Remarks on Relativistic Schrödinger Operators and their Extensions

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Abstract

The operator $H = f(-\Delta) + V(x)$ is considered in $\mathbb{R}^n$, where f is a general real increasing function on $\mathbb{R}_+$, and $V(x)$ is a real potential with singularities. In particular, the relativistic Schrödinger operator $\sqrt{-\Delta+1} + V(x)$ is included.

Limiting absorption principles are proved for H, including the threshold. The smoothness of the extended resolvents is applied to long-time decay for the evolution group $exp(-itH)$.

Key words: Relativistic Schrödinger operators, Limiting absorption principle, threshold regularity, long-time decay.

Résumé

L'opérateur $H = f(-\Delta) + V(x)$ est considéré sur $\mathbb{R}^n$, où f est une fonction réelle croissante sur $\mathbb{R}_+$, et $V(x)$ est un potential réel présentant des singularés.

L'opérateur de Schrödinger relativiste $\sqrt{-\Delta+1} + V(x)$ est un cas particulier.

Les principes d'absorption limite sont montrés pour H, jusqu'au seuil inclus. La régularité des résolventes étendues est appliquée à la décroissance en temps long du groupe d'évolution $exp(-itH)$.

1. Introduction

In a recent paper by T. Umeda [9] some spectral properties of the relativistic Schrödinger operator are studied. This operator is defined by

$$K = \sqrt{-\Delta + 1} + V(x) \ , \quad x \in \mathbb{R}^n \ , \tag{1.1}$$

where $V(x)$ is a short-range sufficiently small real potential. The main tool in the spectral study is a "Limiting Absorption Principle" (henceforth LAP), which states that in a suitable operator topology, the resolvent operator can be extended continuously up to the absolutely continuous spectrum (usually excluding "thresholds").

Agmon [1] proved the LAP for short-range perturbations of elliptic (and principal-type) operators. Indeed, Agmon's method, with minor modifications, can be used in the proof of the LAP for K.

An abstract theory for the LAP for partial differential operators was developed in [4], and applied in various studies, such as the spectral theory of the acoustic propagator [3] and global regularity and decay of time-dependent equations [2],[5],[6]. We refer the reader to references in [4],[9] for a variety of methods and results concerning the LAP.

In this paper we use the general method of [4] in order to deal with a whole class of operators (including K) of the form,

$$H = H_0 + V(x) \ , \quad x \in \mathbb{R}^n \ , \tag{1.2}$$

$$H_0 = f(-\Delta) \ , \tag{1.3}$$

where $f(\theta) \, , \quad \theta \in \mathbb{R}_+ = (0, \infty)$, is a real-valued nonnegative continuous function.

We shall prove several results, which will require increasingly restrictive hypotheses on f. We use the terminology "g is locally Hölder continuous in an open interval $\Omega \subseteq \mathbb{R}_+$" to mean that for every compact $S \subset\subset \Omega$, there exist constants $C > 0 \, , \quad 0 < \alpha \leq 1$, such that

$$| \, g(\theta_1) - g(\theta_2) \, | \leq C \, | \, \theta_1 - \theta_2 \, |^\alpha \ , \quad \theta_1, \theta_2 \in S \ . \tag{1.4}$$

Our basic hypothesis on f, which will be assumed throughout the paper is the following.

Assumption 1.1. $f(\theta)$ is continuously differentiable in $\mathbb{R}_+$, and its derivative $f'(\theta)$ is positive and locally Hölder continuous.

Clearly, this assumption renders H_0 to be a self-adjoint operator in $L^2(\mathbb{R}^n)$ (we identify $-\Delta$ with its natural self-adjoint realization in L^2). The spectrum $\sigma(H_0)$ is absolutely continuous and satisfies $\sigma(H_0) = \overline{\left[f(0),\ f(\infty)\right)}$ (we use $f(\infty) = \lim_{\theta\to\infty} f(\theta)$).

