Writing the two scalar unknowns of the system (12) as Taylor polynomials in ω, we rewrite the system (12) as (20). The identification of the different terms in (20) following increasing powers of the frequency ω leads to the systems (21), (22), (23) and (24) which are uniquely solvable. With the a priori-estimates (14) we show (in Lemma 4.4) that the expansions of the unknowns following increasing powers of the frequency ω are valid at the first order. An induction provides expansions at any order. This leads to an iterative procedure for solving the Drude-Born-Fedorov equations for small values of frequency. From these systems, the representation formulae (8) and (9), we construct the first three terms of the asymptotic expansions of the electric and magnetic fields given by (41) – (46) and we prove the optimal error estimates (31) and (32).
The case of a non simply connected boundary is studied in the final section of this paper.

From the representation theorem proved in [2] (Theorem 4.3), it follows that

$$
(8)\quad E^{(\omega)} = E^{in}_{\omega} + \frac{i}{2}\sqrt{\frac{\mu}{\varepsilon}}\Big\{\nabla \wedge G^{-}_{\omega}\, u^{(\omega)} + c(\omega)\nabla\,\Gamma^{+}_{\omega}\,(\mathrm{div}_{\partial\Omega}\, u^{(\omega)}) + \omega\tilde{G^{+}_{\omega}}\, u^{(\omega)}\Big\},
$$

and

$$
(9)\qquad H^{(\omega)} = H^{in}_{\omega} + \frac{1}{2}\Big\{\nabla \wedge G^{+}_{\omega}\, u^{(\omega)} + c(\omega)\,\nabla\,\Gamma^{-}_{\omega}\,(\mathrm{div}_{\partial\Omega}\, u^{(\omega)}) + \omega\,\tilde{G^{-}_{\omega}}\, u^{(\omega)}\Big\},
$$

where

$$
\begin{aligned}
u^\omega &= n \wedge H^\omega|_{\partial\Omega} \in TH^{-1/2}(\mathrm{div},\partial\Omega),\\
G^\pm_\omega\, u(x) &= \int_{\partial\Omega} (g_1 \pm g_2)(\omega, x-y)\; u(y)\; ds(y),\\
\Gamma^\pm_\omega\, \varphi(x) &= \int_{\partial\Omega} (\gamma_2 g_1 \pm \gamma_1 g_2)(\omega, x-y)\; \varphi(y)\; ds(y),\\
\tilde{G}^\pm_\omega\, u(x) &= \int_{\partial\Omega} (\gamma_1 g_1 + \gamma_2 g_2)(\omega, x-y)\; u(y)\; ds(y),\\
c(\omega) &= \frac{1-\omega^2\beta^2\varepsilon\mu}{\omega\varepsilon\mu}.
\end{aligned}
$$

Multiplying the limit of (8) as x tends to $\partial\Omega$ by a test field $v \in TH^{-1/2}(\mathrm{div},\partial\Omega)$ and integrating by parts gives

$$
a^{(\omega)}(u^{(\omega)}, v) = 2\,i\,\sqrt{\frac{\varepsilon}{\mu}}\,\langle E^{in}_\omega, v\rangle, \tag{10}
$$

where

$$
\begin{aligned}
a^{(\omega)}(u,v) &= \langle G^1_\omega u, \overrightarrow{\mathrm{curl}}_{\partial\Omega}\, v\rangle + \langle G^2_\omega u, v\rangle + \langle G^3_\omega\, \mathrm{div}_{\partial\Omega}\, u, \mathrm{div}_{\partial\Omega}\, v\rangle,\\
G^1_\omega u(x) &= \int_{\partial\Omega} (g_1 - g_2)(\omega, x-y)\; u^{(\omega)}(y)\,.\,(n(x)-n(y))\; ds(y)\\
G^2_\omega u(x) &= \omega \int_{\partial\Omega} (\gamma_1 g_1 + \gamma_2 g_2)(\omega, x-y)\; u^{(\omega)}(y)\; ds(y)\\
&\quad + (R(x) - 2c(x)) \Big(\int_{\partial\Omega} (g_1 - g_2)(\omega, x-y)\; u^{(\omega)}(y) \wedge n(x)\; ds(y) \Big)\\
&\quad - \int_{\partial\Omega} \frac{\partial (g_1 - g_2)}{\partial n_x}(\omega, x-y)\; (u^{(\omega)}(y) \wedge n(x))\; ds(y)\\
G^3_\omega \varphi(x) &= -c(\omega) \int_{\partial\Omega} (\gamma_2 g_1 + \gamma_1 g_2)(\omega, x-y)\; \varphi(y)\, ds(y).
\end{aligned}
$$

The above variational equation is the starting point for our asymptotic analysis of the Drude-Born-Fedorov equations at low frequencies.

4.2. The Case of a Simply Connected Boundary. In this section we assume that $\partial\Omega$ is simply connected. Under this assumption, we have the following orthogonal Hodge decomposition of the space $TH^{-1/2}(\mathrm{div},\partial\Omega)$: (see [12]).

LEMMA 4.1. $TH^{-1/2}(\mathrm{div},\partial\Omega) = \overrightarrow{\mathrm{curl}}_{\partial\Omega}\, H^{1/2}(\partial\Omega) \oplus \nabla_{\partial\Omega}\, H^{3/2}(\partial\Omega)$.

Let us then decompose the tangential fields $u^{(\omega)}$ and v which are in $TH^{-1/2}(\mathrm{div},\partial\Omega)$ as follows

$$
\begin{cases}
u^{(\omega)} = \overrightarrow{\mathrm{curl}}_{\partial\Omega}\, \varphi^{(\omega)} + \nabla_{\partial\Omega}\, \psi^{(\omega)}\\
v = \overrightarrow{\mathrm{curl}}_{\partial\Omega}\, \varphi_t + \nabla_{\partial\Omega}\, \psi_t
\end{cases} \tag{11}
$$

where $(\varphi^{(\omega)}, \psi^{(\omega)})$ and (φ_t, ψ_t) are in $H_0^{1/2}(\partial\Omega) \times H_0^{3/2}(\partial\Omega)$.
Therefore, we can split the variational equation (10) into the system of the two distinct

integral equations :

$$(12)\qquad \left\{\begin{array}{rcl} a_1^{(\omega)}(\varphi^{(\omega)},\varphi_t)+b_1^{(\omega)}(\psi^{(\omega)},\varphi_t) &=& i\sqrt{\dfrac{\varepsilon}{\mu}}\,\langle E_\omega^{in},\,\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi_t\,\rangle,\\ a_2^{(\omega)}(\psi^{(\omega)},\psi_t)+b_2^{(\omega)}(\varphi^{(\omega)},\psi_t) &=& i\sqrt{\dfrac{\varepsilon}{\mu}}\,\langle E_\omega^{in},\,\nabla_{\partial\Omega}\,\psi_t\,\rangle, \end{array}\right.$$

where

$$\begin{aligned} a_1^{(\omega)}(\varphi^{(\omega)},\varphi_t) &= a^{(\omega)}(\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi^{(\omega)},\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi_t),\\ b_1^{(\omega)}(\psi^{(\omega)},\varphi_t) &= a^{(\omega)}(\nabla_{\partial\Omega}\,\psi^{(\omega)},\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi_t),\\ a_1^{(\omega)}(\psi^{(\omega)},\psi_t) &= a^{(\omega)}(\nabla_{\partial\Omega}\,\psi^{(\omega)},\nabla_{\partial\Omega}\,\psi_t),\\ b_2^{(\omega)}(\varphi^{(\omega)},\psi_t) &= a^{(\omega)}(\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi^{(\omega)},\nabla_{\partial\Omega}\,\psi_t). \end{aligned}$$

