Time-Harmonic Electromagnetic Fields in Chiral Media

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Abstract

This paper deals with the scattering problem by an inhomogeneous object embedded in a chiral medium. A combined Hilbert space and integral equation method is employed to prove the existence and uniqueness of the solutions to the scattering problem. It is shown that the classical Silver-Müller radiation condition remains valid in chiral media.

Key words. Integral Equations, Chiral Media, Drude-Born-Fedorov Equations, Representation Theorems, Fredholm Equations, Silver-Müller Radiation Condition.

AMS(MOS) subject classifications. 35C15, 35Q60, 45B05.

1 Introduction

The interaction of electromagnetic fields at microwave frequencies with the optically active chiral media has attracted the attention of the electromagnetic community due to its potential applications in the field of antennas, microwave devices, waveguide propagation, scattering, etc. Chiral media can be characterized by a generalized set of constitutive relations in which the electric and magnetic fields are coupled. The coupling strength is given by the magnitude of a quantity known as the chirality admittance which determines the bulk electromagnetic properties of these materials. In scattering applications it may be possible to use chiral material to coat a scatterer and thus control its scattering properties more efficiently than a dielectric coating due to the extra degree of freedom offered by the presence of chiral parameter. The original investigations of optical activity, or the effect of chirality on the polarization of light, also date back to the nineteenth century. Arago [2], Biot [4], Cauchy [8], Pasteur [28] and Fresnel [13] all examined optical activity in solid and liquid chiral medium. The concept of chirality has also played an increasingly important role in chemistry (see Prelog [30]).

The aim of this paper is to prove uniqueness and existence of a solution to the diffraction problem in a chiral medium. The starting point of the present paper is the constitutive relations for the chiral medium. Different expressions exist for the constitutive relations. The Drude-Born-Fedorov constitutive equations are used here. For interesting explanation of these equations and various physical aspects of the propagation inside chiral media, we refer to [12], [19], [18], [5], [6], [7], [3], [16], [17], [20], [31], [15]. We employ the technique of boundary integral equations to discuss the existence and uniqueness of solutions to the Drude-Born-Fedorov constitutive equations. The use of integral equations in problems of electromagnetics is a well-known technique; a current account of the method may be found in Colton and Kress [10]. For the problem of scattering of time-harmonic Maxwell's equations in an inhomogeneous medium, the boundary integral equation method was applied by Müller [23]. There are significant differences in the analysis needed to study the Drude-Born-Fedorov equations and the Maxwell equations in an inhomogeneous medium. The present paper, devoted to present some results and approaches dealing with the rigorous scattering in a chiral medium, may be considered as a generalization of Müller's results. After a brief discussion of the physical background of electromagnetic wave propagation in chiral media, we derive representation theorems for solutions to the Drude-Born-Fedorov equations. We then introduce the radiation condition for these equations and we show their connection with the Maxwell equations by making use of the Bohren decomposition. After that, we shall provide a detailed proof for the existence and uniqueness of a solution to the scattering problem of electromagnetic waves by an inhomogeneous object embedded in a chiral media. This will be accomplished by using the boundary integral equation method and extending the unique continuation principle for the Drude-Born-Fedorov equations.

2 Physical Background and the Fundamental Solution

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set, with $\mathcal{C}^{1,1}$ boundary $\Gamma = \partial\Omega$. We recall that an open set $\Omega \subset \mathbb{R}^n$ is called a $C^{m,1}$ domain with m nonnegative integer if for each $\mathbf{p} \in \partial\Omega$ there exist a neighborhood $U(\mathbf{p})$ of $\mathbf{p}$ in $\mathbb{R}^n$, and a new orthogonal coordinates $(\mathbf{x}', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, such that :

(i) $U(\mathbf{p})$ is an hypercube in the new coordinates $U(\mathbf{p}) = \Big\{ (\mathbf{x}', x_n),\ -a_j < x_j < a_j \Big\}$;

(ii) there exists a function φ of class $C^{m,1}$, defined in $U'(\mathbf{p}) = \Big\{ \mathbf{x}',\ -\dfrac{a_j}{2} < x_j < \dfrac{a_j}{2} \Big\}$,

and such that

$$|\varphi(\mathbf{x}')| \le \frac{a_n}{2} \quad \text{for every } \mathbf{x}' \in U'(\mathbf{p}), \quad U \cap \Omega = \Big\{ (\mathbf{x}', x_n) : x_n < \varphi(\mathbf{x}') \Big\},$$

and

$$U \cap \partial\Omega = \Big\{ (\mathbf{x}', x_n) : x_n = \varphi(\mathbf{x}') \Big\}.$$

We denote by

$$\nabla . = \nabla^t = (\partial_1, \partial_2, \partial_3),$$

and

$$\nabla \wedge = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}$$

in Cartesian components.

Optical activity of the chiral medium can be explained by the direct substitution of the Drude-Born-Fedorov constitutive equations :

$$\left\{ \begin{array}{lcl} D &=& \varepsilon(x)\Big(E + \beta(x)\nabla \wedge E\Big) \quad \text{in } \mathbb{R}^3, \\ B &=& \mu(x)\Big(H + \beta(x)\nabla \wedge H\Big) \quad \text{in } \mathbb{R}^3, \end{array} \right. \tag{1}$$

into Maxwell's equations

$$\left\{ \begin{array}{lcl} \nabla \wedge E &=& i\omega B \quad \text{in } \mathbb{R}^3, \\ \nabla \wedge H &=& -i\omega D \quad \text{in } \mathbb{R}^3, \end{array} \right. \tag{2}$$

where ε is the electric permittivity, μ the magnetic permeability and β the chirality admittance. We assume that ε, μ and β are real-valued functions in $\mathcal{C}^2(\mathbb{R}^3)$, and they have the following properties :
there are constants $\varepsilon_m, \mu_m > 0$, and ε_M, μ_M such that $\varepsilon_m \leq \varepsilon(x) \leq \varepsilon_M, \mu_m \leq \mu(x) \leq \mu_M$ when $x \in \overline{\Omega}$;
outside the bounded region $\Omega, \mu(x) = \mu_0, \varepsilon(x) = \varepsilon_0$ and $\beta(x) = \beta_0$, for some known constants ε_0, μ_0, and β_0. It follows the equivalent monochromatic constitutive relations :

$$\left\{ \begin{array}{lcl} \Big(1 - (k(x)\beta(x))^2\Big)\, D &=& \varepsilon(x)E + \dfrac{i\beta(x)}{\omega}(k(x))^2\, H \quad \text{in } \mathbb{R}^3, \\ \Big(1 - (k(x)\beta(x))^2\Big)\, B &=& \mu(x)H - \dfrac{i\beta(x)}{\omega}(k(x))^2\, E \quad \text{in } \mathbb{R}^3, \end{array} \right. \tag{3}$$

where

$$k(x) = \omega\sqrt{\varepsilon(x)\,\mu(x)} \tag{4}$$

is simply a shorthand notation and does not represent any wavenumber inside the chiral medium. Combining (2) to (3) gives

$$\left\{ \begin{array}{lcl} \nabla \wedge E &=& (\gamma(x))^2\, \beta(x)E + i\omega\mu(x)(\dfrac{\gamma(x)}{k(x)})^2\, H \quad \text{in } \mathbb{R}^3, \\ \nabla \wedge H &=& (\gamma(x))^2\, \beta(x)\, H - i\omega\varepsilon(x)(\dfrac{\gamma(x)}{k(x)})^2\, E \quad \text{in } \mathbb{R}^3. \end{array} \right. \tag{5}$$

In these equations, the parameter $\gamma(x)$ is defined as :

$$(\gamma(x))^2 \;=\; \frac{(k(x))^2}{1-(\,k(x)\,\beta(x)\,)^2}, \tag{6}$$

and throughout what follows, we assume that $(\,k(x)\,\beta(x)\,)^2 \neq 1, x \in \mathbb{R}^3$.
For more details on the physical background of electromagnetic wave propagation in chiral media, we refer to the book by Lakhtakia et al [19].
Let us introduce

$$\left\{\begin{array}{l} \gamma_1 \;=\; \dfrac{\omega\sqrt{\varepsilon_0\mu_0}}{1-\omega\sqrt{\varepsilon_0\mu_0}\beta_0}, \quad \omega_1 \;=\; \dfrac{\omega}{1-\omega\sqrt{\varepsilon_0\mu_0}\beta_0}, \\ \gamma_2 \;=\; \dfrac{\omega\sqrt{\varepsilon_0\mu_0}}{1+\omega\sqrt{\varepsilon_0\mu_0}\beta_0}, \quad \omega_2 \;=\; \dfrac{\omega}{1+\omega\sqrt{\varepsilon_0\mu_0}\beta_0}, \end{array}\right. \tag{7}$$

and

$$\tilde{\omega} \;=\; \frac{\omega}{1-\omega^2\varepsilon_0\mu_0\beta_0^2}. \tag{8}$$

Assume the incident field is given by the time-harmonic electromagnetic plane wave

$$\left\{\begin{array}{l} E^{in}(x) \;=\; q_1\, e^{i\gamma_1 p_1\,\cdot\,x} + q_2\, e^{i\gamma_2 p_2\,\cdot\,x}, \\ H^{in}(x) \;=\; -i\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\Big(q_1\, e^{i\gamma_1 p_1\,\cdot\,x} + q_2\, e^{i\gamma_2 p_2\,\cdot\,x}\Big), \end{array}\right. \tag{9}$$

where the complex vectors p_1, p_2, q_1 and q_2 are such that

$$p_1 \cdot q_1 \;=\; 0, p_1 \wedge q_1 \;=\; -iq_1, p_2 \cdot q_2 \;=\; 0, \text{ and } p_2 \wedge q_2 \;=\; iq_2. \tag{10}$$

The electromagnetic scattering problem corresponding to the Drude-Born-Fedorov equations (5) is to find the electric and magnetic fields (E, H) such that $(E - E^{in}, H - H^{in})$ satisfy an appropriate radiation condition at infinity. Let us first note that under the assumptions (10) we have

$$\left\{\begin{array}{l} \nabla \wedge (q_1\, e^{i\gamma_1 p_1\,\cdot\,x}) \;=\; \gamma_1\, q_1\, e^{i\gamma_1 p_1\,\cdot\,x}, \\ \nabla \wedge (q_2\, e^{i\gamma_1 p_2\,\cdot\,x}) \;=\; -\gamma_2\, q_2\, e^{i\gamma_1 p_2\,\cdot\,x}. \end{array}\right.$$