Next we list our additional assumptions, which will be used in our refinements of the basic LAP theorem (Theorem 2A).

Assumption 1.2. $f'(\theta)$ satisfies a uniform Hölder condition near $\theta = 0$, namely, there exist constants $C > 0\,, \quad 0 < \alpha \leq 1$, such that

$$| \, f'(\theta_1) - f'(\theta_2) \, | \leq C \, | \, \theta_1 - \theta_2 \, |^\alpha \, , \quad 0 < \theta_1, \, \theta_2 \leq 1 \; , \tag{1.5}$$

and in addition $f'(0) > 0$.
(Remark that under (1.5) $f'(\theta)$ can be extended continuously to $\theta = 0$).

Assumption 1.3. $f'(\theta)$ satisfies a uniform Hölder condition near $\theta = +\infty$, namely, there exists a constant $M > 0$ such that (1.5) holds for $\theta_1,\, \theta_2 > M$.

Assumption 1.4. $f(\theta) \in C^2(\mathbb{R}_+)$.

Remark that Assumption 1.4 implies Assumption 1.1 and Assumption 1.2 (resp. Assumption 1.3) follows from Assumption 1.4 if $f''(\theta)$ is assumed bounded in $(0,1)$ (resp. (M,∞)).

The regularity conditions imposed on f are valid for a large class of Schrödinger operators. The method used here avoids the smoothness assumptions (and the growth conditions at infinity) imposed on f by the use of *phase space* techniques (cf. [7],[8]).

We emphasize that all the above assumptions, as well as some growth conditions imposed on f in the following sections, are satisfied in the case of the relativistic Schrödinger operator $f(\theta) = \sqrt{1+\theta}$.

In Section 2 we prove (Theorem 2A) that Assumption 1.1 suffices to yield a LAP for H_0 in the interior of $\sigma(H_0)$, thus generalizing the result of [9]. Observe, in particular, that f satisfies a rather limited smoothness condition. Then by adding Assumption 1.2 we show (Theorem 2B) that the LAP can be extended to all of $\mathbb{R}$, thus crossing the threshold $f(0)$.

Our method of proof is very different from that of [9], requiring minimal smoothness and yielding the Hölder continuity of the limiting values of the resolvent.

In Section 3 we use the results (and techniques) of Section 2 in order to derive a decay result (as $| t | \to \infty$) of solutions to the time-dependent equation $i\partial_t u = H_0 u$, for all initial conditions $u(t=0)$ in $L^2(\mathbb{R}^n)$, $n \geq 3$. The analogy to the case of the Klein-Gordon equation is clear, and we refer the reader to [2] for similar results concerning the latter (as well as the wave) equation.

Finally in Section 4 we study the case of $H = H_0 + V(x)$. In particular, we remove the uniform boundedness (and smallness) imposed on V in [9], at the price of having (possibly) a discrete sequence of imbedded eigenvalues.

Notation.

We denote by $L^{2,s} = L^{2,s}(\mathbb{R}^n)$ – $(s \in \mathbb{R})$ the weighted – L^2 spaces normed by

$$\| g \|_s^2 := \int_{\mathbb{R}^n} (1+ | x |^2)^s \, | g(x) |^2 \, dx \, . \tag{1.6}$$

We write $\| g \|$ for $\| g \|_0$ and denote by $(\;,\;)$ the $L^2(\mathbb{R}^n)$ inner-product. Denoting by $\hat{g}$ the Fourier transform of g, the Sobolev space H_s $(s \in \mathbb{R})$ can be defined as

$$H_s = \left\{ \hat{g} \mid g \in L^{2,s}(\mathbb{R}^n) \right\} ,$$

with the norm given by

$$\| \hat{g} \|_{H_s} := \| g \|_s \ . \tag{1.7}$$

For any two Hilbert spaces X, Y, we denote by $B(X, Y)$ the space of bounded linear operators from X to Y, equipped with the norm $\| T \|_{B(X,Y)}$.

