GAS DYNAMICS SYSTEM: TWO SPECIAL CASES

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Abstract. We consider the compressible Euler system of conservation laws for the mass, momentum, and energy of a gas. We prove that Murat–Tartar's compensated compactness method applies to this system in two cases:

(1) either the equations of state has a special form (specifically, the Lagrangian sound speed is a function of the pressure only) that leads to the existence of a large family of entropy pairs;
(2) or the conservation law of energy is replaced by the conservation law for the specific entropy. The latter indeed is the physically meaningful formulation when the system is derived by relaxation from an isentropic two-phase mixture.

The paper primarily investigates the structure of the equations of gas dynamics, and includes a complete description of the (i) mathematical entropies, (ii) decoupling properties, (iii) invariant domains after Chuey-Conley-Smoller theory, and (iv) Tartar's commutation relations. Existence of weak solutions in the cases (1) and (2) above is deduced from the pioneering work by DiPerna.

Contents

1. Introduction

This paper considers certain hyperbolic systems of P.D.E.'s that arise in the modeling of compressible fluid flows and multiphase mixtures. We are interested in proving existence of entropy discontinuous solutions based on the vanishing artificial viscosity method. We recall that smooth solutions to hyperbolic conservation laws do not exist globally in time even if the initial data are smooth. Weak solutions in the sense of distributions are seeked, and must be further selected with the so-called entropy criterion. C.f. Lax [17, 18] and Dafermos [12] for background on hyperbolic problems.

Activity concerning the existence of entropy weak solutions to systems of conservation laws with data in L^∞ was initiated in a pioneering work by DiPerna [13, 14, 15]. In particular, in [14], the existence of globally defined in time, weak solutions to the isentropic Euler system was established. This system is composed of two conservation laws for the mass and the momentum of the gas assuming (formally) that the specific entropy is a constant. DiPerna's proof can be viewed as one of the main

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success of the compensated compactness method introduced by Murat and Tartar [22, 30]. The latter is an efficient tool to study nonlinear composite limits of weakly convergent sequences. The proof by DiPerna uses the fact that the system of isentropic gas dynamics is genuinely nonlinear in the sense of Lax, and admits a large family of mathematical entropy pairs, i.e. additional conservation laws. The latter are necessary to take advantage of Tartar's commutation relations.

DiPerna's results have been extended in several directions for systems of two (and more) conservation laws; we refer to the works by Serre [26, 28, 29] and Chen et al. [4, 6, 7, 8] and the references cited therein. C.f. also the recent contributions by Lions et al. [19, 20].

We focus here on the full system of gas dynamics composed of three conservation laws of mass, momentum, and energy. Generally speaking, this system shares little properties with the isentropic system due to the lack of mathematical entropies, and it has been recognized that the compensated compactness method should fail in general [3]. The present work intends to tackle this system in two *special* instances. In both cases it will be established that the 3×3 system of gas dynamics has a very similar structure as the one of the isentropic 2×2 system. Our results therefore can be viewed as a direct corollary of DiPerna's fundamental work [13, 14]. The purpose of this paper, above all, is to investigate the mathematical properties of the gas dynamics system. It contains a full description of the mathematical entropies, decoupling properties, bounded invariant domains, and properties of Tartar's commutation relations.

Our motivation in this paper has been to search for (necessary and) sufficient conditions that allows one to use the compensated compactness technique on the gas dynamics system. In Section 2, we investigate a *special class of equations of state*, specifically the case that the Lagrangian sound speed of the gas depends upon the pressure only. This assumption leads to many interesting properties for the Euler system: it admits a large family of mathematical entropy pairs and Chuey-Conley-Smoller's theory applies to obtain bounded invariant domains. It follows that under the assumption above the system belongs to the class of rich systems introduced by Serre [27]. We prove the strong convergence of the vanishing artificial viscosity method, which implies existence of entropy weak solutions. We recall that the 3×3 system under consideration possesses two genuinely nonlinear characteristic fields and one linearly degenerate one. Observe that in our main theorem in Section 2 (Theorem 2.1) all of the variables (say, density, velocity, and entropy) are proven to be L^∞ functions; furthermore the entropy variable can propagate oscillations as it was expected. Note that Theorem 2.1 allows vacuum states in the solution: the system is not strictly hyperbolic at those points and we use [14].

When this work came under completion, the authors learned about a result similar to our Theorem 2.1, however in the Lagrangian framework, by Chen-Dafermos [5].

Section 3 of the paper considers a coupled set of two isentropic Euler systems that arise in the modeling of two-phase mixtures. By relaxation from this 4×4 model, a system of three conservation laws is derived. We observe that, somewhat surprisingly and for smooth solutions only, the latter is equivalent to the gas dynamics system. However the systems are distinct for discontinuous solutions. Instead of the classical formulation that is composed of the conservation laws of mass, momentum, and energy supplemented with the entropy inequality, we arrive at a set of conservation laws for the mass, momentum, and *specific entropy* supplemented with an *energy* inequality. In this new form, the gas dynamics system happens to share many properties with the isentropic 2×2 system. We check that the linearly degenerate characteristic field is a Temple field, so that the method of Benzoni-Gavage-Serre [1] applies. We prove existence of entropy solutions for this rather non-classical formulation of the gas dynamics system. The key of the proof is that a uniform estimate for the total variation of the specific entropy can be derived.

For further properties of the Euler equations, we refer to Bonnetier-LeFloch [2, 3].

2. A Class of Equations of State

We consider the compressible Euler system which consists of the conservation laws for the mass, momentum, and total energy of a gas:

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0, \\ &\partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ &\partial_t(\rho E) + \partial_x(\rho u E + pu) = 0. \end{aligned} \tag{2.1}$$

The main unknowns are the specific density ρ, the velocity u, and the total energy E. We define the internal energy e as $e = E - u^2/2$, and the specific entropy S and the temperature $T > 0$ by the classical relation $TdS = pd(1/\rho) + de$. The equation of state yields the pressure p as a (smooth enough) function of ρ and S:

$$p = p(\rho, S) > 0, \quad p_\rho > 0 \quad \text{for all } \rho > 0. \tag{2.2a}$$

We also make the standard assumption that

$$p_{\rho\rho} > 0, \quad \text{and} \quad p_S > 0 \quad \text{for all } \rho > 0. \tag{2.2b}$$

The Eulerian sound speed c is classically defined by

$$c^2 = p_\rho(\rho, S) = \big(p_\rho + \rho^{-2} p p_e\big)(\rho, e).$$

It will be convenient to express our results in term of the variables u, p, and S. Therefore we set $\rho = \tilde\rho(p, S)$, and $e = \tilde e(p, S)$ with the natural constraint $\tilde e_p + p\big(1/\tilde\rho\big)_p = 0$. We introduce the specific volume $v = 1/\rho$ (we shall use the notation $v = \tilde v(p, S)$ as well) and the Lagrangian sound speed $C = \rho c$. The latter is the relevant wave speed when the mass Lagrangian coordinates are being used.

