Small Chirality Behavior of Solutions to Electromagnetic Scattering Problems in Chiral Media

H. Ammari J.-C. Nédélec

Abstract

The boundary integral equation method is used to prove the convergence of the Drude-Born-Fedorov equations with variable coefficients, possibly non-smooth, to Maxwell's equations as chirality admittance tends to zero.

key words. Integral Equations, Chiral Media, Drude-Born-Fedorov Equations, Fredholm System, Chirality Admittance, Maxwell's Equations.

AMS subject classifications. 35C15, 35B40, 35Q60, 45B05.

1 Introduction

Chiral media are examples of media responding with both magnetic and electric polarization to electric or magnetic excitation. They thus belong to the general class of bianisotropic media. Chiral media can be characterized by a generalized set of constitutive relations in which the electric and magnetic fields are coupled. The coupling strength is given by the magnitude of a quantity known as the chirality admittance which determines the bulk electromagnetic properties of these materials. Chiral materials are used in many optics applications. Antennas coated with chiral materials have significantly interesting radiation characteristics due to the extra degree of freedom offered by the presence of the chirality admittance. Different expressions exist for the constitutive relations. The Drude-Born-Fedorov constitutive equations are used here. For interesting explanation of these equations and various physical aspects of the propagation inside the chiral media, we refer to [16], [15] and the list of references therein which is representative of the work that is presently being conducted in this area. The original investigations of the effect of chirality on the polarization of light, also date back to the nineteenth century. Arago [6], Biot [8], Cauchy [9], Pasteur [18] and Fresnel [11] all examined optical activity in chiral medium. The concept of chirality has also played an increasingly important role in chemistry with the works of Prelog [21].

In [3], the present authors investigated the scattering problem by a non-homogeneous object embedded in a chiral media. They derived representation theorems and introduced the radiation condition for the Drude-Born-Fedorov equations. The existence and uniqueness of a solution for this problem was proved by using the boundary integral method.

The purpose of the present paper is to prove the convergence of the Drude-Born-Fedorov equations with variable coefficients to Maxwell's equations as the chirality admittance tends to zero. The crucial point for our method is a new integral approach which was originally developed in [2] to study the impedance boundary conditions for thin curved chiral layers and a saddle point formulation of the interior Drude-Born-Fedorov equations. The order of convergence for small chirality asymptotic of exterior Drude-Born-Fedorov equations with variable coefficients, possibly non-smooth, is shown to be the chirality admittance.

This paper is organized as follows. Having introduced some standard notations in section 2 and recalled the Drude-Born-Fedorov equations in section 3 we outline the approach and formulate the result in section 4. In sections 5 and 6, proofs of the convergence of the solution to the interior boundary-value problem for the Drude-Born-Fedorov equations and the solution to the exterior scattering problem for the homogeneous Drude-Born-Fedorov equations are given. Section 7 is devoted to prove our main result.

2 Definitions, Notations and Remarkable Identities

Let Ω be an exterior smooth domain in $I\!R^3$ (i.e., an open set such that its boundary is a smooth closed surface of $I\!R^3$ and the complementary of its closure, denoted by Ω^i, is bounded, connected and has $\partial\Omega$ as boundary), and $n(x)$ the outward pointing normal at $x \in \partial\Omega$. For such a set, there exists a neighborhood of $\partial\Omega$ denoted by $\mathcal{O}$ such that, for any $x \in \mathcal{O}$, if $x_{\partial\Omega}$ is the orthogonal projection of x on $\partial\Omega$ and $s = (x - x_{\partial\Omega}) \,.\, n(x_{\partial\Omega})$, The couple $(x_{\partial\Omega}, s)$ is a local parameterization of $\partial\Omega$. Let Ω_1 be defined as $\partial\Omega\times\ ]0, h'[$ with $h' > 0$ and $\Omega_2 = \Omega \setminus \overline{\Omega_1}$. We recall the following usual differential operators on $(\mathcal{D}'(I\!R^3))^3$ in Cartesian components, the rotational and the divergence defined by :

$$\nabla \wedge = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}, \quad \nabla \,.\, = \nabla^t = (\partial_1, \partial_2, \partial_3),$$

and we introduce the following boundary differential operators defined on tangential fields of $\partial\Omega_2$: the tangential gradient and divergence respectively denoted by $\nabla_{\partial\Omega_2}$ and $\mathrm{div}_{\partial\Omega_2}$, the vector and scalar valued tangential rotational operators respectively denoted by $\overrightarrow{curl}_{\partial\Omega_2}$ and $\mathrm{curl}_{\partial\Omega_2}$, the Laplace-Beltrami operator on $\partial\Omega_2$ denoted by $\Delta_{\partial\Omega_2}$ and defined on scalar functions by

$$\Delta_{\partial\Omega_2} = \mathrm{div}_{\partial\Omega_2}\,\nabla_{\partial\Omega_2} = -\mathrm{curl}_{\partial\Omega_2}\,\overrightarrow{curl}_{\partial\Omega_2}.$$

We recall the mean curvature $c(x)$ defined as $2c(x) = \nabla\,.\,n(x)$ and the tensor of curvature $R(x)$ defined by $R(x) = \nabla\,n(x)$.
To state our boundary-value problems in suitable mathematical forms, we shall use the following notations for the usual functional spaces : $\mathcal{D}'(\Omega_1)$ is the space of distributions in Ω_1.
$L^2(\Omega_1)$ is the space of complex integrable functions defined in Ω_1.
$L^\infty(I\!R^3)$ is the space of bounded functions defined in $I\!R^3$.
$W^{1,\infty}(I\!R^3) = \Big\{\varphi \in L^\infty(I\!R^3), \nabla\,\varphi \in L^\infty(I\!R^3)\Big\}$.
$H(\mathrm{curl},\Omega_1) = \Big\{u \in (L^2(\Omega_1))^3, \nabla\,\wedge\,u \in (L^2(\Omega_1))^3\Big\}$.
$H_0(\mathrm{curl},\Omega_1) = \Big\{\,u \in H(\mathrm{curl},\Omega_1), u\,\wedge\,n\,=\,0 \text{ on } \partial\Omega_1\,\Big\}$.
$\overline{H}^{loc}(\mathrm{curl},\Omega)$ is the Fréchet space of fields which are locally in $H(\mathrm{curl},\Omega)$.
$H^s(\partial\Omega_2)$ is the usual Sobolev space of order $s \in I\!R$ on $\partial\Omega_2$.
$H_0^s(\partial\Omega_2) = \Big\{\varphi \in H^s(\partial\Omega_2), \langle\varphi, 1\rangle_{H^s(\partial\Omega_2),H^{-s}(\partial\Omega_2)} = 0\Big\}$.
$H_0^1(\Omega_1) = \Big\{\varphi \in L^2(\Omega_1), \nabla\,\varphi \in (L^2(\Omega_1))^3, \varphi = 0 \,\mathrm{on}\, \partial\Omega_1\Big\}$.
$TH^{-1/2}(\mathrm{curl},\partial\Omega_2) = \Big\{c \in (H^{-1/2}(\partial\Omega_2))^3, c\,.\,n = 0, \mathrm{curl}_{\partial\Omega_2}\,c \in H^{-1/2}(\partial\Omega_2)\Big\}$.
$TH^{-1/2}(\mathrm{div},\partial\Omega_2) = \Big\{c \in (H^{-1/2}(\partial\Omega_2))^3, c\,.\,n = 0, \mathrm{div}_{\partial\Omega_2}\,c \in H^{-1/2}(\partial\Omega_2)\Big\}$.
$TH^{-1/2}(\mathrm{curl},\partial\Omega_2)$ and $TH^{-1/2}(\mathrm{div},\partial\Omega_2)$ are Hilbert spaces. We recall without proof the well-known following duality result

$$TH^{-1/2}(\mathrm{curl},\partial\Omega_2) = \Big(TH^{-1/2}(\mathrm{div},\partial\Omega_2)\Big)',$$

and the following identities. Let u be a smooth field defined in a neighborhood of $\partial\Omega_2$ and f a smooth function on $\partial\Omega_2$. We have

$$\mathrm{div}_{\partial\Omega_2}(u\,\wedge\,n) = n\,.\Big(\nabla\,\wedge\,u\Big)|_{\partial\Omega_2} = -\mathrm{curl}_{\partial\Omega_2}(n\,\wedge\,(n\,\wedge\,u)).$$

$$\mathrm{div}_{\partial\Omega_2}\,\overrightarrow{curl}_{\partial\Omega_2}\,f = 0, \mathrm{curl}_{\partial\Omega_2}\,\nabla_{\partial\Omega_2}\,f = 0.$$

$$\nabla\,\wedge\,u = \mathrm{curl}_{\partial\Omega_2}\,u_{\partial\Omega_2}\,n + \overrightarrow{curl}_{\partial\Omega_2}(u\,.\,n) + (R - 2c)(u\,\wedge\,n) - \frac{\partial}{\partial s}(u\,\wedge\,n),$$

where $u_{\partial\Omega_2}$ is the tangential component of u on $\partial\Omega_2$: $u_{\partial\Omega_2} = -n \wedge (n \wedge u)$.

$$\begin{aligned}(\nabla\wedge)^2 u &= (\mathrm{curl}_{\partial\Omega_2} \overrightarrow{curl}_{\partial\Omega_2} (u\,.\,n))\, n + \mathrm{div}_{\partial\Omega_2} (R\, u_{\partial\Omega_2})\, n + \mathrm{div}_{\partial\Omega_2} (\frac{\partial}{\partial s} u_{\partial\Omega_2}) \\ &\quad +(R - 2c - \frac{\partial}{\partial s})\, (-\nabla_{\partial\Omega_2}(u\,.\,n) + R\, u_{\partial\Omega_2} + \frac{\partial}{\partial s} u_{\partial\Omega_2}) \\ &\quad + \overrightarrow{curl}_{\partial\Omega_2} \mathrm{curl}_{\partial\Omega_2} u_{\partial\Omega_2}.\end{aligned}$$

$$\mathrm{div}\, u = \mathrm{div}_{\partial\Omega_2} u_{\partial\Omega_2} + 2c\, u\,.\,n + \frac{\partial}{\partial s}(u\,.\,n).$$

$$\nabla_{\partial\Omega_2} (u\,.\,n) = \frac{\partial}{\partial s}(u_{\partial\Omega_2}) + n \wedge (\nabla \wedge u) + R(u_{\partial\Omega_2}).$$

3 The Drude-Born-Fedorov Equations

Electromagnetic fields in inhomogeneous chiral media satisfy the Drude-Born-Fedorov constitutive relations which may be written in the following form [16]

$$(1) \qquad \begin{cases} D^{(\beta)} = \varepsilon \left(E^{(\beta)} + \beta\, \nabla \wedge E^{(\beta)}\right), \\ B^{(\beta)} = \mu \left(H^{(\beta)} + \beta\, \nabla \wedge H^{(\beta)}\right), \end{cases}$$

where the constant β is the chirality admittance, ε the electric permittivity and μ the magnetic permeability. We assume that ε and μ are two real-valued functions in $W^{1,\infty}(I\!R^3) \times L^\infty(I\!R^3)$ and there exist strictly positive constants $\varepsilon_m, \varepsilon_0, \mu_m$ and μ_0 such that

$$\varepsilon(x) \geq \varepsilon_m, \mu(x) \geq \mu_m,$$

$\varepsilon(x) = \varepsilon_0$ and $\mu(x) = \mu_0$ outside the dielectric layer defined by $\partial\Omega \times\,]0, h[$ where $0 < h < h'$.
Direct substitution of the constitutive relations (1) into Maxwell's equations

$$\begin{cases} \nabla \wedge E^{(\beta)} = i\omega\, B^{(\beta)}, \\ \nabla \wedge H^{(\beta)} = -i\omega\, D^{(\beta)}, \end{cases}$$

gives the monochromatic constitutive relations

$$(2) \qquad \begin{cases} \left(1 - \beta^2\omega^2\varepsilon\mu\right) D^{(\beta)} = \varepsilon E^{(\beta)} + i\beta\omega\varepsilon\mu\, H^{(\beta)} \quad \text{in } \Omega, \\ \left(1 - \beta^2\omega^2\varepsilon\mu\right) B^{(\beta)} = \mu H^{(\beta)} - i\beta\omega\varepsilon\mu\, E^{(\beta)} \quad \text{in } \Omega. \end{cases}$$

Combining (1) to (2) yields to the Drude-Born-Fedorov system

$$(3) \qquad \left\{ \begin{array}{lcl} \nabla \wedge E^{(\beta)} & = & \beta\,\gamma^2\,E^{(\beta)} + i\dfrac{\omega\mu}{1-\beta^2\omega^2\varepsilon\mu}\,H^{(\beta)} \quad \text{in } \Omega, \\ \nabla \wedge H^{(\beta)} & = & \beta\,\gamma^2\,H^{(\beta)} - i\dfrac{\omega\varepsilon}{1-\beta^2\omega^2\varepsilon\mu}\,E^{(\beta)} \quad \text{in } \Omega, \end{array} \right.$$

where the real-valued function $\gamma^2(x)$ is defined by :

$$(4) \qquad \gamma^2(x) \;=\; \frac{\omega^2\varepsilon(x)\mu(x)}{1-\beta^2\omega^2\varepsilon(x)\mu(x)}.$$

For more details on the physical background of electromagnetic wave propagation in chiral media, we refer to the book by Lakhtakia et al [16].
Let us finally denote

$$\left\{ \begin{array}{l} \gamma_1 \;=\; \dfrac{\omega\sqrt{\varepsilon_0\mu_0}}{1-\omega\sqrt{\varepsilon_0\mu_0}\beta}, \quad \omega_1 \;=\; \dfrac{\omega}{1-\omega\sqrt{\varepsilon_0\mu_0}\beta}, \\ \gamma_2 \;=\; \dfrac{\omega\sqrt{\varepsilon_0\mu_0}}{1+\omega\sqrt{\varepsilon_0\mu_0}\beta}, \quad \omega_2 \;=\; \dfrac{\omega}{1+\omega\sqrt{\varepsilon_0\mu_0}\beta}, \end{array} \right.$$

and introduce the scalar three-dimensional outgoing fundamental solutions corresponding to γ_j and $\omega\sqrt{\varepsilon_0\mu_0}$ as

$$g_j(\beta, x-y) \;=\; e^{i\gamma_j|x-y|}/4\pi|x-y|,$$

and

$$g(x-y) \;=\; e^{i\omega\sqrt{\varepsilon_0\mu_0}|x-y|}/4\pi|x-y|,$$

for $x-y \neq 0$.

4 Formulation of the Result and Main Steps of the Proof

The purpose of this paper is to prove the convergence of the solution $(E^{(\beta)}, H^{(\beta)})$ to the scattering problem for the Drude-Born-Fedorov equations

$$(5) \qquad \left\{ \begin{array}{l} \nabla \wedge E^{(\beta)} \;=\; \gamma^2\,\beta E^{(\beta)} + i\,\dfrac{\omega\mu}{1-\omega^2\varepsilon\mu\beta^2}\,H^{(\beta)} \quad \text{in } \Omega, \\ \nabla \wedge H^{(\beta)} \;=\; \gamma^2\,\beta\,H^{(\beta)} - i\,\dfrac{\omega\varepsilon}{1-\omega^2\varepsilon\mu\beta^2}\,E^{(\beta)} \quad \text{in } \Omega, \\ n \wedge \left(n \wedge E^{(\beta)}\right) \;=\; f^{(\beta)} \in TH^{-1/2}(\text{curl}, \partial\Omega), \quad \text{on } \partial\Omega, \\ |E^{(\beta)}(x) - \sqrt{\dfrac{\mu_0}{\varepsilon_0}}\,H^{(\beta)}(x) \wedge \dfrac{x}{|x|}| \;\leq\; \dfrac{C}{|x|^2}, \quad |x| \to +\infty, \end{array} \right.$$

to the solution of the scattering problem for the Maxwell equations

$$
\left\{
\begin{array}{l}
\nabla \wedge E^{(0)} = i\,\omega\mu\, H^{(0)} \quad \text{in } \Omega, \\
\nabla \wedge H^{(0)} = -i\,\omega\varepsilon\, E^{(0)} \quad \text{in } \Omega, \\
n \wedge \left(n \wedge E^{(0)}\right) = f^{(0)}, \quad \text{on } \partial\Omega, \\
|E^{(0)}(x) - \sqrt{\dfrac{\mu_0}{\varepsilon_0}}\, H^{(0)}(x) \wedge \dfrac{x}{|x|}| \leq \dfrac{C}{|x|^2}, \quad |x| \to +\infty,
\end{array}
\right.
\tag{6}
$$

where

$$
f^{(0)} = \lim_{\beta \to 0} f^{(\beta)} \quad \text{in } TH^{-1/2}(\text{curl}, \partial\Omega),
$$

and to show that the order of convergence is the chirality admittance β.
Let us briefly describe the essential steps of our method. We begin our analysis by investigating separately the interior and the exterior boundary-value problems of the Drude-Born-Fedorov equations. We first prove with the help of a saddle point formulation the convergence of the solution of the interior boundary-value problem of the Drude-Born-Fedorov equations to the solution of the interior boundary-value problem for the Maxwell equations as the chirality admittance goes to zero. We shall show uniform estimates with respect to the chirality admittance satisfied by the interior electromagnetic boundary operator for the Drude-Born-Fedorov equations mapping the tangential component of the electric field solution of the interior boundary-value problem of the Drude-Born-Fedorov equations onto the tangential magnetic field. These uniform estimates are crucial for our method. They permit the uniform estimate of the inverse of the interior electromagnetic operator in terms of a Neumann series. Then, we show the convergence of the solution to the exterior scattering problem for the homogeneous Drude-Born-Fedorov equations. To this end, a new integral approach is used. This integral approach was originally developed by the authors in [2] to study the impedance boundary conditions for thin curved chiral layers. The main advantage of this integral approach lies in the fact that it reduces the scattering vector problem to a system of two scalar integral equations. Since this system of two integral equations appears as a compact perturbation of an invertible system, the rate of the convergence of the homogeneous exterior Drude-Born-Fedorov equations to Maxwell's equations is obtained by a classical argument. Finally, by using the interior electromagnetic operator, we reduce the Drude-Born-Fedorov equations (5) to a Fredholm system of two integral equations written on $\partial\Omega_2$ analogous to the one obtained in the case of homogeneous Drude-Born-Fedorov equations. From the preliminary asymptotic estimates proved for the interior boundary-value problem and the exterior scattering problem, and by using the uniform estimates with respect to the chirality admittance satisfied by the interior electromagnetic operator we will prove the claim. Note that our method can be used to analyze the behavior

of the solutions of the time harmonic Maxwell problem with variable coefficients as frequency ω tends to zero and to obtain the exact order of convergence. Different methods for studying the Maxwell equations at low frequencies can be found in [22], [24], [13], [14], [19], [28].
The goal of the next section is to analyze the behavior of the solution of the interior boundary value problem in Ω_1 for the Drude-Born-Fedorov equations as chirality admittance tends to zero.
To simplify the presentation, we shall use throughout this paper two positive constants C and C' as generalized constants which precise values are not required and may change line by line. Of course, we must point out any dependence of C and C' on the chirality admittance β.