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2. The Operator H_0 – Two Limiting Absorption Principles

In this section we prove two theorems concerning the LAP for H_0, as defined by (1.3).

Let $\mathbb{C}^{\pm} = \{z \mid \pm \mathrm{Im}\, z > 0\}$ and let $R_0(z) = (H_0 - z)^{-1}$, $\mathrm{Im}\, z \neq 0$, be the resolvent of H_0. In our first theorem we assume only that f satisfies Assumption 1.1.

Theorem 2A. *For every $s > \frac{1}{2}$, the operator-valued function*

$$z \to R_0(z)\,, \quad z \in \mathbb{C}^+\,, (\text{resp. } z \in \mathbb{C}^-)$$

can be extended continuously to the real axis, avoiding the points $f(0)$, $f(\infty)$, in the uniform operator topology of $B(L^{2,s},\, L^{2,-s})$.

Furthermore, the limiting values

$$R_0^{\pm}(\lambda) = \lim_{\varepsilon \to 0^+} R_0(\lambda \pm i\varepsilon)\,, \quad \lambda \in \mathbb{R}\,, \quad \lambda \neq f(0),\ f(\infty)\,, \tag{2.1}$$

are locally Hölder continuous in $B(L^{2,s},\, L^{2,-s})$.

Proof: It suffices to prove the statement for $f(0) < \lambda < f(\infty)$.

Denote by $\{E_0(\lambda)\}$, $\{E_{H_0}(\lambda)\}$, the spectral families associated with $-\Delta$, H_0 respectively. In view of the monotonicity of f,

$$E_{H_0}(\lambda) = E_0\Big(f^{-1}(\lambda)\Big)\,, \quad f(0) < \lambda < f(\infty)\,. \tag{2.2}$$

As was shown in [4] the operator-valued function $E_0(\mu)$, $\mu > 0$, is weakly differentiable in $B(L^{2,s},\, L^{2,-s})$, and its weak derivative $A_0(\mu) \in B(L^{2,s},\, L^{2,-s})$ satisfies

$$\Big[A_0(\mu)\varphi,\, \psi\Big] = \frac{d}{d\mu}\Big(E_0(\mu)\varphi,\, \psi\Big) = \frac{1}{2\sqrt{\mu}} \int_{|\xi|^2=\mu} \hat{\varphi}(\xi)\,\overline{\hat{\psi}(\xi)}\, d\sigma_\xi\,, \tag{2.3}$$

where $\varphi, \psi \in L^{2,s}$ and $[\ ,\ ]$ denotes the $(L^{2,-s},\, L^{2,s})$ pairing. Hence, by (2.2) $E_{H_0}(\lambda)$ is also weakly differentiable in $B(L^{2,s},\, L^{2,-s})$ and its weak derivative $A_{H_0}(\lambda) \in$

$B(L^{2,s},\, L^{2,-s})$ satisfies,

$$A_{H_0}(\lambda) = \left[f'\left(f^{-1}(\lambda)\right)\right]^{-1} \cdot A_0\left(f^{-1}(\lambda)\right), \quad f(0) < \lambda < f(\infty)\,. \tag{2.4}$$

It follows from [4] that $A_0(\mu)$ is locally Hölder continuous in the uniform operator topology of $B(L^{2,s},\, L^{2,-s})$ (with exponent depending only on s, n). Combining this with Assumption 1.1 we conclude from (2.4) that $A_{H_0}(\lambda)$ is locally Hölder continuous in $B(L^{2,s},\, L^{2,-s})$. As shown in [4], this yields the statements of the theorem. □