Now, let us study the kernels

$$(g_1-g_2)(\omega,x-y),(\gamma_1 g_1+\gamma_2 g_2)(\omega,x-y),(\gamma_2 g_1+\gamma_1 g_2)(\omega,x-y),$$

and $\partial_{n_x}(g_1-g_2)(\omega,x-y)$. Let K be an integer. It is easy to see that there exist continuous kernels $(\mathcal{G}_K^j)_{j=1}^3$ and $(\mathcal{L}_K^j)_{j=1}^4$ which are uniformly bounded with respect to the frequency ω such that

$$\begin{aligned} 2\pi(g_1+g_2)(\omega,x-y) &= \frac{1}{|x-y|}+\omega\sum_{k=0}^{K-1}\omega^k\,\frac{P_k^1(|x-y|)}{|x-y|}\\ &\quad+\omega^{K+1}\,\frac{\mathcal{G}_K^1(\omega,x-y)}{|x-y|}, \end{aligned}$$

$$2\pi(g_1-g_2)(\omega,x-y)=\omega^2\sum_{k=0}^{K-2}\omega^k\,\frac{Q_k^1(|x-y|)}{|x-y|}+\omega^{K+1}\,\frac{\mathcal{L}_K^1(\omega,x-y)}{|x-y|},$$

$$\begin{aligned} 2\pi(\gamma_1 g_1+\gamma_2 g_2)(\omega,x-y) &= \frac{\sqrt{\varepsilon\mu}}{|x-y|}+\omega\sum_{k=0}^{K-1}\omega^k\,\frac{P_k^2(|x-y|)}{|x-y|}\\ &\quad+\omega^{K+1}\,\frac{\mathcal{G}_K^2(\omega,x-y)}{|x-y|}, \end{aligned}$$

$$2\pi(\gamma_1 g_1-\gamma_2 g_2)(\omega,x-y)=\omega\sum_{k=0}^{K-1}\omega^k\,\frac{Q_k^2(|x-y|)}{|x-y|}+\omega^{K+1}\,\frac{\mathcal{L}_K^2(\omega,x-y)}{|x-y|},$$

$$\begin{aligned} 2\pi(\gamma_2 g_1+\gamma_1 g_2)(\omega,x-y) &= \frac{\sqrt{\varepsilon\mu}}{|x-y|}+\omega\sum_{k=0}^{K-1}\omega^k\,\frac{P_k^3(|x-y|)}{|x-y|}\\ &\quad+\omega^{K+1}\,\frac{\mathcal{G}_K^3(\omega,x-y)}{|x-y|}, \end{aligned}$$

$$2\pi(\gamma_2 g_1-\gamma_1 g_2)(\omega,x-y)=\omega\sum_{k=0}^{K-1}\omega^k\,\frac{Q_k^3(|x-y|)}{|x-y|}+\omega^{K+1}\,\frac{\mathcal{L}_K^3(\omega,x-y)}{|x-y|},$$

and

$$\partial_{n_x}(g_1-g_2)(\omega,x-y) \;=\; \omega^3\sum_{k=0}^{K-3}\omega^k\,\frac{Q_k^4(|x-y|)}{|x-y|}+\omega^{K+1}\,\frac{\mathcal{L}_K^4(\omega,x-y)}{|x-y|}.$$

Since the kernels $(\mathcal{G}_K^j)_{j=1}^3$ and $(\mathcal{L}_K^j)_{j=1}^4$ are continuous when $|x-y|$ tends to zero, the integral operators with kernels $\mathcal{G}_K^j/|x-y|$ and $\mathcal{L}_K^j/|x-y|$ are at least of order one. From these asymptotic expansions we get the following expansions :

$$(13)\quad\begin{cases} a_1^{(\omega)}(\varphi,\varphi_t) \;=\; \omega a_1^{(1)}(\varphi,\varphi_t)+\omega^3\sum_{k=0}^{K-3}\omega^k\,a_1^{(k+3)}(\varphi,\varphi_t)+\omega^{K+1}\,\mathcal{A}_{1,K}^{(\omega)}(\varphi,\varphi_t),\\ a_2^{(\omega)}(\psi,\psi_t) \;=\; \dfrac{1}{\omega}a_2^{(-1)}(\psi,\psi_t)+\omega\sum_{k=0}^{K-1}\omega^k\,a_2^{(k+1)}(\psi,\psi_t)+\omega^{K+1}\mathcal{A}_{2,K}^{(\omega)}(\psi,\psi_t),\\ b_1^{(\omega)}(\psi,\varphi_t) \;=\; \omega b_1^{(1)}(\psi,\varphi_t)+\omega^3\sum_{k=0}^{K-3}\omega^k\,b_1^{(k+3)}(\psi,\varphi_t)+\omega^{K+1}\,\mathcal{B}_{1,K}^{(\omega)}(\psi,\varphi_t),\\ b_2^{(\omega)}(\varphi,\psi_t) \;=\; \omega b_2^{(1)}(\varphi,\psi_t)+\omega^3\sum_{k=0}^{K-3}\omega^k\,b_2^{(k+3)}(\varphi,\psi_t)+\mathcal{B}_{2,K}^{(\omega)}(\varphi,\psi_t), \end{cases}$$

where the bilinear forms $\mathcal{A}_{j,K}^{(\omega)},\mathcal{B}_{j,K}^{(\omega)}$ are uniformly bounded with respect to the parameter ω, and the forms $a_j^{(k)},b_j^{(k)}$ are continuous. The bilinear forms $a_1^{(1)}$ and $a_2^{(-1)}$ are defined by

$$a_1^{(1)}(\varphi,\varphi_t) \;=\; \sqrt{\varepsilon\mu}\int_{\partial\Omega}\int_{\partial\Omega}\frac{1}{4\pi|x-y|}\;\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi(y)\,.\,\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi_t(x)\;ds(x)\,ds(y),$$

and

$$a_2^{(-1)}(\psi,\psi_t) \;=\; -\frac{1}{\sqrt{\varepsilon\mu}}\int_{\partial\Omega}\int_{\partial\Omega}\frac{1}{4\pi|x-y|}\;\Delta_{\partial\Omega}\,\psi(y)\,\Delta_{\partial\Omega}\,\psi_t(x)\;ds(x)\,ds(y).$$

We have the following a priori-estimates.

LEMMA 4.2. *There exist two strictly positive constants ω_0 and C such that the estimates*

$$(14)\qquad\begin{cases} ||\varphi^{(\omega)}||_{H^{1/2}(\partial\Omega)} \;\le\; C,\\ ||\psi^{(\omega)}||_{H^{3/2}(\partial\Omega)} \;\le\; C\,\omega, \end{cases}$$

hold for $\omega\in]0,\omega_0[$.

Proof. Let us prove by contradiction that $c^{(\omega)} \;=\; ||\varphi^{(\omega)}||_{H^{1/2}(\partial\Omega)}+\omega^{-1}||\psi^{(\omega)}||_{H^{3/2}(\partial\Omega)}$ is uniformly bounded with respect to the frequency ω on an inteval $]0,\omega_0[$. Suppose that we have a subsequence, still denoted by $c^{(\omega)}$ such that it converges to $+\infty$ as ω tends to zero. Let us define $x^{(\omega)}$ and $y^{(\omega)}$ by

$$x^{(\omega)}=\frac{\varphi^{(\omega)}}{c^{(\omega)}},y^{(\omega)}=\frac{\psi^{(\omega)}}{c^{(\omega)}}.$$

There exist $x^{(0)}\in H_0^{1/2}(\partial\Omega)$ and $y^{(0)}\in H_0^{3/2}(\partial\Omega)$ such that $x^{(\omega)}$ tends to $x^{(0)}$ in $H^{1/2}(\partial\Omega)$ weakly and $\dfrac{1}{\omega}y^{(\omega)}$ to $y^{(0)}$ in $H^{3/2}(\partial\Omega)$ weakly. From (12) it follows that

the pair $(x^{(\omega)}, y^{(\omega)})$ satisfies

$$(15)\qquad \begin{cases} a_1^{(\omega)}(x^{(\omega)}, \varphi_t) + b_1^{(\omega)}(y^{(\omega)}, \varphi_t) &= \dfrac{i}{c^{(\omega)}}\sqrt{\dfrac{\varepsilon}{\mu}}\,\langle E_\omega^{in},\, \overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi_t\,\rangle, \\ a_2^{(\omega)}(y^{(\omega)}, \psi_t) + b_2^{(\omega)}(x^{(\omega)}, \psi_t) &= \dfrac{i}{c^{(\omega)}}\sqrt{\dfrac{\varepsilon}{\mu}}\,\langle E_\omega^{in},\, \nabla_{\partial\Omega}\,\psi_t\,\rangle, \end{cases}$$