The incident field E^{in} is then a combination of left-circularly polarized plane wave and right-circularly polarized one and it satisfies the isotropic Drude-Born-Fedorov equations in $\mathbb{R}^3$.
Now, we show how to derive in a natural way a fundamental solution of the isotropic Drude-Born-Fedorov equations. If we define the fields $(\mathcal{E}_1, \mathcal{E}_2)$ in the homogeneous domain $\mathbb{R}^3 \setminus \overline{\Omega}$ by

$$\begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{pmatrix} = \begin{pmatrix} 1 & i\sqrt{\dfrac{\mu_0}{\varepsilon_0}} \\ 1 & -i\sqrt{\dfrac{\mu_0}{\varepsilon_0}} \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix}, \tag{11}$$

and the fields $(\mathcal{H}_1, \mathcal{H}_2)$ by

$$\mathcal{H}_j = \frac{1}{i\omega_j\mu_0}\nabla \wedge \mathcal{E}_j, \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega},$$

for $j = 1, 2$, making use of the equations (5), we verify without any difficulty that

$$\nabla \wedge \mathcal{H}_j = -i\omega_j\varepsilon_0\mathcal{E}_j, \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega},$$

for $j = 1, 2$. The fields $(\mathcal{E}_j, \mathcal{H}_j)$, for $j = 1, 2$, are then solutions of Maxwell's equations

$$\begin{cases} \nabla \wedge \mathcal{E}_j = i\omega_j\mu_0\mathcal{H}_j, & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ \nabla \wedge \mathcal{H}_j = -i\omega_j\varepsilon_0\mathcal{E}_j & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{12}$$

Furthermore, from (5), we have the following two identities

$$\begin{cases} \nabla \wedge \mathcal{E}_1 = \gamma_1\,\mathcal{E}_1, \\ \nabla \wedge \mathcal{E}_2 = -\gamma_2\,\mathcal{E}_2, \end{cases} \tag{13}$$

and then,

$$\begin{cases} \mathcal{H}_1 = -i\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\,\mathcal{E}_1, \\ \mathcal{H}_2 = i\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\,\mathcal{E}_2. \end{cases} \tag{14}$$

Let g_j, for $j = 1, 2$, be the outgoing fundamental solution of $-(\Delta + \gamma_j^2)$ which is given by

$$g_j(x, y) = \frac{e^{i\gamma_j}|x - y|}{4\pi|x - y|}, \quad x \neq y.$$

It is well known that the fundamantal outgoing solution of the Maxwell equations (12) is given by

$$\mathcal{G}^j = \frac{\gamma_j^2}{\omega_j}\begin{pmatrix} 1 + \dfrac{\nabla\nabla\,.}{\gamma_j^2} & i\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\dfrac{1}{\gamma_j}\nabla\wedge \\ -i\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\dfrac{1}{\gamma_j}\nabla\wedge & 1 + \dfrac{\nabla\nabla\,.}{\gamma_j^2} \end{pmatrix} g_j.$$

From (11) and (14), we see that

$$\begin{pmatrix} E \\ H \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathcal{E}_1 \\ \mathcal{H}_1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \mathcal{E}_2 \\ \mathcal{H}_2 \end{pmatrix},$$

$$\begin{pmatrix} \mathcal{E}_1 \\ \mathcal{H}_1 \end{pmatrix} = \begin{pmatrix} 1 & i\sqrt{\dfrac{\mu_0}{\varepsilon_0}} \\ -i\sqrt{\dfrac{\varepsilon_0}{\mu_0}} & 1 \end{pmatrix}\begin{pmatrix} E \\ H \end{pmatrix},$$

and

$$\begin{pmatrix} \mathcal{E}_2 \\ \mathcal{H}_2 \end{pmatrix} = \begin{pmatrix} 1 & -i\sqrt{\dfrac{\mu_0}{\varepsilon_0}} \\ i\sqrt{\dfrac{\varepsilon_0}{\mu_0}} & 1 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix}.$$

Motivated by the identities above, we introduce the kernel G_1 defined by

$$G_1 = \mathcal{G}^1 \begin{pmatrix} 1 & i\sqrt{\dfrac{\mu_0}{\varepsilon_0}} \\ -i\sqrt{\dfrac{\varepsilon_0}{\mu_0}} & 1 \end{pmatrix},$$

the kernel G_2 defined by

$$G_2 = \mathcal{G}^2 \begin{pmatrix} 1 & -i\sqrt{\dfrac{\mu_0}{\varepsilon_0}} \\ i\sqrt{\dfrac{\varepsilon_0}{\mu_0}} & 1 \end{pmatrix},$$

and finally, the kernel G defined by

$$G = \frac{1}{2}(G_1 + G_2). \tag{15}$$

Now, if we denote by $\mathcal{L}_0$ the isotropic Drude-Born-Fedorov operator

$$\mathcal{L}_0 = \begin{pmatrix} 0 & -i\dfrac{\gamma^2\beta_0}{\varepsilon_0} + \dfrac{i}{\varepsilon_0}\nabla\wedge \\ i\dfrac{\gamma^2\beta_0}{\mu_0} - \dfrac{i}{\mu_0}\nabla\wedge & 0 \end{pmatrix},$$

we can easily verify that

$$\Big(\mathcal{L}_0 - \tilde{\omega}\Big)\, G = \delta\, I,$$

where $\tilde{\omega}$ is given by (8), δ is the Dirac function and I is the unit matrix. The kernel G is then a fundamental solution for the isotropic Drude-Born-Fedorov chiral medium. Let us observe that G is given in terms of only outgoing waves. Furthermore, $\nabla \wedge G_1 = \gamma_1 G_1$ and $\nabla \wedge G_2 = -\gamma_2 G_2$, for $x \neq y$, which show that the isotropic chiral media are circularly birefringent. G involves two modes, one being an outgoing wave of frequency ω_1, and the other, an outgoing wave of frequency ω_2. If $\beta_0 = 0$, then (15) leads to the outgoing fundamental solution of the free-space Maxwell's equations

$$\omega\varepsilon_0\mu_0 \begin{pmatrix} 1 + \dfrac{\nabla\nabla\,.}{\omega^2\varepsilon_0\mu_0} & \dfrac{i}{\omega\varepsilon_0}\nabla\wedge \\ -\dfrac{i}{\omega\,\mu_0}\nabla\wedge & 1 + \dfrac{\nabla\nabla\,.}{\omega^2\varepsilon_0\mu_0} \end{pmatrix} g,$$

where g is the fundamental outgoing solution of $-(\Delta + \omega^2\varepsilon_0\mu_0)$.
Using the operator $\mathcal{L}_0$, we can rewrite the Drude-Born-Fedorov equations (5) in the abbreviated form

$$\left(\mathcal{L}_0 - \tilde{\omega}\right)\begin{pmatrix} E \\ H \end{pmatrix} = \omega \begin{pmatrix} \tilde{\varepsilon}(x) & i\omega\tilde{\tilde{\mu}}(x) \\ -i\omega\tilde{\tilde{\varepsilon}}(x) & \tilde{\mu}(x) \end{pmatrix}\begin{pmatrix} E \\ H \end{pmatrix},$$

where the real-valued functions $\tilde{\varepsilon}, \tilde{\mu}, \tilde{\tilde{\varepsilon}}$ and $\tilde{\tilde{\mu}}$ are defined by

$$\begin{cases} \tilde{\varepsilon}(x) \;=\; \dfrac{\varepsilon(x)}{\varepsilon_0(1-\omega^2\varepsilon(x)\mu(x)\beta^2(x))} - \dfrac{1}{1-\omega^2\varepsilon_0\mu_0\beta_0^2}, \\ \tilde{\mu}(x) \;=\; \dfrac{\mu(x)}{\mu_0(1-\omega^2\varepsilon(x)\mu(x)\beta^2(x))} - \dfrac{1}{1-\omega^2\varepsilon_0\mu_0\beta_0^2}, \\ \tilde{\tilde{\varepsilon}}(x) \;=\; \dfrac{\varepsilon_0\beta_0}{1-\omega^2\varepsilon_0\mu_0\beta_0^2} - \dfrac{\varepsilon(x)\mu(x)\beta(x)}{\mu_0(1-\omega^2\varepsilon(x)\mu(x)\beta^2(x))}, \\ \tilde{\tilde{\mu}}(x) \;=\; \dfrac{\mu_0\beta_0}{1-\omega^2\varepsilon_0\mu_0\beta_0^2} - \dfrac{\varepsilon(x)\mu(x)\beta(x)}{\varepsilon_0(1-\omega^2\varepsilon(x)\mu(x)\beta^2(x))}. \end{cases} \tag{16}$$

Formally, we derive an analogue of the Lippman-Schwinger equation :

$$\begin{pmatrix} E - E^{in} \\ H - H^{in} \end{pmatrix} = \omega \int_\Omega G(.-y) \begin{pmatrix} \tilde{\varepsilon}(y) & i\omega\tilde{\tilde{\mu}}(y) \\ -i\omega\tilde{\tilde{\varepsilon}}(y) & \tilde{\mu}(y) \end{pmatrix}\begin{pmatrix} E(y) \\ H(y) \end{pmatrix} dy. \tag{17}$$

The proof of this formula will be given in section 4.

3 Boundary Integral Operators

We first introduce the Sobolev spaces needed to write our boundary integral equation. we denote by $H^s(\Omega)$ the usual L^2- based Sobolev spaces on Ω. Also, if $|s| \leq 2$, the Sobolev spaces $H^s(\Gamma)$ are well defined (*cf.* [14]). The space of all distributions in $(H^s(\Gamma))^3$ which are orthogonal to the unit normal vector field of Γ is denoted by $TH^s(\Gamma)$. By $\overline{H}^s_{loc}(\mathbb{R}^3 \setminus \overline{\Omega})$ we denote the spaces of all distributions $u \in \mathcal{D}'(\mathbb{R}^3 \setminus \overline{\Omega})$ such that, for all open balls B containing $\overline{\Omega}$ in its interior, the restriction of u to $B \cap (\mathbb{R}^3 \setminus \overline{\Omega})$ belongs to $H^s(B \cap \mathbb{R}^3 \setminus \overline{\Omega})$. Equipped with the family of semi-norms

$$||u||_B \;=\; ||u|_{B\cap(\mathbb{R}^3\setminus\overline{\Omega})}||_{H^1(B\cap(\mathbb{R}^3\setminus\overline{\Omega}))},$$

$\overline{H}^1_{loc}(\mathbb{R}^3 \setminus \overline{\Omega})$ becomes a Fréchet space. Finally, we define

$$TH^{1/2}(\mathrm{div}, \Gamma) \;= \Big\{ u \in TH^{1/2}(\Gamma), \text{ such that } \mathrm{div}_\Gamma\, u \in H^{1/2}(\Gamma) \Big\},$$

$$H^1(\mathrm{div},\Omega) \;=\Big\{u\in (H^1(\Omega))^3,\ \text{such that } \mathrm{div}_\Gamma(n\,\wedge\,u)\in H^{1/2}(\Gamma)\Big\},$$

and

$$\overline{H}^1_{loc}(\mathrm{div},\mathbb{R}^3\setminus\overline{\Omega}) \;=\Big\{u\in (\overline{H}^1_{loc}(\mathbb{R}^3\setminus\overline{\Omega}))^3,\ \text{such that } \mathrm{div}_\Gamma(n\,\wedge\,u)\in H^{1/2}(\Gamma)\Big\},$$

where div_Γ is the surface divergence on Γ.
These spaces are equipped with the following norms

$$||u||^2_{TH^{1/2}(\mathrm{div},\Gamma)} \;=\; ||u||^2_{(H^{1/2}(\Gamma))^3}+||\mathrm{div}_\Gamma u||^2_{H^{1/2}(\Gamma)},$$

and

$$||u||^2_{H^1(\mathrm{div},\Omega)} \;=\; ||u||^2_{(H^1(\Omega))^3}+||\mathrm{div}_\Gamma(n\,\wedge\,u)||^2_{H^{1/2}(\Gamma)}.$$