Corollary 2.1 *(Regularity of $R_0^{\pm}(\lambda)$). Let $L^{2,-s}_{H_0}$ be the graph-norm space of H_0 in $L^{2,-s}$ (i.e., the completion of $D(H_0)$ with respect to the norm $\| \varphi \|_{-s} + \| H_0\varphi \|_{-s}$). Then the operator space $B(L^{2,s},\, L^{2,-s})$ in the statements of Theorem 2A can be replaced by $B(L^{2,s},\, L^{2,-s}_{H_0})$.*

Proof: Follows simply from $H_0 R_0(z) = I + zR_0(z)$. □

Example 2.2. If $f(\theta) \geq C\theta^{\gamma/2}$ as $\theta \to +\infty$, for some $\gamma > 0$, then the norm of $L^{2,-s}_{H_0}$ is stronger than that of the weighted Sobolev space H^{-s}_{γ} (which is the graph-norm space of $(-\Delta)^{\gamma/2}$). Corollary 2.1 then implies that $B(L^{2,s},\, L^{2,-s})$ can be replaced by $B(L^{2,s},\, H^{-s}_{\gamma})$. For the (free) relativistic Schrödinger operator $f(\theta) = \sqrt{1+\theta}$ and $\gamma = 1$, which is the result obtained in [9].

In general (even for $-\Delta$) the limiting values $R_0^{\pm}(\lambda)$ are unbounded (in $B(L^{2,s},\, L^{2,-s})$) as $\lambda \to f(0)$. However, by strengthening the assumptions on f, taking $s > 1$ and restricting the dimension to $n \geq 3$ we can actually obtain the smooth behavior of $R_0^{\pm}(\lambda)$ near $\lambda = f(0)$.

Theorem 2B. *Let $n \geq 3$, $s > 1$, and let f satisfy Assumptions 1.1, 1.2, and assume in addition that $f(\infty) = \infty$. Then the operator-valued function,*

$$z \to R_0(z), \quad z \in \mathbb{C}^+ \quad (\text{resp.} \quad z \in \mathbb{C}^-)\,,$$

can be extended continuously to $\overline{\mathbb{C}}^+ = \mathbb{C}^+ \cup \mathbb{R}$ (resp. $\overline{\mathbb{C}}^- = \mathbb{C}^- \cup \mathbb{R}$) in the uniform *operator topology of* $B\Big(L^{2,s}(\mathbb{R}^n),\ L^{2,-s}(\mathbb{R}^n)\Big)$. *The limiting values* $R_0^{\pm}(\lambda)$ *are locally Hölder continuous in* $\lambda \in \mathbb{R}$, *with values in* $B(L^{2,s},\ L^{2,-s})$.

Proof: In view of Theorem 2A, it suffices to deal with a neighborhood of $\lambda = f(0)$. We recall the following trace lemma, the proof of which can be found in [6, Appendix].

Trace Lemma. *Let* $\hat{\varphi} \in H_s(\mathbb{R}^n)$, $\quad n \geq 3$, $\frac{1}{2} < s < \frac{3}{2}$. *Then for any* $r > 0$,

$$\Big(\int_{|\xi|=r} |\ \hat{\varphi}(\xi)\ |^2\ d\sigma_r\Big)^{\frac{1}{2}} \leq C \cdot \mathrm{Min}\ \Big(\mathrm{r}^{\mathrm{s}-\frac{1}{2}}\ ,\ 1\Big) \cdot \|\ \hat{\varphi}\ \|_{\mathrm{H_s}}\ , \tag{2.5}$$

where C *depends only on* s, n.