It follows that

$$(16)\qquad \frac{1}{\omega}a_1^{(\omega)}(x^{(\omega)}, \varphi_t) + b_1^{(\omega)}(\frac{1}{\omega}y^{(\omega)}, \varphi_t) \;=\; \frac{i}{c^{(\omega)}}\sqrt{\frac{\varepsilon}{\mu}}\,\frac{1}{\omega}\,\langle\, E_\omega^{in},\, \overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi_t\,\rangle,$$

$$(17)\qquad \omega\, a_2^{(\omega)}(y^{(\omega)}, \psi_t) + \omega\, b_2^{(\omega)}(x^{(\omega)}, \psi_t) \;=\; \frac{i\omega}{c^{(\omega)}}\sqrt{\frac{\varepsilon}{\mu}}\,\langle E_\omega^{in},\, \nabla_{\partial\Omega}\,\psi_t\,\rangle.$$

The quantity

$$\frac{1}{\omega}\,\langle\, E_\omega^{in},\, \overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi_t\,\rangle$$

remains bounded when ω tends to zero (integrate by parts and use the asymptotic expansion (18) with respect to ω of the term $\mathrm{curl}_{\partial\Omega}E_\omega^{in}$). Using the following equalities obtained from (13) :

$$\begin{cases} \dfrac{1}{\omega}a_1^{(\omega)}(x^{(\omega)}, \varphi_t) \;=\; a_1^{(1)}(x^{(\omega)}, \varphi_t) + 0(\omega), \\ a_2^{(\omega)}(y^{(\omega)}, \psi_t) \;=\; a_2^{(-1)}(\dfrac{1}{\omega}y^{(\omega)}, \psi_t) + 0(\omega), \\ b_1^{(\omega)}(\dfrac{1}{\omega}y^{(\omega)}, \varphi_t) \;=\; 0(\omega), b_2^{(\omega)}(x^{(\omega)}, \psi_t) \;=\; 0(\omega), \end{cases}$$

making $\omega \to 0$ in the equations (16) and (17), we get

$$a_1^{(1)}(x^{(0)}, \varphi_t) \;=\; 0, \quad \forall\, \varphi_t \in H_0^{1/2}(\partial\Omega)$$

$$a_2^{(-1)}(y^{(0)}, \psi_t) \;=\; 0, \quad \forall\, \psi_t \in H_0^{3/2}(\partial\Omega),$$

hence $x^{(0)} = y^{(0)} = 0$. Now, making $\varphi_t = \overline{x^{(\omega)}}, \psi_t = \dfrac{1}{\omega}\overline{y^{(\omega)}}$ in (16) and (17) gives the two limits

$$\lim_{\omega\to 0} a_1^{(1)}(x^{(\omega)}, \overline{x^{(\omega)}}) \;=\; 0, \lim_{\omega\to 0} a_2^{(-1)}(\frac{y^{(\omega)}}{\omega}, \frac{\overline{y^{(\omega)}}}{\omega}) \;=\; 0,$$

so, the pair $(x^{(\omega)}, \dfrac{1}{\omega}y^{(\omega)})$ converges strongly to $(0,0)$ in $H^{1/2}(\partial\Omega) \times H^{3/2}(\partial\Omega)$. This follows from the fact that the bilinear forms $a_1^{(1)}(.,.), a_2^{(-1)}(.,.)$ are coercive on $H_0^{1/2}(\partial\Omega)$ and $H_0^{3/2}(\partial\Omega)$ respectively : there exists a strictly positive constant C such that for any $\varphi \in H_0^{1/2}(\partial\Omega)$

$$\int_{\partial\Omega}\int_{\partial\Omega} \frac{1}{|x-y|}|\overrightarrow{\mathrm{curl}}_{\partial\Omega}\varphi(x)|^2\; ds(x)\; ds(y) \;\geq\; C\,||\varphi||^2_{H_0^{1/2}(\partial\Omega)}$$

(see [20] for a proof) and for any $\psi \in H_0^{3/2}(\partial\Omega)$

$$\int_{\partial\Omega}\int_{\partial\Omega} \frac{1}{|x-y|}|\Delta_{\partial\Omega}\psi(x)|^2\; ds(x)\; ds(y) \;\geq\; C\,||\psi||^2_{H_0^{3/2}(\partial\Omega)}.$$

This leads to a contradiction since

$$||x^{(\omega)}||_{H^{1/2}(\partial\Omega)} + \frac{1}{\omega}||y^{(\omega)}||_{H^{3/2}(\partial\Omega)} \;=\; 1,$$

for any ω. The proof of the lemma is then over. □

In the following part of this section, we construct the asymptotic expansions in powers of the frequency ω of the electric field $E^{(\omega)}$ and the magnetic field $H^{(\omega)}$ solutions of the exterior Drude-Born-Fedorov equations.
Firstly, the incident plane wave admits the following asymptotic expansion :

$$E^{in}_{\omega}(x) \;=\; E^{in}_0(x) + \omega E^{in}_1(x) + \omega^2 E^{in}_2(x) + \ldots,$$

where E^{in}_0 is independent of x. We denote by $E^{in}_{\omega,\partial\Omega}$ the tangential component of $E^{in}_{\partial\Omega}$ on $\partial\Omega$. The asymptotic expansions of $\mathrm{div}_{\partial\Omega}\, E^{in}_{\omega,\partial\Omega}$ and $\mathrm{curl}_{\partial\Omega}\, E^{in}_{\omega,\partial\Omega}$ with respect to the frequency ω are given by

$$\left\{\begin{array}{l} 2i\sqrt{\dfrac{\varepsilon}{\mu}}\mathrm{div}_{\partial\Omega}\, E^{in}_{\omega,\partial\Omega} \;=\; \psi^{in}_0 + \omega\psi^{in}_1 + \omega^2\psi^{in}_2 + \ldots, \\ 2i\sqrt{\dfrac{\varepsilon}{\mu}}\mathrm{curl}_{\partial\Omega}\, E^{in}_{\omega,\partial\Omega} \;=\; \omega\varphi^{in}_1 + \omega^2\varphi^{in}_2 + \omega^3\varphi^{in}_3 + \ldots. \end{array}\right. \tag{18}$$

Let us suppose that the functions $(\psi^{(\omega)}, \varphi^{(\omega)})$ admit the following expansions :

$$\left\{\begin{array}{l} \psi^{(\omega)} \;=\; \psi^{(0)} + \omega\psi^{(1)} + \omega^2\psi^{(2)} + \ldots, \\ \varphi^{(\omega)} \;=\; \varphi^{(0)} + \omega\varphi^{(1)} + \omega^2\varphi^{(2)} + \ldots, \end{array}\right. \tag{19}$$

where $(\varphi^{(i)}, \psi^{(i)}) \in H^{1/2}_0(\partial\Omega) \times H^{3/2}_0(\partial\Omega)$ (further, we prove that the previous expansions are unique, and valid at any order), we look then for an asymptotic expansion for $u^{(\omega)}$ of the form

$$u^{(\omega)} \;=\; u^{(0)} + \omega u^{(1)} + \omega^2\, u^{(2)} + \ldots$$

where for each integer $i, u^{(i)} = \overrightarrow{\mathrm{curl}}_{\partial\Omega}\varphi^{(i)} + \nabla_{\partial\Omega}\psi^{(i)}$ is in $TH^{-1/2}(\mathrm{div}, \partial\Omega)$.
Therefore, using the a priori estimates (14) we rewrite the system (12) as

$$\left\{\begin{array}{l} a^{(1)}_1(\varphi^{(\omega)}, \varphi_t) + b^{(1)}_1(\psi^{(\omega)}, \varphi_t) + \omega^2 a^{(3)}_1(\varphi^{(\omega)}, \varphi_t) \\ \;=\; \langle\varphi^{in}_1, \varphi_t\rangle + \omega\langle\varphi^{in}_2, \varphi_t\rangle + \omega^2\langle\varphi^{in}_3, \varphi_t\rangle + 0(\omega^3)\,||\varphi_t||_{H^{1/2}(\partial\Omega)}, \\ \dfrac{1}{\omega}a^{(-1)}_2(\psi^{(\omega)}, \psi_t) + \omega a^{(1)}_2(\psi^{(\omega)}, \psi_t) + \omega b^{(1)}_2(\varphi^{(\omega)}, \psi_t) + \omega^2 a^{(2)}_2(\psi^{(\omega)}, \psi_t) \\ \;=\; \langle\psi^{in}_0, \psi_t\rangle + \omega\langle\psi^{in}_1, \psi_t\rangle + \omega^2\langle\psi^{in}_2, \psi_t\rangle + 0(\omega^3)\,||\psi_t||_{H^{3/2}(\partial\Omega)}. \end{array}\right. \tag{20}$$