They become Hilbert spaces. The topology on $\overline{H}^1_{loc,\mathrm{div}}(\mathbb{R}^3\setminus\overline{\Omega})$ is defined by the family of semi-norms

$$||u||^2_{\mathrm{div},B} \;=\; ||u||^2_{H^1(B\cap(\mathbb{R}^3\setminus\overline{\Omega}))}+||\mathrm{div}_\Gamma(n\wedge u)||^2_{H^{1/2}(\Gamma)},\quad B \text{ an open ball},\quad \overline{\Omega}\subset B.$$

This makes $\overline{H}^1_{loc,\mathrm{div}}(\mathbb{R}^3\setminus\overline{\Omega})$ a Fréchet space. We shall also introduce the Sobolev spaces

$$\begin{aligned}
H_0(\mathrm{curl},\Omega) &= \Big\{v\in (L^2(\Omega))^3, \nabla\,\wedge\,v\in (L^2(\Omega))^3, n\,\wedge\,v\;=\;0 \text{ on } \Gamma\Big\},\\
H(\mathrm{div},\Omega) &= \Big\{v\in (L^2(\Omega))^3, \nabla\,.\,v\in L^2(\Omega)\Big\},\\
H^1_0(\Omega) &= \Big\{\psi\in H^1(\Omega), \psi=0 \text{ on } \Gamma\Big\}.
\end{aligned}$$

Now, we define the potential operators $V^{\pm}_\Omega, K^{\pm}_\Omega$ and $D^{\pm}_\Omega$ by the following formulae. If $\phi\in(\mathcal{C}^{1,1}(\Gamma))^3$, we set

$$V^{\pm}_\Omega\phi(x) \;=\; \int_\Gamma(\gamma_1 g_1\pm\gamma_2 g_2)(x-y)\,\phi(y)\,ds(y),\quad x\in\Omega,$$

$$K^{\pm}_\Omega\phi(x) \;=\; \nabla\,\wedge\,\int_\Gamma(g_1\pm g_2)(x-y)\,\phi(y)\,ds(y),\quad x\in\Omega,$$

and finally,

$$D^{\pm}_\Omega\phi(x) \;=\; (\nabla\wedge)^2\int_\Gamma(\gamma_2 g_1\pm\gamma_1 g_2)(x-y)\,\phi(y)\,ds(y),\quad x\in\Omega.$$

Similarly, we define the potential operators for the exterior domain $\mathbb{R}^3\setminus\overline{\Omega}$. These operators are denoted by $V^{\pm}_{\mathbb{R}^3\setminus\overline{\Omega}}, K^{\pm}_{\mathbb{R}^3\setminus\overline{\Omega}}$ and $D^{\pm}_{\mathbb{R}^3\setminus\overline{\Omega}}$, respectively. We will also need the traces of these operators on Γ. For $\phi\in(\mathcal{C}^{1,1}(\Gamma))^3$, let

$$V^{\pm}\phi(x) \;=\; \int_\Gamma(\gamma_1 g_1\pm\gamma_2 g_2)(x-y)\,\phi(y)\,ds(y),\quad x\in\Gamma,$$

and

$$K^{\pm}\phi(x) \;=\; v.p \int_{\Gamma} \nabla_x \,(g_1 \pm g_2)(x-y) \,\wedge\, \phi(y)\, ds(y), \quad x \in \Gamma.$$

By using the properties of layer potentials when the boundary is $\mathcal{C}^{1,1}$ proved by Nachman in [24], one can show that the following propositions hold. Since the proofs are straightforward, we omit them here.

Proposition 3.1 *Let $\phi \in (\mathcal{C}^{1,1}(\Gamma))^3$. Then $V_{\Omega}^{\pm}\phi$ and $V_{\mathbb{R}^3\setminus\overline{\Omega}}^{\pm}\phi$ can be extended continuously up to Γ, and for the boundary values we have*

$$V^{\pm}\phi(x) \;=\; V_{\Omega}^{\pm}\phi(x) \;=\; V_{\mathbb{R}^3\setminus\overline{\Omega}}^{\pm}\phi(x), \quad x \in \Gamma, \tag{18}$$

$$\begin{aligned} K^{+}\phi(x) \;&=\; K_{\Omega}^{+}\phi(x) + n(x) \,\wedge\, \phi(x), \quad x \in \Gamma, \\ &=\; K_{\mathbb{R}^3\setminus\overline{\Omega}}^{+}\phi(x) - n(x) \,\wedge\, \phi(x), \quad x \in \Gamma, \end{aligned} \tag{19}$$

$$\begin{aligned} K^{-}\phi(x) \;&=\; K_{\Omega}^{-}\phi(x), \quad x \in \Gamma, \\ &=\; K_{\mathbb{R}^3\setminus\overline{\Omega}}^{-}\phi(x), \quad x \in \Gamma, \end{aligned} \tag{20}$$

where $n(x)$ is the outward normal at $x \in \Gamma$.
Furthermore, if ϕ is tangential

$$\begin{aligned} D^{\pm}\phi(x) \;&=\; D_{\Omega}^{\pm}\phi(x) + (\gamma_2 \pm \gamma_1)\, n(x)\, \mathrm{div}_{\Gamma}\, \phi(x)/2, \quad x \in \Gamma, \\ &=\; D_{\mathbb{R}^3\setminus\overline{\Omega}}^{\pm}\phi(x) - (\gamma_2 \pm \gamma_1)\, n(x)\, \mathrm{div}_{\Gamma}\, \phi(x)/2, \quad x \in \Gamma, \end{aligned} \tag{21}$$

where the operator $D^{\pm}$ is defined by

$$D^{\pm}\phi(x) \;=\; \nabla V^{\pm}(\mathrm{div}_{\Gamma}\, \phi)(x) + \gamma^2 V^{\pm}\phi(x), \quad x \in \Gamma. \tag{22}$$

In the following proposition, we summarize the mapping properties of the above integral operators in Sobolev spaces. We refer to Nachman [24] for the proofs.

Proposition 3.2 *We have the continuous extensions*

$$\begin{aligned} V_{\Omega}^{\pm} \;&: (H^{1/2}(\Gamma))^3 \longmapsto (H^2(\Omega))^3 \\ K_{\Omega}^{\pm} \;&: (H^{1/2}(\Gamma))^3 \longmapsto (H^1(\Omega))^3 \\ D_{\Omega}^{\pm} \;&: (H^{1/2}(\Gamma))^3 \longmapsto (L^2(\Omega))^3 \\ D_{\Omega}^{\pm}, K_{\Omega}^{\pm} \;&: TH^{1/2}(\mathrm{div}, \Gamma) \longmapsto H^1(\mathrm{div}, \Omega) \end{aligned}$$

From proposition 3.2, it follows that the operators $n \wedge K^{\pm}$ and $n \wedge D^{\pm}$ extend to bounded operators $TH^{1/2}(\mathrm{div}, \Gamma) \longmapsto TH^{1/2}(\mathrm{div}, \Gamma)$. We have similar results for exterior potential operators

$$V_{\mathbb{R}^3\setminus\overline{\Omega}}^{\pm} \;: (H^{1/2}(\Gamma))^3 \longmapsto (\overline{H}_{loc}^{2}(\mathbb{R}^3 \setminus \overline{\Omega}))^3$$

$$\begin{aligned} K^{\pm}_{\mathbb{R}^3\setminus\overline{\Omega}} &: (H^{1/2}(\Gamma))^3 \longmapsto (\overline{H}^1_{loc}(\mathbb{R}^3 \setminus \overline{\Omega}))^3 \\ D^{\pm}_{\mathbb{R}^3\setminus\overline{\Omega}} &: (H^{1/2}(\Gamma))^3 \longmapsto (L^2_{loc}(\mathbb{R}^3 \setminus \overline{\Omega}))^3 \\ D^{\pm}_{\mathbb{R}^3\setminus\overline{\Omega}}, K^{\pm}_{\mathbb{R}^3\setminus\overline{\Omega}} &: TH^{1/2}(\mathrm{div}, \Gamma) \longmapsto \overline{H}^1_{loc}(\mathrm{div}, \mathbb{R}^3 \setminus \overline{\Omega}). \end{aligned}$$

4 Representation Theorem and Radiation Condition

This section is probably the most important section of this paper because essentially all what follows is either based on or motivated by the results we are now about to discuss. Our basic aim is to introduce a radiation condition for the Drude-Born-Fedorov equations. We first give a generalization of the Stratton-Chu representation theorem for Maxwell's equations in a bounded domain. To accomplish this, we calculate the commutator of $\mathcal{L}_0 - \tilde{\omega}$ and the convolution with the kernel G over Ω where $\tilde{\omega}$ is defined by (8).