In view of (2.2), system (2.1) is strictly hyperbolic and possesses two genuinely nonlinear characteristic fields associated with the wave speeds $u \pm c$, and one linearly degenerate field associated with u. Strict hyperbolicity however may be lost at the vacuum where $\rho = 0$. It is well-known that discontinuities form in finite time in initially smooth solutions to (2.1). We are interested in proving existence of weak solutions based on the vanishing artificial viscosity method. So we consider the following approximation:

$$\begin{aligned} &\partial_t \rho^\varepsilon + \partial_x(\rho^\varepsilon u^\varepsilon) = \varepsilon\, \partial^2_{xx}(\rho^\varepsilon), \\ &\partial_t(\rho^\varepsilon u^\varepsilon) + \partial_x(\rho^\varepsilon (u^\varepsilon)^2 + p^\varepsilon) = \varepsilon\, \partial^2_{xx}(\rho^\varepsilon u^\varepsilon), \\ &\partial_t(\rho^\varepsilon E^\varepsilon) + \partial_x(\rho u^\varepsilon E^\varepsilon + p^\varepsilon u^\varepsilon) = \varepsilon\, \partial^2_{xx}(\rho^\varepsilon E^\varepsilon). \end{aligned} \tag{2.3}$$

We shall assume that smooth solutions to (2.3) assuming a smooth initial data exist. For a proof in the isentropic case, see [14]. Initial data for u, p, and S, say u_0, p_0, and S_0, respectively are assumed to be given functions in $L^\infty(\boldsymbol{R})$ and, with obvious notation, we assume that ρ_0 and $\rho_0 E_0$ belong to $L^1(\boldsymbol{R})$. The entropy inequality is classically stated as

$$\partial_t(\rho S) + \partial_x(\rho u S) \le 0, \tag{2.4}$$

which, at least formally, can be checked for the limit of the $(u^\varepsilon, p^\varepsilon, S^\varepsilon)$'s as $\varepsilon \to 0$.

We consider in this section the case that the Lagrangian sound speed is a function of the pressure, i.e.

$$C(p, S) = C_0(p) \quad \text{or, equivalently,} \quad \rho c(p, S) = C_0(p). \tag{2.5}$$

As we prove it below, (2.5) is satisfied if, for some positive constants a and b,

$$\frac{1}{\tilde\rho(p, S)} - aS \quad \text{and} \quad \tilde e(p, S) - bS \quad \text{are functions of } p \text{ only.} \tag{2.6}$$

Our main result of existence and convergence is now stated.

Theorem 2.1. *Consider the compressible Euler system* (2.1)-(2.2) *with an equation of state of the form* (2.6). *Then we have the following uniform bounds for the approximate solutions* $(u^\varepsilon, p^\varepsilon, S^\varepsilon)$ *generated by* (2.3)*:*

$$\begin{aligned} &0 \le p^\varepsilon(t,x), |u^\varepsilon(t,x)| \le const\big(\|p_0\|_{L^\infty(\boldsymbol{R})} + \|u_0\|_{L^\infty(\boldsymbol{R})}\big), \\ &0 \le S^\varepsilon(t,x) \le const \sup_{\boldsymbol{R}} S_0, \end{aligned} \tag{2.7a}$$

where the constants are independent of t, x, *and* ε, *and*

$$\|\rho^\varepsilon(t)\|_{L^1(\boldsymbol{R})} = \|\rho_0^\varepsilon\|_{L^1(\boldsymbol{R})}, \quad \|\rho^\varepsilon(t)E^\varepsilon(t)\|_{L^1(\boldsymbol{R})} = \|\rho_0^\varepsilon E_0\|_{L^1(\boldsymbol{R})}. \tag{2.7b}$$

Let us make one of the following two assumptions:

(1) *the approximations are bounded away from the vacuum, i.e.*

$$0 < const \le p^\varepsilon(t,x) \le const \tag{2.8a}$$

with uniform constants;

(2) *the equation of state has the form*

$$p = k\,\rho^\gamma \quad \text{with} \quad \gamma > 1 \quad \text{and} \quad k > 0. \tag{2.8b}$$

Then there is a triplet (u,p,S) *in* $L^\infty(\boldsymbol{R}_+ \times \boldsymbol{R})^3$ *with* ρ *and* ρE *in* $L^\infty(\boldsymbol{R}_+, L^1(\boldsymbol{R}))$ *and a subsequence of* $(u^\varepsilon, p^\varepsilon, S^\varepsilon)$ *so that:*

$$\begin{aligned} &u^\varepsilon \to u, \quad p^\varepsilon \to p \quad \text{in the strong } L^q \text{ norm for all finite } q, \\ &\rho^\varepsilon = \tilde\rho(p^\varepsilon, S^\varepsilon) \rightharpoonup \rho = \tilde\rho(p,S) \quad \text{in the weak-}\star\ L^\infty \text{ topology,} \end{aligned} \tag{2.9}$$

and (u,p,S) *is an entropy weak solution to* (2.1), (2.4).

The proof of Theorem 2.1 is based on four lemmas dealing respectively with

(1) the mathematical entropies (Lemma 2.1),
(2) a decoupling property (Lemma 2.2),
(3) Tartar's equation (Lemma 2.3),
(4) and the invariant regions (Lemma 2.4).

We shall state those four lemmas, give the proofs of the lemmas, and conclude with a proof of Theorem 2.1. First of all, we provide a complete description of the set of entropies. A pair (U,F) is a mathematical entropy for the system

$$\partial_t w + \partial_x f(w) = 0$$

if every smooth solution w satisfies the additional conservation law

$$\partial_t U(w) + \partial_x F(w) = 0.$$

Equivalently, the pair (U,F) should satisfies

$$\nabla F(w) = \nabla U(w) \nabla f(w). \tag{2.10}$$

We prove that (2.5) is a necessary and sufficient condition for the existence of a large family of entropies for (2.1).