Now, we shall show how to obtain the functions $\varphi^{(i)}$ and $\psi^{(i)}$. Upon inserting the expansions (19) of $(\varphi^{(\omega)}, \psi^{(\omega)})$ in the previous equations, we get two equalities between asymptotics in ω. Necessarily, the terms of the same order are equal, so the system (20) implies the 0following equations

$$\left\{\begin{array}{l} a^{(-1)}_2(\psi^{(0)}, \psi_t) \;=\; 0, \\ a^{(1)}_1(\varphi^{(0)}, \varphi_t) + b^{(1)}_1(\psi^{(0)}, \psi_t) \;=\; \langle\varphi^{in}_1, \varphi_t\rangle, \end{array}\right. \tag{21}$$

$$\left\{\begin{array}{l} a_2^{(-1)}(\psi^{(1)},\psi_t) \,=\, \langle \psi_0^{in},\psi_t\rangle, \\ a_1^{(1)}(\varphi^{(1)},\varphi_t)+b_1^{(1)}(\psi^{(1)},\varphi_t) \,=\, \langle \varphi_2^{in},\varphi_t\rangle, \end{array}\right. \tag{22}$$

$$\left\{\begin{array}{l} a_2^{(-1)}(\psi^{(2)},\psi_t)+b_2^{(1)}(\varphi^{(0)},\psi_t)+a_2^{(1)}(\psi^{(0)},\psi_t) \,=\, \langle \psi_1^{in},\psi_t\rangle, \\ a_1^{(1)}(\varphi^{(2)},\varphi_t)+b_1^{(1)}(\psi^{(2)},\varphi_t)+a_1^{(3)}(\varphi^{(0)},\varphi_t)+b_1^{(3)}(\psi^{(0)},\varphi_t) \,=\, \langle \varphi_3^{in},\varphi_t\rangle, \end{array}\right. \tag{23}$$

and more generaly,

$$\left\{\begin{array}{l} a_2^{(-1)}(\psi^{(K)},\psi_t)+\displaystyle\sum_{k=0}^{K-4} b_2^{(k+3)}(\varphi^{(K-k-4)},\psi_t)+\sum_{k=0}^{K-2} a_2^{(k+1)}(\psi^{(K-k-2)},\psi_t) \,=\, (\psi_{K-1}^{in},\psi_t), \\ a_1^{(1)}(\varphi^{(K)},\varphi_t)+b_1^{(1)}(\psi^{(K)},\varphi_t)+\displaystyle\sum_{k=0}^{K-2} a_1^{(k+3)}(\varphi^{(K-k-2)},\varphi_t) \\ +\displaystyle\sum_{k=0}^{K-2} b_1^{(k+3)}(\psi^{(K-k-2)},\varphi_t) \,=\, \langle \varphi_{K+1}^{in},\varphi_t\rangle, \end{array}\right. \tag{24}$$

for any (ψ_t,φ_t) in $H^{3/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)$. Let us first prove that the above systems are uniquely solvable in $H_0^{3/2}(\partial\Omega)\times H_0^{1/2}(\partial\Omega)$.

LEMMA 4.3. *The systems* (21), (22), (23) *and more generaly* (24) *are uniquely solvable in* $H_0^{3/2}(\partial\Omega)\times H_0^{1/2}(\partial\Omega)$.

Proof. Since the forms $a_2^{(-1)}$ and $a_1^{(1)}$ are coercive, the first equation of (21) implies $\psi^{(0)}=0$ and the second is uniquely solvable thus $\varphi^{(0)}$ is determined. For the same reason the first equation of (22) is uniquely solvable so, $\psi^{(1)}$ is determined and the second too. Inserting the solution of the previous systems in (23), we solve the first equation of (23) and after the second. So at the end of the process, $(\varphi^{(i)},\psi^{(i)})$ are determined for $i\in\{0,1,2\}$. This naturally leads to an iterative process to determine the unknowns $(\varphi^{(i)},\psi^{(i)})$ at any order. An immediate induction shows that : given the functions $(\varphi^{(i)},\psi^{(i)})$ for $i\le n$, the next unknowns $(\varphi^{(n+1)},\psi^{(n+1)})$ are solutions of a system similar to (23), and appear respectively in the coercive forms $a_1^{(1)}$ and $a_2^{(-1)}$ only. The other terms are well-known continuous linear forms, depending on $(\varphi^{(i)},\psi^{(i)})$ for i smaller than $n+1$. □

The next stage of our study is to estimate the difference between $\varphi^{(\omega)}$ (resp. $\psi^{(\omega)}$) and $\displaystyle\sum_{k=0}^{K}\omega^k\,\varphi^{(k)}$ (resp. $\displaystyle\sum_{k=0}^{K-1}\omega^{k+1}\psi^{(k+1)}$) for small values of ω. For this, we begin by proving the following lemma.

LEMMA 4.4. *There exists a strictly positive constant* C *such that for small values of* ω *the following inequalities hold.*

$$\left\{\begin{array}{l} ||\varphi^{(\omega)}-\varphi^{(0)}||_{H^{1/2}(\partial\Omega)} \,\le\, C\,\omega, \\ ||\psi^{(\omega)}-\omega\psi^{(1)}||_{H^{3/2}(\partial\Omega)} \,\le\, C\,\omega^2. \end{array}\right. \tag{25}$$

Proof. In order to estimate the norm of $\varphi^{(\omega)}-\varphi^{(0)}$ in $H^{1/2}(\partial\Omega)$ we first observe that

$$a_1^{(1)}(\varphi^{(\omega)}-\varphi^{(0)},\varphi_t) \,=\, \langle \varphi_1^{in},\varphi_t\rangle - a_1^{(1)}(\varphi^{(0)},\varphi_t) - b_1^{(1)}(\psi^{(\omega)},\varphi_t) + 0(\omega)\,||\varphi_t||_{H^{1/2}(\partial\Omega)},$$

for any φ_t in $H^{1/2}(\partial\Omega)$. Using the estimates (14), we get

$$a_1^{(1)}(\varphi^{(\omega)} - \varphi^{(0)}, \varphi_t) \;=\; 0(\omega)\,||\varphi_t||_{H^{1/2}(\partial\Omega)}.$$

Therefore from ([20]) we obtain the claim.
For the estimation of $||\psi^{(\omega)} - \omega\psi^{(1)}||_{H^{3/2}(\partial\Omega)}$, it is easy to see that

$$\begin{aligned} a_2^{(-1)}(\psi^{(\omega)} - \omega\psi^{(1)}, \psi_t) &= \omega\langle\psi_0^{in}, \psi_t\rangle - \omega\, a_2^{(-1)}(\psi^{(1)}, \psi_t) + 0(\omega^2)\,||\psi_t||_{H^{3/2}(\partial\Omega)} \\ &= 0(\omega^2)\,||\psi_t||_{H^{3/2}(\partial\Omega)}, \end{aligned}$$

and therefore we have

$$||\psi^{(\omega)} - \omega\psi^{(1)}||_{H^{3/2}(\partial\Omega)} \;\leq\; C\,\omega^2.$$

□

Now we are in position to prove the following lemma.