Theorem 4.1 (**Representation Theorem**) *Let Ω be a bounded domain in $\mathbb{R}^3$ with $\mathcal{C}^{1,1}$ boundary Γ. Let u and v be in $(\mathcal{C}^{1,1}(\overline{\Omega}))^3$. Then we have the following representation formula :*

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix} &= \int_\Omega G(. - y)\, (\mathcal{L}_0 - \tilde{\omega}) \begin{pmatrix} u(y) \\ v(y) \end{pmatrix} dy \\ -\frac{1}{2} &\begin{pmatrix} K_\Omega & i\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\, D_\Omega \\ -i\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\, D_\Omega & K_\Omega \end{pmatrix} \begin{pmatrix} n \wedge u|_\Gamma \\ n \wedge v|_\Gamma \end{pmatrix}, \end{aligned}$$

where the integral operator K_Ω is given by

$$K_\Omega \;=\; K^+_\Omega + \frac{1}{\gamma^2}\, D^-_\Omega,$$

and the integral operator D_Ω is defined as

$$D_\Omega \;=\; K^-_\Omega + \frac{1}{\gamma^2}\, D^+_\Omega.$$

Proof. The proof is a straightforward integration by parts argument. By $w(x, y)$ we denote the first component of $G(x - y)\, (\mathcal{L}_0 - \tilde{\omega}) \begin{pmatrix} u(y) \\ v(y) \end{pmatrix}$. If we denote by $w^{(1)}$ the sum of the terms where u appears and $w^{(2)}$ the sum of the terms where v appears, it is clear that

$$w(x, y) \;=\; w^{(1)}(x, y) + w^{(2)}(x, y).$$

Straightforward computations show that

$$\begin{aligned} 2\,w^{(1)}(x,y) &= 2\,\delta(x-y)\,u(y) + \nabla_y \wedge (\gamma_1 g_1 - \gamma_2 g_2)\,u \\ &\quad + \nabla_x \wedge \nabla_y \wedge (g_1+g_2)\,u + \frac{1}{\gamma^2}\nabla_x\,\nabla_x\,.\,(\gamma_2 g_1 - \gamma_2 g_1)\,\nabla_y \wedge u. \end{aligned}$$

Integrating by parts, we obtain that

$$\begin{aligned} 2\int_\Omega w^{(1)}(x,y)\,dy &= 2\,u(x) + \nabla \wedge \int_\Gamma (g_1+g_2)(x-y)\; n \wedge u(y)\,ds(y) \\ &\quad + \frac{1}{\gamma^2}\,(\nabla\wedge)^2 \int_\Gamma (\gamma_2 g_1 - \gamma_1 g_2)\,\nabla_y \wedge u(y)\,ds(y). \end{aligned}$$

For $w^{(2)}$, it follows that

$$-2i\sqrt{\frac{\varepsilon_0}{\mu_0}}\int_\Omega w^{(2)}(x,y)\,dy = \nabla \wedge \int_\Gamma (g_1-g_2)(x-y)\; n \wedge v(y)\,ds(y)$$
$$+\frac{1}{\gamma^2}\,(\nabla\wedge)^2 \int_\Gamma (\gamma_2 g_1 + \gamma_1 g_2)(x-y)\; n \wedge v(y)\,ds(y).$$

A similar computation for the second component yields the claim. □

Now, we are in position to derive a radiation condition for the Drude-Born-Fedorov equations. We prove the following theorem.

Theorem 4.2 (**Radiation condition**) *If E^{sc} and H^{sc} are solutions of* (5) *such that the Silver-Müller radiation condition :*

$$|E^{sc}(x) - \sqrt{\frac{\mu_0}{\varepsilon_0}}\,H^{sc}(x) \wedge n(x)\,| \le \frac{C}{|x|^2}, \quad \text{for } |x| \to +\infty, \tag{23}$$

holds, where $n(x) = \dfrac{x}{|x|}$. *Then, we have*

$$\lim_{R\to+\infty} \begin{pmatrix} K_{B_R} & i\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\,D_{B_R} \\ -i\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\,D_{B_R} & K_{B_R} \end{pmatrix} \begin{pmatrix} n \wedge E^{sc}|_{\partial B_R} \\ n \wedge H^{sc}|_{\partial B_R} \end{pmatrix} = 0, \tag{24}$$

in the sense that the left-hand side together with all its derivatives goes uniformly to zero on compact sets as $R \to +\infty$.

Proof. The first component of

$$\begin{pmatrix} K_{B_R} & i\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\,D_{B_R} \\ -i\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\,D_{B_R} & K_{B_R} \end{pmatrix} \begin{pmatrix} n \wedge E^{sc}|_{\partial B_R} \\ n \wedge H^{sc}|_{\partial B_R} \end{pmatrix},$$

can be rewritten as

$$\sum_{j=1,2} \nabla \wedge \int_{\partial B_R} g_j(x-y)\, n \wedge \mathcal{E}_j \, ds(y) + \frac{1}{\omega_j \sqrt{\varepsilon_0 \mu_0}} (\nabla \wedge)^2 \int_{\partial B_R} g_j(x-y)\, n \wedge \mathcal{E}_j \, ds(y), \tag{25}$$

where the fields $\mathcal{E}_j$ are defined for $j = 1, 2$ by

$$\mathcal{E}_j \;=\; E^{sc} + (-1)^{j+1}\, i \sqrt{\frac{\mu_0}{\varepsilon_0}}\, H^{sc}. \tag{26}$$

Let $\mathcal{H}_j$ be defined by the following relation

$$\nabla \,\wedge\, \mathcal{E}_j \;=\; i\omega_j \,\mu_0 \,\mathcal{H}_j. \tag{27}$$

From (5), it follows that $(\mathcal{E}_j, \mathcal{H}_j)$ satisfy the Maxwell system

$$\begin{cases} \nabla \,\wedge\, \mathcal{E}_j \;=\; i\omega_j \,\mu_0 \,\mathcal{H}_j, \\ \nabla \,\wedge\, \mathcal{H}_j \;=\; -i\omega_j \,\varepsilon_0 \,\mathcal{E}_j. \end{cases} \tag{28}$$

It is obvious that

$$E^{sc} \;=\; \frac{1}{2}\Big(\mathcal{E}_1 + \mathcal{E}_2\Big). \tag{29}$$

One of the basic tools in studying the uniqueness of a solution to the scattering problem for the Drude-Born-Fedorov equations is provided by the decomposition (26). We wish to point out that if $(\mathcal{E}_j, \mathcal{H}_j)$ satisfy the Maxwell equations (28), (E^{sc}, H^{sc}) satisfy the homogeneous Drude-Born-Fedorov equations. The converse is also true.
Now, a straightforward computation gives

$$\begin{aligned} \mathcal{E}_j - \sqrt{\frac{\mu_0}{\varepsilon_0}}\, \mathcal{H}_j \,\wedge\, n \;\; &= \;\; \Big(E^{sc} - \sqrt{\frac{\mu_0}{\varepsilon_0}}\, H^{sc} \,\wedge\, n\Big) \\ &\quad + (-1)^{j+1} i \sqrt{\frac{\mu_0}{\varepsilon_0}}\, \Big(H^{sc} + \sqrt{\frac{\varepsilon_0}{\mu_0}}\, E^{sc} \,\wedge\, n\Big). \end{aligned} \tag{30}$$

Since (E^{sc}, H^{sc}) satisfy (23), we obtain that $(\mathcal{E}_j, \mathcal{H}_j)$ satisfy the Silver-Müller radiation condition

$$|\mathcal{E}_j(x) - \sqrt{\frac{\mu_0}{\varepsilon_0}}\, \mathcal{H}_j(x) \,\wedge\, n(x)\,| \;\leq\; \frac{C}{|x|^2}, \quad \text{for } |x| \to +\infty.$$

From [10], it follows that

$$\nabla \wedge \int_{\partial B_R} g_j(x-y)\, n \wedge \mathcal{E}_j \, ds(y) + \frac{1}{\omega_j \sqrt{\varepsilon_0 \mu_0}} (\nabla \wedge)^2 \int_{\partial B_R} g_j(x-y)\, n \wedge \mathcal{E}_j \, ds(y) \longrightarrow 0,$$

where R goes to infinity and then, the first component (25) goes to 0. The second component is treated similarly. $\square$

Now, Let $R > 0$ be such that $\overline{\Omega} \subset B_R = \Big\{x \text{ such that } |x| < R\Big\}$. Assume E and H satisfy the Drude-Born-Fedorov equations in the isotropic medium $\mathbb{R}^3 \setminus \overline{\Omega}$. It follows that

$$\Big(\mathcal{L}_0 - \tilde{\omega}\Big)\begin{pmatrix} E^{sc} \\ H^{sc} \end{pmatrix} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega},$$

where $E^{sc} = E - E^{in}$ and $H^{sc} = H - H^{in}$. Then, by theorem 4.1, we obtain

$$\begin{aligned}\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} E^{in} \\ H^{in} \end{pmatrix} &- \frac{1}{2}\begin{pmatrix} K_{\mathbb{R}^3\setminus\overline{\Omega}} & i\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\, D_{\mathbb{R}^3\setminus\overline{\Omega}} \\ -i\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\, D_{\mathbb{R}^3\setminus\overline{\Omega}} & K_{\mathbb{R}^3\setminus\overline{\Omega}} \end{pmatrix}\begin{pmatrix} n \wedge E^{in}|_\Gamma \\ n \wedge H^{in}|_\Gamma \end{pmatrix} \\ &+\frac{1}{2}\begin{pmatrix} K_{\mathbb{R}^3\setminus\overline{\Omega}} & i\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\, D_{\mathbb{R}^3\setminus\overline{\Omega}} \\ -i\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\, D_{\mathbb{R}^3\setminus\overline{\Omega}} & K_{\mathbb{R}^3\setminus\overline{\Omega}} \end{pmatrix}\begin{pmatrix} n \wedge E|_\Gamma \\ n \wedge H|_\Gamma \end{pmatrix} \\ &-\frac{1}{2}\begin{pmatrix} K_{B_R} & i\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\, D_{B_R} \\ -i\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\, D_{B_R} & K_{B_R} \end{pmatrix}\begin{pmatrix} n \wedge E^{sc}|_{\partial B_R} \\ n \wedge H^{sc}|_{\partial B_R} \end{pmatrix}.\end{aligned} \tag{31}$$

In the exterior domain, (31) gives a representation formula in terms of the tangential boundary values on Γ and fundamental outgoing solutions g_1 and g_2. Let us first verify that

$$\begin{pmatrix} K_{\mathbb{R}^3\setminus\overline{\Omega}} & i\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\, D_{\mathbb{R}^3\setminus\overline{\Omega}} \\ -i\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\, D_{\mathbb{R}^3\setminus\overline{\Omega}} & K_{\mathbb{R}^3\setminus\overline{\Omega}} \end{pmatrix}\begin{pmatrix} n \wedge E^{in}|_\Gamma \\ n \wedge H^{in}|_\Gamma \end{pmatrix} = 0.$$

Since E^{in} and H^{in} satisfy the isotropic Drude-Born-Fedorov equations in the whole space, then applying the representation theorem 4.1 for them in B_R and in $B_R \setminus \overline{\Omega}$, respectively, gives

$$\begin{pmatrix} E^{in} \\ H^{in} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} K_{B_R} & i\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\, D_{B_R} \\ -i\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\, D_{B_R} & K_{B_R} \end{pmatrix}\begin{pmatrix} n \wedge E^{in}|_{\partial B_R} \\ n \wedge H^{in}|_{\partial B_R} \end{pmatrix},$$

for $x \in B_R$ and

$$\begin{pmatrix} E^{in} \\ H^{in} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} K_{B_R} & i\sqrt{\frac{\mu_0}{\varepsilon_0}}\, D_{B_R} \\ -i\sqrt{\frac{\varepsilon_0}{\mu_0}}\, D_{B_R} & K_{B_R} \end{pmatrix} \begin{pmatrix} n \wedge E^{in}|_{\partial B_R} \\ n \wedge H^{in}|_{\partial B_R} \end{pmatrix}$$

$$+\frac{1}{2} \begin{pmatrix} K_{\mathbb{R}^3\setminus\overline{\Omega}} & i\sqrt{\frac{\mu_0}{\varepsilon_0}}\, D_{\mathbb{R}^3\setminus\overline{\Omega}} \\ -i\sqrt{\frac{\varepsilon_0}{\mu_0}}\, D_{\mathbb{R}^3\setminus\overline{\Omega}} & K_{\mathbb{R}^3\setminus\overline{\Omega}} \end{pmatrix} \begin{pmatrix} n \wedge E^{in}|_{\partial B_R} \\ n \wedge H^{in}|_{\partial B_R} \end{pmatrix},$$

for $x \in B_R \setminus \overline{\Omega}$, where

$$K_{\mathbb{R}^3\setminus\overline{\Omega}} = K^{+}_{\mathbb{R}^3\setminus\overline{\Omega}} + \frac{1}{\gamma^2} D^{-}_{\mathbb{R}^3\setminus\overline{\Omega}},$$

and

$$D_{\mathbb{R}^3\setminus\overline{\Omega}} = K^{-}_{\mathbb{R}^3\setminus\overline{\Omega}} + \frac{1}{\gamma^2} D^{+}_{\mathbb{R}^3\setminus\overline{\Omega}}.$$