5 The Interior Boundary-Value Problem for the Drude-Born-Fedorov Equations

In this section we consider the interior boundary-value problem for the Drude-Born-Fedorov equations

$$(7) \qquad \begin{cases} \nabla \wedge e_1^{(\beta)} = \gamma^2 \beta e_1^{(\beta)} + i \dfrac{\omega\mu}{1-\omega^2\varepsilon\mu\beta^2} h_1^{(\beta)} & \text{in } \Omega_1, \\ \nabla \wedge h_1^{(\beta)} = \gamma^2 \beta h_1^{(\beta)} - i \dfrac{\omega\varepsilon}{1-\omega^2\varepsilon\mu\beta^2} e_1^{(\beta)} & \text{in } \Omega_1, \\ n \wedge \left(n \wedge e_1^{(\beta)}\right) = f^{(\beta)} & \text{on } \partial\Omega, \\ n \wedge \left(n \wedge e_1^{(\beta)}\right) = w^{(\beta)} & \text{on } \partial\Omega_2, \end{cases}$$

where $w^{(\beta)} \in TH^{-1/2}(\mathrm{curl}, \partial\Omega_2)$ is a known vector field and we assume that the limit of $w^{(\beta)}$ when β tends to zero exists in $TH^{-1/2}(\mathrm{curl}, \partial\Omega_2)$ and we denote it by $w^{(0)}$. Our basic aim is to study the behavior of the solution to (7) as the chirality admittance tends to zero. We shall prove the convergence of $e_1^{(\beta)}$ to the unique solution $e^{(0)}$ of the interior boundary-value problem of the Maxwell equations

$$(8) \qquad \begin{cases} \nabla \wedge \left(\dfrac{1}{\mu} \nabla \wedge e_1^{(0)}\right) - \omega^2 \varepsilon\, e_1^{(0)} = 0 & \text{in } \Omega_1, \\ n \wedge \left(n \wedge e_1^{(0)}\right) = w^{(0)} & \text{on } \partial\Omega_2, \\ n \wedge \left(n \wedge e_1^{(0)}\right) = f^{(0)} & \text{on } \partial\Omega. \end{cases}$$

One of our principal motivations in this section is to establish uniform estimates with respect to the chirality admittance satisfied by the interior electromagnetic operator for the Drude-Born-Fedorov equations defined by

$$T_1^{(\beta)} \; : w^{(\beta)} \in TH^{-1/2}(\mathrm{curl}, \partial\Omega_2) \longmapsto n \wedge \left(n \, \wedge \, h_1^{(\beta)}\right) \tag{9}$$

in $TH^{-1/2}(\mathrm{curl}, \partial\Omega_2)$, where $h_1^{(\beta)}$ is the magnetic field solution of the boundary-value problem (7). These estimates are fundamental for our method. Let the operator $T_1^{(0)}$ be defined by

$$w^{(0)} \in TH^{-1/2}(\mathrm{curl}, \partial\Omega_2) \longmapsto -\frac{i}{\omega\,\mu_0}\, n \wedge \left(n \wedge \left(\nabla \, \wedge \, e_1^{(0)}\right)\right), \tag{10}$$

in $TH^{-1/2}(\mathrm{curl}, \partial\Omega_2)$ where $e_1^{(0)}$ is the electric field solution of the boundary-value problem (8). We will prove the following important results.

Theorem 5.1 *There exists $\beta_0 > 0$, such that for $0 < \beta \leq \beta_0$ the following estimate holds for any $w \in TH^{-1/2}(\mathrm{curl}, \partial\Omega_2)$*

$$\begin{aligned} ||T_1^{(\beta)}(w) - T_1^{(0)}(w)||_{TH^{-1/2}(\mathrm{curl},\partial\Omega_2)} \quad &\leq \quad C\,\beta\,||w||_{TH^{-1/2}(\mathrm{curl},\partial\Omega_2)} \\ &\qquad +C'\,||f^{(\beta)} - f^{(0)}||_{TH^{-1/2}(\mathrm{curl},\partial\Omega)}, \end{aligned} \tag{11}$$

where the two strictly positive constants C and C' are independent of β.

Theorem 5.2 *There exists $\beta_0 > 0$, such that for $0 < \beta \leq \beta_0$ the following estimate holds*

$$\begin{aligned} ||e_1^{(\beta)} - e_1^{(0)}||_{H(\mathrm{curl},\Omega_1)} \quad &\leq \quad C\,\beta + C'\,\Big(||f^{(\beta)} - f^{(0)}||_{TH^{-1/2}(\mathrm{curl},\partial\Omega)} \\ &\qquad +||w^{(\beta)} - w^{(0)}||_{TH^{-1/2}(\mathrm{curl},\partial\Omega_2)}\Big), \end{aligned}$$

where the two strictly positive constants C and C' are independent of β, $e_1^{(\beta)}$ the unique solution of (7) and $e_1^{(0)}$ the unique solution of (8).

We begin our analysis by proving Theorem 5.2. The crucial point in our proof is the use of a saddle point formulation of the interior boundary-value problem of the Drude-Born-Fedorov equations.

Before proving our convergence result, we shall show the existence and uniqueness of a solution to (7). First, we prove that we can take $\omega > 0$ such that zero is not an eigenvalue of the Drude-Born-Fedorov operator in Ω_1 for $0 \leq \beta < \beta_0$ with the boundary conditions $n \wedge . = 0$ on $\partial\Omega_1$ and then, the boundary-value problem (7) is uniquely solvable. We can show the following lemma.

Lemma 5.1 *There exist $\omega > 0$ and $\beta_0 > 0$, such that for any β in the interval $[\,0, \beta_0\,]$, 0 is not an eigenvalue of the Drude-Born-Fedorov operator in Ω_1 with the chirality admittance β and the boundary conditions $n \wedge . = 0$ on $\partial\Omega_1$.*

Proof. We give an outline of the basic steps of the proof of this lemma. We denote by $L^{(\beta)}$ the interior Drude-Born-Fedorov operator given by

$$L^{(\beta)} \;=\; \nabla\wedge\Big(\frac{1-\omega^2\beta^2\varepsilon\mu}{\mu}\nabla\,\wedge\,.\Big) - \omega^2\nabla\,\wedge\,(\varepsilon\beta.) - \omega^2\varepsilon\beta\,\nabla\,\wedge\,-\omega^2\varepsilon.$$

We can write $L^{(\beta)}$ as $L^{(\beta)} - L^{(0)} + L^{(0)}$ where $L^{(0)}$ is the interior Maxwell operator given by

$$L^{(0)} \;=\; \nabla\wedge\Big(\frac{1}{\mu}\nabla\,\wedge\,.\Big) - \omega^2\varepsilon.$$

It is known that there exist $\omega > 0$, such that the interior Maxwell operator $L^{(0)}$ is invertible in $\Omega_1 = \partial\Omega\times\,]0,h'[$ (see for instance [4]). Let v be in $H_0(\mathrm{curl},\Omega_1)$. From

$$\begin{aligned}\Big((L^{(\beta)} - L^{(0)})v, v\Big) \;&=\; -\beta^2\omega^2\int_{\Omega_1}\mu|\nabla\,\wedge\,v|^2 - \beta\omega^2\int_{\Omega_1}\varepsilon\,\nabla\,\wedge\,v\,.\,\overline{v}\\ &\quad +\beta\omega^2\int_{\Omega_1}\varepsilon v\,.\,\nabla\,\wedge\,\overline{v},\end{aligned}$$

we deduce that

$$|\Big((L^{(\beta)} - L^{(0)})v, v\Big)| \;\le\; C\,(\beta+\beta^2)||\nabla\,\wedge\,v||^2_{(L^2(\Omega_1))^3} + C'\,\beta\,||v||^2_{(L^2(\Omega_1))^3}.$$

It follows that the norm of the operator $L^{(\beta)} - L^{(0)}$ in $H_0(\mathrm{curl},\Omega_1)$ goes to zero as β tends to zero. Therefore, there exists $\beta_0 > 0$, such that for $0 < \beta < \beta_0$, the interior Drude-Born-Fedorov operator $L^{(\beta)}$ in $\partial\Omega\times\,]0,h'[$ is invertible. □

After this brief discussion about the unique solvability of (7), we observe that the weak form of the Drude-Born-Fedorov equations (7) in a neighborhood of the interface $\partial\Omega\times\Big\{s = h\Big\}$ gives the following jump relations on $\partial\Omega\times\Big\{s = h\Big\}$:

$$\text{(12)}\qquad \left\{\begin{array}{l} [\,e_1^{(\beta)}\,\wedge\,n\,] \;=\; 0,\\ [\,\dfrac{1}{\mu}(1-\beta^2\omega^2\varepsilon\mu)\Big(\nabla\,\wedge\,e_1^{(\beta)}\Big)\,\wedge\,n\,] \;=\; \beta\omega^2\,[\,\varepsilon\,]\,e_1^{(\beta)}\,\wedge\,n.\end{array}\right.$$

Now, eliminating the magnetic field $h_1^{(\beta)}$ from the interior Drude-Born-Fedorov system (7) yields

$$\text{(13)}\qquad \begin{aligned}&\nabla\wedge\Big(\frac{1}{\mu}\,(1-\beta^2\omega^2\varepsilon\mu)\,\nabla\,\wedge\,e_1^{(\beta)}\Big) - \beta\,\omega^2\,\varepsilon\nabla\,\wedge\,e_1^{(\beta)} - \beta\,\omega^2\,\nabla\,\wedge\,\varepsilon\,e_1^{(\beta)}\\ &-\omega^2\,\varepsilon\,e_1^{(\beta)} \;=\; 0,\end{aligned}$$

in $\mathcal{D}'(\Omega_1)$. It follows that

$$\text{(14)}\qquad \beta\,\nabla\,.\Big(\varepsilon\,\nabla\,\wedge\,e_1^{(\beta)}\Big) + \nabla\,.\,\Big(\varepsilon\,e_1^{(\beta)}\Big) \;=\; 0.$$

Following [3], we decompose the electric field $e_1^{(\beta)}$ as follows

$$e_1^{(\beta)} \;=\; e^{(\beta)} - v^{(\beta)} - \nabla\, p^{(\beta)},$$

where $p^{(\beta)}$ is a scalar function in $H^1(\Omega_1)$ and $e^{(\beta)}$ is such that

$$\mathrm{div}(\varepsilon\, e^{(\beta)}) \;=\; 0, \quad \text{in } \Omega_1$$

and $e^{(\beta)} \wedge n = 0$ on $\partial\Omega_1$. Then, we introduce the scalar function $p^{(\beta)}$ and the fields $v^{(\beta)}$ and $v^{(0)}$ as unique solutions of the following problems :

$$\left\{\begin{array}{l} \nabla\,.\left(\varepsilon \nabla\, p^{(\beta)}\right) \;=\; \beta\,\nabla\,.\left(\varepsilon\,\nabla\,\wedge\, e_1^{(\beta)}\right) \in H^{-1}(\Omega_1), \quad \text{in}\,\Omega_1, \\ p^{(\beta)} \;=\; 0 \quad \text{on}\,\partial\Omega_1. \end{array}\right. \tag{15}$$

$$\left\{\begin{array}{l} (\nabla\wedge)^2\, v^{(\beta)} + \omega^2\,\varepsilon\, v^{(\beta)} \;=\; 0 \text{ in}\,\Omega_1, \\ n\wedge\left(n\,\wedge\, v^{(\beta)}\right) \;=\; -w^{(\beta)} \text{ on}\,\partial\Omega_2, \\ n\wedge\left(n\,\wedge\, v^{(\beta)}\right) \;=\; -f^{(\beta)} \text{ on}\,\partial\Omega, \end{array}\right. \quad \left\{\begin{array}{l} (\nabla\wedge)^2\, v^{(0)} + \omega^2\,\varepsilon\, v^{(0)} \;=\; 0 \text{ in}\,\Omega_1, \\ n\wedge\left(n\,\wedge\, v^{(0)}\right) \;=\; -w^{(0)} \text{ on}\,\partial\Omega_2, \\ n\wedge\left(n\,\wedge\, v^{(0)}\right) \;=\; -f^{(0)} \text{ on}\,\partial\Omega. \end{array}\right. \tag{16}$$

The proof of the convergence of $e_1^{(\beta)}$ to $e_1^{(0)}$ as β tends to zero is divided into three steps. We first show that $e_1^{(\beta)}$ is uniformly bounded in $H(\mathrm{curl},\Omega_1)$. Then, we prove the convergence of $p^{(\beta)} \to 0$, in $H^1(\Omega_1)$. Finally, we prove that $e^{(\beta)}$ converges to $e_1^{(0)} - v^{(0)}$ in $H(\mathrm{curl},\Omega_1)$ and show that the rate of convergence is $0(\beta)$.

It is easy to show that we have the following lemma.

Lemma 5.2 *There exists $\beta_0 > 0$, such that for $0 < \beta \le \beta_0$ the following estimate holds*

$$||v^{(\beta)}-v^{(0)}||^2_{H(\mathrm{curl},\Omega_1)} \;\le\; C\Big(||w^{(\beta)}-w^{(0)}||^2_{TH^{-1/2}(\mathrm{curl},\partial\Omega_2)}+||f^{(\beta)}-f^{(0)}||^2_{TH^{-1/2}(\mathrm{curl},\partial\Omega)}\Big),$$

where the constant C is independent of β.

We now start our analysis by proving that the electric field $e_1^{(\beta)}$ is uniformly bounded with respect to β in $H(\mathrm{curl},\Omega_1)$.

Lemma 5.3 *There exists $\beta_0 > 0$, such that the estimate*

$$||e_1^{(\beta)}||_{H(\mathrm{curl},\Omega_1)} \;\le\; C,$$

holds for any $0 < \beta < \beta_0$, where the constant C is independent of β and $e_1^{(\beta)}$ is the unique solution of the interior Drude-Born-Fedorov system (7).