Combining the trace lemma with (2.3)–(2.4) we obtain for $\lambda > f(0)$, $\varphi,\ \psi \in L^{2,s}(\mathbb{R}^n)$,

$$\Big|\Big[A_{H_0}(\lambda)\varphi,\ \psi\Big]\Big| \leq C\Big[f'\Big(f^{-1}(\lambda)\Big)\Big]^{-1} \cdot \Big(f^{-1}(\lambda)\Big)^{-\frac{1}{2}} \cdot \mathrm{Min}\ \Big(\mathrm{f}^{-1}(\lambda)^{\mathrm{s}-\frac{1}{2}}\ ,\ 1\Big)\ \|\ \varphi\ \|_{\mathrm{s}} \|\ \psi\ \|_{\mathrm{s}}\ . \tag{2.6}$$

In particular, $A_{H_0}(\lambda)$ extends to a locally Hölder continuous function on $\mathbb{R}$ by setting $A_{H_0}(\lambda) \equiv 0$ for $\lambda \leq f(0)$. The conclusions now follow as in the case of the proof of Theorem 2A. $\square$

Remark 2.3 Note that if $s = 1$, then by (2.6), $A_{H_0}(\lambda)$ is bounded in $B(L^{2,1},\ L^{2,-1})$ for λ near $f(0)$, but does not necessarily vanish as $\lambda \to f(0)$.

3. Large-Time Decay of the Unitary Group $\exp(-itH_0)$

If $\varphi \in L^2(\mathbb{R}^n)$, then $u(t) = \exp(-itH_0)\varphi$ can be expressed as

$$u(t) = \int_{f(0)}^{f(\infty)} e^{-it\lambda} A_{H_0}(\lambda)\, \varphi \, d\lambda \ , \tag{3.1}$$

which must be interpreted in the appropriate weak sense.

We say that φ has "compact energy support" if $\hat{\varphi}(\xi)$ vanishes for ξ in a neighborhood of $\xi = 0$ and ∞. This is equivalent, by (2.3)–(2.4), to the fact that $A_{H_0}(\lambda)\,\varphi$ is compactly supported in $\big(f(0),\ f(\infty)\big)$.

Clearly, if φ has compact energy support, then for every $t \in \mathbb{R}$ and every $\beta \in \mathbb{R}$, $u(t) \in H_\beta(\mathbb{R}^n)$. In fact, the norm $\| \ u(t) \ \|_{H_\beta}$ decays, as $|\ t \ | \to \infty$, in the following sense,

$$\int_{\mathbb{R}} \| \left(1 + |\ x\ |^2\right)^{-\frac{s}{2}} u \ \|^2_{H_\beta} \ dt \le C \ \| \ \varphi \ \|^2 \ , \tag{3.2}$$

where $s > \frac{1}{2}$ and C depends only on s, n, β and supp $\hat{\varphi}$. The proof of (3.2) is similar to proofs of analogous statements in [5] and is omitted.

In order to prove a global decay result, avoiding the "compact energy support" hypothesis, we need to strengthen somewhat the requirements imposed on f in Theorem 2B.

Theorem 3A. *Let $n \ge 3$ and let f satisfy Assumptions 1.1, 1.2. Assume in addition that for some constant $L > 0$,*

$$f'(\mu) \ge \frac{L}{\sqrt{\mu}} \quad \text{as } \mu \to +\infty \ . \tag{3.3}$$

Then there exists a constant $C > 0$, depending only on n, f, such that for every $\varphi \in L^2(\mathbb{R}^n)$, the L^2-valued function $u(t) = \exp(-itH_0)\varphi$ satisfies

$$\int_{\mathbb{R}} \| \left(1 + |\ x\ |^2\right)^{-\frac{1}{2}} u(t) \ \|^2 \ dt \le C \ \| \ \varphi \ \|^2 \ . \tag{3.4}$$

Proof: It is seen from (2.6) and (3.3) that

$$\sup \Big\{ \| A_{H_0}(\lambda) \|_{B(L^{2,1}, L^{2,-1})},\ \lambda > f(0) \Big\} < \infty \ . \tag{3.5}$$