From subtracting the two representations above at a point $x \in B_R \setminus \overline{\Omega}$, we obtain the claim. We have then the following representation formula :

$$\begin{pmatrix} E(x) \\ H(x) \end{pmatrix} = \begin{pmatrix} E^{in}(x) \\ H^{in}(x) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} K_{\mathbb{R}^3\setminus\overline{\Omega}} & i\sqrt{\frac{\mu_0}{\varepsilon_0}}\, D_{\mathbb{R}^3\setminus\overline{\Omega}} \\ -i\sqrt{\frac{\varepsilon_0}{\mu_0}}\, D_{\mathbb{R}^3\setminus\overline{\Omega}} & K_{\mathbb{R}^3\setminus\overline{\Omega}} \end{pmatrix} \begin{pmatrix} n \wedge E|_{\Gamma} \\ n \wedge H|_{\Gamma} \end{pmatrix},$$

for $x \in \mathbb{R}^3 \setminus \overline{\Omega}$. We are now in position to prove the following result which plays a central role in this article.

Theorem 4.3 *Let u and v be in $(\mathcal{C}^{1,1}(\Gamma))^3$ and define E and H by*

$$\begin{pmatrix} E - E^{in} \\ H - H^{in} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} K_{\mathbb{R}^3\setminus\overline{\Omega}} & i\sqrt{\frac{\mu_0}{\varepsilon_0}}\, D_{\mathbb{R}^3\setminus\overline{\Omega}} \\ -i\sqrt{\frac{\varepsilon_0}{\mu_0}}\, D_{\mathbb{R}^3\setminus\overline{\Omega}} & K_{\mathbb{R}^3\setminus\overline{\Omega}} \end{pmatrix} \begin{pmatrix} n \wedge u \\ n \wedge v \end{pmatrix}. \quad (32)$$

Then E and H satisfy the Drude-Born-Fedorov equations in the exterior domain $\mathbb{R}^3 \setminus \overline{\Omega}$ and the Silver-Müller radiation condition (23) where $E^{sc} = E - E^{in}$ and $H^{sc} = H - H^{in}$. Conversely, if E and H satisfy the Drude-Born-Fedorov equations in $\mathbb{R}^3 \setminus \overline{\Omega}$ and (24), then E and H have the representation (32) with $u = E$ and $v = H$.

Proof. The proof of the converse was already given above. Let $(\mathcal{E}_j, \mathcal{H}_j)$ be defined by (26) and (27). From (32), we have

$$\begin{pmatrix} \mathcal{E}_j(x) \\ \mathcal{H}_j(x) \end{pmatrix} = \begin{pmatrix} \mathcal{K}_j & i\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\mathcal{D}_j \\ -i\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\mathcal{D}_j & \mathcal{K}_j \end{pmatrix} \begin{pmatrix} n \wedge e_j \\ n \wedge h_j \end{pmatrix}.$$

where the integral operators $\mathcal{K}_j$ and $\mathcal{D}_j$ are given by

$$\begin{cases} \mathcal{K}_j\,\phi(x) \;=\; \nabla \wedge \displaystyle\int_\Gamma g_j(x-y)\,\phi(y)\,ds(y), \\ \mathcal{D}_j\,\phi(x) \;=\; \dfrac{1}{\gamma_j}\,(\nabla\wedge)^2 \displaystyle\int_\Gamma g_j(x-y)\,\phi(y)\,ds(y), \end{cases}$$

and

$$\begin{cases} e_j \;=\; u + (-1)^{j+1} i\sqrt{\dfrac{\mu_0}{\varepsilon_0}}v, \\ h_j \;=\; (-1)^j i\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\Big(u + (-1)^{j+1}\sqrt{\dfrac{\mu_0}{\varepsilon_0}}v\Big). \end{cases}$$

From [10], it follows that $(\mathcal{E}_j, \mathcal{H}_j)$ satisfy the Maxwell equations (28) and the Silver-Müller conditions

$$\begin{cases} |\mathcal{E}_j - \sqrt{\dfrac{\mu_0}{\varepsilon_0}}\mathcal{H}_j \wedge n| \;\le\; \dfrac{C}{|x|^2}, \quad |x| \longrightarrow +\infty, \\ |\mathcal{H}_j + \sqrt{\dfrac{\varepsilon_0}{\mu_0}}\mathcal{E}_j \wedge n| \;\le\; \dfrac{C}{|x|^2}, \quad |x| \longrightarrow +\infty, \end{cases}$$

and by using (30) we obtain the claim. □

Finally, we shall show how to derive the Lippman-Schwinger equation (17). Since $\tilde{\varepsilon}, \tilde{\tilde{\varepsilon}}, \tilde{\mu}, \tilde{\tilde{\mu}}$ are supported in $\overline{\Omega}$, then by applying the representation theorem 4.1 in B_R and using the weak radiation condition (24), (17) is obvious.
To investigate the unique solvability of the integral equation (32), we proceed in the following way. First, we express on Γ, $n \wedge H$ in terms of $n \wedge E$ by making use of the electromagnetic operator. To define this operator, we have to study the interior Drude-Born-Fedorov equations. Then, we write a single integral equation on Γ satisfied by $n \wedge E$ which is of Fredholm type.

5 The Existence and Uniqueness of Solutions to the Drude-Born-Fedorov Equations

We shall show in this section that we can reduce the Drude-Born-Fedorov equations to a single integral equation on Γ for $n \wedge E$. The integral equation is

established by making use of the interior electromagnetic operator mapping the tangential component of the electric field onto the tangential component of the magnetic field. Since the interior problem for the Drude-Born-Fedorov equations is in general not uniquely solvable, the interior electromagnetic operator can not be defined. Throughout what follows, we assume that we can take Ω' containing Ω such that the interior Drude-Born-Fedorov equations are uniquely solvable in Ω'. Assuming this, we show the equivalence between the integral equation and the scattering problem and the existence of a solution to the scattering problem for the Drude-Born-Fedorov equations will follow from uniqueness by Fredholm's alternative.
Let us first consider the interior problem :

$$\left\{\begin{array}{rcll} \nabla \wedge E &=& (\gamma(x))^2\,\beta(x)E + i\omega\mu(x)(\dfrac{\gamma(x)}{k(x)})^2\,H & \text{in } \Omega, \\ \nabla \wedge H &=& (\gamma(x))^2\,\beta(x)\,H - i\omega\varepsilon(x)(\dfrac{\gamma(x)}{k(x)})^2\,E & \text{in } \Omega, \end{array}\right. \tag{33}$$

with the boundary condition

$$n \wedge E = u \in TH^{1/2}(\mathrm{div}, \Gamma). \tag{34}$$

We have the following lemma.

Lemma 5.1 *Assume that* (33) *with the boundary condition* $E \wedge n = 0$ *on* Γ *has only the trivial solution. Then the boundary value problem* $(33) - (34)$ *has a unique solution in* $(H^1(\Omega))^3 \times (H^1(\Omega))^3$.

Proof. Existence and uniqueness for this problem follow the arguments given by the authors in [1]. Here, for the sake of brevity, we give only an outline of the basic steps of our proof which is valid in the general case $\varepsilon = \varepsilon(x), \mu = \mu(x)$ and $\beta = \beta(x)$. Let $\tilde{u} \in (H^1(\Omega))^3$ be such that $n \wedge \tilde{u} = u$ on Γ.
Following [1], we try to find the electric field in the form

$$E = \tilde{u} + e + \nabla\,\varphi,$$

where $e \in H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega)$ and $\varphi \in H^1_0(\Omega)$.
The interior Drude-Born-Fedorov equations (33) lead to a saddle point problem involving the form a defined by

$$\begin{array}{rcl} a(e, w) &=& \displaystyle\int_\Omega \frac{1}{\mu}\,(1 - \omega^2\varepsilon\mu\beta^2)\,(\nabla \wedge e)\,.\,(\nabla \wedge \overline{w}) - \omega^2 \int_\Omega \nabla \wedge (\varepsilon\beta e)\,.\,\overline{w} \\ && \displaystyle -\omega^2 \int_\Omega \varepsilon\,\beta\,(\nabla \wedge e)\,.\,\overline{w} - \omega^2 \int_\Omega \varepsilon\,e\,.\,\overline{w}, \text{ for all } w \in H_0(\mathrm{curl}, \Omega), \end{array}$$

and the form b given as

$$b(\varphi, w) = -\omega^2 \int_\Omega \nabla \wedge (\varepsilon\,\beta\,\nabla\,\varphi)\,.\,\overline{w} - \omega^2 \int_\Omega \varepsilon\,\nabla\,\varphi\,.\,\overline{w}, \text{ for all } \varphi \in H^1_0(\Omega).$$

The boundary value problem $(33) - (34)$ is equivalent to

$$\left\{ \begin{array}{l} a(e,w) + b(\varphi, w) \ = \ a(\tilde{u}, w), \quad \text{for all } w \in H_0(\text{curl}, \Omega), \\ b(\psi, e) \ = \ 0, \quad \text{for all } \psi \in H^1_0(\Omega). \end{array} \right.$$

It is clear that there exist two strictly positive constants C_1 and C_2 such that :

$$|a(e,e)| \ \geq \ C_1 \, ||\nabla \ \wedge \ e||^2_{(L^2(\Omega))^3} - C_2 \, ||e||^2_{(L^2(\Omega))^3}. \tag{35}$$

Now, for each ψ in $H^1_0(\Omega)$, it is clear that $\nabla \psi$ satisfies $\nabla \psi \ \wedge \ n \ = \ 0$ on Γ and there exists a strictly positive constant C_3 such that

$$|b(\psi, \nabla \psi)| \ \geq \ C_3 \ \ ||\psi||^2_{H^1_0(\Omega)}.$$

The form b satisfies then the inf-sup condition. On the other hand, the embedding $N \hookrightarrow (L^2(\Omega))^3$ is compact, where

$$N \ = \Big\{ v \in H_0(\text{curl}, \Omega) \text{ such that } \text{div} \, \varepsilon \, v \ = \ \nabla(\varepsilon \, \beta) \, . \, (\nabla \ \wedge \ v) \in L^2(\Omega) \ \Big\}$$

is the kernel of the form b. This important compactness result has been proved by Leis [21] for smooth domains, by Weck [33] for cone like domains and by Weber [32] and Picard [29] for domains with strict cone property. By combining this result with the inequality (35), we can deduce that a is a compact perturbation of a coercive form on the kernel N of the form b. It follows that Fredholm's alternative holds and then, uniqueness implies existence of a solution to (33). The proof of lemma 5.1 is then over. □