LEMMA 4.5. *Let K be an integer. There exists a strictly positive constant C_K such that for small values of ω the following inequalities hold.*

$$\text{(26)} \qquad \begin{cases} ||\psi^{(\omega)} - \omega \sum_{k=0}^{K-1} \omega^k\,\psi^{(k+1)}||_{H^{3/2}(\partial\Omega)} \;\leq\; C_K\,\omega^{K+1}, \\ ||\varphi^{(\omega)} - \sum_{k=0}^{K} \omega^k\,\varphi^{(k)}||_{H^{1/2}(\partial\Omega)} \;\leq\; C_K\,\omega^{K+1}. \end{cases}$$

Proof. We proceed in the same way as in the above lemma. Assume that there exists a constant C_K such that

$$\text{(27)} \qquad ||\psi^{(\omega)} - \omega \sum_{k=0}^{K-1} \omega^k\,\psi^{(k+1)}||_{H^{3/2}(\partial\Omega)} \;\leq\; C_K\,\omega^{K+1}.$$

Using the a priori estimates (14), we deduce from

$$\text{(28)} \qquad \begin{aligned} a_1^{(1)}(\varphi^{(\omega)} - \sum_{k=0}^{K} \omega^k\,\varphi^{(k)}, \varphi_t) &= -b_1^{(1)}(\psi^{(\omega)} - \sum_{k=0}^{K-1} \omega^{k+1}\,\psi^{(k+1)}, \varphi_t) \\ &\quad +0(\omega^{K+1})\,||\varphi_t||_{H^{1/2}(\partial\Omega)} \\ &= 0(\omega^{K+1})\,||\varphi_t||_{H^{1/2}(\partial\Omega)}, \end{aligned}$$

that there exists a constant C'_K such that

$$\text{(29)} \qquad ||\varphi^{(\omega)} - \sum_{k=0}^{K} \omega^k\,\varphi^{(k)}||_{H^{1/2}(\partial\Omega)} \;\leq\; C'_K\,\omega^{K+1}.$$

It is easy to see that

$$\begin{aligned} &|a_2^{(-1)}(\psi^{(\omega)} - \sum_{k=0}^{K} \omega^{k+1}\,\psi^{(k+1)}, \psi_t)| \;\leq\; \omega\,||\psi^{(\omega)} - \sum_{k=0}^{K-1} \omega^{k+1}\,\psi^{(k+1)}||_{H^{3/2}(\partial\Omega)}\,||\psi_t||_{H^{3/2}(\partial\Omega)} \\ &+0(\omega^{K+2})\,||\psi_t||_{H^{3/2}(\partial\Omega)}. \end{aligned}$$

Taking into account (27), we get

$$a_2^{(-1)}(\psi^{(\omega)} - \sum_{k=0}^{K} \omega^{k+1}\,\psi^{(k+1)}, \psi_t) \;=\; 0(\omega^{K+2})\,||\psi_t||_{H^{3/2}(\partial\Omega)},$$

and then, the estimation

$$||\psi^{(\omega)} - \sum_{k=0}^{K} \omega^{k+1}\,\psi^{(k+1)}|| \;\leq\; C_{K+1}\,\omega^{K+1},$$

holds where C_{K+1} is a constant independent of ω. □

The above lemma allows us to estimate the difference between $u^{(\omega)}$ and $\sum_{k=0}^{K} \omega^k\,u^{(k)}$. We have the

LEMMA 4.6.

$$||u^{(\omega)} - \sum_{k=0}^{K} \omega^k\,u^{(k)}||_{TH^{-1/2}(\mathrm{div},\partial\Omega)} \;\leq\; C\,\omega^{K+1}. \tag{30}$$

Proof. Since the operators

$$\begin{aligned}
&\overrightarrow{\mathrm{curl}}_{\partial\Omega} \;: H^{1/2}(\partial\Omega) \longrightarrow TH^{-1/2}(\mathrm{div},\partial\Omega),\\
&\nabla_{\partial\Omega} \;: H^{3/2}(\partial\Omega) \longrightarrow TH^{-1/2}(\mathrm{div},\partial\Omega),
\end{aligned}$$

are continuous, there exists a strictly positive constant C independent of ω such that the following inequality

$$\begin{aligned}
||u^{(\omega)} - \sum_{k=0}^{K} \omega^k\,u^{(k)}||_{TH^{-1/2}(\mathrm{div},\partial\Omega)} \;\leq\; C\,\Big(&||\varphi^{(\omega)} - \sum_{k=0}^{K} \omega^k\,\varphi^{(k)}||_{H^{1/2}(\partial\Omega)}\\
&+ ||\psi^{(\omega)} - \omega \sum_{k=0}^{K-1} \omega^k\,\psi^{(k+1)}||_{H^{3/2}(\partial\Omega)}\Big),
\end{aligned}$$

holds for small values of ω, and then Lemma 3.6 follows from (26). □

The main result of this section is as follows :

THEOREM 4.7. *We have the following estimates :*

$$||E^{(\omega)} - \sum_{k=0}^{K} \omega^k\,E^{(k)}||_{H(curl,\mathcal{O})} \;\leq\; C(\mathcal{O})\,\omega^{K+1}, \tag{31}$$

and

$$||H^{(\omega)} - \sum_{k=0}^{K} \omega^k\,H^{(k)}||_{H(curl,\mathcal{O})} \;\leq\; C(\mathcal{O})\,\omega^{K+1}, \tag{32}$$

for any bounded smooth open set $\mathcal{O}$ included in Ω.

Proof. This result is easily proved by combining the a priori-estimates (14), the representation formulae (8) and (10) and the optimal error estimate (30). □

4.3. The Case of a Non-Simply Connected Boundary. The aim of this section is to extend the above analysis of the Drude-Born-Fedorov equations at frequency ω tends to zero to the case of a non simply connected boundary. We consider the Drude-Born-Fedorov equations in a three-dimensional exterior domain with boundary $\partial\Omega$, where $\partial\Omega$ is a non simply connected smooth surface. The question of interest is again the behavior of the electric and magnetic fields as frequency tends to zero.
Under this assumption on $\partial\Omega$, we have the following Hodge decomposition of the space $TH^{-1/2}(\mathrm{div},\partial\Omega)$:

LEMMA 4.8. *$TH^{-1/2}(\mathrm{div},\partial\Omega) = \overrightarrow{\mathrm{curl}}_{\partial\Omega} H^{1/2}(\partial\Omega) \oplus \nabla_{\partial\Omega} H^{3/2}(\partial\Omega) \oplus \mathcal{N}(\partial\Omega)$, where*

$$\mathcal{N}(\partial\Omega) = \Big\{ \alpha, \Delta_{\partial\Omega}\,\alpha = 0 \Big\} = \Big\{ \alpha, \mathrm{div}_{\partial\Omega}\alpha = 0, \mathrm{curl}_{\partial\Omega}\alpha = 0 \Big\}.$$

$\mathcal{N}(\partial\Omega)$ is a finite-dimensional vector space.

This decomposition holds from the fact that the the elements of $\mathcal{N}(\partial\Omega)$ are in $\mathcal{C}^\infty(\partial\Omega)$ if $\partial\Omega$ is of class $\mathcal{C}^\infty$. Let us then decompose the unknown tangential field $u^{(\omega)}$ and the test tangential field v, both in $TH^{-1/2}(\mathrm{div},\partial\Omega)$ as follows

$$\begin{cases} u^{(\omega)} = \overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi^{(\omega)} + \nabla_{\partial\Omega}\,\psi^{(\omega)} + \alpha^{(\omega)}, \\ v = \overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi_t + \nabla_{\partial\Omega}\,\psi_t + \alpha_t, \end{cases}$$

where the two couples $(\varphi^{(\omega)},\psi^{(\omega)})$ and (φ_t,ψ_t) are in $H_0^{1/2}(\partial\Omega)\times H_0^{3/2}(\partial\Omega)$, and $(\alpha^{(\omega)},\alpha_t)$ in $\mathcal{N}(\partial\Omega)\times\mathcal{N}(\partial\Omega)$.
Therefore, we can split the variational equation (10) into three distinct equations given by

$$(33)\quad \begin{cases} a_1^{(\omega)}(\varphi^{(\omega)},\varphi_t) + b_1^{(\omega)}(\psi^{(\omega)},\varphi_t) + c_1^{(\omega)}(\alpha^{(\omega)},\varphi_t) = i\sqrt{\dfrac{\varepsilon}{\mu}}\,\langle E_\omega^{in},\, \overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi\rangle, \\ a_2^{(\omega)}(\psi^{(\omega)},\psi_t) + b_2^{(\omega)}(\varphi^{(\omega)},\psi_t) + c_2^{(\omega)}(\alpha^{(\omega)},\psi_t) = i\sqrt{\dfrac{\varepsilon}{\mu}}\,\langle E_\omega^{in},\, \nabla_{\partial\Omega}\,\psi_t\,\rangle, \\ c_3^{(\omega)}(\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi^{(\omega)} + \nabla_{\partial\Omega}\,\psi^{(\omega)} + \alpha^{(\omega)},\alpha_t) = i\sqrt{\dfrac{\varepsilon}{\mu}}\,\langle E_\omega^{in},\, \alpha_t\,\rangle, \end{cases}$$