By an obvious density argument, it suffices to establish (3.4) for $\varphi \in C_0^\infty(\mathbb{R}^n)$. Let $w(x,t) \in C_0^\infty(\mathbb{R}^{n+1})$ and use (3.1) to write

$$\begin{aligned} \int_{\mathbb{R}} \Big(u(t),\, w(\cdot,t)\Big)\, dt &= \int_{\mathbb{R}} \int_{f(0)}^{\infty} e^{-it\lambda} \Big[A_{H_0}(\lambda)\, \varphi, w(\cdot,t)\Big] d\lambda\, dt \\ &= \int_{f(0)}^{\infty} \Big[A_{H_0}(\lambda)\, \varphi,\ \int_{\mathbb{R}} e^{+it\lambda}\, w(\cdot,t)\ dt\Big]\ d\lambda \ . \end{aligned} \tag{3.6}$$

Denoting by $\tilde{w}(\cdot,\lambda) = \displaystyle\int_{\mathbb{R}} e^{+it\lambda}\, w(\cdot,t)\, dt$ the Fourier transform of $w(\cdot,t)$ with respect to t, we rewrite (3.6) as,

$$\int_{\mathbb{R}} \Big(u(t),\, w(\cdot,t)\Big)\, dt = \int_{f(0)}^{\infty} \Big[A_{H_0}(\lambda)\, \varphi,\ \tilde{w}(\cdot,\lambda)\Big]\ d\lambda \ . \tag{3.7}$$

Since $A_{H_0}(\lambda) = \dfrac{d}{d\lambda} E_{H_0}(\lambda)$, the bilinear form $\Big[A_{H_0}(\lambda)\cdot,\cdot\Big]$ is nonnegative and, for any

$$\psi \in L^{2,s}(\mathbb{R}^n)\,,\ \int_{f(0)}^{\infty} \Big[A_{H_0}(\lambda)\, \psi, \psi\Big]\ d\lambda = \| \psi \|^2 \ .$$

Using these considerations and the Cauchy-Schwartz inequality in (3.7) we obtain

$$\begin{aligned} \Big| \int_{\mathbb{R}} \Big(u(t),\, w(\cdot,t)\Big)\, dt \Big| &\le \Big(\int_{f(0)}^{\infty} \Big[A_{H_0}(\lambda)\, \varphi, \varphi\Big]\ d\lambda \Big)^{\frac{1}{2}} \cdot \\ &\quad \cdot \left(\int_{f(0)}^{\infty} \Big[A_{H_0}(\lambda)\, \tilde{w}(\cdot,\lambda),\ \tilde{w}(\cdot,\lambda)\Big]\ d\lambda \right)^{\frac{1}{2}} \\ &\le C \,\| \varphi \| \cdot \Big(\int_{f(0)}^{\infty} \| \tilde{w}(\cdot,\lambda) \|_1^2 \ d\lambda \Big)^{\frac{1}{2}} \ , \end{aligned} \tag{3.8}$$

where we have used (3.5) in the last step. The Plancherel theorem now yields,

$$\Big| \int_{\mathbb{R}} \Big(u(t),\, w(\cdot,t)\Big)\, dt \Big| \le C \,\| \varphi \| \ \Big(\int_{\mathbb{R}} \| w(\cdot,t) \|_1^2 \ dt \Big)^{\frac{1}{2}} \ , \tag{3.9}$$

which can be rewritten as,

$$\Big|\int_{\mathbb{R}} \Big((1+ \mid x \mid^2)^{-\frac{1}{2}}\ u(t),\, w(\cdot,t)\Big)\, dt\Big| \leq C \parallel \varphi \parallel \Big(\int_{\mathbb{R}^{n+1}} \mid w(x,t) \mid^2\ dx\, dt\Big)^{\frac{1}{2}}\ , \quad (3.10)$$

and which is equivalent to (3.4) by duality. □

Corollary 3.1. *The function $f(\theta) = \sqrt{1+\theta}$ satisfies all the hypotheses of Theorem 3A, so that (3.4) applies in the case of the relativistic (free) Schrödinger operator $H_0 = \sqrt{-\Delta+1}$.*