Lemma 2.1. *Consider the compressible Euler system* (2.1). *Let us view C as a function of the variables* (p, S).

(1) *Suppose first that $\partial_S C$ is identically zero, so* (2.5) *holds. Then the mathematical entropies U are of the form*
$$U(u, p, S) = \rho h(S) + \rho h_1(u, p) \tag{2.11a}$$
(with ρ viewed as a function of (p, S)), where h is an arbitrary function and h_1 satisfies the following wave equation:
$$\partial^2_{uu} h_1 - \partial_p \big(C_0(p)^2\, \partial_p h_1\big) = 0. \tag{2.11b}$$

(2) *Suppose that $\partial_S c$ does not vanish identically on any interval. Then the mathematical entropies U are of the form*
$$U(u, p, S) = \rho h(S), \tag{2.12}$$
where h is an arbitrary function.

The existence of entropies that depend upon u for particular equations of state was discovered by Schochet [25] and Croisille-Villedieu [11]. For simplicity, we did not include in Case (2) of Lemma 2.1 the trivial entropies ρ, ρu, and ρE. However the latter are recovered in Case (1) by a suitable choice of h_1.

Lemma 2.2. *Assume that* (2.5) *holds. We denote by g an antiderivative of $-1/C_0(p)^2$ and G an antiderivative of pg', i.e.*
$$g'(p) = -1/C_0(p)^2, \quad G'(p) = pg'(p).$$
Then there exist two functions $g_1 = g_1(S)$ and $g_2 = g_2(S)$, one of them being non-constant, satisfying
$$g_1'(S) \geq 0 \quad \text{and} \quad g_2'(S) \geq 0, \tag{2.13}$$
so that
$$\frac{1}{\rho(p, S)} = g(p) + g_1(S), \qquad e(p, S) = -G(p) + g_2(S). \tag{2.14}$$

The assumption (2.6) made in Theorem 2.1 corresponds to the choice $g_1(S) = aS$ and $g_2(S) = bS$ in (2.14). We continue our discussion with the more general assumption, i.e. (2.5). We shall return to (2.6) in the course of the proof of Theorem 2.1. Assuming (2.5), we now show that (2.1) can be formally decoupled into a system of two equations for u and p and a scalar equation for S. As explained now, the decoupling holds for smooth solutions in Lagrangian coordinates but also in a weaker sense for weak solutions in Eulerian coordinates.

For smooth solutions, system (2.1) is equivalent to
$$\begin{aligned}
&\partial_t p + u \partial_x p + \frac{C_0(p)^2}{\rho} \partial_x u = 0,\\
&\partial_t u + u \partial_x u + \frac{1}{\rho} \partial_x p = 0,\\
&\partial_t S + u \partial_x S = 0.
\end{aligned} \tag{2.15}$$
The decoupling becomes clear when using Lagrangian coordinates. Consider the change of variables $(t, x) \to (t, y)$ defined by
$$\partial_t y = \rho u, \quad \partial_x y = -\rho. \tag{2.16}$$

For smooth solutions again, (2.15) transforms into

$$\begin{aligned}
&\partial_t p + C_0(p)^2 \partial_y u = 0,\\
&\partial_t u + \partial_y p = 0,\\
&\partial_t S = 0.
\end{aligned} \tag{2.17}$$

The first two equations in (2.17) can be solved independently from the last one. This is not true when (2.5) is not satisfied and C_0 depends on S as well.

Consider next the case of discontinuous solutions. The change of variables $(t, x) \to (t, y)$ is Lipschitz continuous if ρ and u are measurable and bounded functions. We recall that the Lagrangian- Eulerian transformation preserves the notion of entropy weak solution [32]. In view of (2.14), the equations (2.1) take the form

$$\begin{aligned}
&\partial_t \frac{1}{g+g_1} + \partial_x \frac{u}{g+g_1} = 0,\\
&\partial_t \frac{u}{g+g_1} + \partial_x \big(\frac{u^2}{g+g_1} + p\big) = 0,\\
&\partial_t \frac{-G(p) + g_2(S) + u^2/2}{g+g_1} + \partial_x \big(\frac{(-G(p) + g_2(S) + u^2/2)u}{g+g_1} + pu\big) = 0.
\end{aligned} \tag{2.18}$$

Passing to Lagrangian coordinates, we are left with

$$\begin{aligned}
&\partial_t (g(p) + g_1(S)) - \partial_y u = 0,\\
&\partial_t u + \partial_y p = 0,\\
&\partial_t (-G(p) + u^2/2 + g_2(S)) + \partial_y (pu) = 0,
\end{aligned} \tag{2.19}$$

i.e.

$$\begin{aligned}
&\partial_t g(p) - \partial_y u = -\partial_t g_1(S),\\
&\partial_t u + \partial_y p = 0,\\
&\partial_t \big(- G(p) + u^2/2\big) + \partial_y (pu) = -\partial_t g_2(S).
\end{aligned} \tag{2.20}$$

Observe that the right hand side of (2.20) depends on u and p but not on S. Consider the first two equations in (2.17) and (2.20): the left hand sides in (2.17) and (2.20) coincide (up to multiplying the first equation in (2.17) by $g'(p)$); the right hand side of (2.17) is identically zero, whereas the right hand side of (2.20) contains a term due to the fact that the specific entropy need not be a smooth function.

Possible oscillations in the sequence $(u_\varepsilon, p_\varepsilon, S_\varepsilon)$ are classically described by a Young measure $\nu_{t,x}$, i.e. a probability measure for almost every (t, x) such that

$$f(u_\varepsilon, p_\varepsilon, S_\varepsilon) \rightharpoonup < \nu_{t,x}, f > \quad L^\infty \text{ weak-}\star$$

for every continuous function f. Uniform bounds on the amplitude of the $(u_\varepsilon, p_\varepsilon, S_\varepsilon)$'s are assumed for the time being (C.f. Lemma 2.4 below). The following lemma is concerned with the reduction of the Young measure to a Dirac mass measure.