Proof. We begin the proof by recalling the following arithmetic-geometric mean inequality : Let $(a,b) \in I\!\!R^2$, for any $\eta > 0$ the inequality

$$ab \,\le\, \eta a^2 + \frac{1}{\eta} b^2,$$

holds. From (13), it is easy to see that

$$\begin{aligned}
&\int_{\Omega_1} \frac{1}{\mu}\,(1-\beta^2\omega^2\varepsilon\mu)\,|\nabla\wedge\left(e_1^{(\beta)}+v^{(\beta)}\right)|^2 \\
&-\beta\,\omega^2\int_{\Omega_1}\varepsilon\nabla\wedge\left(e_1^{(\beta)}+v^{(\beta)}\right).\left(\overline{e}_1^{(\beta)}+\overline{v}^{(\beta)}\right) \\
&-\beta\,\omega^2\int_{\Omega_1}\varepsilon\left(e_1^{(\beta)}+v^{(\beta)}\right).\nabla\wedge\left(\overline{e}_1^{(\beta)}+\overline{v}^{(\beta)}\right)-\omega^2\,\varepsilon\int_{\Omega_1}|e_1^{(\beta)}+v^{(\beta)}|^2 \\
&=\int_{\Omega_1}\frac{1}{\mu}\,(1-\beta^2\omega^2\varepsilon\mu)\;\nabla\,\wedge\,v^{(\beta)}\,.\,\nabla\wedge\left(\overline{e}_1^{(\beta)}+\overline{v}^{(\beta)}\right) \\
&-\beta\,\omega^2\int_{\Omega_1}\varepsilon\nabla\,\wedge\,v^{(\beta)}\,.\left(\overline{e}_1^{(\beta)}+\overline{v}^{(\beta)}\right) \\
&-\beta\,\omega^2\int_{\Omega_1}\varepsilon\,v^{(\beta)}\,.\,\nabla\wedge\left(\overline{e}_1^{(\beta)}+\overline{v}^{(\beta)}\right)-\omega^2\int_{\Omega_1}\varepsilon\,v^{(\beta)}\,.\left(\overline{e}_1^{(\beta)}+\overline{v}^{(\beta)}\right).
\end{aligned}$$

By using the η-inequality, it follows from the above identity that

$$\int_{\Omega_1}|\nabla\wedge\left(e_1^{(\beta)}+v^{(\beta)}\right)|^2-C\int_{\Omega_1}|e_1^{(\beta)}+v^{(\beta)}|^2\,\le\,C'\,||v^{(\beta)}||_{H(\mathrm{curl},\Omega_1)}\,||e_1^{(\beta)}+v^{(\beta)}||_{H(\mathrm{curl},\Omega_1)}.$$

Applying the η-inequality again yields

$$\begin{aligned}
||\nabla\wedge\left(e_1^{(\beta)}+v^{(\beta)}\right)||^2_{(L^2(\Omega_1))^3} \quad &\le \quad C\,||e_1^{(\beta)}+v^{(\beta)}||^2_{(L^2(\Omega_1))^3} \\
&\quad +C'\,||v^{(\beta)}||^2_{H(\mathrm{curl},\Omega_1)},
\end{aligned} \tag{17}$$

and so, in order to prove the lemma, it suffices to show that $||e_1^{(\beta)}+v^{(\beta)}||_{(L^2(\Omega_1))^3}$ is uniformly bounded with respect to β since $||v^{(\beta)}||_{H(\mathrm{curl},\Omega_1)}$ is uniformly bounded from Lemma 5.2. Let us suppose that we have a subsequence of $e_1^{(\beta)}+v^{(\beta)}$, still denoted by $e_1^{(\beta)}+v^{(\beta)}$, such that $||e_1^{(\beta)}+v^{(\beta)}||_{(L^2(\Omega_1))^3}$ converges to $+\infty$ as β tends to zero. Set

$$\mathcal{E}^{(\beta)} \;=\; \frac{e_1^{(\beta)}+v^{(\beta)}}{||e_1^{(\beta)}+v^{(\beta)}||_{(L^2(\Omega_1))^3}}.$$

The last inequality implies that $\mathcal{E}^{(\beta)}$ is bounded in $H(\mathrm{curl}, \Omega_1)$. Furthermore, it satisfies the variational equation

$$\int_{\Omega_1} \frac{1}{\mu}(1-\beta^2\omega^2\varepsilon\mu)\,\nabla\wedge\mathcal{E}^{(\beta)}\,.\,\nabla\wedge v - \beta\,\omega^2\int_{\Omega_1}\varepsilon\nabla\wedge\mathcal{E}^{(\beta)}\,.\,v$$
$$-\beta\,\omega^2\int_{\Omega_1}\varepsilon\,\mathcal{E}^{(\beta)}\,.\,\nabla\wedge v - \omega^2\int_{\Omega_1}\varepsilon\,\mathcal{E}^{(\beta)}\,.\,v$$
$$= \frac{1}{||e_1^{(\beta)}+v^{(\beta)}||_{(L^2(\Omega_1))^3}}\Big(\int_{\Omega_1}\frac{1}{\mu}(1-\beta^2\omega^2\varepsilon\mu)\,\nabla\wedge v^{(\beta)}\,.\,\nabla\wedge v$$
$$-\beta\,\omega^2\int_{\Omega_1}\varepsilon\nabla\wedge v^{(\beta)}\,.\,v - \beta\,\omega^2\int_{\Omega_1}\varepsilon\,v^{(\beta)}\,.\,\nabla\wedge v$$
$$-\omega^2\int_{\Omega_1}\varepsilon\,v^{(\beta)}\,.\,v\Big),$$

for any $v \in H_0(\mathrm{curl}, \Omega_1)$. The integral on the right-hand side tends to zero for any $v \in H_0(\mathrm{curl}, \Omega_1)$ as β tends to zero. It follows from the unique solvability of the interior Maxwell equations (Lemma 5.1) that $\mathcal{E}^{(\beta)}$ converges, at least for a subsequence, to zero in $H(\mathrm{curl}, \Omega_1)$ weakly. Since

$$\nabla\,.\left(\varepsilon\,\mathcal{E}^{(\beta)}\right) = \frac{\beta}{||e_1^{(\beta)}+v^{(\beta)}||_{(L^2(\Omega_1))^3}}\,\nabla\,.\left(\varepsilon\nabla\wedge e_1^{(\beta)}\right),$$

and $\varepsilon \in W^{1,\infty}(I\!R^3)$, we obtain that $\nabla\,.\left(\varepsilon\,\mathcal{E}^{(\beta)}\right)$ is uniformly bounded in $L^2(\Omega_1)$ with respect to β. Note that the assumption on the regularity of the electric permittivity ε is essential in our proof.
It is well-known that $\Big\{u \in H_0(\mathrm{curl}, \Omega_1), \mathrm{div}\, u \in L^2(\Omega_1)\Big\}$ is compactly embedded in $(L^2(\Omega_1))^3$. This compactness result has been proved by Leis [17] for smooth domains, by Weck [24] for cone like domains and, by Weber [23] and Picard [20] for domains with the strict cone property. The strong convergence of $\mathcal{E}^{(\beta)}$ to zero in $(L^2(\Omega_1))^3$ is deduced from the above compactness result. $||\mathcal{E}^{(\beta)}||_{(L^2(\Omega_1))^3} = 1$ leads then to a contradiction and the proof of the lemma is then over. □

We are now in position to prove the strong convergence of $p^{(\beta)}$ to zero in $H_0^1(\Omega_1)$ when β tends to zero.

Lemma 5.4 *The unique solution $p^{(\beta)}$ of (15) converges to zero, $H_0^1(\Omega_1)$ strongly. Furthermore, there exists β_0, such that we have*

$$||p^{(\beta)}||_{H^1(\Omega_1)} \leq C\,\beta,$$

for any $0 < \beta < \beta_0$, where the constant C is independent of β

Proof. Multiplying (15) by $\overline{p}^{(\beta)}$ and integrating by parts over Ω_1 yield

$$\int_{\Omega_1} \varepsilon \, |\nabla \, p^{(\beta)}|^2 \; = \; \beta \int_{\Omega_1} \varepsilon \left(\nabla \, \wedge \, e_1^{(\beta)} \right) . \, \nabla \, p^{(\beta)}.$$

Therefore, from

$$||p^{(\beta)}||^2_{H^1(\Omega_1)} \; \leq \; C \, \beta \, ||e_1^{(\beta)}||_{H(\mathrm{curl},\Omega_1)} \, ||p^{(\beta)}||_{H^1(\Omega_1)},$$

and by Lemma 5.3, we obtain the claim.

□

To complete the proof of Theorem 5.2, we must show that

$$e^{(\beta)} \rightarrow e_1^{(0)} + v^{(0)} \quad \text{in } H(\mathrm{curl}, \Omega_1)$$

as $0(\beta)$.
We first observe that from (14), (15) and (16) we have

$$\nabla \, . \left(\varepsilon \, e^{(\beta)} \right) \; = \; 0,$$

since

$$e^{(\beta)} \; = \; e_1^{(\beta)} + \nabla \, p^{(\beta)} + v^{(\beta)}.$$

Now, from (13), it is easily seen that the field $e^{(\beta)}$ satisfies

$$\begin{aligned} &\nabla \wedge \Big(\frac{1}{\mu} \, (1 - \beta^2 \omega^2 \varepsilon \mu) \, \nabla \, \wedge \, e^{(\beta)} \Big) - \beta \, \omega^2 \, \varepsilon \nabla \, \wedge \, e^{(\beta)} - \beta \, \omega^2 \, \nabla \, \wedge \, \varepsilon \, e^{(\beta)} \\ &-\omega^2 \, \varepsilon \, e^{(\beta)} \, = \, -\nabla \wedge \Big(\frac{1}{\mu} \, (1 - \beta^2 \omega^2 \varepsilon \mu) \, \nabla \, \wedge \, v^{(\beta)} \Big) + \beta \, \omega^2 \, \varepsilon \nabla \, \wedge \, v^{(\beta)} \\ &+\beta \, \omega^2 \, \nabla \, \wedge \, \varepsilon \, v^{(\beta)} + \omega^2 \, \varepsilon \, v^{(\beta)} + \beta \omega^2 \nabla \, \wedge \, \varepsilon \, \nabla \, p^{(\beta)} + \omega^2 \varepsilon \, \nabla \, p^{(\beta)} \text{ in } (\mathcal{D}'(\Omega_1))^3. \end{aligned} \tag{18}$$

Now, if we multiply (18) by a test field $v \in H_0(\mathrm{curl}, \Omega_1)$, integrate over Ω_1, and use integration by parts and the jump relations (12), we obtain

$$\begin{aligned} &\int_{\Omega_1} \frac{1}{\mu} \, (1 - \beta^2 \omega^2 \varepsilon \mu) \, \nabla \, \wedge \, e^{(\beta)} \, . \, \nabla \, \wedge \, v \; + \beta \, \omega^2 \int_{\Omega_1} \varepsilon \nabla \, \wedge \, e^{(\beta)} \, . \, v \\ &-\beta \omega^2 \int_{\Omega_1} \varepsilon \, e^{(\beta)} \, . \, \nabla \, \wedge \, v - \omega^2 \int_{\Omega_1} \varepsilon \, e^{(\beta)} \, . \, v \\ &= -\int_{\Omega_1} \frac{1}{\mu} \, (1 - \beta^2 \omega^2 \varepsilon \mu) \, \nabla \, \wedge \, v^{(\beta)} \, . \, \nabla \, \wedge \, v - \beta \, \omega^2 \int_{\Omega_1} \varepsilon \nabla \, \wedge \, v^{(\beta)} \, . \, v \\ &+\beta \, \omega^2 \int_{\Omega_1} \varepsilon \, v^{(\beta)} \, . \, \nabla \, \wedge \, v + \omega^2 \int_{\Omega_1} \varepsilon \, v^{(\beta)} \, . \, v \\ &+\beta \omega^2 \int_{\Omega_1} \nabla \, \wedge \, \varepsilon \, \nabla \, p^{(\beta)} \, . \, v + \omega^2 \int_{\Omega_1} \varepsilon \, \nabla \, p^{(\beta)} \, . \, v. \end{aligned} \tag{19}$$

Following [3], we try to find the field $e^{(\beta)} \in H(\mathrm{curl}, \Omega_1)$ in the form :

$$e^{(\beta)} \;=\; z^{(\beta)} + \nabla\, \varphi^{(\beta)}, \tag{20}$$

where $z^{(\beta)} \in H(\mathrm{curl}, \Omega_1)$ satisfies

$$\nabla\,.\left(\varepsilon\, z^{(\beta)}\right) \;=\; 0 \quad \text{in } \Omega_1, \tag{21}$$

$z^{(\beta)} \wedge n = 0$ on $\partial\Omega_1$ and $\varphi^{(\beta)} \in H^1_0(\Omega_1)$. By using the above decomposition of the field $e^{(\beta)}$, we rewrite the variational equation (19) and the identity (21) as

$$\left\{ \begin{array}{l} a^{(\beta)}(z^{(\beta)}, v) + b^{(\beta)}(v, \varphi^{(\beta)}) \;=\; -a^{(\beta)}(v^{(\beta)} + \nabla\, p^{(\beta)}, v), \forall\, v \in H_0(\mathrm{curl}, \Omega_1), \\ b^{(0)}(z^{(\beta)}, \psi) \;=\; 0, \quad \forall\, \psi \in H^1_0(\Omega_1), \end{array} \right. \tag{22}$$

where $a^{(\beta)}, b^{(\beta)}$ and $b^{(0)}$ are defined by

$$\begin{aligned} a^{(\beta)}(w, v) \;&=\; \int_{\Omega_1} \frac{1}{\mu}\,(1 - \beta^2\omega^2\varepsilon\mu)\,\nabla\,\wedge\,w\,.\,\nabla\,\wedge\,\overline{v} + \beta\,\omega^2 \int_{\Omega_1} \varepsilon\, w\,.\,\nabla\,\wedge\,\overline{v} \\ &\quad -\beta\,\omega^2 \int_{\Omega_1} \varepsilon\,\nabla\,\wedge\,w\,.\,\overline{v} - \omega^2 \int_{\Omega_1} \varepsilon\, w\,.\,\overline{v}, \end{aligned}$$

$$b^{(\beta)}(v, \psi) \;=\; \beta\,\omega^2 \int_{\Omega_1} \varepsilon\,\nabla\,\psi\,.\,\nabla\,\wedge\,\overline{v} - \omega^2 \int_{\Omega_1} \varepsilon\,\nabla\,\psi\,.\,\overline{v},$$

and

$$b^{(0)}(v, \psi) \;=\; -\omega^2 \int_{\Omega_1} \varepsilon\,\nabla\,\psi\,.\,\overline{v},$$

for all $(v, w, \psi) \in H_0(\mathrm{curl}, \Omega_1) \times H_0(\mathrm{curl}, \Omega_1) \times H^1_0(\Omega_1)$.
Let us introduce $(z^{(0)}, \varphi^{(0)}) \in H_0(\mathrm{curl}, \Omega_1) \times H^1_0(\Omega_1)$ as the unique solution of the variational problem :

$$\left\{ \begin{array}{l} a^{(0)}(z^{(0)}, v) + b^{(0)}(v, \varphi^{(0)}) \;=\; -a^{(0)}(v^{(0)}, v), \quad \forall\, v \in H_0(\mathrm{curl}, \Omega_1), \\ b^{(0)}(z^{(0)}, \psi) \;=\; 0, \quad \forall\, \psi \in H^1_0(\Omega_1), \end{array} \right. \tag{23}$$

where

$$a^{(0)}(w, v) \;=\; \int_{\Omega_1} \frac{1}{\mu}\,\nabla\,\wedge\,w\,.\,\nabla\,\wedge\,\overline{v} - \omega^2 \int_{\Omega_1} \varepsilon\, w\,.\,\overline{v}.$$