Remark 3.2 (Asymptotic behavior of $R_0^{\pm}(\lambda)$ as $\lambda \to +\infty$). If f satisfies Assumption 1.3 and (3.3) is satisfied, then we deduce from (2.3)–(2.4) that $A_{H_0}(\lambda)$ is uniformly bounded and uniformly Hölder continuous for large λ. This implies, in view of the Privaloff-Korn theorem, that, for every $s > \frac{1}{2}$

$$\limsup_{\lambda\to+\infty}\ \parallel R_0^{\pm}(\lambda) \parallel_{B(L^{2,s},L^{2,-s})} < \infty\ . \quad (3.11)$$

In particular, this holds in the case of $\sqrt{-\Delta+1}$. However, as the example in [9] shows, it is not true that $R_0^{\pm}(\lambda) \to 0$ as $\lambda \to +\infty$ in the operator norm.

4. The Operator $H = H_0 + V(x)$

In order to allow (some) singular behavior of $V(x)$, we need to assume some regularity of $R_0^{\pm}(\lambda)$, as expressed by Corollary 2.1 and Example 2.2. Also, to exploit the general method of [4], more smoothness (locally) is required for f. Thus, we now assume that f satisfies Assumption 1.4 and, in addition, for some $\gamma > 0$,

$$f(\theta) > C\theta^{\gamma/2} \quad \text{as} \quad \theta \to +\infty \, . \tag{4.1}$$

In view of Example 2.2, H_0 satisfies the LAP in $B(L^{2,s},\, H_\gamma^{-s})$, for $f(0) < \lambda < \infty\,,\ s > \frac{1}{2}\,.$

From the general theory in [4, Section 3] we now obtain:

Theorem 4A. *Let $f \in C^2(\mathbb{R}_+)$ satisfy (4.1). Let $V(x)$ be a real potential such that, for some $\varepsilon > 0$, the multiplication operator $(1+ \mid x \mid)^{1+\varepsilon} V(x)$ is compact from $H_\gamma(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. Then:*

(i) The continuous spectrum $\sigma_c(H) = \Big[f(0), \infty\Big)$ is absolutely continuous, except possibly for a discrete sequence Λ_i of imbedded eigenvalues, which can accumulate only at $f(0), \infty$.

(ii) The resolvent $R(z) = (H - z)^{-1}\,, \quad z \in \mathbb{C}^+$ (resp. $z \in \mathbb{C}^-$) can be extended continuously, with respect to the operator norm topology of $B(L^{2,s},\, H_\gamma^{-s})\,, \quad s > \frac{1}{2}$, to $\mathbb{C}^+ \cup \Big(\sigma_c(H) \setminus \Lambda\Big)$ (resp. $\mathbb{C}^- \cup \Big(\sigma_c(H) \setminus \Lambda\Big)$), where $\Lambda = \Lambda_i \cup \{f(0)\}$.

Furthermore, the limiting values $R^{\pm}(\lambda)\,, \quad \lambda \in \sigma_c(H) \setminus \Lambda$, are locally Hölder continuous in the same topology.

Remark 4.1. Following the proof in [4, Section 5] we obtain here that if $\varphi \in L^{2,s}$ satisfies $\varphi = -VR_0^+(\lambda)\varphi$ for some $\lambda > f(0)$, then

$$\Big|\Big[\Big(A_{H_0}(\mu) - A_{H_0}(\lambda)\Big)\, \varphi, \varphi\Big]\Big| \leq C\Big|\mu - \lambda\Big|^{1+\delta}\,, \quad \delta > 0\,,$$

for μ in a neighborhood of λ. It is here that the assumption $f \in C^2(\mathbb{R}_+)$ is required.