Lemma 2.3. *Let $\nu = \nu(u, p, S)$ be a Young measure with compact support satisfying Tartar's commutation relations*

$$< \nu, U_1 F_2 - U_2 F_1 > = < \nu, U_1 >< \nu, F_2 > - < \nu, U_2 >< \nu, F_1 > \tag{2.21}$$

for any (U_1, F_1) *and* (U_2, F_2) *in the following family of entropy pairs:*

$$\begin{aligned} U(u,p,S) &= \rho h(S) + \rho h_1(u,p), \\ F(u,p,S) &= \rho u h(S) + \rho u h_1(u,p) + k_1(u,p), \end{aligned} \tag{2.22}$$

with h *arbitrary,* h_1 *a solution to* (2.11b), *and* k_1 *given by solving* $k_{1,u} = C_0(p)^2 h_{1,p}$ *and* $k_{1,p} = h_{1,p}$. *Suppose that either the closure of the support of* ν *contains no point with* $\rho = 0$, *or* C_0 *is defined from the equation of state* (2.8b). *Then* ν *is a tensor product with the form*

$$\nu(u,p,S) = \delta_{u_*} \otimes \delta_{p_*} \otimes \mu(S), \tag{2.23}$$

where u_* *and* p_* *are constants and* $\mu = \mu(S)$ *is a probability measure.*

Let us complete Lemma 2.3 with additional comments. We recall that Tartar obtained reduction of the Young measure in the case of scalar equations using only one entropy function. Next DiPerna proved a similar result for the elasticity system [13]. Following [13] and [8, 31], it is not difficult to check the following result for the 3×3 system. If the Young measure ν has a small support in the variables (u,p) about a constant state (u_*, p_*), then one extra entropy pair is enough to obtain the reduction of the Young measure to the form (2.23). Namely one can consider the mathematical entropy

$$U(u,p,S) = \frac{u g(p)}{g(p) + g_1(S)}, \quad F(u,p,S) = p + \frac{u^2}{2} \frac{g(p) - g_1(S)}{g(p) + g_1(S)}. \tag{2.24}$$

It remains to derive the required a priori estimates. The following result was mentionned in [27] first.

Lemma 2.4. *Each domain*

$$\Big\{ \pm u + \int^p \frac{dp}{C_0(p)} \le const \Big\}$$

is an invariant domain for (2.1) and (2.3). The uniform bounds

$$\begin{aligned} &0 \le p^\varepsilon(t,x), |u^\varepsilon(t,x)| \le const, \\ &0 \le \sup_{x \in \boldsymbol{R}} S^\varepsilon(t,x) \le \sup_{x \in \boldsymbol{R}} S^\varepsilon(0,x), \end{aligned} \tag{2.25}$$

hold for the approximate solutions $(u^\varepsilon, p^\varepsilon, S^\varepsilon)$ *generated by* (2.3) *with a constant independent of* t, x, *and* ε.

We now give the proofs of the lemmas stated above.

Proof of Lemma 2.1. It is convenient to use the Lagrangian coordinates (2.16) which transform (2.1) into

$$\begin{aligned} \partial_t u + \partial_y p &= 0, \\ \partial_t p + C(p,S)^2 \partial_y u &= 0, \\ \partial_t S &= 0. \end{aligned}$$

Searching for conservation laws of the form $\partial_t \varphi(u,p,S) + \partial_y \psi(u,p,S) = 0$, we arrive at the following necessary and sufficient conditions:

$$\begin{aligned} &\psi_u = C(p,S)^2 \varphi_p, \\ &\psi_p = \varphi_u, \\ &\psi_S = 0. \end{aligned} \tag{2.26}$$

Eliminating ψ in (2.26) we obtain

$$\begin{aligned} \varphi_{uu} &= \big(C(p,S)^2\varphi_p\big)_p, \\ \varphi_{uS} &= 0 \\ (C(p,S)^2\varphi_p)_S &= 0. \end{aligned} \tag{2.27}$$

Suppose first that $C = C_0(p)$. Then the third equation in (2.27) reduces to $\varphi_{pS} = 0$. It follows that φ has the form

$$\varphi(u,p,S) = h(S) + h_1(u,p),$$

where h is an arbitrary function of S and h_1 is a solution to the wave equation (2.11b). The associated entropy-flux ψ is obtained by integration from (2.26). We can recover the conservation laws (2.1) with suitable choices of h and h_1. We also observe that the entropy-flux does not depend on S. In particular, this implies that the pair (h_1, ψ) is also an entropy pair for the 2×2 system composed of the first two equations in (2.17).

Consider next the case that $C_S \neq 0$. Taking the u-derivative of the third equation in (2.27) and combining with the second equation in (2.27) yield $(C^2)_S\varphi_{up} = 0$. Therefore φ can be decomposed into

$$\varphi(u,p,S) = h_2(p,S) + h_3(u).$$

From the first equation in (2.27), it follows that $h_3''(u) = \partial_p\big(C^2\partial_p h_2\big)(p,S)$, which therefore equals some constant, say α. One deduces the general form of h_2 and h_3 by integrating out the latter:

$$\varphi(u,p,S) = \alpha u^2/2 + \beta u + \int^p \frac{\alpha p + \gamma}{C^2(p,S)} dp + h(S),$$

α, β, and γ being arbitrary constants and h an arbitrary function of S. Choosing the constant in a proper way, we see that the only available entropies are the specific volume, the velocity, the energy, and any function of the physical entropy S. In other words, there is no extra conservation law but

$$\partial_t S = 0.$$

The above conclusion are easily rewritten in the Eulerian setting by observing that $U = \rho\varphi$, $F = \rho u \varphi + \psi$ is an entropy pair for (2.1) if (φ, ψ) is an entropy pair in the Lagrangian framework. Finally the conditions that guarantee the convexity of the entropies are derived from straightforward computations. C.f. also Harten [16] for the entropy (2.12). The proof of Lemma 2.1 is completed. □