Lemma 5.2 *Assume that the domain Ω has a $\mathcal{C}^{1,1}-$boundary and that ε, μ and β are in $\mathcal{C}^2(\mathbb{R}^3)$ and they have the following properties : there exist constants $\varepsilon_m, \mu_m > 0$, and ε_M, μ_M such that $\varepsilon_m \leq \varepsilon(x) \leq \varepsilon_M, \mu_m \leq \mu(x) \leq \mu_M$ when $x \in \overline{\Omega}$; in $\mathbb{R}^3 \setminus \Omega$,*
$\mu(x) = \mu_0, \varepsilon(x) = \varepsilon_0$ and $\beta(x) = \beta_0$, for some known constants ε_0, μ_0, and β_0. Then the interior electromagnetic operator

$$\Lambda \ : u \longmapsto n \ \wedge \ H \,|_\Gamma$$

is a well defined isomorphism

$$\Lambda \ : TH^{1/2}(\text{div}, \Gamma) \longmapsto TH^{1/2}(\text{div}, \Gamma).$$

Proof. Applying the trace theorem, we show that

$$n \ \wedge \ H \,|_\Gamma \in TH^{1/2}(\Gamma),$$

and furthermore

$$\begin{aligned}
\mathrm{div}_\Gamma\,(n \wedge H) &= -n\,.\,(\nabla \wedge H)\,|_\Gamma \\
&= -n\,.\,(\gamma^2 \beta H - i\omega\varepsilon(\frac{\gamma}{k})^2 E)\,|_\Gamma \\
&= -\gamma^2 \beta_0\, H\,.\,n\,|_\Gamma + i\tilde{\omega}\varepsilon_0 E\,.\,n\,|_\Gamma \\
&\in H^{1/2}(\Gamma).
\end{aligned}$$

The invertibility of Λ follows from the unique solvability of the equations (33) with the tangential boundary data. □

Now, we show the equivalence of the exterior problem

$$\left\{\begin{array}{lcl}
\nabla \wedge E &=& \gamma^2\,\beta_0 E + i\tilde{\omega}\mu_0\,H \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\
\nabla \wedge H &=& \gamma^2\,\beta_0\,H(x) - i\tilde{\omega}\,\varepsilon_0\,E \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\
n \wedge H &=& \Lambda(n \wedge E) \quad \text{on } \Gamma, \\
\multicolumn{3}{l}{(E^{sc}, H^{sc}) \text{ satisfy the weak radiation condition (24),}}
\end{array}\right. \tag{36}$$

and a single boundary integral equation for $n \wedge E$. Let us define the integral operators

$$K = K^+ + \frac{1}{\gamma^2} D^-, \tag{37}$$

and

$$D = K^- + \frac{1}{\gamma^2} D^+. \tag{38}$$

We have the following lemma.

Lemma 5.3 *Assume that* $(E,H) \in \overline{H}^1_{loc}(\mathrm{div}, \mathbb{R}^3 \setminus \overline{\Omega})$ *solves* (36) *then*

$$u = n \wedge E\,|_\Gamma \in TH^{1/2}(\mathrm{div}, \Gamma)$$

satisfies the integral equation

$$\frac{1}{2} u \,=\, n \wedge E^{in} + \frac{1}{2} n \wedge K\,u + \frac{1}{2} i \sqrt{\frac{\mu_0}{\varepsilon_0}}\, n \wedge D\,\Lambda\,u. \tag{39}$$

Conversely, assume $u \in TH^{1/2}(\mathrm{div}, \Gamma)$ *solves the integral equation* (39). *Then if the interior problem*

$$\left\{\begin{array}{lcl}
\nabla \wedge E &=& (\gamma(x))^2\,\beta(x) E + i\omega\mu(x)(\dfrac{\gamma(x)}{k(x)})^2\,H \quad \text{in } \Omega, \\
\nabla \wedge H &=& (\gamma(x))^2\,\beta(x)\,H(x) - i\omega\varepsilon(x)(\dfrac{\gamma(x)}{k(x)})^2\,E \quad \text{in } \Omega, \\
n \wedge E &=& 0 \quad \text{on } \Gamma,
\end{array}\right. \tag{40}$$

has only the trivial solution in $H^1(\mathrm{div},\Omega) \times H^1(\mathrm{div},\Omega)$*, the fields*

$$\begin{cases} E &= E^{in} + \frac{1}{2} K_{\mathbb{R}^3\setminus\overline{\Omega}}\, u + \frac{1}{2} i \sqrt{\frac{\mu_0}{\varepsilon_0}}\, D_{\mathbb{R}^3\setminus\overline{\Omega}}\, \Lambda\, u, \\ H &= H^{in} - \frac{1}{2} i \sqrt{\frac{\varepsilon_0}{\mu_0}}\, D_{\mathbb{R}^3\setminus\overline{\Omega}}\, u + \frac{1}{2} K_{\mathbb{R}^3\setminus\overline{\Omega}}\, \Lambda\, u, \end{cases}$$

satisfy the Drude-Born-Fedorov equations (36)*.*

Proof. The integral equation (39) follows from Theorem 4.3 and Proposition 3.1. Conversely, assume that $u \in TH^{1/2}(\mathrm{div},\Gamma)$ is a solution of the integral equation (39). From the mapping properties of $D_{\mathbb{R}^3\setminus\overline{\Omega}}$ and $K_{\mathbb{R}^3\setminus\overline{\Omega}}$ (*cf.* Proposition 3.2), both E and H are in $\overline{H}^1_{loc}(\mathrm{div},\mathbb{R}^3 \setminus \overline{\Omega})$. From Theorem 4.3, we can deduce that (E,H) satisfy the isotropic Drude-Born-Fedorov equations and the Silver-Müller radiation condition. We only have to show that the boundary condition

$$n \wedge H = \Lambda(n \wedge E) \quad \text{on } \Gamma,$$

holds. To accomplish this, let us define (E_0, H_0) by

$$\begin{cases} E_0 &= E^{in} + \frac{1}{2} K_{\Omega}\, u + \frac{1}{2} i \sqrt{\frac{\mu_0}{\varepsilon_0}}\, D_{\Omega}\, \Lambda\, u, \text{ in } \Omega, \\ H_0 &= H^{in} - \frac{1}{2} i \sqrt{\frac{\varepsilon_0}{\mu_0}}\, D_{\Omega}\, u + \frac{1}{2} K_{\Omega}\, \Lambda\, u, \text{ in } \Omega. \end{cases}$$

Now, we use the identities (19) and (21) to find

$$n \wedge E_0 = n \wedge E^{in} + \frac{1}{2} n \wedge Ku + \frac{1}{2} i \sqrt{\frac{\mu_0}{\varepsilon_0}}\, n \wedge D\,\Lambda u - \frac{1}{2} u = 0 \quad \text{on } \Gamma.$$

It follows that $E_0 = H_0 = 0$ in Ω. From this we can easily obtain by the jump relations that

$$n \wedge H - n \wedge H_0 = \Lambda u = \Lambda(n \wedge E), \quad \text{on } \Gamma.$$

The proof of the lemma is then complete. □

Lemma 5.4 *Assume that the interior problem* (40) *has only the trivial solution in* $H^1(\mathrm{div},\Omega) \times H^1(\mathrm{div},\Omega)$*. Then the integral operator*

$$n \wedge K + Id + i \sqrt{\frac{\mu_0}{\varepsilon_0}}\, n \wedge D\,\Lambda$$

is compact in $TH^{1/2}(\mathrm{div},\Gamma)$*.*

Proof. Let u be in $TH^{1/2}(\mathrm{div}, \Gamma)$ and assume that E and H satisfy (33) with the boundary condition (34). Then E and H are in $(H^1(\Omega))^3$. From

$$\begin{cases} \nabla \,.\, (\gamma^2 \beta E) + i\omega \nabla \,.\, (\mu \dfrac{\gamma^2}{k^2} H) \;=\; 0, \\ \nabla \,.\, (\gamma^2 \beta H) - i\omega \nabla \,.\, (\mu \dfrac{\gamma^2}{k^2} E) \;=\; 0, \end{cases}$$

it follows that $\nabla \,.\, E$ and $\nabla \,.\, H$ are in $H^1(\Omega)$, $\nabla \,.\, E\,|_\Gamma \;=\; 0$ and $\nabla \,.\, H\,|_\Gamma \;=\; 0$. E and H are then in $H^1(\mathrm{curl}, \Omega) \cap H_0^1(\mathrm{div}, \Omega)$, where

$$H^1(\mathrm{curl}, \Omega) \;= \Big\{ u \in (H^1(\Omega))^3 \text{ such that } \nabla \wedge u \in (H^1(\Omega))^3 \Big\},$$

and

$$H_0^1(\mathrm{div}, \Omega) \;= \Big\{ u \in (H^1(\Omega))^3 \text{ such that } \nabla \,.\, u\,|_\Gamma \;=\; 0 \Big\}.$$

Now, by applying theorem 4.1, we obtain that

$$\begin{aligned} \begin{pmatrix} E \\ H \end{pmatrix} \;&=\; \int_\Omega G(x-y)\, (\mathcal{L}_0 - \tilde{\omega}) \begin{pmatrix} E(y) \\ H(y) \end{pmatrix} dy \\ &\quad -\frac{1}{2} \begin{pmatrix} K_\Omega & i\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\, D_\Omega \\ -i\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\, D_\Omega & K_\Omega \end{pmatrix} \begin{pmatrix} u \\ \Lambda\, u \end{pmatrix} (x), \end{aligned} \tag{41}$$

for $x \in \Omega$. From

$$\Big(\mathcal{L}_0 - \tilde{\omega}\Big) \begin{pmatrix} E(y) \\ H(y) \end{pmatrix} = \omega \begin{pmatrix} \tilde{\varepsilon}(y) & i\omega\tilde{\tilde{\mu}}(y) \\ -i\omega\tilde{\tilde{\varepsilon}}(y) & \tilde{\mu}(y) \end{pmatrix} \begin{pmatrix} E(y) \\ H(y) \end{pmatrix},$$

we have

$$\begin{aligned} \begin{pmatrix} E(x) \\ H(x) \end{pmatrix} \;&=\; \omega \int_\Omega G(x-y) \begin{pmatrix} \tilde{\varepsilon}(y) & i\omega\tilde{\tilde{\mu}}(y) \\ -i\omega\tilde{\tilde{\varepsilon}}(y) & \tilde{\mu}(y) \end{pmatrix} \begin{pmatrix} E(y) \\ H(y) \end{pmatrix} dy \\ &\quad -\frac{1}{2} \begin{pmatrix} K_\Omega & i\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\, D_\Omega \\ -i\sqrt{\dfrac{\varepsilon_0}{\mu_0}}\, D_\Omega & K_\Omega \end{pmatrix} \begin{pmatrix} u \\ \Lambda\, u \end{pmatrix} (x), \quad x \in \Omega. \end{aligned}$$