It is known from [3] that the variational system (23) is a saddle point formulation of the interior Maxwell equations (8). The form $b^{(0)}$ satisfies the inf-sup condition and $a^{(0)}$ is a compact perturbation of a coercive form

$$\int_{\Omega_1} \frac{1}{\mu}\,\nabla\,\wedge\,w\,.\,\nabla\,\wedge\,\overline{v} + \omega^2 \int_{\Omega_1} \varepsilon\, w\,.\,\overline{v},$$

on the kernel $N^{(0)}$ of $b^{(0)}$ given by

$$N^{(0)} = \Big\{ v \in H(\mathrm{curl}, \Omega_1), \nabla \,.\, \Big(\varepsilon\, v\Big) \, = \, 0 \,\mathrm{in}\, \Omega_1, v \,\wedge\, n \, = \, 0 \,\mathrm{on}\, \partial\Omega_1 \,\Big\},$$

which is compactly embedded in $(L^2(\Omega_1))^3$.
Now, from (22) and (23), a direct computation yields

$$a^{(0)}(z^{(\beta)} - z^{(0)}, v) + b^{(0)}(v, \varphi^{(\beta)} - \varphi^{(0)}) \, = \, a^{(0)}(v^{(0)}, v) - a^{(\beta)}(v^{(\beta)} + \nabla\, p^{(\beta)}, v)$$
$$+ \Big(a^{(\beta)} - a^{(0)}\Big)(z^{(\beta)}, v) + \Big(b^{(0)} - b^{(\beta)}\Big)(v, \varphi^{(\beta)}),$$
(24)
for any $v \in H_0(\mathrm{curl}, \Omega_1)$, and

$$b^{(0)}(z^{(\beta)} - z^{(0)}, \psi) \, = \, 0,$$

for any $\psi \in H_0^1(\Omega_1)$. It follows from Lemma 5.3 and 5.4 that

$$||z^{(\beta)}||_{H(\mathrm{curl},\Omega_1)} + ||\varphi^{(\beta)}||_{H^1(\Omega_1)}$$

is bounded uniformly with respect to β. Then, we deduce from (24) that $(z^{(\beta)}, \varphi^{(\beta)})$ satisfy

$$\left\{ \begin{array}{l} a^{(0)}(z^{(\beta)} - z^{(0)}, v) + b^{(0)}(v, \varphi^{(\beta)} - \varphi^{(0)}) \, = \, 0(\beta + ||v^{(\beta)} - v^{(0)}||_{H(\mathrm{curl},\Omega_1)})\, ||v||_{H(\mathrm{curl},\Omega_1)}, \\ b^{(0)}(z^{(\beta)} - z^{(0)}, \psi) \, = \, 0, \end{array} \right.$$

for all $(v, \psi) \in H_0(\mathrm{curl}, \Omega_1) \times H_0^1(\Omega_1)$. From the inequalities

$$|a^{(0)}(z^{(\beta)} - z^{(0)}, z^{(\beta)} - z^{(0)})| \, \leq \, 0(\beta)\, ||z^{(\beta)} - z^{(0)}||_{H(\mathrm{curl},\Omega_1)},$$

and

$$|a^{(0)}(z^{(\beta)} - z^{(0)}, z^{(\beta)} - z^{(0)})| \, \geq \, C\, ||z^{(\beta)} - z^{(0)}||^2_{H(\mathrm{curl},\Omega_1)} - C'\, ||z^{(\beta)} - z^{(0)}||^2_{L^2(\Omega_1)},$$

it follows that

$$\begin{aligned} C\, ||z^{(\beta)} - z^{(0)}||^2_{H(\mathrm{curl},\Omega_1)} \quad \leq \quad & C'\, ||z^{(\beta)} - z^{(0)}||^2_{L^2(\Omega_1)} \\ & + 0(\beta + ||v^{(\beta)} - v^{(0)}||_{H(\mathrm{curl},\Omega_1)})\, ||z^{(\beta)} - z^{(0)}||_{H(\mathrm{curl},\Omega_1)}. \end{aligned}$$

We are now ready to prove Theorem 5.2. We first prove by contradiction that

$$||z^{(\beta)} - z^{(0)}||_{H(\mathrm{curl},\Omega_1)} \, = \, 0(\beta + ||v^{(\beta)} - v^{(0)}||_{H(\mathrm{curl},\Omega_1)}).$$

Suppose that we have a subsequence, still denoted by $z^{(\beta)}$ such that

$$(\beta + ||v^{(\beta)} - v^{(0)}||_{H(\mathrm{curl},\Omega_1)})^{-1} ||z^{(\beta)} - z^{(0)}||_{H(\mathrm{curl},\Omega_1)}$$

converges to $+\infty$ as β tends to zero. We introduce

$$x^{(\beta)} \;=\; \frac{z^{(\beta)} - z^{(0)}}{||z^{(\beta)} - z^{(0)}||_{H(\mathrm{curl},\Omega_1)}},$$

and

$$y^{(\beta)} \;=\; \frac{\varphi^{(\beta)} - \varphi^{(0)}}{||z^{(\beta)} - z^{(0)}||_{H(\mathrm{curl},\Omega_1)}}.$$

Combining

$$b^{(0)}(v, \varphi^{(\beta)} - \varphi^{(0)}) \;=\; 0(\beta)\, ||v||_{H(\mathrm{curl},\Omega_1)} - a^{(0)}(z^{(\beta)} - z^{(0)}, v),$$

and the inf-sup condition

$$||\varphi^{(\beta)} - \varphi^{(0)}||_{H^1(\Omega_1)} \;\leq\; \sup_{v\in H(\mathrm{curl},\Omega_1)} \frac{|b^{(0)}(v, \varphi^{(\beta)} - \varphi^{(0)})|}{||v||_{H(\mathrm{curl},\Omega_1)}},$$

allows us to obtain the following estimate

$$\begin{aligned} &||\varphi^{(\beta)} - \varphi^{(0)}||_{H^1(\Omega_1)} \;\leq\; C\, ||z^{(\beta)} - z^{(0)}||_{H(\mathrm{curl},\Omega_1)} \\ &+0(\beta + ||v^{(\beta)} - v^{(0)}||_{H(\mathrm{curl},\Omega_1)}), \end{aligned} \tag{25}$$

and then, $y^{(\beta)}$ is uniformly bounded with respect to β in $H_0^1(\Omega_1)$. It follows that there exists $(x^*, y^*) \in H_0(\mathrm{curl}, \Omega_1) \times H_0^1(\Omega_1)$ such that $(x^{(\beta)}, y^{(\beta)})$ converges to $(x^*, y^*), H(\mathrm{curl}, \Omega_1) \times H^1(\Omega_1)$, weakly. The pair (x^*, y^*) satisfies the following variational problem

$$\left\{ \begin{aligned} &a^{(0)}(x^*, v) + b^{(0)}(v, y^*) \;=\; 0, \quad \forall\, v \in H_0(\mathrm{curl}, \Omega_1), \\ &b^{(0)}(y^*, \psi) \;=\; 0, \quad \forall\, \psi \in H_0^1(\Omega_1). \end{aligned} \right. \tag{26}$$

The unique solvability of the interior Maxwell equations (Lemma 5.1) leads to

$$(x^*, y^*) = (0, 0).$$

Since the embedding $N^{(0)} \hookrightarrow (L^2(\Omega_1))^3$ is compact, we obtain that $x^{(\beta)} \in N^{(0)}$ converges to $0, (L^2(\Omega_1))^3$, strongly. By

$$a^{(0)}(x^{(\beta)}, x^{(\beta)}) \longrightarrow 0, \quad \beta \longrightarrow 0,$$

the strong convergence of $x^{(\beta)}$ to in $H(\mathrm{curl}, \Omega_1)$ is proved, but $||x^{(\beta)}||_{H(\mathrm{curl},\Omega_1)} = 1$ which leads to a contradiction. Hence, the estimate

$$||z^{(\beta)} - z^{(0)}||_{H(\mathrm{curl},\Omega_1)} \;=\; 0(\beta + ||v^{(\beta)} - v^{(0)}||_{H(\mathrm{curl},\Omega_1)}), \tag{27}$$

holds and we also obtain from (25) that

$$||\varphi^{(\beta)} - \varphi^{(0)}||_{H^1(\Omega_1)} \;=\; 0(\beta + ||v^{(\beta)} - v^{(0)}||_{H(\mathrm{curl},\Omega_1)}). \tag{28}$$

Therefore, the assertion of Theorem 5.2 follows.

We conclude this section by proving Theorem 5.1. In order to achieve this aim, we need to look more carefully at (24) since estimate (11) follows from Theorem 5.2 by taking $w^{(\beta)} = w^{(0)}$ and making use of the continuity of

$$e \in H(\mathrm{curl}, \Omega_1) \mapsto e \wedge n \in TH^{-1/2}(\mathrm{div}, \partial\Omega_1).$$

In fact we have

$$\begin{aligned} |a^{(0)}(v^{(0)}, v) - a^{(\beta)}(v^{(\beta)} + \nabla p^{(\beta)}, v)| \quad &\leq \quad C\,\beta\,||v^{(\beta)}||_{H(\mathrm{curl},\Omega_1)}\,||v||_{H(\mathrm{curl},\Omega_1)} \\ &\quad +C'\,\beta\,||v||_{H(\mathrm{curl},\Omega_1)}. \end{aligned}$$

From

$$||v^{(\beta)}||_{H(\mathrm{curl},\Omega_1)} \leq ||v^{(\beta)} - v^{(0)}||_{H(\mathrm{curl},\Omega_1)} + ||v^{(0)}||_{H(\mathrm{curl},\Omega_1)},$$

it follows that

$$\begin{aligned} ||v^{(\beta)}||_{H(\mathrm{curl},\Omega_1)} \quad &\leq \quad C\,||f^{(\beta)} - f^{(0)}||_{TH^{-1/2}(\mathrm{curl},\Omega_1)} \\ &\quad +C'\,||w||_{TH^{-1/2}(\mathrm{curl},\Omega_1)}, \end{aligned}$$

which leads to

$$\begin{aligned} |a^{(0)}(v^{(0)}, v) - a^{(\beta)}(v^{(\beta)} + \nabla p^{(\beta)}, v)| \quad &\leq \quad C\,\beta\,||w||_{TH^{-1/2}(\mathrm{curl},\Omega_1)}\,||v||_{H(\mathrm{curl},\Omega_1)} \\ &\quad +C'\,\beta\,||v||_{H(\mathrm{curl},\Omega_1)}. \end{aligned}$$

From this, we can conclude that (11) holds and the proof of Theorem 5.1 is then over.

6 The Exterior Boundary-Value Problem for the Homogeneous Drude-Born-Fedorov Equations

We now turn our attention to the exterior scattering problem for the homogeneous Drude-Born-Fedorov equations. Of particular interest to us will be the use of a new integral approach. We shall analyze the behavior of the solution $e_2^{(\beta)}$ to the exterior scattering problem for the homogeneous Drude-Born-Fedorov equations

$$(29) \qquad \left\{ \begin{aligned} \nabla \wedge e_2^{(\beta)} &= \gamma^2\,\beta e_2^{(\beta)} + i\,\frac{\omega\mu_0}{1-\omega^2\varepsilon_0\mu_0\beta^2}\,h_2^{(\beta)} \quad \text{in } \Omega_2, \\ \nabla \wedge h_2^{(\beta)} &= \gamma^2\,\beta\,h_2^{(\beta)} - i\,\frac{\omega\varepsilon_0}{1-\omega^2\varepsilon_0\mu_0\beta^2}\,e_2^{(\beta)} \quad \text{in } \Omega_2, \\ n \wedge \left(n \wedge e_2^{(\beta)}\right) &= w^{(\beta)} \quad \text{on } \partial\Omega_2, \\ |e_2^{(\beta)}(x) - \sqrt{\frac{\mu_0}{\varepsilon_0}}\,h_2^{(\beta)}(x) \wedge \frac{x}{|x|}| &\leq \frac{C}{|x|^2}, \quad |x| \to +\infty, \end{aligned} \right.$$

where $w^{(\beta)} \in TH^{-1/2}(\mathrm{curl}, \partial\Omega_2)$ is a known vector field which tends to $w^{(0)}$ in $TH^{-1/2}(\mathrm{curl}, \partial\Omega_2)$ when β tends to zero.

We first recall shortly how to prove the existence and the uniqueness of a solution to (29). We then proceed to the proof of the convergence of $e^{(\beta)}$ as the chirality admittance tends to zero.
Let us define $e_2^{(0)}$ as the outgoing solution of the Maxwell equations

$$\left\{ \begin{array}{l} (\nabla \wedge)^2 \, e_2^{(0)} - \omega^2 \, \varepsilon_0 \, \mu_0 \, e_2^{(0)} \; = \; 0 \quad \mathrm{in}\, \Omega_2, \\ n \wedge \left(n \, \wedge \, e_2^{(0)} \right) \; = \; w^{(0)} \quad \mathrm{on}\, \partial\Omega_2. \end{array} \right. \tag{30}$$

Our purpose in this section is to prove the following convergence result.

Theorem 6.1 *For any $\mathcal{O} \subset\subset \Omega_2$, there exists $\beta_0 > 0$, such that for $0 < \beta < \beta_0$ the following estimate holds*

$$\begin{array}{rcl} ||e_2^{(\beta)} - e_2^{(0)}||_{H(\mathrm{curl}, \mathcal{O})} & \leq & C \, ||w^{(\beta)} - w^{(0)}||_{TH^{-1/2}(\mathrm{curl}, \partial\Omega_2)} \\ & & + C' \, \beta, \end{array} \tag{31}$$

where C and C' are two strictly positive constants independent of β.

To prove (31) we proceed as we said in section 4 by reducing the solution of (29) to a Fredholm system of two integral equations. Then, we establish the limit and prove the estimate (31) with the help of a classical result that we will formulate in an abstract manner.
From the representation theorem proved in [3] (Theorem 4.3), it follows that

$$\begin{array}{rcl} e_2^{(\beta)}(x) & = & f_2^{(\beta)}(x) + \dfrac{i}{2} \sqrt{\dfrac{\mu_0}{\varepsilon_0}} \, \nabla \wedge \displaystyle\int_{\partial\Omega_2} (g_1 - g_2)(\beta, x - y) \; u^{(\beta)}(y) \; ds(y) \\ & & + \dfrac{i}{2\gamma^2} \sqrt{\dfrac{\mu_0}{\varepsilon_0}} \, \nabla \displaystyle\int_{\partial\Omega_2} (\gamma_2 g_1 + \gamma_1 g_2)(\beta, x - y) \; \mathrm{div}_{\partial\Omega_2} \, u^{(\beta)}(y) \; ds(y) \\ & & + \dfrac{i}{2} \sqrt{\dfrac{\mu_0}{\varepsilon_0}} \displaystyle\int_{\partial\Omega_2} (\gamma_1 g_1 + \gamma_2 g_2)(\beta, x - y) \; u^{(\beta)}(y) \; ds(y), \quad \forall x \in \Omega_2, \end{array} \tag{32}$$

where the tangential field $u^{(\beta)} \in TH^{-1/2}(\mathrm{div}, \partial\Omega_2)$ is defined as $u^{(\beta)} = n \wedge h_2^{(\beta)}$ and $f_2^{(\beta)}$ is given by

$$\begin{array}{rcl} f_2^{(\beta)}(x) & = & \dfrac{1}{2} \nabla \wedge \displaystyle\int_{\partial\Omega_2} (g_1 + g_2)(\beta, x - y) \, (w^{(\beta)} \, \wedge \, n)(y) \; ds(y) \\ & & + \dfrac{1}{2\gamma^2} \, \nabla \displaystyle\int_{\partial\Omega_2} (\gamma_2 g_1 - \gamma_1 g_2)(\beta, x - y) \, \mathrm{div}_{\partial\Omega_2} \, (w^{(\beta)} \, \wedge \, n)(y) \; ds(y), \end{array}$$

for x in Ω_2 and $-n \wedge (n \wedge f_2^{(\beta)})(x)$ by

$$-n \wedge \Big(n \wedge \frac{1}{2}\int_{\partial\Omega_2} \nabla_x (g_1+g_2)(\beta, x-y) \wedge (w^{(\beta)} \wedge n)(y)\, ds(y)$$
$$+\frac{1}{2\gamma^2}\nabla \int_{\partial\Omega_2} (\gamma_2 g_1 - \gamma_1 g_2)(\beta, x-y)\, \mathrm{div}_{\partial\Omega_2} (w^{(\beta)} \wedge n)(y)\, ds(y)\Big) - \frac{1}{2} w^{(\beta)}(x),$$

for x on the boundary $\partial\Omega_2$. In the above integral representation of the electric field $e_2^{(\beta)}$, $f_2^{(\beta)}$ depends only on the tangential component of the electric field $e_2^{(\beta)}$ on $\partial\Omega_2$ which is given and the other terms depend on the tangential component of the magnetic field $h_2^{(\beta)}$ on $\partial\Omega_2$.
Note that in order to solve the exterior homogeneous equations (29) it suffices to determine the tangential component of the magnetic field on $\partial\Omega_2$.
Multiplying the limit of (32) as x tends to $\partial\Omega_2$ by a test field $v \in TH^{-1/2}(\mathrm{div}, \partial\Omega_2)$ and integrating by parts using the expression of $\nabla \wedge$ in local coordinates gives

$$\int_{\partial\Omega_2}\int_{\partial\Omega_2} (g_1-g_2)(\beta, x-y)\, u^{(\beta)}(y) \,.\, (n(x)-n(y))\, \mathrm{curl}_{\partial\Omega_2} v(x)\, ds(y)\, ds(x)$$
$$+\int_{\partial\Omega_2} (R(x) - 2c(x)) \Big(\int_{\partial\Omega_2} (g_1-g_2)(\beta, x-y)\, u^{(\beta)}(y) \wedge n(x)\, ds(y)\Big) \,.\, v(x)\, ds(x)$$
$$-\frac{1}{\gamma^2}\int_{\partial\Omega_2}\int_{\partial\Omega_2} (\gamma_2 g_1 + \gamma_1 g_2)(\beta, x-y)\, \mathrm{div}_{\partial\Omega_2} u^{(\beta)}(y)\, \mathrm{div}_{\partial\Omega_2} v(x)\, ds(x)\, ds(y)$$
$$-\int_{\partial\Omega_2}\int_{\partial\Omega_2} \frac{\partial (g_1-g_2)}{\partial n_x}(\beta, x-y)\, (u^{(\beta)}(y) \wedge n(x)) \,.\, v(x)\, ds(y)\, ds(x)$$
$$+\int_{\partial\Omega_2}\int_{\partial\Omega_2} (\gamma_1 g_1 + \gamma_2 g_2)(\beta, x-y)\, u^{(\beta)}(y) \,.\, v(x)\, ds(x)\, ds(y)$$
$$= 2i\sqrt{\frac{\varepsilon_0}{\mu_0}}\int_{\partial\Omega_2} \Big(w^{(\beta)} + f_2^{(\beta)}\Big)(x) \,.\, v(x)\, ds(x).$$

As the boundary $\partial\Omega_2$ is assumed to be simply connected, we can use the following Hodge decomposition of the space $TH^{-1/2}(\mathrm{div}, \partial\Omega_2)$ proved in [12].