Proof of Lemma 2.2. Using the specific volume $v = 1/\rho$ instead of ρ, the condition (2.5) reads $-p_v = C_0(p)^2$, which is readily integrated (in view of the definition of g, i.e. $g'(p) = -1/C_0(p)^2$) into $g(p) = v - g_1(S)$ for some function g_1. This leads to the expression for the density ρ in (2.14). The expression for the energy e follows immediately by integration from the constraint $\tilde{e}_p + p\tilde{v}_p = 0$. Observe that all the standard thermodynamics constraints are satisfied. In particular we have

$$p\,dv + de = (pg_1'(S) + g_2'(S))\,dS \equiv T dS$$

with $T > 0$ if (2.13) holds and not both functions g_1 and g_2 are constants. □

Proof of Lemma 2.3. Consider first the relation (2.21) for the family of entropies of the form (2.11) with $h = 0$. We observe that the latter reduces to the classical commutation relations for the isentropic system. Namely define a two-variable young measure $\tilde{\nu} = \tilde{\nu}(u,p)$ by the relation

$$< \tilde{\nu}, \theta > = \frac{< \nu, \rho(p,S)\theta(u,p) >}{< \nu, \rho(p,S) >} \quad \text{for every } \theta \text{ continuous.}$$

For every continuous h_1 and h_2, we have

$$\begin{aligned} &< \nu, \rho h_1(\rho u h_2 + k_2) - \rho h_2(\rho u h_1 + k_1) > \\ &= < \nu, \rho h_1 >< \nu, (\rho u h_2 + k_2) > - < \nu, \rho h_2 >< \nu, (\rho u h_1 + k_1) > . \end{aligned} \tag{2.28}$$

Using (2.28) with the choice $h_2 = 1, k_2 = 0$, we get the following relation for the new measure $\tilde{\nu}$:

$$< \tilde{\nu}, \rho u h_1 > - < \tilde{\nu}, (\rho u h_1 + k_1) > = < \tilde{\nu}, h_1 >< \tilde{\nu}, u >< \nu, \rho > - < \nu, \rho >< \tilde{\nu}, u h_1 + k_1/\rho >,$$

which combined with (2.28) easily yields:

$$< \tilde{\nu}, h_1 k_2 - k_2 h_1 > = < \tilde{\nu}, h_1 >< \tilde{\nu}, k_2 > - < \tilde{\nu}, k_2 >< \tilde{\nu}, h_1 > . \tag{2.29}$$

Therefore we can apply the classical reduction theorem to $\tilde{\nu}$ and (2.29). That is either the result in [13] in the strictly hyperbolic case (if the support of ν does not meet the vaccum), or the result in [14] if the equation of state is given by (2.8b) and vacuum is allowed. It follows that the Young measure $\tilde{\nu}$ is either a Dirac mass in (u,p) or has its support on the vacuum line $\{p = 0\}$. We observe that we may modify the measure ν at the vacuum points to meet our purpose and (2.23) is established. □

Proof of Lemma 2.4. Under the assumption (2.5), the system (2.1) admits the three Riemann invariants $\pm u + \int^p \frac{dp}{C_0(p)}$ and S as it is clear from the decoupled formulation (2.17). The desired estimates in Lemma 2.4 are a consequence of Chuey- Conley- Smoller's theory about bounded invariant domains for non-degenerate parabolic equations [9]. The relevant properties for each of the above functions, say $G(u,p,S) = G(\rho, \rho u, \rho E)$ are

$$\nabla_{(\rho,\rho u,\rho E)} G \text{ is a left eigenvector} \tag{2.30a}$$

of the Jacobian matrix of the conservative system (2.1), and G is a quasiconvex function that is, for every $\xi \in \boldsymbol{R}^3$,

$$\xi \cdot \nabla_{(\rho,\rho u,\rho E)} G = 0 \Longrightarrow \nabla^2_{(\rho,\rho u,\rho E)} G \geq 0. \tag{2.30b}$$

Those properties are easily checked for $G = \pm u + \int^p \frac{dp}{C_0(p)}$ or $G = S$.

Let us also mention brieffly that the assumption (2.5) is not necessary to derive the a priori estimate for S^ε. C.f. [3] for details. Consider for simplicity a sequence of entropy solutions to (2.1). Consider the entropy inequality associated with the entropy (2.12) fopr functions satisfying $h' > 0$ and $h'' > 0$. Integrating the inequality $\partial_t \rho^\varepsilon\, h(S^\varepsilon) \leq 0$ with respect to the space variable, we get

$$\frac{d}{dt} \int_{\boldsymbol{R}} \rho^\varepsilon\, h(S^\varepsilon)\, dx \leq 0. \tag{2.31}$$

Writing (2.31) for the family of functions $h_q(S) = S^q$ with $q \to \infty$ yields at the limit

$$\frac{d}{dt} \sup_{x \in \boldsymbol{R}} S^\varepsilon(t,x) \leq 0$$

at least away from the vacuum. We observe that, without loss of generality the value of S can be arbitrarily modified at a point where $\rho = 0$. Namely in the equations (2.1) and the entropy inequalities, all the terms containing the variable S are multiplied by ρ. The proof of Lemma 2.4 is completed. □

Proof of Theorem 2.1. In view of Lemma 2.4, there exists a Young measure to represent the composite weak-$\star$ limits of the sequence $(u^\varepsilon, p^\varepsilon, S^\varepsilon)$. In order to apply the compensated compactness method to (2.3), we recall [13] that the entropy dissipation measures corresponding to the entropies (2.22)

$$\partial_t U(u,p,S) + \partial_x F(u,p,S)$$

remain in a compact subset of the negative Sobolev space $H^{-1}_{\rm loc}$. This follows from an energy type estimate for (2.3) and Murat's lemma [21]. Therefore the classical div-rot lemma applies and provide us with (2.21). Lemma 2.3 shows that the Young measure is a Dirac mass in the variables (u,p) but not necessarily in S. It follows that there is a pair (u,p) and a subsequence such that $(u^\varepsilon, p^\varepsilon)$ converges strongly in all L^q $(q<\infty)$ and S^ε converges weakly. Extracting another subsequence, if necessary, we can assume that ρ^ε converges weakly to some function, say ρ. We then *define* S by the relation $\tilde\rho(p,S)=\rho$. We emphasize that, with the choice of variables (u,p,ρ), the conservative variables and fluxes in (2.1) are linear functions of ρ. Namely this is obvious for the conservation laws of mass and momentum. On the other hand, for the energy equation, we observe that assumption (2.6) used with (2.14) implies

$$\rho E = \rho\big(u^2/2 - G(p) - bg(p)/a\big),$$

so that the conservative variable and the flux in the conservation law of energy are linear in ρ. Note that the assumption (2.6) is required in this very step of the proof only. Since u^ε and p^ε converge strongly and ρ^ε converges weakly but we need deal with *linear functions* of ρ^ε only, one can pass to the limit in (2.3) and get (2.1). The proof of Theorem 2.1. is completed. □