where

$$\begin{aligned} c_1^{(\omega)}(\alpha^{(\omega)},\varphi_t) &= a^{(\omega)}(\alpha^{(\omega)},\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi_t) \\ c_2^{(\omega)}(\alpha^{(\omega)},\psi_t) &= a^{(\omega)}(\alpha^{(\omega)},\nabla_{\partial\Omega}\,\psi_t) \\ c_3^{(\omega)}(\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi^{(\omega)} + \nabla_{\partial\Omega}\,\psi^{(\omega)} + \alpha^{(\omega)},\alpha_t) &= a^{(\omega)}(\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi^{(\omega)} + \nabla_{\partial\Omega}\,\psi^{(\omega)} + \alpha^{(\omega)},\alpha_t). \end{aligned}$$

Analogously to $a_1^{(\omega)}$ and $a_2^{(\omega)}$, we have

$$(34)\quad \begin{cases} c_1^{(\omega)}(\alpha,\varphi_t) = \omega c_1^{(1)}(\alpha,\varphi_t) + \omega^3 c_1^{(3)}(\alpha,\varphi_t) + \dots \\ c_2^{(\omega)}(\alpha,\psi_t) = \omega c_2^{(1)}(\alpha,\psi_t) + \omega^3 c_2^{(3)}(\alpha,\psi_t) + \dots \\ c_3^{(\omega)}(\overrightarrow{\mathrm{curl}}_{\partial\Omega}\varphi + \nabla_{\partial\Omega}\psi + \alpha,\alpha_t) = \Big(\omega c_3^{(1)} + \omega^3 c_3^{(3)}\Big)(\overrightarrow{\mathrm{curl}}_{\partial\Omega}\varphi + \nabla_{\partial\Omega}\psi + \alpha,\alpha_t) + \dots, \end{cases}$$

where the bilinear forms $c_j^{(i)}$ introduced above are continuous and

$$c_1^{(1)}(\alpha,\varphi_t) = \sqrt{\varepsilon\mu}\int_{\partial\Omega}\int_{\partial\Omega}\frac{1}{4\pi|x-y|}\;\alpha(y)\,.\,\overrightarrow{\mathrm{curl}}_{\partial\Omega}\varphi_t(x)\;ds(x)\,ds(y).$$

As in the previous section, let us suppose that the unknowns $(\varphi^{(\omega)}, \psi^{(\omega)}, \alpha^{(\omega)})$ admit the following expansions :

$$
(35) \qquad \left\{ \begin{array}{l} \varphi^{(\omega)} \;=\; \varphi^{(0)} + \omega\varphi^{(1)} + \omega^2\varphi^{(2)} + \dots \\ \psi^{(\omega)} \;=\; \psi^{(0)} + \omega\psi^{(1)} + \omega^2\psi^{(2)} + \dots \\ \alpha^{(\omega)} \;=\; \alpha^{(0)} + \omega\alpha^{(1)} + \omega^2\alpha^{(2)} + \dots \end{array} \right.
$$

where for each integer i, $(\varphi^{(i)}, \psi^{(i)}, \alpha^{(i)}) \in H^{1/2}(\partial\Omega) \times H^{3/2}(\partial\Omega) \times \mathcal{N}(\partial\Omega)$.
We write the system (33) as

$$
(36) \qquad \left\{ \begin{array}{l} a_1^{(1)}(\varphi^{(\omega)}, \varphi_t) + b_1^{(1)}(\psi^{(\omega)}, \varphi_t) + \omega^2 a_1^{(3)}(\varphi^{(\omega)}, \varphi_t) \\ + c_1^{(1)}(\alpha^{(\omega)}, \varphi_t) + \omega^2 c_1^{(3)}(\alpha^{(\omega)}, \varphi_t) \\ = \langle \varphi_1^{in}, \varphi_t \rangle + \omega \langle \varphi_2^{in}, \varphi_t \rangle + \omega^2 \langle \varphi_3^{in}, \varphi_t \rangle + 0(\omega^3)\, ||\varphi_t||_{H^{1/2}(\partial\Omega)}, \\ \dfrac{1}{\omega} a_2^{(-1)}(\psi^{(\omega)}, \psi_t) + \omega a_2^{(1)}(\psi^{(\omega)}, \psi_t) + \omega b_2^{(1)}(\varphi^{(\omega)}, \psi_t) \\ + \omega^2 a_2^{(2)}(\psi^{(\omega)}, \psi_t) + \omega c_2^{(1)}(\alpha^{(\omega)}, \psi_t) \;=\; \langle \psi_0^{in}, \psi_t \rangle + \omega \langle \psi_1^{in}, \psi_t \rangle \\ + \omega^2 \langle \psi_2^{in}, \psi_t \rangle + 0(\omega^3)\, ||\psi_t||_{H^{3/2}(\partial\Omega)}, \\ \left(c_3^{(1)} + \omega c_3^{(2)} + \omega^2 c_3^{(3)} \right) (\overrightarrow{\mathrm{curl}}_{\partial\Omega} \varphi^{(\omega)} + \nabla_{\partial\Omega} \psi^{(\omega)} + \alpha^{(\omega)}, \alpha_t) \\ = i\sqrt{\dfrac{\varepsilon}{\mu}} \langle E_1^{in} + \omega E_2^{in} + \omega^2 E_3^{in},\, \alpha_t \rangle + 0(\omega^3)\, ||\alpha_t||_{T H^{-1/2}(\partial\Omega)}. \end{array} \right.
$$

In the above system, we have used that

$$
\langle\, E_0^{in}, \alpha_t \,\rangle \;=\; 0, \quad \forall\, \alpha_t \in \mathcal{N}(\partial\Omega),
$$

since E_0^{in} is a constant vector.
Upon inserting (35) into the above system, and identifying different terms following an increasing powers of ω, we obtain

$$
(37) \qquad \left\{ \begin{array}{l} a_2^{(-1)}(\psi^{(0)}, \psi_t) \;=\; 0, \\ a_1^{(1)}(\varphi^{(0)}, \varphi_t) + b_1^{(1)}(\psi^{(0)}, \varphi_t) + c_1^{(1)}(\alpha^{(0)}, \varphi_t) \;=\; \langle \varphi_1^{in}, \varphi_t \rangle, \\ c_3^{(1)}(\alpha^{(0)} + \overrightarrow{\mathrm{curl}}_{\partial\Omega}\, \varphi^{(0)} + \nabla_{\partial\Omega} \psi^{(0)}, \alpha_t) \;=\; i\sqrt{\dfrac{\varepsilon}{\mu}} \langle E_1^{in}, \alpha_t \,\rangle, \end{array} \right.
$$

$$
(38) \qquad \left\{ \begin{array}{l} a_2^{(-1)}(\psi^{(1)}, \psi_t) \;=\; \langle \psi_0^{in}, \psi_t \rangle, \\ a_1^{(1)}(\varphi^{(1)}, \varphi_t) + b_1^{(1)}(\psi^{(1)}, \varphi_t) + c_1^{(1)}(\alpha^{(1)}, \varphi_t) \;=\; \langle \varphi_2^{in}, \varphi_t \rangle, \\ c_3^{(1)}(\alpha^{(1)} + \overrightarrow{\mathrm{curl}}_{\partial\Omega}\, \varphi^{(1)} + \nabla_{\partial\Omega} \psi^{(1)}, \alpha_t) \\ + c_3^{(2)}(\alpha^{(0)} + \overrightarrow{\mathrm{curl}}_{\partial\Omega}\, \varphi^{(0)} + \nabla_{\partial\Omega} \psi^{(0)}, \alpha_t) \;=\; i\sqrt{\dfrac{\varepsilon}{\mu}}\, \langle E_2^{in}, \alpha_t \,\rangle, \end{array} \right.
$$