Lemma 6.1 $TH^{-1/2}(\mathrm{div}, \partial\Omega_2) = \overrightarrow{curl}_{\partial\Omega_2} H^{1/2}(\partial\Omega_2) \oplus \nabla_{\partial\Omega_2} H^{3/2}(\partial\Omega_2)$.

Making use of this decomposition, we decompose the tangential fields $u^{(\beta)}$ and v which are in $TH^{-1/2}(\mathrm{div}, \partial\Omega_2)$ as follows

$$\left\{\begin{array}{l} u^{(\beta)} \;=\; \overrightarrow{curl}_{\partial\Omega_2}\, \varphi^{(\beta)} + \nabla_{\partial\Omega_2}\, \psi^{(\beta)} \\ v \;=\; \overrightarrow{curl}_{\partial\Omega_2}\, \varphi_t + \nabla_{\partial\Omega_2}\, \psi_t \end{array}\right.$$

where $(\varphi^{(\beta)}, \psi^{(\beta)})$ and (φ_t, ψ_t) are in $H^{1/2}(\partial\Omega_2) \times H^{3/2}(\partial\Omega_2)$. Therefore, we can split the above variational equation into two distinct equations given by

$$\begin{aligned} & a_1^{(\beta)}(\varphi^{(\beta)}, \varphi_t) + b_1^{(\beta)}(\psi^{(\beta)}, \varphi_t) \ = \\ & i\sqrt{\frac{\varepsilon_0}{\mu_0}} \int_{\partial\Omega_2} \Big(w^{(\beta)} + f_2^{(\beta)}\Big)(x) \,.\, \overrightarrow{curl}_{\partial\Omega_2}\, \varphi_t(x)\, ds(x), \end{aligned} \tag{33}$$

and

$$\begin{aligned} & a_2^{(\beta)}(\psi^{(\beta)}, \psi_t) + b_2^{(\beta)}(\varphi^{(\beta)}, \psi_t) \ = \\ & i\sqrt{\frac{\varepsilon_0}{\mu_0}} \int_{\partial\Omega_2} \Big(w^{(\beta)} + f_2^{(\beta)}\Big)(x) \,.\, \nabla_{\partial\Omega_2}\, \psi_t(x)\, ds(x), \end{aligned} \tag{34}$$

where $2a_1^{(\beta)}(\varphi^{(\beta)}, \varphi_t)$ is given by

$$\begin{aligned} & \int_{\partial\Omega_2}\int_{\partial\Omega_2} (\gamma_1 g_1 + \gamma_2 g_2)(\beta, x-y)\ \overrightarrow{curl}_{\partial\Omega_2}\, \varphi^{(\beta)}(y) \,.\, \overrightarrow{curl}_{\partial\Omega_2}\, \varphi_t(x)\ ds(x)\, ds(y) \\ & - \int_{\partial\Omega_2}\int_{\partial\Omega_2} (g_1 - g_2)(\beta, x-y)\ \overrightarrow{curl}_{\partial\Omega_2}\, \varphi^{(\beta)}(y) \,.\, (n(x) - n(y))\, \Delta_{\partial\Omega_2}\, \varphi_t(x)\ ds(y)\, ds(x) \\ & + \int_{\partial\Omega_2} (R(x) - 2c(x)) \\ & \Big(\int_{\partial\Omega_2} (g_1 - g_2)(\beta, x-y)\ \overrightarrow{curl}_{\partial\Omega_2}\, \varphi^{(\beta)}(y) \wedge n(x)\ ds(y) \Big) \,.\, \overrightarrow{curl}_{\partial\Omega_2}\, \varphi_t(x)\ ds(x) \\ & - \int_{\partial\Omega_2}\int_{\partial\Omega_2} \frac{\partial(g_1 - g_2)}{\partial n_x}(\beta, x-y)\ \Big(\overrightarrow{curl}_{\partial\Omega_2}\, \varphi^{(\beta)}(y) \wedge n(x)\Big) \,.\, \overrightarrow{curl}_{\partial\Omega_2}\, \varphi_t(x)\ ds(y)\, ds(x), \end{aligned}$$

$2b_1^{(\beta)}(\psi^{(\beta)}, \varphi_t)$ by

$$\begin{aligned} & - \int_{\partial\Omega_2}\int_{\partial\Omega_2} (g_1 - g_2)(\beta, x-y)\ \nabla_{\partial\omega_2}\, \psi^{(\beta)} \,.\, (n(x) - n(y))\, \Delta_{\partial\Omega_2}\, \varphi_t(x)\ ds(y)\, ds(x) \\ & + \int_{\partial\Omega_2} (R(x) - 2c(x)) \Big(\int_{\partial\Omega_2} (g_1 - g_2)(\beta, x-y)\ \nabla_{\partial\Omega_2}\, \psi^{(\beta)}(y) \,.\, \overrightarrow{curl}_{\partial\Omega_2}\, \varphi_t(x)\ ds(x) \\ & - \int_{\partial\Omega_2}\int_{\partial\Omega_2} \frac{\partial(g_1 - g_2)}{\partial n_x}(\beta, x-y)\ \Big(\nabla_{\partial\Omega_2}\, \psi^{(\beta)}(y) \wedge n(x)\Big) \,.\, \overrightarrow{curl}_{\partial\Omega_2}\, \varphi_t(x)\ ds(y)\, ds(x) \\ & + \int_{\partial\Omega_2}\int_{\partial\Omega_2} (\gamma_1 g_1 + \gamma_2 g_2)(\beta, x-y)\ \nabla_{\partial\Omega_2}\, \psi^{(\beta)}(y) \,.\, \overrightarrow{curl}_{\partial\Omega_2}\, \varphi_t(x)\ ds(x)\, ds(y), \end{aligned}$$

$2a_2^{(\beta)}(\psi^{(\beta)}, \psi_t)$ by

$$-\frac{1}{\gamma^2}\int_{\partial\Omega_2}\int_{\partial\Omega_2}(\gamma_2 g_1+\gamma_1 g_2)(\beta, x-y)\ \Delta_{\partial\Omega_2}\,\psi^{(\beta)}(y)\,\Delta_{\partial\Omega_2}\,\psi_t(x)\ ds(x)\,ds(y)$$

$$+\int_{\partial\Omega_2}(R(x)-2c(x))$$

$$\Big(\int_{\partial\Omega_2}(g_1-g_2)(\beta, x-y)\ \nabla_{\partial\Omega_2}\,\psi^{(\beta)}(y)\ \wedge\ n(x)\,ds(y)\Big)\,.\,\nabla_{\partial\Omega_2}\psi_t(x)\,ds(x)$$

$$-\int_{\partial\Omega_2}\int_{\partial\Omega_2}\frac{\partial(g_2-g_1)}{\partial n_x}(\beta, x-y)\ \Big(\nabla_{\partial\Omega_2}\,\psi^{(\beta)}(y)\wedge n(x)\Big)\,.\,\nabla_{\partial\Omega_2}\,\psi_t(x)\ ds(y)\,ds(x)$$

$$+\int_{\partial\Omega_2}\int_{\partial\Omega_2}(\gamma_1 g_1+\gamma_2 g_2)(\beta, x-y)\ \nabla_{\partial\Omega_2}\,\psi^{(\beta)}(y)\,.\,\nabla_{\partial\Omega_2}\,\psi_t(x)\ ds(x)\,ds(y),$$

and $2b_2^{(\beta)}(\varphi^{(\beta)}, \psi_t)$ is equal to

$$\int_{\partial\Omega_2}(R(x)-2c(x))$$

$$\Big(\int_{\partial\Omega_2}(g_1-g_2)(\beta, x-y)\ \overrightarrow{curl}_{\partial\Omega_2}\,\varphi^{(\beta)}(y))\ \wedge\ n(x)\,ds(y)\Big)\,.\,\nabla_{\partial\Omega_2}\psi_t(x)\,ds(x)$$

$$-\int_{\partial\Omega_2}\int_{\partial\Omega_2}\frac{\partial(g_2-g_1)}{\partial n_x}(\beta, x-y)\ \Big(\overrightarrow{curl}_{\partial\Omega_2}\,\varphi^{(\beta)}(y)\ \wedge\ n(x)\Big)\,.\,\nabla_{\partial\Omega_2}\,\psi_t(x)\ ds(y)\,ds(x)$$

$$+\int_{\partial\Omega_2}\int_{\partial\Omega_2}(\gamma_1 g_1+\gamma_2 g_2)(\beta, x-y)\ \overrightarrow{curl}_{\partial\Omega_2}\,\varphi^{(\beta)}(y)\,.\,\nabla_{\partial\Omega_2}\,\psi_t(x)\ ds(x)\,ds(y)$$

It is easy to show that the system $(33)-(34)$ is of Fredholm type. The existence of a solution is then deduced from uniqueness by Fredholm's alternative. The uniqueness follows from the Bohren decomposition (see [2] and [5]).
Now, consider the system of the following two integral equations :

$$(35)\ a_1^{(0)}(\varphi^{(0)}, \varphi_t)+b_1^{(0)}(\psi^{(0)}, \varphi_t)\ =\ i\sqrt{\frac{\varepsilon_0}{\mu_0}}\int_{\partial\Omega_2}v^{(0)}(x)\,.\,\overrightarrow{curl}_{\partial\Omega_2}\,\varphi_t(x)\ ds(x),$$

and

$$(36)\ a_2^{(0)}(\psi^{(0)}, \psi_t)+b_2^{(0)}(\varphi^{(0)}, \psi_t)\ =\ i\sqrt{\frac{\varepsilon_0}{\mu_0}}\int_{\partial\Omega_2}v^{(0)}(x)\,.\,\nabla_{\partial\Omega_2}\,\psi_t(x)\ ds(x).$$

where

$$v^{(0)}(x)\ =\Big(w^{(0)}+f_2^{(0)}\Big)(x),$$

$$a_1^{(0)}(\varphi^{(0)}, \varphi_t)\ =\ \omega\sqrt{\varepsilon_0\mu_0}\int_{\partial\Omega_2}\int_{\partial\Omega_2}g(x-y)\ \overrightarrow{curl}_{\partial\Omega_2}\,\varphi^{(0)}(y)\,.\,\overrightarrow{curl}_{\partial\Omega_2}\,\varphi_t(x)\ ds(x)\,ds(y),$$

$$b_1^{(0)}(\psi^{(0)},\varphi_t) \;=\; \omega\sqrt{\varepsilon_0\mu_0}\int_{\partial\Omega_2}\int_{\partial\Omega_2} g(x{-}y)\,\nabla_{\partial\Omega_2}\,\psi^{(0)}(y)\,.\,\overrightarrow{curl}_{\partial\Omega_2}\,\varphi_t(x)\;ds(y)\;ds(x)$$

$$\begin{aligned} a_2^{(0)}(\psi^{(0)},\psi_t) \;&=\; -\frac{1}{\omega\sqrt{\varepsilon_0\mu_0}}\int_{\partial\Omega_2}\int_{\partial\Omega_2} g(x-y)\;\Delta_{\partial\Omega_2}\,\psi^{(0)}(y)\,\Delta_{\partial\Omega_2}\,\psi_t(x)\;ds(x)\,ds(y)\\ &\quad+\omega\sqrt{\varepsilon_0\mu_0}\int_{\partial\Omega_2}\int_{\partial\Omega_2} g(x-y)\;\nabla_{\partial\Omega_2}\,\psi^{(0)}(y)\,.\,\nabla_{\partial\Omega_2}\,\psi_t(x)\;ds(x)\,ds(y),\end{aligned}$$

$$b_2^{(0)}(\varphi^{(0)},\psi_t) \;=\; \omega\sqrt{\varepsilon_0\mu_0}\int_{\partial\Omega_2}\int_{\partial\Omega_2} g(x{-}y)\;\overrightarrow{curl}_{\partial\Omega_2}\,\varphi^{(0)}(y)\,.\,\nabla_{\partial\Omega_2}\,\psi_t(x)\;ds(x)\,ds(y),$$

$$f_2^{(0)}(x) \;=\; \nabla\,\wedge\int_{\partial\Omega_2} g(x-y)\,(w^{(0)}\;\wedge\,n)(y)\;ds(y), x\in\Omega_2,$$

$$-n\wedge(n\wedge f_2^{(0)})(x) \;=\; -\frac{1}{2}w^{(0)}(x){-}n\wedge\Big(n\wedge\int_{\partial\Omega_2}\nabla_x\,g(x{-}y)\wedge(w^{(0)}\wedge n)(y)\;ds(y)\Big),$$

for $x\in\partial\Omega_2$.
Let us define the operator $S^{(\beta)}$ from $TH^{-1/2}(\mathrm{curl},\partial\Omega_2)$ into itself by

$$S^{(\beta)}\;:w^{(\beta)}\longmapsto -n\wedge\Big(n\;\wedge\;f_2^{(\beta)}\Big)|_{\partial\Omega_2}, \tag{37}$$

and introduce the operator $S^{(0)}$ from $TH^{-1/2}(\mathrm{curl},\partial\Omega_2)$ into itself as

$$S^{(0)}\;:w^{(0)}\longmapsto -n\wedge\Big(n\;\wedge\;f_2^{(0)}\Big)|_{\partial\Omega_2}. \tag{38}$$

These operators will be used in the next section.
An essential step in the fact that the pair $(\psi^{(\beta)},\varphi^{(\beta)})$ converges to $(\psi^{(0)},\varphi^{(0)})$ the unique solution of $(35)-(36)$, is the following lemma.

Lemma 6.2 *The system* $(35)-(36)$ *is of Fredholm type.*

Proof. Since the system $(35)-(36)$ is a compact perturbation of the coercive one on $H_0^{1/2}(\partial\Omega_2)\times H_0^{3/2}(\partial\Omega_2)$, given by

$$\begin{pmatrix} \omega\sqrt{\varepsilon_0\mu_0}\displaystyle\int_{\partial\Omega_2}\int_{\partial\Omega_2} g(x-y)\;\overrightarrow{curl}_{\partial\Omega_2}\,\varphi^{(0)}(y)\,.\,\overrightarrow{curl}_{\partial\Omega_2}\,\varphi_t(x)\;ds(x)\,ds(y) & 0\\ 0\quad -\dfrac{1}{\omega\sqrt{\varepsilon_0\mu_0}}\displaystyle\int_{\partial\Omega_2}\int_{\partial\Omega_2} g(x-y)\;\Delta_{\partial\Omega_2}\,\psi^{(0)}(y)\,\Delta_{\partial\Omega_2}\,\psi_t(x)\;ds(x)\,ds(y), & \end{pmatrix}. \tag{39}$$

it suffices to show that the other terms in $(35)-(36)$ are compact. We get, after some easy manipulations,

$$\begin{aligned} &\int_{\partial\Omega_2}\int_{\partial\Omega_2} g(x-y)\,\nabla_{\partial\Omega_2}\,\psi^{(0)}(y)\,.\,\overrightarrow{curl}_{\partial\Omega_2}\,\varphi_t(x)\;ds(y)\;ds(x)\\ &=\int_{\partial\Omega_2}\varphi_t(x)\;\mathrm{curl}_{\partial\Omega_2}\int_{\partial\Omega_2} g(x-y)\,\nabla_{\partial\Omega_2}\,\psi^{(0)}(y)\;ds(y)\;ds(x),\\ &=\int_{\partial\Omega_2}\int_{\partial\Omega_2}(n_x-n_y)\;\wedge\;\nabla_x\,g(x-y)\,.\,\nabla_{\partial\Omega_2}\,\psi^{(0)}(y)\,\varphi_t(x)\;ds(y)\;ds(x),\end{aligned}$$

and consequently,

$$\begin{aligned}&|\int_{\partial\Omega_2}\int_{\partial\Omega_2} g(x-y)\,\nabla_{\partial\Omega_2}\,\psi^{(0)}(y)\,.\,\overrightarrow{curl}_{\partial\Omega_2}\,\varphi_t(x)\;ds(y)\;ds(x)|\\ &\leq\; C\,||\psi^{(0)}||_{H^{1/2}(\partial\Omega_2)}\,||\varphi_t||_{H^{-1/2}(\partial\Omega_2)}.\end{aligned}$$

The compactness of all the other operators in (36) is also easy to prove. However, we have

$$\begin{aligned}&|\int_{\partial\Omega_2}\int_{\partial\Omega_2} g(x-y)\;\nabla_{\partial\Omega_2}\,\psi^{(0)}(y)\,.\,\nabla_{\partial\Omega_2}\,\psi_t(x)\;ds(x)\,ds(y)|\\ &\leq\; C\,||\psi^{(0)}||_{H^{1/2}(\partial\Omega_2)}\,||\psi_t||_{H^{1/2}(\partial\Omega_2)},\end{aligned}$$

and

$$\begin{aligned}&|\int_{\partial\Omega_2}\int_{\partial\Omega_2} g(x-y)\;\overrightarrow{curl}_{\partial\Omega_2}\,\varphi^{(0)}(y)\,.\,\nabla_{\partial\Omega_2}\,\psi_t(x)\;ds(x)\,ds(y)|\\ &\leq\; C\,||\varphi^{(0)}||_{L^2(\partial\Omega_2)}\,||\psi_t||_{H^1(\partial\Omega_2)}.\end{aligned}$$

The assertion of the lemma follows. □

Now, the uniqueness of a solution to $(35)-(36)$ follows from the fact that

$$\begin{aligned}&i\omega\mu_0\int_{\partial\Omega_2} g(x-y)\,\Big(\overrightarrow{curl}_{\partial\Omega_2}\,\varphi^{(0)}+\nabla_{\partial\Omega_2}\,\psi^{(0)}\Big)(y)\;ds(y)\\ &+\frac{i}{\omega\varepsilon_0}\,\nabla\int_{\partial\Omega_2} g(x-y)\,\Delta_{\partial\Omega_2}\,\psi^{(0)}(y)\;ds(y)\\ &+\nabla\,\wedge\int_{\partial\Omega_2} g(x-y)\,(w^{(0)}\,\wedge\,n)(y)\;ds(y),\end{aligned}\tag{40}$$

satisfies the Maxwell equations (30) in Ω_2 and then, we can deduce from the unique solvability of (30) in $\overline{H}^{loc}(\mathrm{curl},\Omega_2)$ that

$$i\omega\,\varepsilon_0\left(\nabla\,\wedge\,e_2^{(0)}\right)\,\wedge\,n|_{\partial\Omega_2}\;=\;\overrightarrow{curl}_{\partial\Omega_2}\,\varphi^{(0)}+\nabla_{\partial\Omega_2}\,\psi^{(0)}.$$

By Fredholm's alternative, the system $(35)-(36)$ is then uniquely solvable.