3. Relaxed Model For Two-Phase Mixtures

Let us begin with the following system of four conservation laws $(i=1,2)$:

$$\begin{aligned}&\partial_t\rho_i + \partial_x(\rho_i u_i) = 0,\\ &\partial_t(\rho_i u_i) + \partial_x(\rho_i u_i^2 + p_i) = s_i\,\frac{\rho_2}{\delta}(u_1-u_2),\end{aligned} \tag{3.1}$$

which describes a mixture of two fluids, say of gas and water bubbles. Here $s_1=-1$ and $s_2=1$. The main unknowns are the mass density ρ_i and the velocity u_i of the phases $i=1,2$. We supplement the system with the equations of state for the pressures p_i. We shall assume that

$$p_i = \tilde p_i(\rho_i), \quad \frac{d\tilde p_i}{d\rho} > 0, \quad \frac{d^2\tilde p_i}{d\rho^2} > 0,$$

which ensures the hyperbolicity and genuine nonlinearity of (3.1). The internal energy $e_i=\tilde e_i(\rho_i)$ is defined by the relation

$$\frac{d\tilde e_i'}{d\rho} = \frac{\tilde p_i}{\rho_i^2} \tag{3.2}$$

and the total energy for the phase i is $E_i = e_i + u^2/2$. In view of

$$\partial_t(\rho_i E_i) + \partial_x(\rho_i u_i E_i + p_i u_i) = s_i u_i\,\frac{\rho_2}{\delta}(u_1-u_2), \tag{3.3}$$

which follows from (3.1) (at least for classical solutions), it is natural to supplement (3.1) with the entropy inequality

$$\partial_t(\rho_1E_1+\rho_2E_2) + \partial_x(\rho_1u_1E_1+\rho_2u_2E_2+p_1u_1+p_2u_2) = -\frac{\rho_2}{\delta}(u_1-u_2)^2 \,\le\, 0. \tag{3.4}$$

This model describes a mixture of gas and liquid (or solid) dropplets. The source term in the momentum equation is called the drag force. The parameter δ can be identified with a relaxation time, i.e. the time for the small dropplets to acquire the same velocity as the one of the gas. C.f. Sainsaulieu [24] for the derivation of this model from a more microscopic physical description.

Our focus here is on analyzing the zero-relaxation limiting system derived from (3.1). When $\delta \to 0$, we have formally

$$u_1 = u_2 \equiv u \quad \text{(average flow velocity)}, \tag{3.5}$$

and we are left with the 3×3 relaxed system

$$\begin{aligned} &\partial_t \rho_i + \partial_x(\rho_i u) = 0, \quad i = 1, 2 \\ &\partial_t\big((\rho_1+\rho_2)u\big) + \partial_x\big((\rho_1+\rho_2)u^2 + p_1 + p_2\big) = 0, \end{aligned} \tag{3.6}$$

while the entropy inequality (3.4) becomes

$$\partial_t\big(\rho_1 e_1 + \rho_2 e_2 + (\rho_1+\rho_2)u^2/2\big) + \partial_x\big((\rho_1 e_1 + \rho_2 e_2)u + (p_1+p_2)u\big) \le 0. \tag{3.7}$$

The main unknowns of the new system are taken to be ρ_1, ρ_2, and u.

At this stage it is interesting to observe that –for classical solutions– (3.6), (3.7) coincides with the gas dynamics system studied in Section 2. The weak solutions of both systems do not coincide however. To make our point, we introduce the new variables

$$\rho = \rho_1 + \rho_2, \quad S = \rho_2/\rho,$$

together with the familiar notation

$$p = p_1 + p_2, \quad e = e_1 + e_2, \quad E = e + u^2/2,$$

so that (3.6)-(3.7) takes the form

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0, \\ &\partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ &\partial_t(\rho S) + \partial_x(\rho S u) = 0, \end{aligned} \tag{3.8}$$

and

$$\partial_t(\rho E) + \partial_x(\rho u E + p u) \le 0. \tag{3.9}$$

For smooth solutions, (3.8) is equivalent to the system (2.1). In particular the family of mathematical entropies derived in Lemma 2.1 for (2.1) is the same for (3.8). The entropy weak solutions to both systems are distinct however, since the classical formulation of the gas dynamic system conserves the total energy and lets the entropy decrease, while (3.8)-(3.9) conserves the entropy while decreasing the energy!

We shall prove that the new formulation (3.8)-(3.9) allows one to apply the compensated compactness method in all generality without assuming the restriction (2.5) on the equation of state. For the sake of generality, we disregarde the special form of p that could be derived from the above relation $p = p_1 + p_2$. Consider the approximate solutions ρ^ε, u^ε, and S^ε given by solving

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = \varepsilon\, \partial_{xx}\rho, \\ &\partial_t(\rho u) + \partial_x(\rho u^2 + p) = \varepsilon\, \partial_{xx}(\rho u), \\ &\partial_t(\rho S) + \partial_x(\rho S u) = \varepsilon\, \partial_{xx}(\rho S) \end{aligned} \tag{3.10}$$

from initial data ρ_0, u_0, and S_0 in L^∞. Observe that the inequality (3.9) is automatically satisfied in the limit. For simplicity we restrict ourselves to solutions that are bounded away from the vacuum. We denote by $BV(\boldsymbol{R})$ the space of all Lebesgue measurable, scalar-valued functions of bounded variation in one variable.

Theorem 3.1. *Suppose that ρ^ε, u^ε, and S^ε are smooth solutions to* (3.10) *satisfying the uniform bound*

$$\begin{aligned} &0 < const \le \rho^\varepsilon(t,x) \le const, \\ &|u^\varepsilon(t,x)| \le const, \quad |S^\varepsilon(t,x)| \le const, \end{aligned} \tag{3.11}$$

where the constants are independent of t, x, and ε. Suppose that the initial data S_0 belongs to the space $BV(\boldsymbol{R})$. Then there is a triplet (ρ, u, S) in $L^\infty(\boldsymbol{R}_+ \times \boldsymbol{R})^3$ and a subsequence of $(\rho^\varepsilon, u^\varepsilon, S^\varepsilon)$ so that

$$\rho^\varepsilon \to \rho, \quad u^\varepsilon \to u, \quad S^\varepsilon \to S \tag{3.12}$$

in the strong L^q norm for all finite q, and (ρ, u, S) is an entropy weak solution to (3.8)-(3.9).