Thus, it suffices to prove that the quantity

$$\int_\Omega G(x-y) \begin{pmatrix} \tilde{\varepsilon}(y) & i\omega\tilde{\tilde{\mu}}(y) \\ -i\omega\tilde{\tilde{\varepsilon}}(y) & \tilde{\mu}(y) \end{pmatrix} \begin{pmatrix} E(y) \\ H(y) \end{pmatrix} dy$$

is in $(H^2(\Omega))^3$. This result will be deduced from the fact that for

$$u = \Big(-i\omega\tilde{\tilde{\varepsilon}}(y)E(y) + \tilde{\mu}(y)H(y)\Big)$$

or $\Big(\tilde{\varepsilon}(y)E(y) + i\omega\tilde{\tilde{\mu}}(y)H(y)\Big)$, we have

$$\nabla . u \in H^1(\Omega), \nabla \wedge u \in (H^1(\Omega))^3 \text{ and } u \wedge n = 0 \text{ on } \Gamma.$$

To see this, let $v^{(1)}(x,y)$ be the first component of

$$G(x-y) \begin{pmatrix} \tilde{\varepsilon}(y) & i\omega\tilde{\tilde{\mu}}(y) \\ -i\omega\tilde{\tilde{\varepsilon}}(y) & \tilde{\mu}(y) \end{pmatrix} \begin{pmatrix} E(y) \\ H(y) \end{pmatrix}$$

and $v^{(1)}(x)$ defined in $\mathbb{R}^3$ by

$$v^{(1)}(x) \; = \; \int_\Omega v^{(1)}(x,y)\, dy, \quad x \in \mathbb{R}^3.$$

It is easy to see that

$$\begin{aligned}
v^{(1)}(x) \; &= \; \frac{\gamma_1^2}{2\omega_1}\,(1 + \frac{\nabla_x \nabla_x \cdot}{\gamma_1^2}) \int_\Omega g_1(x-y) \Big(\tilde{\varepsilon}(y)E(y) + i\omega\tilde{\tilde{\mu}}(y)H(y)\Big)\, dy \\
&+ \frac{1}{\gamma_1}\nabla_x \wedge \int_\Omega g_1(x-y) \Big(\tilde{\varepsilon}(y)E(y) + i\omega\tilde{\tilde{\mu}}(y)H(y)\Big)\, dy \\
&+ i\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{\gamma_1^2}{\omega_1}(1 + \frac{\nabla_x \nabla_x \cdot}{\gamma_1^2}) \int_\Omega g_1(x-y) \Big(-i\omega\tilde{\tilde{\varepsilon}}(y)E(y) + \tilde{\mu}(y)H(y)\Big)\, dy \\
&+ \frac{1}{\gamma_1}\nabla_x \wedge \int_\Omega g_1(x-y) \Big(-i\omega\tilde{\tilde{\varepsilon}}(y)E(y) + \tilde{\mu}(y)H(y)\Big)\, dy \\
&+ \frac{\gamma_2^2}{2\omega_2}\,(1 + \frac{\nabla_x \nabla_x \cdot}{\gamma_2^2}) \int_\Omega g_2(x-y) \Big(\tilde{\varepsilon}(y)E(y) + i\omega\tilde{\tilde{\mu}}(y)H(y)\Big)\, dy \\
&- \frac{1}{\gamma_2}\nabla_x \wedge \int_\Omega g_2(x-y) \Big(\tilde{\varepsilon}(y)E(y) + i\omega\tilde{\tilde{\mu}}(y)H(y)\Big)\, dy \\
&- i\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{\gamma_2^2}{\omega_2}(1 + \frac{\nabla_x \nabla_x \cdot}{\gamma_2^2}) \int_\Omega g_2(x-y) \Big(-i\omega\tilde{\tilde{\varepsilon}}(y)E(y) + \tilde{\mu}(y)H(y)\Big)\, dy \\
&- \frac{1}{\gamma_2}\nabla_x \wedge \int_\Omega g_2(x-y) \Big(-i\omega\tilde{\tilde{\varepsilon}}(y)E(y) + \tilde{\mu}(y)H(y)\Big)\, dy.
\end{aligned}$$

Using the expression above for $v^{(1)}(x)$, we have

$$\omega\,\nabla . v^{(1)} \; = \; -\nabla . (\tilde{\varepsilon}E + i\omega\tilde{\tilde{\mu}}H) + i\omega\beta_0\mu_0\nabla . (-i\omega\tilde{\tilde{\varepsilon}}E + \tilde{\mu}H),$$

and hence, from the fact that $\nabla \,.\, E$ and $\nabla \,.\, H$ are in $H^1(\Omega)$, we can deduce that

$$\nabla \,.\, v^{(1)} \in H^1(\Omega'), \tag{42}$$

where Ω' is any bounded smooth domain containing Ω. We also have

$$\begin{aligned}
\frac{1}{\omega}\nabla \wedge v^{(1)}(x) \;=\; & \frac{\gamma_1^2}{2\omega_1}\nabla \wedge \int_\Omega g_1(x-y)\Big(\tilde{\varepsilon}(y)E(y) + i\omega\tilde{\tilde{\mu}}(y)H(y)\Big)\,dy \\
&+\frac{1}{\gamma_1}(\nabla\wedge)^2 \int_\Omega g_1(x-y)\Big(\tilde{\varepsilon}(y)E(y) + i\omega\tilde{\tilde{\mu}}(y)H(y)\Big)\,dy \\
&+i\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{\gamma_1^2}{\omega_1}\nabla \wedge \int_\Omega g_1(x-y)\Big(-i\omega\tilde{\tilde{\varepsilon}}(y)E(y) + \tilde{\mu}(y)H(y)\Big)\,dy \\
&+\frac{1}{\gamma_1}(\nabla\wedge)^2 \int_\Omega g_1(x-y)\Big(-i\omega\tilde{\tilde{\varepsilon}}(y)E(y) + \tilde{\mu}(y)H(y)\Big)\,dy \\
&+\frac{\gamma_2^2}{2\omega_2}\nabla \wedge \int_\Omega g_2(x-y)\Big(\tilde{\varepsilon}(y)E(y) + i\omega\tilde{\tilde{\mu}}(y)H(y)\Big)\,dy \\
&-\frac{1}{\gamma_2}(\nabla\wedge)^2 \int_\Omega g_2(x-y)\Big(\tilde{\varepsilon}(y)E(y) + i\omega\tilde{\tilde{\mu}}(y)H(y)\Big)\,dy \\
&-i\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{\gamma_2^2}{\omega_2}\nabla \wedge \int_\Omega g_2(x-y)\Big(-i\omega\tilde{\tilde{\varepsilon}}(y)E(y) + \tilde{\mu}(y)H(y)\Big)\,dy \\
&-\frac{1}{\gamma_2}(\nabla\wedge)^2 \int_\Omega g_2(x-y)\Big(-i\omega\tilde{\tilde{\varepsilon}}(y)E(y) + \tilde{\mu}(y)H(y)\Big)\,dy.
\end{aligned}$$

Since for $u = \Big(-i\omega\tilde{\tilde{\varepsilon}}(y)E(y) + \tilde{\mu}(y)H(y)\Big)$ or $\Big(\tilde{\varepsilon}(y)E(y) + i\omega\tilde{\tilde{\mu}}(y)H(y)\Big)$, we have $\nabla \,.\, u \in H^1(\Omega), \nabla \wedge u \in (H^1(\Omega))^3$ and $u \wedge n = 0$ on Γ, the above expression gives

$$\nabla \wedge v^{(1)} \in (H^1(\Omega'))^3. \tag{43}$$

Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ such that $\varphi|_\Omega = 1$. Using (43) together with (42) gives

$$\Delta(\varphi v^{(1)}) \;=\; -(\nabla\wedge)^2(\varphi v^{(1)}) + \nabla\nabla\,.\,(\varphi v^{(1)}) \in L^2(\mathbb{R}^3).$$

Therefore,

$$v^{(1)} \in (H^2(\Omega))^3,$$

and so,

$$n \wedge v^{(1)} \in TH^{3/2}(\Gamma).$$

Applying the tangential trace on the both sides of equation (41), the first component yields

$$(n\wedge K + Id + i\sqrt{\frac{\mu_0}{\varepsilon_0}}\, n\wedge D\,\Lambda)\, u \;=\; 2\,\omega\, n\wedge v^{(1)}|_\Gamma \in TH^{3/2}(\Gamma) \hookrightarrow TH^{1/2}(\Gamma) \text{ compactly}.$$

In order to complete the proof of this lemma, we should verify that

$$\mathrm{div}_\Gamma\,(n \wedge v^{(1)}) \in H^{3/2}(\Gamma).$$

If we define $w^{(1)}$ as $\nabla \wedge v^{(1)}$, it is clear that $w^{(1)}$ satisfies

$$\begin{aligned}
\mathrm{div}_\Gamma\,(n \wedge v^{(1)}) &= -n\,.\,\nabla \wedge v^{(1)}\,|_\Gamma,\\
&= -n\,.\,w^{(1)}|_\Gamma,
\end{aligned}$$

$$\nabla\,.\,w^{(1)} = 0,$$

and

$$\begin{aligned}
\nabla \wedge w^{(1)} &= \frac{\gamma_1^2}{2\omega_1}\,(\nabla\wedge)^2 \int_\Omega g_1(x-y)\left(\tilde{\varepsilon}(y)E(y) + i\omega\tilde{\tilde{\mu}}(y)H(y)\right) dy\\
&+\frac{1}{\gamma_1}\,\nabla \wedge \int_\Omega g_1(x-y)\left(\tilde{\varepsilon}(y)E(y) + i\omega\tilde{\tilde{\mu}}(y)H(y)\right) dy\\
&+i\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{\gamma_1^2}{\omega_1}\,(\nabla\wedge)^2 \int_\Omega g_1(x-y)\left(-i\omega\tilde{\tilde{\varepsilon}}(y)E(y) + \tilde{\mu}(y)H(y)\right) dy\\
&+\frac{1}{\gamma_1}\,\nabla \wedge \int_\Omega g_1(x-y)\left(-i\omega\tilde{\tilde{\varepsilon}}(y)E(y) + \tilde{\mu}(y)H(y)\right) dy\\
&+\frac{\gamma_2^2}{2\omega_2}\,(\nabla\wedge)^2 \int_\Omega g_2(x-y)\left(\tilde{\varepsilon}(y)E(y) + i\omega\tilde{\tilde{\mu}}(y)H(y)\right) dy\\
&-\frac{1}{\gamma_2}\,\nabla \wedge \int_\Omega g_2(x-y)\left(\tilde{\varepsilon}(y)E(y) + i\omega\tilde{\tilde{\mu}}(y)H(y)\right) dy\\
&-i\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{\gamma_2^2}{\omega_2}(\nabla\wedge)^2 \int_\Omega g_2(x-y)\left(-i\omega\tilde{\tilde{\varepsilon}}(y)E(y) + \tilde{\mu}(y)H(y)\right) dy\\
&-\frac{1}{\gamma_2}\,\nabla \wedge \int_\Omega g_2(x-y)\left(-i\omega\tilde{\tilde{\varepsilon}}(y)E(y) + \tilde{\mu}(y)H(y)\right) dy.
\end{aligned}$$

$\nabla\,.\,w^{(1)} \in H^1(\Omega)$ together with $\nabla \wedge w^{(1)} \in (H^1(\Omega))^3$ show by the same arguments as for $v^{(1)}$ that $w^{(1)} \in (H^2(\Omega))^3$ and so

$$\mathrm{div}_\Gamma(n \wedge v^{(1)}) \in H^{3/2}(\Gamma).$$

The proof of Lemma 5.4 is then over. □

Now, under the assumptions $\varepsilon, \mu, \beta \in \mathcal{C}^2(\mathbb{R}^3)$, we shall show that there exists at most one solution to the scattering problem (5). Our analysis is based on the Bohren decomposition and the unique continuation principle for the Drude-Born-Fedorov equations.