We are ready to prove the convergence of $(\varphi^{(\beta)},\psi^{(\beta)})$ to $(\varphi^{(0)},\psi^{(0)})$ in $H_0^{1/2}(\partial\Omega_2)\times H_0^{3/2}(\partial\Omega_2)$ strongly. We have the following lemma.

Lemma 6.3 *There exists $\beta_0>0$ such that for $0<\beta\leq\beta_0$ we have*

$$\begin{aligned}||\varphi^{(\beta)}-\varphi^{(0)}||_{H^{1/2}(\partial\Omega_2)}+||\psi^{(\beta)}-\psi^{(0)}||_{H^{3/2}(\partial\Omega_2)}\quad&\leq\quad C\,||w^{(\beta)}-w^{(0)}||_{TH^{-1/2}(\mathrm{curl},\partial\Omega_2)}\\ &\quad+C'\,\beta,\end{aligned}\tag{41}$$

where the two strictly positive constants C and C' are independent of β.

To prove (41), we shall need a classical result that we can formulate in an abstract manner in the following lemma.

Lemma 6.4 *Let V_1 and V_2 be two Hilbert spaces. Let a_i be a compact perturbation of a bilinear coercive form on $V_i \times V_i$ and b_i a compact form on $V_1 \times V_2$, for $i = 1, 2$. We consider the following variational system on $V_1 \times V_2$:*

$$\left\{ \begin{aligned} a_1(\varphi^\beta, \varphi_t) + b_1(\varphi_t, \psi^\beta) &= (f_1^\beta, \varphi_t), \\ a_2(\psi^\beta, \psi_t) + b_2(\varphi^\beta, \psi_t) &= (f_2^\beta, \psi_t), \end{aligned} \right. \tag{42}$$

where f_i^β is in V_i' which is the dual of the Hilbert space V_i. We assume that $||f_i^\beta||_{V_i'}$ converges to zero as β tends to zero and that the variational system

$$\left\{ \begin{aligned} a_1(\varphi, \varphi_t) + b_1(\varphi_t, \psi) &= 0, \\ a_2(\psi, \psi_t) + b_2(\varphi, \psi_t) &= 0, \end{aligned} \right. \tag{43}$$

has only the trivial solution in $V_1 \times V_2$. Then, there exist two strictly positive constants C and β_0 such that the estimate

$$||\varphi^\beta||_{V_1} + ||\psi^\beta||_{V_2} \leq C\Big(||f_1^\beta||_{V_1'} + ||f_2^\beta||_{V_2'}\Big), \tag{44}$$

holds uniformly for $0 < \beta < \beta_0$.

Proof. This lemma is a straightforward generalization of Lemma 4.5 proved in [1]. The proof follows the same compactness arguments. □

Now, we return to the proof of Lemma 6.3. If we replace $\varphi^{(0)}$ by $\varphi^{(\beta)} - \varphi^{(0)}$ in (35) and $\psi^{(0)}$ by $\psi^{(\beta)} - \psi^{(0)}$ in (36), we get the following system of integral equations :

$$\begin{aligned}
&a_1^{(0)}(\varphi^{(0)} - \varphi^{(\beta)}, \varphi_t) + b_1^{(0)}(\psi^{(0)} - \psi^{(\beta)}, \varphi_t) \\
&= i\sqrt{\frac{\varepsilon_0}{\mu_0}} \int_{\partial\Omega_2} \Big(w^{(\beta)} - w^{(0)} + f_2^{(\beta)} - f_2^{(0)}\Big)(x) \,.\, \overrightarrow{curl}_{\partial\Omega_2}\, \varphi_t(x)\; ds(x) \\
&+ \int_{\partial\Omega_2}\int_{\partial\Omega_2} \Big\{ \frac{1}{2\gamma^2\omega\sqrt{\varepsilon_0\mu_0}}(\gamma_1 g_1 + \gamma_2 g_2)(\beta, x-y) - g(x-y) \Big\} \\
&\overrightarrow{curl}_{\partial\Omega_2}\, \varphi^{(\beta)}(y) \,.\, \overrightarrow{curl}_{\partial\Omega_2}\, \varphi_t(x)\; ds(x)\, ds(y) \\
&- \frac{1}{2\omega\sqrt{\varepsilon_0\mu_0}} \int_{\partial\Omega_2}\int_{\partial\Omega_2} (g_1 - g_2)(\beta, x-y)\, \Big(\nabla_{\partial\Omega_2}\, \psi^{(\beta)}(y) \\
&+ \overrightarrow{curl}_{\partial\Omega_2}\, \varphi^{(\beta)}(y)\Big) \,.\, (n(x) - n(y))\, \Delta_{\partial\Omega_2}\, \varphi_t(x)\; ds(y)\, ds(x) \\
&+ R_1,
\end{aligned} \tag{45}$$

where the quantity R_1 is given by

$$R_1 = \frac{1}{2\omega\sqrt{\varepsilon_0\mu_0}} \int_{\partial\Omega_2} (R(x) - 2c(x)) \Big(\int_{\partial\Omega_2} (g_1 - g_2)(\beta, x-y) \Big(\nabla_{\partial\Omega_2} \psi^{(\beta)}(y)$$

$$+ \overrightarrow{curl}_{\partial\Omega_2} \varphi^{(\beta)}(y)) \wedge n(x)\, ds(y) \Big) \,.\, \overrightarrow{curl}_{\partial\Omega_2} \varphi_t(x)\, ds(x)$$

$$- \frac{1}{2\omega\sqrt{\varepsilon_0\mu_0}} \int_{\partial\Omega_2}\int_{\partial\Omega_2} \frac{\partial(g_1 - g_2)}{\partial n_x}(\beta, x-y) \Big((\nabla_{\partial\Omega_2} \psi^{(\beta)}(y)$$

$$+\overrightarrow{curl}_{\partial\Omega_2} \varphi^{(\beta)}(y)) \wedge n(x)\Big) \,.\, \overrightarrow{curl}_{\partial\Omega_2} \varphi_t(x)\, ds(y)\, ds(x)$$

$$+ \int_{\partial\Omega_2}\int_{\partial\Omega_2} \Big\{ \frac{1}{2\gamma^2\omega\sqrt{\varepsilon_0\mu_0}} (\gamma_1 g_1 + \gamma_2 g_2)(\beta, x-y) - g(x-y) \Big\}$$

$$\nabla_{\partial\Omega_2} \psi^{(\beta)}(y) \,.\, \overrightarrow{curl}_{\partial\Omega_2} \varphi_t(x)\, ds(x)\, ds(y),$$

and

$$a_2^{(0)}(\psi^{(0)} - \psi^{(\beta)}, \psi_t) + b_2^{(0)}(\varphi^{(0)} - \varphi^{(\beta)}, \psi_t)$$

$$= i\sqrt{\frac{\varepsilon_0}{\mu_0}} \int_{\partial\Omega_2} \Big(w^{(\beta)} - w^{(0)} + f_2^{(\beta)} - f_2^{(0)} \Big)(x) \,.\, \nabla_{\partial\Omega_2} \psi_t(x)\, ds(x)$$

$$+ \int_{\partial\Omega_2}\int_{\partial\Omega_2} \Big\{ \frac{1}{\gamma^2} (\gamma_2 g_1 + \gamma_1 g_2)(\beta, x-y) - \frac{2}{\omega\sqrt{\varepsilon_0\mu_0}} g(x-y) \Big\} \tag{46}$$

$$\Delta_{\partial\Omega_2} \psi^{(\beta)}(y)\, \Delta_{\partial\Omega_2} \psi_t(x)\, ds(x)\, ds(y)$$

$$+R_2,$$

where

$$R_2 = -\int_{\partial\Omega_2} (R(x) - 2c(x)) \Big(\int_{\partial\Omega_2} (g_1 - g_2)(\beta, x-y)\, (\nabla_{\partial\Omega_2} \psi^{(\beta)}(y)$$

$$+\overrightarrow{curl}_{\partial\Omega_2} \varphi^{(\beta)}(y)) \wedge n(x)\, ds(y) \Big) \,.\, \nabla_{\partial\Omega_2} \psi_t(x)\, ds(x)$$

$$+ \int_{\partial\Omega_2}\int_{\partial\Omega_2} \frac{\partial(g_2 - g_1)}{\partial n_x}(\beta, x-y) \Big((\nabla_{\partial\Omega_2} \psi^{(\beta)}(y)$$

$$+\overrightarrow{curl}_{\partial\Omega_2} \varphi^{(\beta)}(y)) \wedge n(x)\Big) \,.\, \nabla_{\partial\Omega_2} \psi_t(x)\, ds(y)\, ds(x)$$

$$- \int_{\partial\Omega_2}\int_{\partial\Omega_2} \Big\{ (\gamma_1 g_1 + \gamma_2 g_2)(\beta, x-y) - 2\omega\sqrt{\varepsilon_0\mu_0}\, g(x-y) \Big\}$$

$$\nabla_{\partial\Omega_2} \psi^{(\beta)}(y) \,.\, \nabla_{\partial\Omega_2} \psi_t(x)\, ds(x)\, ds(y)$$

$$- \int_{\partial\Omega_2}\int_{\partial\Omega_2} \Big\{ (\gamma_1 g_1 + \gamma_2 g_2)(\beta, x-y) - 2\omega\sqrt{\varepsilon_0\mu_0}\, g(x-y) \Big\}$$

$$\overrightarrow{curl}_{\partial\Omega_2} \varphi^{(\beta)}(y) \,.\, \nabla_{\partial\Omega_2} \psi_t(x)\, ds(x)\, ds(y).$$

In order to prove Lemma 6.3, we shall observe that

$$\begin{aligned}
&\int_{\partial\Omega_2} \left(f_2^{(\beta)} - f_2^{(0)}\right)(x) \,.\, \overrightarrow{curl}_{\partial\Omega_2}\, \varphi_t(x)\; ds(x) \\
&= \int_{\partial\Omega_2} \frac{1}{2}\nabla \wedge \Big(\int_{\partial\Omega_2} (g_1+g_2)(\beta, x-y)\,(w^{(\beta)} \;\wedge n)(y) \\
&-g(x-y)\,(w^{(0)} \;\wedge n)(y)\; ds(y)\Big) \,.\, \overrightarrow{curl}_{\partial\Omega_2}\, \varphi_t(x)\; ds(x) \\
&-\frac{1}{2}\int_{\partial\Omega_2} \left(w^{(\beta)}(x) - w^{(0)}(x)\right) \,.\, \overrightarrow{curl}_{\partial\Omega_2}\, \varphi_t(x)\; ds(x),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\partial\Omega_2} \left(f_2^{(\beta)} - f_2^{(0)}\right)(x) \,.\, \nabla_{\partial\Omega_2}\, \psi_t(x)\; ds(x) \\
&= \frac{1}{2\gamma^2}\int_{\partial\Omega_2} \nabla \int_{\partial\Omega_2} (\gamma_2 g_1 - \gamma_1 g_2)(\beta, x-y)\, \mathrm{div}_{\partial\Omega_2}\,(w^{(\beta)} \;\wedge n)(y)\; ds(y) \\
&-\frac{1}{2}\int_{\partial\Omega_2} \left(w^{(\beta)}(x) - w^{(0)}(x)\right) \,.\, \nabla_{\partial\Omega_2}\, \psi_t(x)\; ds(x).
\end{aligned}$$

The integral operator with kernel $\gamma_2 g_1 - \gamma_1 g_2$ is of order -2. We shall also need to examine the kernels

$$\frac{1}{\gamma^2}\,(\gamma_2 g_1 + \gamma_1 g_2) - \frac{2}{\omega\sqrt{\varepsilon_0\mu_0}} g, \,(\gamma_1 g_1 + \gamma_2 g_2) - 2\omega\sqrt{\varepsilon_0\mu_0} g,$$

and $g_1 - g_2$. It is easily seen that there exist three kernels $(\mathcal{G}_j)_{j=1}^3$ which are uniformly bounded with respect to chirality admittance β such that

$$\frac{1}{\gamma^2}\,(\gamma_2 g_1 + \gamma_1 g_2)(\beta, x-y) - \frac{2}{\omega\sqrt{\varepsilon_0\mu_0}} g(x-y) \;=\; \beta\,\mathcal{G}_1(\beta, x-y),$$

$$(\gamma_1 g_1 + \gamma_2 g_2)(\beta, x-y) - 2\omega\sqrt{\varepsilon_0\mu_0} g(x-y) \;=\; \beta\,\mathcal{G}_2(\beta, x-y),$$

and

$$(g_1 - g_2)(\beta, x-y) \;=\; \beta\,\mathcal{G}_3(\beta, x-y).$$

Indeed, since $\mathcal{G}_j(\beta, x-y) = 0(1)$ as $|x-y|$ tends to zero the integral operators with kernels $(\mathcal{G})_j$ are of order -2. It follows that

$$\begin{aligned}
&|a_1^{(0)}(\varphi^{(0)} - \varphi^{(\beta)}, \varphi_t) + b_1^{(0)}(\psi^{(0)} - \psi^{(\beta)}, \varphi_t)| \\
&\leq C\,||w^{(0)} - w^{(\beta)}||_{TH^{-1/2}(\mathrm{curl}, \partial\Omega_2)}\, ||\varphi_t||_{H^{1/2}(\partial\Omega_2)} \\
&+0(\beta)\left(||\varphi^{(0)} - \varphi^{(\beta)}||_{H^{1/2}(\partial\Omega_2)} + ||\psi^{(0)} - \psi^{(\beta)}||_{H^{3/2}(\partial\Omega_2)}\right) ||\varphi_t||_{H^{1/2}(\partial\Omega_2)} \qquad (47) \\
&+0(\beta)\, ||\varphi_t||_{H^{1/2}(\partial\Omega_2)},
\end{aligned}$$

and

$$
\begin{aligned}
&|a_2^{(0)}(\psi^{(0)} - \psi^{(\beta)}, \psi_t) + b_2^{(0)}(\varphi^{(0)} - \varphi^{(\beta)}, \psi_t)| \\
&\leq C\,||w^{(0)} - w^{(\beta)}||_{TH^{-1/2}(\mathrm{curl},\partial\Omega_2)}\,||\psi_t||_{H^{3/2}(\partial\Omega_2)} \\
&+0(\beta)\left(||\varphi^{(0)} - \varphi^{(\beta)}||_{H^{1/2}(\partial\Omega_2)} + ||\psi^{(0)} - \psi^{(\beta)}||_{H^{3/2}(\partial\Omega_2)}\right)||\psi_t||_{H^{1/2}(\partial\Omega_2)} \\
&+0(\beta)\,||\psi_t||_{H^{3/2}(\partial\Omega_2)}.
\end{aligned} \tag{48}
$$

Recalling that the system

$$
\left\{
\begin{aligned}
a_1^{(0)}(\varphi, \varphi_t) + b_1^{(0)}(\psi, \varphi_t) &= 0, \quad \forall\, \varphi_t \in H^{1/2}(\partial\Omega_2), \\
a_2^{(0)}(\psi, \psi_t) + b_2^{(0)}(\varphi, \psi_t) &= 0, \quad \forall\, \psi_t \in H^{3/2}(\partial\Omega_2),
\end{aligned}
\right.
$$

has only the trivial solution in $H_0^{1/2}(\partial\Omega_2) \times H_0^{3/2}(\partial\Omega_2)$, we obtain by using Lemma 6.4 the claim.