This convergence and existence result can be extended to the cases that either one of the conservation laws of mass, momentum, and energy is used as an "entropy inequality", while the rest of the equations are written as conservation laws (including the specific entropy). In other words, Theorem 3.1 holds for any of the following two formulations:

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) = 0, &\quad \partial_t \rho + \partial_x(\rho u) \le 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) \le 0, &\quad \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t(\rho E) + \partial_x(\rho u E + pu) = 0, &\quad \partial_t(\rho E) + \partial_x(\rho u E + pu) = 0, \\ \partial_t(\rho S) + \partial_x(\rho S u) = 0, &\quad \partial_t(\rho S) + \partial_x(\rho S u) = 0. \end{aligned} \tag{3.13}$$

Proof of Theorem 3.1. The heart of the proof is deriving a a priori estimate for the total variation of S^ε can be derived. A straightforward calculation using the first and third equations in (3.11) shows that S is a solution to the scalar advection-diffusion equation:

$$\partial_t S^\varepsilon + V^\varepsilon \, \partial_x S^\varepsilon = \varepsilon \partial_{xx} S^\varepsilon \tag{3.14}$$

with

$$V^\varepsilon = u^\varepsilon - 2\varepsilon \partial_x (\log \rho^\varepsilon).$$

We use a classical approach to estimate the total variation of S as follows. Set $w = \partial_x S^\varepsilon$ and take $q > 1/2$. From (3.14) we deduce

$$\partial_t |w|^{2q} + \partial_x(V^\varepsilon |w|^{2q}) + (2q-1)|w|^{2q} \partial_x V^\varepsilon \le \varepsilon \partial_{xx}\big(|w|^{2q+1}/(2q+1)\big).$$

Letting $q \to 1/2$, we obtain

$$\partial_t |w| + \partial_x(V^\varepsilon |w|) \le \varepsilon \partial_{xx}(|w|^2/2).$$

Integrating the latter over $x \in \boldsymbol{R}$ yields an estimate for $\partial_x S^\varepsilon$:

$$\int_{\boldsymbol{R}} |\partial_x S^\varepsilon(t,x)| \, dx \le \int_{\boldsymbol{R}} |S_0'(x)| \, dx \tag{3.15}$$

when S_0 belongs to $W^{1,1}(\boldsymbol{R})$. The case that $S_0 \in \mathrm{BV}(\boldsymbol{R})$ is treated by a standard regularization process. By Helly's theorem, the sequence $\{S^\varepsilon\}$ (or a subsequence of it) converges in any L^q norm, $1 \le q < \infty$.

Next we turn to applying the compensated compactness method to (3.10). We shall use functions that are entropy functions for the isentropic gas dynamics system but not for the full system following here the method proposed by Benzoni-Gavage-Serre. We give the main lines of the proof and refer to [1] for details. The above total variation estimate allows us to control the nonconservative terms that arise. Let C denote as before the Lagrangian sound speed and consider the class $\mathcal{A}$ of all pairs (U, F) of the form $U = \rho h_1(u, p)$ that satisfies the wave equation (2.11b) with C_0 replaced by C, i.e.

$$\partial^2_{uu} h_1 - \partial_p \big(C(p, S)^2 \, \partial_p h_1 \big) = 0 \tag{3.16}$$

for each value of S. For an entropy pair (U, F) in the class $\mathcal{A}$, it is not hard to check by using (3.15) that

$$\partial_t U + \partial_x F \in \text{ compact set of } H^{-1}_{\mathrm{loc}}(\boldsymbol{R}_+ \times \boldsymbol{R}),$$

so that the commutation equation (2.21) is satisfied by every entropy pairs in $\mathcal{A}$. From the reduction theorem by DiPerna [13] for genuinely nonlinear systems, we deduce that u^ε and p^ε converge in a strong sense. The proof of Theorem 3.1 is completed.

Remark 3.1. Chuey-Conley-Smoller theory does not apply to system (3.10), and tere are in general no available invariant regions. In the special case that $p = a^2 S \dfrac{\rho}{1-\rho}$, Peng [23] shows the existence of invariant regions for the Riemann problem, andobatin global existence using the Glimm scheme.

It would be interesting to show that, as the relaxation parameter δ tends to 0, the solutions of (3.1) converge strongly to a solution of the relaxed system (3.8). One can anticipate that the relaxation terms have the same effect in cancelling oscillations, as we proved it for the vanishing viscosity method. Rigourous convergence results were established by Chen-Levermore-Liu [6, 7] for genuinely nonlinear 2×2 systems with relaxation. Their argument is based on the fact that, from every convex entropy of the relaxed equations, stems a *compatible* entropy of the full system: the scalar product of the entropy gradient with the relaxation term has a constant sign, which yields the required H^{-1}_{loc} compactness property.

Even though (3.1) consists in two genuinely nonlinear systems, only coupled through the lower order source terms, a rigorous proof of convergence of the solutions $(\rho_i^\delta, u_i^\delta)_{i=1,2}$ to a solution of the relaxed system (3.8)-(3.9) seems extremely difficult. This is mainly due to the form of the relaxation terms, as we now explain it.

Let us begin with the vanishing viscosity regularization for (3.1), i.e. for $i = 1, 2$:

$$\begin{aligned} &\partial_t \rho_i + \partial_x (\rho_i u_i) = \varepsilon \, \partial_{xx} \rho_i, \\ &\partial_t (\rho_i u_i) + \partial_x (\rho_i u_i^2 + p_i) = s_i \, \frac{\rho_2}{\delta} (u_1 - u_2) + \varepsilon \, \partial_{xx} (\rho_i u_i). \end{aligned} \tag{3.17}$$

We show in Proposition 3.2 below, that (3.17) does not admit any invariant domain in the sense of Chuey-Conley-Smoller. Nevertheless, assuming a uniform L^∞ bound on the solutions of (3.17), a standard energy method yields the following a priori estimate:

$$\delta^{-1/2} \| u_1^{\varepsilon,\delta} - u_2^{\varepsilon,\delta} \|_{L^2(\boldsymbol{R})} + \varepsilon^{1/2} \Sigma_{i=1,2} \big(\| \partial_x \rho_i^{\varepsilon,\delta} \|_{L^2(\boldsymbol{R})} + \| \partial_x u_i^{\varepsilon,\delta} \|_{L^2(\boldsymbol{R})} \big) \le Const. \tag{3.18}$$

Letting δ tend to 0 first, it is easy to see that the solutions of (3.17) converge strongly in L^2_{loc} to a solution of (3.8), (3.9), and Theorem 3.1 applies to the resulting 3×3 system.