$$
(39)\begin{cases}
a_2^{(-1)}(\psi^{(2)},\psi_t)+b_2^{(1)}(\varphi^{(0)},\psi_t)+c_2^{(1)}(\alpha^{(0)},\psi_t)+b_2^{(1)}(\varphi^{(0)},\psi_t) \;=\; \langle \psi_1^{in},\psi_t\rangle,\\
a_1^{(1)}(\varphi^{(2)},\varphi_t)+b_1^{(1)}(\psi^{(2)},\varphi_t)+a_1^{(3)}(\varphi^{(0)},\varphi_t)+c_1^{(1)}(\alpha^{(2)},\varphi_t) \;=\; \langle \varphi_3^{in},\varphi_t\rangle,\\
c_3^{(1)}(\alpha^{(2)}+\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi^{(2)}+\nabla_{\partial\Omega}\psi^{(2)},\alpha_t)\\
+c_3^{(2)}(\alpha^{(1)}+\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi^{(1)}+\nabla_{\partial\Omega}\psi^{(1)},\alpha_t)\\
+c_3^{(3)}(\alpha^{(0)}+\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi^{(0)}+\nabla_{\partial\Omega}\psi^{(0)},\alpha_t) \;=\; i\sqrt{\dfrac{\varepsilon}{\mu}}\,\langle\, E_3^{in},\alpha_t\,\rangle,
\end{cases}
$$

for any $(\psi_t,\varphi_t,\alpha_t)\in H^{3/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)\times\mathcal{N}(\partial\Omega)$.
In addition we obtain $\psi^{(3)}\in H_0^{3/2}(\partial\Omega)$ as the unique solution of the variational equation

$$
\begin{aligned}
&a_2^{(-1)}(\psi^{(3)},\psi_t)+a_2^{(1)}(\psi^{(1)},\psi_t)+b_2^{(1)}(\varphi^{(1)},\psi_t)+a_2^{(2)}(\psi^{(0)},\psi_t)\\
&+c_2^{(1)}(\alpha^{(1)},\psi_t) \;=\; (\psi_2^{in},\psi_t),\quad \forall\,\psi_t\in H^{3/2}(\partial\Omega).
\end{aligned}
\tag{40}
$$

LEMMA 4.9. *The systems* (37), (38), (39) *and the variational equation* (40) *are uniquely solvable.*

Proof. Since the form $a_2^{(-1)}$ is coercive, the variational equation (40) is uniquely solvable in $H_0^{3/2}(\partial\Omega)$. To show that the system (37) has at most one solution, we first observe that the first equation implies that $\psi^{(0)}=0$. Then, by choosing $\varphi_t=\varphi^{(0)},\alpha_t=\alpha_0$ and adding the two last equations of (37) we obtain

$$
\int_{\partial\Omega}\int_{\partial\Omega}\frac{1}{|x-y|}\,|\overrightarrow{\mathrm{curl}}_{\partial\Omega}\varphi^{(0)}+\alpha^{(0)}|^2\;ds(x)\;ds(y) \;=\; 0.
$$

From this, we deduce that $\overrightarrow{\mathrm{curl}}_{\partial\Omega}\varphi^{(0)}+\alpha^{(0)}=0$ and by taking the scalar rotational of the previous identity, it follows that $\varphi^{(0)}=0$ and so, $\alpha^{(0)}=0$. The existence of a solution to (37) can be shown by using the fact that the system (37) is of Fredholm type. □

Finally, using the solutions of the systems (37), (38) and (39), and the variational equation (40), we obtain the following asymptotic expansions in powers of the frequency ω of the electric and magnetic fields

$$
\begin{cases}
E^{(\omega)} \;=\; E^{(0)}+\omega E^{(1)}+\omega^2E^{(2)}+\ldots,\\
H^{(\omega)} \;=\; H^{(0)}+\omega H^{(1)}+\omega^2H^{(2)}+\ldots,
\end{cases}
$$

where

$$
E^{(0)}(x) \;=\; E_0^{in}(x)+\frac{i}{\varepsilon}\nabla\int_{\partial\Omega}\frac{1}{4\pi|x-y|}\,\Delta_{\partial\Omega}\,\psi^{(1)}(y)\;ds(y),
$$

and

$$
H^{(0)}(x) \;=\; H_0^{in}(x)+\nabla\;\wedge\;\int_{\partial\Omega}\frac{1}{4\pi|x-y|}\left(\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi^{(0)}(y)+\alpha^{(0)}\right)ds(y).
$$

The dependence of the asymptotics of $(E^{(\omega)},H^{(\omega)})$ on the topological properties of $\partial\Omega$ is now clear. Using these asymptotics, it is clear that estimates (31) and (32) hold in the case of a non-simply connected boundary.

5. Concluding Remarks. In this paper we have studied the limiting behavior of the exterior Drude-Born-Fedorov equations as frequency tends to zero and we have derived asymptotic expansions of the electric and magnetic fields to the order two with respect to the frequency. We have shown the dependence of these asymptotics on the topological properties of the boundary of the domain via the Hodge decomposition theorem. We have proved that the effect of chirality on the scattered fields is of order two at low frequencies.

Appendix A. The Hodge Decomposition Theorem.
Throughout this appendix, $\partial\Omega$ will be a compact oriented Riemannian manifold of dimension n and $\Delta_{\partial\Omega}$ the Laplace-Beltrami operator on $\partial\Omega$ which is a generalization of the ordinary Laplacien on differential forms. Let for p with $0 \leq p \leq n$ $\mathcal{F}^p(\partial\Omega)$ be the set of p-forms on $\partial\Omega$. It is to be noted that $\mathcal{F}^0(\partial\Omega)$ is the space of scalar functions on $\partial\Omega$ and $\mathcal{F}^1(\partial\Omega)$ is the space of the vector tangential functions defined on $\partial\Omega$. $\Delta_{\partial\Omega}$ is a linear operator on $\mathcal{F}^p(\partial\Omega)$ for each p with $0 \leq p \leq n$.
Our main object in this appendix is to state the Hodge decomposition theorem, which says that the equation $\Delta_{\partial\Omega}\,\alpha = f$ has a solution α in the smooth p-forms on $\partial\Omega$ if and only if the p-form f is orthogonal in the inner product on $\mathcal{F}^p(\partial\Omega)$ to the space of those for which $\Delta_{\partial\Omega}\phi = 0$ called the harmonic p-forms. We let

$$\mathcal{N}^p(\partial\Omega) = \Big\{\alpha \in \mathcal{F}^p(\partial\Omega), \Delta_{\partial\Omega}\,\alpha = 0\Big\}.$$

The elements of $\mathcal{N}^p(\partial\Omega)$ are called harmonic p-forms.

The Hodge Decomposition Theorem : For each integer p with $0 \leq p \leq n$, $\mathcal{N}^p(\partial\Omega)$ is a finite dimensional, and we have the following orthogonal direct sum decompositions of the space $\mathcal{F}^p(\partial\Omega)$ of smooth p-forms on $\partial\Omega$:

$$\mathcal{F}^p(\partial\Omega) = \Delta(\mathcal{F}^p(\partial\Omega)) \oplus \mathcal{N}^p(\partial\Omega).$$

Consequently, the equation $\Delta_{\partial\Omega}\,\alpha = f$ has a solution $\alpha \in \mathcal{F}^p(\partial\Omega)$ if and only if the p-form f is orthogonal to the space of harmonic p-forms. We refer to Warner [26] for a proof of this theorem.
The number of linearly independent p-forms in $\mathcal{N}^p(\partial\Omega)$ is called the p-Betti number of $\partial\Omega$ denoted by b_p.

Examples : (see Berger-Gostiaux [8] p.139)
$\partial\Omega$ = sphere S^n: all b_p are zero except $b_0 = b_n = 1$.
$\partial\Omega = T^n$ (torus) : $b_p = C_n^p$ for every p.
$\partial\Omega$ = Klein bottle : $b_0 = b_1 = 1, b_2 = 0$.
$\partial\Omega$ = surface with g holes : $b_0 = b_2 = 1, b_1 = 2g$.