It is now easy to check from Lemma 6.3 together with the integral representation formula (32) and

$$
\begin{aligned}
e_2^{(0)}(x) \;&=\; f_2^{(0)}(x) + i\omega\mu_0 \int_{\partial\Omega_2} g(x-y)\, u^{(0)}(y)\; ds(y) \\
&\quad + \frac{i}{\omega\varepsilon_0}\nabla \int_{\partial\Omega_2} g(x-y)\, \mathrm{div}_{\partial\Omega_2}\, u^{(0)}(y)\; ds(y),
\end{aligned}
$$

where

$$
u^{(0)} \;=\; \overrightarrow{curl}_{\partial\Omega_2}\, \varphi^{(0)} + \nabla_{\partial\Omega_2}\, \psi^{(0)},
$$

that the optimal estimate (31) holds. The proof of Theorem 6.1 is then over.

7 Convergence of the Solution to the Scattering Problem for the Drude-Born-Fedorov Equations with Variable Coefficients

With the analysis of the preceding sections, we are now prepared to study the behavior of the solution to the scattering problem for the Drude-Born-Fedorov equations (5) as the chirality admittance tends to zero. We have the following result which is the main result of this paper.

Theorem 7.1 *Let* $\Omega_{h''}$ *be* $\partial\Omega \times\,]\,0, h''\,[$ *where* $h'' > 0$*. Then, there exists* $\beta_0 > 0$ *such that for* $0 < \beta \leq \beta_0$ *the following estimate holds*

$$
||E^{(\beta)} - E^{(0)}||_{H(\mathrm{curl},\Omega_{h''})} \;\leq C\,\beta + C'\,||f^{(\beta)} - f^{(0)}||_{TH^{-1/2}(\mathrm{curl},\partial\Omega)},
$$

and

$$||H^{(\beta)} - H^{(0)}||_{H(\text{curl},\Omega_{h''})} \;\leq C\,\beta + C'\,||f^{(\beta)} - f^{(0)}||_{TH^{-1/2}(\text{curl},\partial\Omega)},$$

where the two strictly positive constants C and C' are independent of β and $(E^{(\beta)}, H^{(\beta)})$ the solutions of the Drude-Born-Fedorov equations (5) *and* $(E^{(0)}, H^{(0)})$ *are the solutions of the Maxwell system* (6).

We shall expect an integral representation of $E^{(\beta)}$ and $H^{(\beta)}$ by means of the pseudo-differential operator $T_1^{(\beta)}$ defined by (9). As we shall see, to reduce the Drude-Born-Fedorov equations (3) to a system of two integral equations on $\partial\Omega_2$, it suffices to write the tangential component of the electric field $E^{(\beta)}$ as a function of the tangential component of the magnetic field $H^{(\beta)}$ in the integral representation formula (32). In (5) the tangential component of the electric field $E^{(\beta)}$ is unknown on the boundary $\partial\Omega_2$. It will be related to the tangential component of the magnetic field $H^{(\beta)}$ with the help of the interior electromagnetic operator $T_1^{(\beta)}$. We shall need the operator $S^{(\beta)}$ mapping the tangential component of $E^{(\beta)}$ onto the tangential component of $f_2^{(\beta)}$ on $\partial\Omega_2$ to prove the convergence of the solution of the scattering problem for the Drude-Born-Fedorov equations (5) to the solution of the scattering problem for the Maxwell equations (6).

To prove this theorem, let us observe that from the uniform estimates proved in Theorem 5.1 we obtain the following lemma.

Lemma 7.1 *There exists $\beta_0 > 0$, such that for $0 < \beta < \beta_0$, the pseudo-differential operator $T_1^{(\beta)}$ from $TH^{-1/2}(\text{curl}, \partial\Omega_2)$ into itself is invertible and its inverse is given by the Neuman series :*

$$(T_1^{(\beta)})^{-1} \;=\; (T_1^{(0)})^{-1} + \sum_{n\geq 1} (T_1^{(0)})^{-1}\Big(T_1^{(0)} - T_1^{(\beta)}\Big)^n (T_1^{(0)})^{-1}. \tag{49}$$

Let us now decompose $n \wedge H^{(\beta)}$ on $\partial\Omega_2$ which is in the space $TH^{-1/2}(\text{div}, \partial\Omega_2)$ as follows

$$n \,\wedge\, H^{(\beta)} \;=\; \overrightarrow{curl}_{\partial\Omega_2}\,\varphi^{(\beta)} + \nabla_{\partial\Omega_2}\,\psi^{(\beta)},$$

where $(\varphi^{(\beta)}, \psi^{(\beta)}) \in H_0^{1/2}(\partial\Omega_2) \times H_0^{3/2}(\partial\Omega_2)$ and the tangential component of the electric field is then given by $(T_1^{(\beta)})^{-1}\Big(\nabla_{\partial\Omega_2}\,\varphi^{(\beta)} - \overrightarrow{curl}_{\partial\Omega_2}\,\psi^{(\beta)}\Big)$.

By using the operators $S^{(\beta)}$ and $T_1^{(\beta)}$, we obtain that the pair $(\varphi^{(\beta)}, \psi^{(\beta)})$ satisfies the following system of two integral equations on $\partial\Omega_2$ given by

$$\begin{aligned} & a_1^{(\beta)}(\varphi^{(\beta)}, \varphi_t) + b_1^{(\beta)}(\psi^{(\beta)}, \varphi_t) \\ & = i\sqrt{\frac{\varepsilon_0}{\mu_0}}\int_{\partial\Omega_2} (I + S^{(\beta)})\Big((T_1^{(\beta)})^{-1}\Big(\nabla_{\partial\Omega_2}\,\varphi^{(\beta)} - \overrightarrow{curl}_{\partial\Omega_2}\,\psi^{(\beta)}\Big)\Big)(x)\,.\,\overrightarrow{curl}_{\partial\Omega_2}\,\varphi_t(x)\;ds(x), \end{aligned} \tag{50}$$

and

$$a_2^{(\beta)}(\psi^{(\beta)},\psi_t)+b_2^{(\beta)}(\varphi^{(\beta)},\psi_t)$$
$$= i\sqrt{\frac{\varepsilon_0}{\mu_0}}\int_{\partial\Omega_2}(I+S^{(\beta)})\Big((T_1^{(\beta)})^{-1}\Big(\nabla_{\partial\Omega_2}\varphi^{(\beta)}-\overrightarrow{curl}_{\partial\Omega_2}\psi^{(\beta)}\Big)\Big)(x)\,.\,\nabla_{\partial\Omega_2}\psi_t(x)\;ds(x),$$
(51)

where the operator $S^{(\beta)}$ is defined by (37).
Analogously to the system $(50)-(51)$, we can show that the Maxwell equations (6) are equivalent to the system

$$a_1^{(0)}(\varphi^{(0)},\varphi_t)+b_1^{(0)}(\psi^{(0)},\varphi_t)$$
$$= i\sqrt{\frac{\varepsilon_0}{\mu_0}}\int_{\partial\Omega_2}(I+S^{(0)})\Big((T_1^{(0)})^{-1}\Big(\nabla_{\partial\Omega_2}\varphi^{(0)}-\overrightarrow{curl}_{\partial\Omega_2}\psi^{(0)}\Big)\Big)(x)\,.\,\overrightarrow{curl}_{\partial\Omega_2}\varphi_t(x)\;ds(x),$$
(52)

and

$$a_2^{(0)}(\psi^{(0)},\psi_t)+b_2^{(0)}(\varphi^{(0)},\psi_t)$$
$$= i\sqrt{\frac{\varepsilon_0}{\mu_0}}\int_{\partial\Omega_2}(I+S^{(0)})\Big((T_1^{(0)})^{-1}\Big(\nabla_{\partial\Omega_2}\varphi^{(0)}-\overrightarrow{curl}_{\partial\Omega_2}\psi^{(0)}\Big)\Big)(x)\,.\,\nabla_{\partial\Omega_2}\psi_t(x)\;ds(x),$$
(53)

where $(\varphi^{(0)},\psi^{(0)})\in H_0^{1/2}(\partial\Omega_2)\times H_0^{3/2}(\partial\Omega_2)$. Here we recall that the operators $T_1^{(0)}$ and $S^{(0)}$ are defined respectively by (10) and (38).
Note that the electric field $E^{(0)}\in\overline{H}^{loc}(\mathrm{curl},\Omega)$ is given from $(\varphi^{(0)},\psi^{(0)})$ by the integral representation (40) in Ω_2 and by the solution of the interior problem (8) with

$$w^{(0)}=\nabla_{\partial\Omega_2}\varphi^{(0)}+\overrightarrow{curl}_{\partial\Omega_2}\psi^{(0)}.$$

We also observe that in the systems $(50)-(51)$ and $(52)-(53)$ the effect of the dielectric layer Ω_1 where the coefficients are variable is expressed by means of the interior pseudo-differential operators $T_1^{(\beta)}$ and $T_1^{(0)}$ respectively.
Now, our aim is to prove the convergence of $(\varphi^{(\beta)},\psi^{(\beta)})$ to $(\varphi^{(0)},\psi^{(0)})$ in $H_0^{1/2}(\partial\Omega_2)\times H_0^{3/2}(\partial\Omega_2)$ when β tends to zero and to estimate its rate. To this end, we use the same techniques as in Lemma 6.3 for the scattering problem for the homogeneous Drude-Born-Fedorov equations. We must estimate the quantities

$$\begin{aligned}\mathcal{A}_1^{(\beta)} &= a_1^{(0)}(\varphi^{(0)}-\varphi^{(\beta)},\varphi_t)+b_1^{(0)}(\psi^{(0)}-\psi^{(\beta)},\varphi_t)\\ &-i\sqrt{\frac{\varepsilon_0}{\mu_0}}\int_{\partial\Omega_2}(I+S^{(0)})\Big((T_1^{(0)})^{-1}\Big(\nabla_{\partial\Omega_2}\left(\varphi^{(0)}-\varphi^{(\beta)}\right)\\ &-\overrightarrow{curl}_{\partial\Omega_2}\left(\psi^{(0)}-\psi^{(\beta)}\right)\Big)\Big)(x)\,.\,\overrightarrow{curl}_{\partial\Omega_2}\varphi_t(x)\;ds(x),\end{aligned}$$

and

$$\mathcal{A}_2^{(\beta)} = a_2^{(0)}(\psi^{(0)} - \psi^{(\beta)}, \psi_t) + b_2^{(0)}(\varphi^{(0)} - \varphi^{(\beta)}, \psi_t)$$
$$- i\sqrt{\frac{\varepsilon_0}{\mu_0}} \int_{\partial\Omega_2} (I + S^{(0)}) \Big((T_1^{(0)})^{-1} \Big(\nabla_{\partial\Omega_2} \left(\varphi^{(0)} - \varphi^{(\beta)} \right)$$
$$- \overrightarrow{curl}_{\partial\Omega_2} \left(\psi^{(0)} - \psi^{(\beta)} \right) \Big) \Big)(x) \,.\, \nabla_{\partial\Omega_2} \psi_t(x) \, ds(x),$$

and prove that the system (52) – (53) is of Fredholm type. As in Lemma 6.3, by Lemma 6.1 we will obtain the claim.
We get from easy manipulations that the quantity $\mathcal{A}_1^{(\beta)}$ is equal to

$$i\sqrt{\frac{\varepsilon_0}{\mu_0}} \int_{\partial\Omega_2} \Big((T_1^{(\beta)})^{-1} - (T_1^{(0)})^{-1} \Big) \Big(\nabla_{\partial\Omega_2} \varphi^{(\beta)} - \overrightarrow{curl}_{\partial\Omega_2} \psi^{(\beta)} \Big)(x) \,.\, \overrightarrow{curl}_{\partial\Omega_2} \varphi_t(x) \, ds(x)$$
$$+ i\sqrt{\frac{\varepsilon_0}{\mu_0}} \int_{\partial\Omega_2} (S^{(0)} - S^{(\beta)}) \Big((T_1^{(\beta)})^{-1} \Big(\nabla_{\partial\Omega_2} \varphi^{(\beta)} - \overrightarrow{curl}_{\partial\Omega_2} \psi^{(\beta)} \Big) \Big)(x) \,.\, \overrightarrow{curl}_{\partial\Omega_2} \varphi_t(x) \, ds(x)$$
$$+ R_1',$$

where

$$R_1' = \int_{\partial\Omega_2} \int_{\partial\Omega_2} \Big\{ \frac{1}{2\gamma^2\omega\sqrt{\varepsilon_0\mu_0}} (\gamma_1 g_1 + \gamma_2 g_2)(\beta, x-y) - g(x-y) \Big\}$$
$$\overrightarrow{curl}_{\partial\Omega_2} \varphi^{(\beta)}(y) \,.\, \overrightarrow{curl}_{\partial\Omega_2} \varphi_t(x) \, ds(x) \, ds(y)$$
$$- \frac{1}{2\omega\sqrt{\varepsilon_0\mu_0}} \int_{\partial\Omega_2} \int_{\partial\Omega_2} (g_1 - g_2)(\beta, x-y) \, \Big(\nabla_{\partial\Omega_2} \psi^{(\beta)}(y)$$
$$+ \overrightarrow{curl}_{\partial\Omega_2} \varphi^{(\beta)}(y) \Big) \,.\, (n(x) - n(y)) \, \Delta_{\partial\Omega_2} \varphi_t(x) \, ds(y) \, ds(x)$$
$$+ \frac{1}{2\omega\sqrt{\varepsilon_0\mu_0}} \int_{\partial\Omega_2} (R(x) - 2c(x)) \Big(\int_{\partial\Omega_2} (g_1 - g_2)(\beta, x-y) \, \Big(\nabla_{\partial\Omega_2} \psi^{(\beta)}(y)$$
$$+ \overrightarrow{curl}_{\partial\Omega_2} \varphi^{(\beta)}(y)) \wedge n(x) \, ds(y) \Big) \,.\, \overrightarrow{curl}_{\partial\Omega_2} \varphi_t(x) \, ds(x)$$
$$- \frac{1}{2\omega\sqrt{\varepsilon_0\mu_0}} \int_{\partial\Omega_2} \int_{\partial\Omega_2} \frac{\partial (g_1 - g_2)}{\partial n_x}(\beta, x-y) \, \Big((\nabla_{\partial\Omega_2} \psi^{(\beta)}(y)$$
$$+ \overrightarrow{curl}_{\partial\Omega_2} \varphi^{(\beta)}(y)) \wedge n(x) \Big) \,.\, \overrightarrow{curl}_{\partial\Omega_2} \varphi_t(x) \, ds(y) \, ds(x)$$
$$+ \int_{\partial\Omega_2} \int_{\partial\Omega_2} \Big\{ \frac{1}{2\gamma^2\omega\sqrt{\varepsilon_0\mu_0}} (\gamma_1 g_1 + \gamma_2 g_2)(\beta, x-y) - g(x-y) \Big\}$$
$$\nabla_{\partial\Omega_2} \psi^{(\beta)}(y) \,.\, \overrightarrow{curl}_{\partial\Omega_2} \varphi_t(x) \, ds(x) \, ds(y),$$

and the quantity $\mathcal{A}_2^{(\beta)}$ is equal to

$$i\sqrt{\frac{\varepsilon_0}{\mu_0}}\int_{\partial\Omega_2}\Big((T_1^{(\beta)})^{-1}-(T_1^{(0)})^{-1}\Big)\Big(\nabla_{\partial\Omega_2}\varphi^{(\beta)}-\overrightarrow{curl}_{\partial\Omega_2}\psi^{(\beta)}\Big)(x)\,.\,\nabla_{\partial\Omega_2}\psi_t(x)\,ds(x)$$
$$+i\sqrt{\frac{\varepsilon_0}{\mu_0}}\int_{\partial\Omega_2}(S^{(0)}-S^{(\beta)})\Big((T_1^{(\beta)})^{-1}\Big(\nabla_{\partial\Omega_2}\varphi^{(\beta)}-\overrightarrow{curl}_{\partial\Omega_2}\psi^{(\beta)}\Big)\Big)(x)\,.\,\nabla_{\partial\Omega_2}\psi_t(x)\,ds(x)$$
$$+R_2',$$

where the quantity R_2' is given by

$$\begin{aligned}
R_2' =& \int_{\partial\Omega_2}\int_{\partial\Omega_2}\Big\{\frac{1}{\gamma^2}(\gamma_2 g_1+\gamma_1 g_2)(\beta,x-y)-\frac{2}{\omega\sqrt{\varepsilon_0\mu_0}}g(x-y)\Big\}\\
&\Delta_{\partial\Omega_2}\psi^{(\beta)}(y)\,\Delta_{\partial\Omega_2}\psi_t(x)\,ds(x)\,ds(y)\\
&-\int_{\partial\Omega_2}(R(x)-2c(x))\Big(\int_{\partial\Omega_2}(g_1-g_2)(\beta,x-y)\,(\nabla_{\partial\Omega_2}\psi^{(\beta)}(y)\\
&+\overrightarrow{curl}_{\partial\Omega_2}\varphi^{(\beta)}(y))\wedge n(x)\,ds(y)\Big)\,.\,\nabla_{\partial\Omega_2}\psi_t(x)\,ds(x)\\
&+\int_{\partial\Omega_2}\int_{\partial\Omega_2}\frac{\partial(g_2-g_1)}{\partial n_x}(\beta,x-y)\,\Big((\nabla_{\partial\Omega_2}\psi^{(\beta)}(y)\\
&\quad+\overrightarrow{curl}_{\partial\Omega_2}\varphi^{(\beta)}(y))\wedge n(x)\Big)\,.\,\nabla_{\partial\Omega_2}\psi_t(x)\,ds(y)\,ds(x)\\
&\quad-\int_{\partial\Omega_2}\int_{\partial\Omega_2}\Big\{(\gamma_1 g_1+\gamma_2 g_2)(\beta,x-y)-2\omega\sqrt{\varepsilon_0\mu_0}g(x-y)\Big\}\\
&\quad\nabla_{\partial\Omega_2}\psi^{(\beta)}(y)\,.\,\nabla_{\partial\Omega_2}\psi_t(x)\,ds(x)\,ds(y)\\
&\quad-\int_{\partial\Omega_2}\int_{\partial\Omega_2}\Big\{(\gamma_1 g_1+\gamma_2 g_2)(\beta,x-y)-2\omega\sqrt{\varepsilon_0\mu_0}g(x-y)\Big\}\\
&\quad\overrightarrow{curl}_{\partial\Omega_2}\varphi^{(\beta)}(y)\,.\,\nabla_{\partial\Omega_2}\psi_t(x)\,ds(x)\,ds(y).
\end{aligned}$$