Example 4.2. For $f(\theta) = \sqrt{1+\theta}$ all the hypotheses of Theorem 4A are satisfied with $\gamma = 1$. Thus, the operator $K = \sqrt{-\Delta + 1} + V(x)$ satisfies the LAP for any potential $V(x)$ such that

$$| V(x) | \leq C(1 + | x |)^{-1-\varepsilon} . \tag{4.2}$$

We refer to [1] for more general criteria for compactness in $B(H_1, L^2)$.

As in [9], if C in (4.2) is sufficiently small, then there are no imbedded eigenvalues.

In order to extend the LAP for H across the threshold $\lambda = f(0)$, in analogy with Theorem 2B, one needs to ensure that $I + VR_0^{\pm}\big(f(0)\big)$ are invertible in $L^{2,s}$ (compare [6]). Assuming that f satisfies the growth condition (4.1) we recall the following definition [6].

Definition 4.3. *The point $\lambda = f(0)$ is said to be a resonance of H if, for some $s > 1$, there exists $0 \neq \psi \in H_\gamma^{-s}$, such that $H\psi = 0$ in the sense of distributions.*

We can now formulate the following theorem, which supplements Theorem 4A.

Theorem 4B. *Let $n \geq 3$, $s > 1$. Let $f \in C^2(\mathbb{R}_+)$ and assume that f satisfies (4.1) and that $f''(\theta)$ is uniformly bounded in a neighborhood of $\theta = 0$. Assume further that $V(x)$ satisfies, for some C, $\delta > 0$,*

$$| V(x) | \leq C(1 + | x |^2)^{-1-\delta} , \tag{4.3}$$

and that $H = H_0 + V$ has no resonance at $f(0)$.

Then there exists $\eta > 0$ such that the limits

$$R^{\pm}(\lambda) = \lim_{\varepsilon \to 0^+} R(\lambda \pm i\varepsilon) , \quad \lambda \in \big(f(0) - \eta, \sigma(0) + \eta\big) , \tag{4.4}$$

exist and are Hölder continuous in the operator norm topology of $B(L^{2,s}, H_\gamma^{-s})$. In particular, H has no eigenvalues in $\big(f(0) - \eta, f(0) + \eta\big)$.

Proof: It follows from Theorem 2B that $R_0(z)$ can be extended continuously to $\overline{\mathbb{C}^+}$ (resp. $\overline{\mathbb{C}^-}$). By our non-resonance assumption the operators $I + VR_0^+(\lambda)$ (resp. $I + VR_0^-(\lambda)$) are invertible in $L^{2,s}$ for λ near $f(0)$ (note that (4.3) implies the compactness of V). The proof may now be concluded as the proof of Theorem 2 in [6]. $\square$

Remark 4.4. Note that by combining the assumptions of Theorem 4B and Remark 3.2 (i.e., (3.3) and Assumptions 1.1–1.3, or, more simply, (3.3) and the uniform boundedness of $f''(\theta)\,,\quad \theta \in \mathbb{R}_+$), it follows that if $\Lambda_i = \phi$,

$$\sup\left\{\| \ R^{\pm}(\lambda) \ \|_{B(L^{2,s},\, L^{2,-s})}\ ,\quad \lambda \in \sigma_c(H)\right\} < \infty\ . \tag{4.5}$$

Since the weak derivative $A_H(\lambda) = \dfrac{d}{d\lambda}\, E_H(\lambda)$ of the spectral family associated with H, $E_H(\lambda)$, satisfies the equality,

$$A_H(\lambda) = \frac{1}{2\pi i}\left(R^+(\lambda) - R^-(\lambda)\right)\ , \tag{4.6}$$

it follows that $\left\{A_H(\lambda)\,,\quad \lambda \in \sigma_c(H)\right\}$ is also uniformly bounded in $B(L^{2,s},\, L^{2,-s})$. We now proceed as in the proof of Theorem 3A to obtain a large-time decay result for solutions of $iu_t = Hu$, where $u(t=0)$ belongs to the absolutely continuous subspace with respect to H.

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