Lemma 5.5 *The scattering problem for the Drude-Born-Fedorov equations* (5) *has at most one solution.*

Proof. Let E, H denote the difference between two solutions. Then, clearly, E, H satisfy the weak radiation condition (24) and the isotropic Drude-Born-Fedorov equations in $\mathbb{R}^3 \setminus \overline{\Omega}$.
From (5), we derive that

$$\Re e \Big(\int_\Omega H \,.\, (\nabla \wedge \overline{E}) \, dy - \int_\Omega (\nabla \wedge H) \,.\, \overline{E} \, dy \Big) = 0.$$

It follows that

$$\Re e \int_\Gamma (n \wedge \overline{E}) \,.\, H = 0. \tag{44}$$

Now, if we define

$$\begin{cases} \mathcal{E}_1 = E + i \sqrt{\dfrac{\mu_0}{\varepsilon_0}} \, H, \\ \mathcal{E}_2 = E - i \sqrt{\dfrac{\mu_0}{\varepsilon_0}} \, H, \end{cases}$$

the identity (44) yields

$$\Im m \Big(\int_\Gamma (n \wedge \overline{\mathcal{E}}_1) \,.\, \mathcal{E}_1 - \int_\Gamma (n \wedge \overline{\mathcal{E}}_2) \,.\, \mathcal{E}_2 \Big) + \Im m \Big(\int_\Gamma (n \wedge \overline{\mathcal{E}}_2) \,.\, \mathcal{E}_1 - \int_\Gamma (n \wedge \overline{\mathcal{E}}_1) \,.\, \mathcal{E}_2 \Big) = 0.$$

Since

$$\begin{cases} \nabla \wedge \mathcal{E}_1 = \gamma_1 \, \mathcal{E}_1, \\ \nabla \wedge \mathcal{E}_2 = -\gamma_2 \mathcal{E}_2, \end{cases} \tag{45}$$

and

$$\Im m \Big(\int_\Gamma (n \wedge \overline{\mathcal{E}}_2) \,.\, \mathcal{E}_1 - \int_\Gamma (n \wedge \overline{\mathcal{E}}_1) \,.\, \mathcal{E}_2 \Big) = 0,$$

it follows that

$$\Im m \Big(\frac{1}{\gamma_1} \int_\Gamma (n \wedge \overline{\mathcal{E}}_1) \,.\, \nabla \wedge \mathcal{E}_1 + \frac{1}{\gamma_2} \int_\Gamma (n \wedge \overline{\mathcal{E}}_2) \,.\, \nabla \wedge \mathcal{E}_2 \Big) = 0,$$

and so, since $(\mathcal{E}_j)_{j=1,2}$ satisfy Maxwell's equations with the Silver-Müller radiation condition, it follows that (see for instance [1])

$$n \wedge \mathcal{E}_j = n \wedge (\nabla \wedge \mathcal{E}_j) = 0, \quad \text{on } \Gamma.$$

We conclude by the classical unique continuation argument that $\mathcal{E}_j = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$. Therefore, we obtain that $E = H = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$ and so,

$$E \wedge n = (\nabla \wedge E) \wedge n = 0, \quad \text{on } \Gamma.$$

In order to show that $E = H = 0$ in Ω we need to establish the unique continuation principle for the Drude-Born-Fedorov equations. Let Ω' be any smooth bounded domain containing Ω. Since by assumption ε, μ, β are in $\mathcal{C}^2(\overline{\Omega}')$, we first conclude that E and H are in $(\mathcal{C}(B))^3$ for any ball with $\overline{B} \in \Omega'$. Then, from the Drude-Born-Fedorov equations (5), it follows that

$$\left\{ \begin{array}{lcl} (\nabla \wedge)^2 \, E & = & \gamma^2 \beta \, \nabla \, \wedge \, E + \nabla(\gamma^2 \beta) \, \wedge \, E \\ & & \quad + i \omega \mu \dfrac{\gamma^2}{k^2} \, \nabla \, \wedge \, H + i \omega \, \nabla(\mu \dfrac{\gamma^2}{k^2}) \, \wedge \, H, \\ (\nabla \wedge)^2 \, H & = & \gamma^2 \beta \, \nabla \, \wedge \, H + \nabla(\gamma^2 \beta) \, \wedge \, H \\ & & \quad - i \omega \varepsilon \dfrac{\gamma^2}{k^2} \, \nabla \, \wedge \, E - i \omega \nabla(\varepsilon \dfrac{\gamma^2}{k^2}) \, \wedge \, E, \end{array} \right. \tag{46}$$

$$\left\{ \begin{array}{lcl} \nabla \, . \, E & = & \beta \, \nabla(\gamma^2 \beta) \, . \, E - \omega^2 \dfrac{\mu}{k^2} \, \nabla(\dfrac{\varepsilon \gamma^2}{k^2}) \, . \, E \\ & & + i \omega \beta \, \nabla(\mu \dfrac{\gamma^2}{k^2}) \, . \, H - i \omega \dfrac{\mu}{k^2} \, \nabla(\gamma^2 \beta^2) \, . \, H, \\ \nabla \, . \, H & = & \beta \, \nabla(\gamma^2 \beta) \, . \, H - \omega^2 \dfrac{\varepsilon}{k^2} \, \nabla(\dfrac{\mu \gamma^2}{k^2}) \, . \, H \\ & & - i \omega \beta \, \nabla(\varepsilon \dfrac{\gamma^2}{k^2}) \, . \, E + i \omega \dfrac{\varepsilon}{k^2} \, \nabla(\gamma^2 \beta^2) \, . \, E. \end{array} \right. \tag{47}$$

Let $(u_j)_{j=1}^{12}$ be the real and imaginary parts of the Cartesian components of the electric and magnetic fields E and H. Now, using the vector identity

$$(\nabla \wedge)^2 \, = \, -\Delta + \nabla \, \nabla \, .,$$

the inequalities

$$|\Delta u_i| \, \leq \, C \, \sum_{j=1}^{12} \Big(|u_j| + |\nabla u_j| \Big) \quad \text{in } \Omega',$$

for $i = 1, \ldots, 12$ and some constant C follow from (46) and (47) in a trivial manner. From [9] (Lemma 8.5 p.212), we obtain that if $E = H = 0$ in any ball B with $\overline{B} \in \Omega'$, then $E = H = 0$ in Ω'. The proof of Lemma 5.5 is now completed by applying the unique continuation principle that we have established to (E, H). □

We are now in a position to prove the main result of this paper.

Theorem 5.6 *The scattering problem for the Drude-Born-Fedorov equations* (5) *has a unique solution.*

Proof. Let $u \in TH^{1/2}(\mathrm{div}, \Gamma)$ be a solution of the integral equation (39). We define (E^e, H^e) by

$$\left\{ \begin{array}{lcl} E^e & = & E^{in} + \dfrac{1}{2} K_{\mathbb{R}^3 \setminus \overline{\Omega}} \, u + \dfrac{1}{2} i \sqrt{\dfrac{\mu_0}{\varepsilon_0}} \, D_{\mathbb{R}^3 \setminus \overline{\Omega}} \, \Lambda \, u, \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ H^e & = & H^{in} - \dfrac{1}{2} i \sqrt{\dfrac{\varepsilon_0}{\mu_0}} \, D_{\mathbb{R}^3 \setminus \overline{\Omega}} \, u + \dfrac{1}{2} K_{\mathbb{R}^3 \setminus \overline{\Omega}} \, \Lambda \, u, \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}. \end{array} \right.$$

From lemma 5.1, the interior problem

$$\begin{cases} \nabla \wedge E^i &= (\gamma(x))^2 \, \beta(x) E^i + i\omega\mu(x) (\dfrac{\gamma(x)}{k(x)})^2 \, H^i \quad \text{in } \Omega, \\ \nabla \wedge H^i &= (\gamma(x))^2 \, \beta(x) \, H^i - i\omega\varepsilon(x) (\dfrac{\gamma(x)}{k(x)})^2 \, E^i \quad \text{in } \Omega, \end{cases}$$

with the boundary condition

$$n \wedge E^i = n \wedge E^e \quad \text{on } \Gamma,$$

has a unique solution. Furthermore, by the definition of the operator Λ, we have

$$n \wedge H^i = n \wedge H^e \quad \text{on } \Gamma.$$

Let us define $(\mathcal{E}, \mathcal{H})$ by

$$\mathcal{E}|_{\mathbb{R}^3 \setminus \overline{\Omega}} = E^e \text{ and } \mathcal{E}|_{\Omega} = E^i,$$

and similarly for $\mathcal{H}$. We see that

$$(\mathcal{E}, \mathcal{H}) \in \overline{H}^1_{loc}(\text{div}, \mathbb{R}^3) \times \overline{H}^1_{loc}(\text{div}, \mathbb{R}^3).$$

$(\mathcal{E}, \mathcal{H})$ satisfy the Drude-Born-Fedorov equations (5) and the weak radiation condition (24). Now, By Fredholm's alternative and the unique solvability of (5) (lemma 5.5), the integral equation (39) has a unique solution in $TH^{1/2}(\text{div}, \Gamma)$ and so, that the scattering problem for the Drude-Born-Fedorov equations (5) has a unique solution in $\overline{H}^1_{loc}(\text{div}, \mathbb{R}^3) \times \overline{H}^1_{loc}(\text{div}, \mathbb{R}^3)$. □

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