Note that the only difference between this system and $(45)-(46)$ is that, $w^{(\beta)}-w^{(0)}$ is replaced here by

$$\begin{aligned}
&(T_1^{(0)})^{-1}\Big(\nabla_{\partial\Omega_2}\Big(\varphi^{(0)}-\varphi^{(\beta)}\Big)-\overrightarrow{curl}_{\partial\Omega_2}\Big(\psi^{(0)}-\psi^{(\beta)}\Big)\Big)\\
&+\Big((T_1^{(\beta)})^{-1}-(T_1^{(0)})^{-1}\Big)\Big(\nabla_{\partial\Omega_2}\varphi^{(\beta)}-\overrightarrow{curl}_{\partial\Omega_2}\psi^{(\beta)}\Big).
\end{aligned}$$

The estimates of the terms R_1 and R_2 when β tends to zero remain valid for R_1' and R_2'. To prove Theorem 7.1, we only need to estimate the following terms

$$(54)\quad \int_{\partial\Omega_2}\Big((T_1^{(\beta)})^{-1}-(T_1^{(0)})^{-1}\Big)\Big(\nabla_{\partial\Omega_2}\varphi^{(\beta)}-\overrightarrow{curl}_{\partial\Omega_2}\psi^{(\beta)}\Big)\,.\,\overrightarrow{curl}_{\partial\Omega_2}\varphi_t,$$

$$(55)\quad \int_{\partial\Omega_2}\Big((T_1^{(\beta)})^{-1}-(T_1^{(0)})^{-1}\Big)\Big(\nabla_{\partial\Omega_2}\,\varphi^{(\beta)}-\overrightarrow{curl}_{\partial\Omega_2}\,\psi^{(\beta)}\Big)\,.\,\nabla_{\partial\Omega_2}\,\psi_t,$$

$$(56)\quad \int_{\partial\Omega_2}(S^{(\beta)}-S^{(0)})\Big((T_1^{(\beta)})^{-1}\Big(\nabla_{\partial\Omega_2}\,\varphi^{(\beta)}-\overrightarrow{curl}_{\partial\Omega_2}\,\psi^{(\beta)}\Big)\Big)\,.\,\overrightarrow{curl}_{\partial\Omega_2}\,\varphi_t,$$

and

$$(57)\quad \int_{\partial\Omega_2}(S^{(\beta)}-S^{(0)})\Big((T_1^{(\beta)})^{-1}\Big(\nabla_{\partial\Omega_2}\,\varphi^{(\beta)}-\overrightarrow{curl}_{\partial\Omega_2}\,\psi^{(\beta)}\Big)\Big)\,.\,\nabla_{\partial\Omega_2}\,\psi_t.$$

By Lemma 7.1, we see that

$$(58)\qquad (T_1^{(\beta)})^{-1}-(T_1^{(0)})^{-1}\;=\;\sum_{n\geq 1}(T_1^{(0)})^{-1}\Big(T_1^{(0)}-T_1^{(\beta)}\Big)^n(T_1^{(0)})^{-1}.$$

It follows that (54) and (56) are

$$0(\beta)\,\Big(||\varphi^{(\beta)}||_{H^{1/2}(\partial\Omega_2)}+||\psi^{(\beta)}||_{H^{3/2}(\partial\Omega_2)}\Big)\,||\varphi_t||_{H^{1/2}(\partial\Omega_2)}.$$

Similarly, (55) and (57) are

$$0(\beta)\,\Big(||\varphi^{(\beta)}||_{H^{1/2}(\partial\Omega_2)}+||\psi^{(\beta)}||_{H^{3/2}(\partial\Omega_2)}\Big)\,||\psi_t||_{H^{3/2}(\partial\Omega_2)}.$$

Finally, it follows that

$$(59)\qquad \begin{aligned}\mathcal{A}_1^{(\beta)}\;=&\;0(\beta)\,||\varphi_t||_{H^{1/2}(\partial\Omega_2)}\\ &+0(\beta)\,\Big(||\varphi^{(\beta)}||_{H^{1/2}(\partial\Omega_2)}+||\psi^{(\beta)}||_{H^{3/2}(\partial\Omega_2)}\Big)\,||\varphi_t||_{H^{1/2}(\partial\Omega_2)},\end{aligned}$$

and

$$(60)\qquad \begin{aligned}\mathcal{A}_2^{(\beta)}\;=&\;0(\beta)\,||\psi_t||_{H^{3/2}(\partial\Omega_2)}\\ &+0(\beta)\,\Big(||\varphi^{(\beta)}||_{H^{1/2}(\partial\Omega_2)}+||\psi^{(\beta)}||_{H^{3/2}(\partial\Omega_2)}\Big)\,||\psi_t||_{H^{3/2}(\partial\Omega_2)}.\end{aligned}$$

Now, we are able to state and prove the following lemma.

Lemma 7.2 *There exists $\beta_0>0$, such that for $0<\beta<\beta_0$ the following estimate holds*

$$(61)\qquad \begin{aligned}||\varphi^{(\beta)}-\varphi^{(0)}||_{H^{1/2}(\partial\Omega_2)}+||\psi^{(\beta)}-\psi^{(0)}||_{H^{3/2}(\partial\Omega_2)}\;\;\leq\;&\;C\,||f^{(\beta)}-f^{(0)}||_{TH^{-1/2}(\mathrm{curl},\partial\Omega)}\\ &+C'\,\beta,\end{aligned}$$

where the constants C and C' are independent of β.

The proof of this lemma is essentially the same that the proof of the Lemma 6.3. The only difference is that we have a slightly different system. To apply Lemma 6.4, we must extend Lemma 6.2 to this system. More precisely, we must prove that the system of the two integral equations (59) and (60) is a compact perturbation of an invertible one. This result has its own importance and seems to be new. It shows the "compactness" of the effect of a dielectric layer with variable coefficients, possibly non-smooth, on the integral equations related to the Maxwell system.

Lemma 7.3 *Let (f_1, f_2) be in $H^{-1/2}(\partial\Omega_2) \times H^{-3/2}(\partial\Omega_2)$. The system*

$$a_1^{(0)}(\varphi, \varphi_t) + b_1^{(0)}(\psi, \varphi_t)$$
$$-i\sqrt{\frac{\varepsilon_0}{\mu_0}} \int_{\partial\Omega_2} (I + S^{(0)})\Big((T_1^{(0)})^{-1}\Big(\nabla_{\partial\Omega_2}\varphi - \overrightarrow{curl}_{\partial\Omega_2}\psi\Big)(x) \,.\, \overrightarrow{curl}_{\partial\Omega_2}\varphi_t(x)\; ds(x)$$
$$= (f_1, \varphi_t), \forall\, \varphi_t \in H^{1/2}(\partial\Omega_2), \tag{62}$$

and

$$a_2^{(0)}(\psi, \psi_t) + b_2^{(0)}(\varphi, \psi_t)$$
$$-i\sqrt{\frac{\varepsilon_0}{\mu_0}} \int_{\partial\Omega_2} (I + S^{(0)})\Big((T_1^{(0)})^{-1}\Big(\nabla_{\partial\Omega_2}\varphi - \overrightarrow{curl}_{\partial\Omega_2}\psi\Big)(x) \,.\, \nabla_{\partial\Omega_2}\psi_t(x)\; ds(x)$$
$$= (f_2, \psi_t), \forall\, \psi_t \in H^{3/2}(\partial\Omega_2), \tag{63}$$

in $H_0^{1/2}(\partial\Omega_2) \times H_0^{3/2}(\partial\Omega_2)$ is of Fredholm type.

Proof. Let us define the operator $\mathcal{T}_1^{(0)}$ by

$$w^{(0)} \in TH^{-1/2}(\mathrm{curl}, \partial\Omega_2) \longmapsto -\frac{i}{\omega\,\mu_0}\, n \wedge \Big(n \wedge \Big(\nabla \wedge e^{(0)}\Big)\Big), \tag{64}$$

in $TH^{-1/2}(\mathrm{curl}, \partial\Omega_2)$ where $e^{(0)}$ is the unique solution of the interior homogeneous Maxwell equations given by

$$\begin{cases} (\nabla \wedge)^2\, e^{(0)} - \omega^2\, e^{(0)} \;=\; 0 \quad \text{in}\, \Omega_1, \\ n \wedge \Big(n \wedge e^{(0)}\Big) \;=\; w^{(0)} \quad \text{on}\, \partial\Omega_2, \\ n \wedge \Big(n \wedge e^{(0)}\Big) \;=\; f^{(0)} \quad \text{on}\, \partial\Omega. \end{cases} \tag{65}$$

From [5] it follows that the system of the two integral equations obtained by replacing the pseudo-differential operator $T_1^{(0)}$ by $\mathcal{T}_1^{(0)}$ in (62) – (63) is of Fredholm type. Now, since $(\mathcal{T}_1^{(0)})^{-1} = (T_1^{(0)})^{-1} \bmod OPS^{-\infty}$ (see for instance [3]

where a very closed result is proved), it follows that the system of the two integral equations (62) and (63) is of Fredholm type. □
Now, the proof of Theorem 7.1 follows then from Lemma 6.4, Lemma 7.3 and the results of convergence of the interior and exterior Drude-Born-Fedorov system established in the above sections.

References

[1] T. Abboud and H. Ammari, Diffraction at a curved grating : Approximation by an infinite plane grating, *J. Math. Anal. Appl.*, **202**, 995-1026 (1996).

[2] H. Ammari and J.C. Nédélec, Time-harmonic electromagnetic fields in thin chiral curved layers, *SIAM J. Math. Anal.* (to appear).

[3] H. Ammari and J.C. Nédélec, Time-harmonic electromagnetic fields in chiral media, (submitted).

[4] H. Ammari and J.C. Nédélec, Sur les conditions d'impédance généralisées, *C. R. Acad. Sci. Paris, Série I,* **322**, 995-1000, (1996).

[5] H. Ammari and J.C. Nédélec, Équation intégrale et diffraction dans un milieu chiral, *C. R. Acad. Sci. Paris, Série I,* **322**, 1087-1091, (1996).

[6] D.F. Arago, Mémoire sur une modification remarquable qu'éprouvent les rayons lumineux dans leur passage à travers certains corps diaphanes, et sur quelques autres nouveaux phénomènes d'optique, *Mém. Sci. Math. Phys. Inst.*, 93-134 (1811).

[7] A. Bendali, Numerical analysis of the exterior boundary value problem for the time-harmonic Maxwell equations by a boundary finite element, Part I: the continuous problem, *Math. Comp.*, **43**, 29-46 (1984).

[8] J.B. Biot, Mémoire sur un nouveau genre d'oscillation que les molécules de la lumière éprouvent en traversant certains cristaux, *Mém. Sci. Math. Phys. Inst.,* 1-372 (1812).

[9] A.L. Cauchy, *Mémoire sur l'application de l'analyse mathématique à la recherche des lois générales observées par des physiciens, et, en particulier, sur les lois de polarisation circulaire, C. R. Acad. Sci., Paris,* **15**, 910-918 (1842).

[10] D. Colton and R. Kress, Integral equation methods in scattering theory, Wiley-Interscience, 1983.

[11] A. Fresnel, Extrait d'un mémoire sur la double réfraction particulière que présente le cristal de roche dans la direction de son axe, *Bull. Soc. Philomat.,* (1822), 191-198.

[12] A. de La Bourdonnaye, Décomposition de $TH_{\mathrm{div}}^{-1/2}(\partial\Omega_2)$ et nature de l'opérateur de Steklov-Poincaré du problème extérieur de l'électromagnétisme, *C. R. Acad. Sci. Paris, Série I,* **316**, 369-372 (1993).

[13] I.V. Lindell, J. C-E. Sten and R.E. Kleinman, Low frequency image theory for the dielectric sphere, *J. Elect. Waves and Appl.*, **8**, 295-313 (1994).

[14] R. Kress, On the limiting behaviour of solutions to boundary integral equations associated with time harmonic wave equations for small frequencies, *Math. Meth. in Appl. Sci.* **1**, 89-100 (1979).

[15] A. Lakhtakia, Beltrami fields in chiral media, World Scientific Series in Comtemporary Chemical-Physics-Vol.2, 1994.

[16] A. Lakhtakia, V.K. Varadan and V.V. Varadan, Time-Harmonic Electromagnetic Fields in Chiral Media, Lecture Notes in Physics, 355, 1989.

[17] R. Leis, Initial Boundary Value Problems in Mathematical Physics, Teubner and Wiley, Stuttgart, 1986.

[18] L. Pasteur, Sur les relations qui peuvent exister entre la forme cristalline, la composante chimique et le sens de la polarisation rotatoire, *Ann. Chimie et Physique*, **24**, 442-459 (1848).

[19] R. Picard, On the low frequency asymptotics in electromagnetic theory, *J. Reine Angew. Math.,* **354**, 50-73 (1984).

[20] R. Picard, An elementary proof for a compact imbedding result in generalized electromagnetic theory, *Math. Z.,* **187**, 151-164 (1984).

[21] V. Prelog, Chirality in Chemistry, Nobel Lecture (Dec. 12, 1975).

[22] A.G. Ramm, O.L. Weaver, N. Weck and K.J. Witsch, Dissipative Maxwell's equations at low frequencies, *Math. Meth. in Appl. Sci.,* **13**, 305-322 (1990).

[23] Ch. Weber, A local compactness theorem for Maxwell's equations, *Math. Meth. in Appl. Sci.,* **2**, 12-25 (1980).

[24] N. Weck, Maxwell's boundary value problem on Riemannian manifolds with nonsmooth boundaries, *J. Math. Anal. Appl.*, **46**, 410-437, (1974).

[25] N. Weck and K.J. Witsch, Low-frequency asymptotics for dissipative Maxwell's equations in bounded domains, *Math. Meth. in Appl. Sci.,* **13**, 81-93 (1990).

[26] N. Weck and K.J. Witsch, Exterior Dirichlet problem for the reduced wave equation : asymptotic analysis at low frequencies, *Comm. Part. Diff. Equa.*, **16**, 173-195 (1991).

[27] N. Weck and K.J. Witsch, Exact low frequency analysis for a class of exterior boundary value problems for the reduced wave equation in two dimensions, *J. Diff. Equa.*, **100**, 312-340 (1992).

[28] P. Werner, Über das verhalten elektromagnetischer felder für kleine frequenzen in mehrfach zusammenhängenden gebieten I, *J. Reine Angew. Math.,* **278**, 365-397 (1975).

Centre de Mathématiques Appliquées, CNRS URA 756,
École Polytechnique, 91128 Palaiseau Cedex, France.