On the other hand, if we let ε tend to 0 first, the desired convergence result, and therefore the existence of entropy weak solutions to (3.1), (3.4), is a consequence of DiPerna's theorem for 2×2 genuinely non linear systems [14]. Furthermore, for each $\delta > 0$, these solutions satisfy the entropy inequality (3.4). On the other hand, in the terminology of [6, 7], the total energy is a compatible entropy for (3.1). It turns out that there is no other compatible entropy.

Proposition 3.2.

(1) *The system (3.17) does not admit any invariant domain in the sense of Chuey-Conley-Smoller.*
(2) *Every entropy of (3.17) takes the form* $\eta_1(\rho_1, u_1) + \eta_2(\rho_2, u_2)$*, where* η_i *is an entropy for the associated sub-system*

$$\begin{aligned} &\partial_t \rho_i + \partial_x(\rho_i u_i) = 0, \\ &\partial_t(\rho_i u_i) + \partial_x(\rho_i u_i^2 + p_i) = 0. \end{aligned} \tag{3.19}$$

(3) *The only compatible entropy for (3.17) is the total energy* $\rho_1 E_1 + \rho_2 E_2$*.*

Proof of Proposition 3.2. Let R_δ denote the right-hand side of (3.1). To show the first claim, observe that the invariant domains in the sense of [9] have to be in the form $\pm w_i(\rho_1, \rho_1 u_1, \rho_2, \rho_2 u_2) \leq 0$, where $(w_i)_{i=1,4}$ are four Riemann invariants for (3.1). Another necessary condition is that

$$\nabla w_i \cdot R_\delta \leq 0. \tag{3.20}$$

An easy calculation shows that the Riemann invariants of (3.17) are the same as those of the sub-systems, and thus only depend on either (ρ_1, u_1) or (ρ_2, u_2), whereas R_δ is proportional to $u_1 - u_2$. Thus (3.20) cannot hold.

We now examine the structure of the entropy pairs $(U, F)(\rho_1, u_1, \rho_2, u_2)$ for (3.17). Eliminating the entropy flux F in the condition (2.10) yields the following equations for U:

$$\begin{bmatrix} (u_1 - u_2) & -\tilde{p}_2'/\rho_2 & \tilde{p}_1'/\rho_1 & 0 \\ -\rho_2 & (u_1 - u_2) & 0 & \tilde{p}_1'/\rho_1 \\ \rho_1 & 0 & (u_1 - u_2) & -\tilde{p}_2'/\rho_2 \\ 0 & \rho_1 & -\rho_2 & (u_1 - u_2) \end{bmatrix} \begin{bmatrix} U_{\rho_1,\rho_2} \\ U_{\rho_1,u_2} \\ U_{u_1,\rho_2} \\ U_{u_1,u_2} \end{bmatrix} = 0, \tag{3.21}$$

together with

$$U_{u_i,u_i} = \frac{\tilde{p}_i'}{\rho_i} U_{\rho_i,\rho_i} \qquad i = 1, 2. \tag{3.22}$$

If $u_1 \pm (p_1')^{1/2} \neq u_2 \pm (p_2')^{1/2}$, the matrix in (3.21) is invertible and therefore

$$U_{\rho_1,\rho_2} = U_{\rho_1,u_2} = U_{u_1,\rho_2} = U_{u_1,u_2} = 0.$$

The conditions (2.10) induce exactly the same constraints on F. Thus, the entropy pairs for (3.1) have the form $(U, F) = \big(\eta_1(\rho_1, u_1) + \eta_2(\rho_2, u_2), q_1(\rho_1, u_1) + q_2(\rho_2, u_2)\big)$, where η_i satisfies (3.22), i.e. is an entropy for the i-th sub-system, and q_i is the associated entropy flux. For such a pair $(U, F) = (\eta_1 + \eta_2, q_1 + q_2)$ with convex η_1 and η_2, the solutions of (3.1) obtained by the viscosity method satisfy

$$\partial_t U(\rho_1^\delta, u_1^\delta, \rho_2^\delta, u_2^\delta) + \partial_x F(\rho_1^\delta, u_1^\delta, \rho_2^\delta, u_2^\delta) \leq \nabla U \cdot R_\delta. \tag{3.23}$$

We now seek compatible entropies, i.e. pairs (U, F) for which the term on the right-hand side of (3.23) is non-positive. In other words, we require that

$$sign(u_1 - u_2)(g_1 - g_2) \leq 0 \tag{3.24}$$

for all quadruples $(\rho_1, u_1, \rho_2, u_2)$, where $g_i(\rho_i, u_i) = \eta_{i,u}(\rho_i, u_i)/\rho_i$. Condition (3.24) implies that g_i only depends on u_i, and thus

$$\eta_{i,u} = \rho_i g_i(u_i). \tag{3.25}$$

From (3.22) and the definition (3.2) of the internal energy, we get

$$\frac{\partial^2 \eta_i}{\partial \rho^2} = \frac{\tilde{p}_i'}{\rho_i^2} \frac{\partial^2 \eta_i}{\partial u^2} = \tilde{e}_i'' g_i''(u_i).$$

Integrating twice with respect to ρ_i, we find

$$\eta_i = \tilde{e}_i g_i'(u_i) + \rho_i h(u_i) + k(u_i)$$

for some functions h and k. From (3.25), we see that k is a constant and

$$\rho_i^{-1} \tilde{e}_i g_i''(u_i) = \rho_i^{-1} \tilde{e}_i(\rho_i) g_i''(u_i) = g_i(u_i) - h(u_i),$$

which implies that $g_i'' = 0$ since the right-hand side is independent of ρ_i. Thus g_i is a linear function of u_i, while (3.25) again implies that $h' = g_i$. We conclude that the only compatible entropies have the form

$$\eta_i = \alpha(\rho_i u_i^2/2 + e_i) + \beta \rho_i u_i + \gamma \rho_i + \zeta$$

for some constants $\alpha, \beta, \gamma, \zeta$, which completes the proof of Proposition 3.2. □

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