Appendix B. The second order terms.
The aim of this appendix is to give the first terms of the asymptotics of the fields $E^{(\omega)}, H^{(\omega)}$.
• $\partial\Omega$ is simply connected : $E^{(i)}$ in Ω for $i = 0, 1, 2$ are given by

$$E^{(0)}(x) = E_0^{in}(x) + \frac{i}{\varepsilon}\nabla \int_{\partial\Omega} \frac{1}{4\pi|x-y|}\,\Delta_{\partial\Omega}\,\psi^{(1)}(y)\,ds(y), \tag{41}$$

$$\begin{aligned} E^{(1)}(x) &= E_1^{in}(x) + \frac{i}{\varepsilon}\nabla \int_{\partial\Omega} \frac{1}{4\pi|x-y|}\,\Delta_{\partial\Omega}\,\psi^{(2)}(y)\,ds(y) \\ &\quad + i\mu \int_{\partial\Omega} \frac{1}{4\pi|x-y|}\,\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi^{(0)}(y)\,ds(y), \end{aligned} \tag{42}$$

and

$$
\begin{aligned}
E^{(2)}(x) &= E_2^{in}(x) + \frac{i}{\varepsilon}\nabla \int_{\partial\Omega} \frac{1}{4\pi|x-y|}\,\Delta_{\partial\Omega}\,\psi^{(3)}(y)\;ds(y) \\
&\quad + i\mu \int_{\partial\Omega} \frac{1}{4\pi|x-y|}\,\Big(\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi^{(1)}(y) + \nabla_{\partial\Omega}\,\psi^{(1)}(y)\Big) ds(y) \\
&\quad - \frac{i\mu}{8\pi}\,\nabla \int_{\partial\Omega} |x-y|\,\Delta_{\partial\Omega}\,\psi^{(1)}(y)\,ds(y),
\end{aligned} \tag{43}
$$

and $H^{(i)}$ in Ω for $i = 0, 1, 2$ by :

$$
H^{(0)}(x) = H_0^{in}(x) + \nabla \wedge \int_{\partial\Omega} \frac{1}{4\pi|x-y|}\,\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi^{(0)}(y)\;ds(y), \tag{44}
$$

$$
\begin{aligned}
H^{(1)}(x) &= H_1^{in}(x) + \nabla \wedge \int_{\partial\Omega} \frac{1}{4\pi|x-y|}\,\Big(\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi^{(1)}(y) + \nabla_{\partial\Omega}\,\psi^{(1)}(y)\Big)\;ds(y) \\
&\quad - \beta_0 \nabla \int_{\partial\Omega} \frac{1}{4\pi|x-y|}\,\Delta_{\partial\Omega}\,\psi^{(1)}(y)\;ds(y),
\end{aligned} \tag{45}
$$

and

$$
\begin{aligned}
H^{(2)}(x) &= H_2^{in}(x) + \nabla \wedge \int_{\partial\Omega} \frac{1}{4\pi|x-y|}\,\Big(\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi^{(2)}(y) + \nabla_{\partial\Omega}\,\psi^{(2)}(y)\Big)\;ds(y) \\
&\quad - \frac{\varepsilon\mu}{8\pi}\nabla \wedge \int_{\partial\Omega} |x-y|\Big(\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi^{(0)}(y) + \nabla_{\partial\Omega}\,\psi^{(0)}(y)\Big)\;ds(y) \\
&\quad - \beta_1 \nabla \int_{\partial\Omega} \frac{1}{4\pi|x-y|}\Delta_{\partial\Omega}\,\psi^{(1)}(y)\;ds(y) - \beta_0 \nabla \int_{\partial\Omega} \frac{1}{4\pi|x-y|}\Delta_{\partial\Omega}\,\psi^{(2)}(y)\;ds(y) \\
&\quad + \varepsilon\mu\beta_0 \int_{\partial\Omega} \frac{1}{4\pi|x-y|}\,\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi^{(0)}(y)\;ds(y),
\end{aligned} \tag{46}
$$

where

$$
\begin{cases}
H_0^{in}(x) = -i\sqrt{\dfrac{\varepsilon}{\mu}}\Big(q_1 + q_2\Big), \\
H_1^{in}(x) = \varepsilon\Big((p_1\,.\,x)\;q_1 + (p_2\,.\,x)\;q_2\Big), \\
H_2^{in}(x) = -i\,\varepsilon^{3/2}\,\mu^{1/2} \displaystyle\sum_{j=1,2} \Big(i\beta_0\;(p_j\,.\,x) + (-1)^j \frac{1}{2}(p_j\,.\,x)^2\Big)\;q_j.
\end{cases}
$$

• $\partial\Omega$ is non-simply connected : we have

$$
\begin{aligned}
E^{(1)}(x) &= E_1^{in}(x) + \frac{i}{\varepsilon}\nabla \int_{\partial\Omega} \frac{1}{4\pi|x-y|}\,\Delta_{\partial\Omega}\,\psi^{(2)}(y)\;ds(y) \\
&\quad + i\mu \int_{\partial\Omega} \frac{1}{4\pi|x-y|}\,\Big(\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi^{(0)}(y) + \alpha^{(0)}(y)\Big)\;ds(y), \\
E^{(2)}(x) &= E_2^{in}(x) + \frac{i}{\varepsilon}\nabla \int_{\partial\Omega} \frac{1}{4\pi|x-y|}\,\Delta_{\partial\Omega}\,\psi^{(3)}(y)\;ds(y) \\
&\quad + i\mu \int_{\partial\Omega} \frac{1}{4\pi|x-y|}\,\Big(\overrightarrow{\mathrm{curl}}_{\partial\Omega}\,\varphi^{(1)}(y) + \nabla_{\partial\Omega}\,\psi^{(1)}(y) + \alpha^{(1)}(y)\Big)ds(y) \\
&\quad - \frac{i\mu}{8\pi}\,\nabla \int_{\partial\Omega} |x-y|\,\Delta_{\partial\Omega}\,\psi^{(1)}(y)\,ds(y),
\end{aligned}
$$

$$H^{(1)}(x) = H_1^{in}(x) + \nabla \wedge \int_{\partial\Omega} \frac{1}{4\pi|x-y|} \left(\overrightarrow{\mathrm{curl}}_{\partial\Omega}\, \varphi^{(1)}(y) + \nabla_{\partial\Omega}\, \psi^{(1)}(y) + \alpha^{(1)}(y) \right) ds(y)$$
$$-\beta_0 \nabla \int_{\partial\Omega} \frac{1}{4\pi|x-y|}\, \Delta_{\partial\Omega}\, \psi^{(1)}(y)\, ds(y),$$

and

$$H^{(2)}(x) = H_2^{in}(x) + \nabla \wedge \int_{\partial\Omega} \frac{1}{4\pi|x-y|} \left(\overrightarrow{\mathrm{curl}}_{\partial\Omega}\, \varphi^{(2)}(y) + \nabla_{\partial\Omega}\, \psi^{(2)}(y) + \alpha^{(2)}(y) \right) ds(y)$$
$$-\frac{\varepsilon\mu}{8\pi} \nabla \wedge \int_{\partial\Omega} |x-y| \left(\overrightarrow{\mathrm{curl}}_{\partial\Omega}\, \varphi^{(0)}(y) + \nabla_{\partial\Omega}\, \psi^{(0)}(y) \right) ds(y)$$
$$-\beta_1 \nabla \int_{\partial\Omega} \frac{1}{4\pi|x-y|} \Delta_{\partial\Omega}\, \psi^{(1)}(y)\, ds(y)$$
$$-\beta_0 \nabla \int_{\partial\Omega} \frac{1}{4\pi|x-y|} \Delta_{\partial\Omega}\, \psi^{(2)}(y)\, ds(y)$$
$$+\varepsilon\mu\beta_0 \int_{\partial\Omega} \frac{1}{4\pi|x-y|} \left(\overrightarrow{\mathrm{curl}}_{\partial\Omega}\, \varphi^{(0)}(y) + \alpha^{(0)}(y) \right) ds(y).$$

We conclude this paper by making the following important remarks. From the expressions of the three first corrector terms, we note that the dependence of the asymptotic expansions of $E^{(\omega)}$ on the chirality admittance appears for the first time at the second order terms and at the first order terms for the magnetic field $H^{(\omega)}$.

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