Some remarks on measure solutions of the Vlasov equation

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Abstract. The mathematical modeling of electron or ion beams leads to consider boundary value problems for the stationary Vlasov equation where the boundary data are measures and moreover the data depend on a parameter. We give in this paper natural sufficient conditions for such a boundary problem to be well posed and we derive an explicit form of the corresponding measure solution and of its derivative with respect to the parameter.

Key-words. Convection equations, Vlasov equation, measure solutions.

Introduction.

The mathematical modeling of charged particle beams is generally based on the Vlasov-Maxwell system of equations or more simply on the Vlasov-Poisson equations where one can neglect the self-consistent magnetic field. In the extraction of such beams (extraction from a cathode in the case of an electron beam or extraction from a quasi-neutral plasma for an ion beam), the extractor field may be so intense than one can assume that, at emission, the beam behaves as a monokinetic one. In mathematical terms, this means that, at the emission boundary, the distribution function of the beam is indeed a measure. This implies in particular that we have to solve the stationary Vlasov equation with measure boundary data.

Moreover, in a number cases, this measure is not a given datum but is also an unknown of the problem. On one hand, in electron extraction from a cathode, the electron current density is given by the Child-Langmuir law wich expresses that the electric field must vanish at the cathode. This leads to solve a Vlasov equation where at the cathode the emission distribution function depends on a parameter. On the other hand, in ion extraction from a neutral plasma, the emission surface is not known : we obtain a free-boundary problem for the Vlasov-Poisson equations. A Newton iteration procedure of solution of this problem again leads to solve a Vlasov equation where the location of the emission boundary plays the role of a parameter. We refer to [2],[3] for a description of these problems.

Thus, we need to solve boundary value problems for the Vlasov equation which present the two following difficulties :
(i) the boundary data are measures ;
(ii) the data depend on a parameter which belongs to some infinite-dimensional space.
This paper is devoted to the study of such problems and is organized as follows. In Section 1, we consider convection equations with measure boundary data : we give sufficient conditions for these boundary value problems to be well posed and we derive an explicit form of the measure solution. In Section 2, we assume in addition that the data of the convection problem depend on a parameter and we consider the associated linearized problem (with respect to the parameter). Again we derive an explicit form of the solution which is now a distribution of order 1. Finally, in Section 3, we apply the results of the two previous sections to the Vlasov equation with a monokinetic boundary emission.

1. Measure solutions of convection equations.

Let Ω be an open subset of $\mathbb{R}^d$ with a C^1 boundary Γ. Denote by $\boldsymbol{\nu}$ the outward unit normal vector to Γ. We are given a vector field $\boldsymbol{a} \in C^1(\bar{\Omega})^d$ which satisfies for simplicity $\boldsymbol{\nabla}.\boldsymbol{a}=0$ and we introduce the following subsets of Γ

$$\Gamma\pm=\{\boldsymbol{x}\in\Gamma;(\boldsymbol{a}.\boldsymbol{\nu})(\boldsymbol{x})\gtrless 0\}\ ,\ \Gamma_0=\{\boldsymbol{x}\in\Gamma;(\boldsymbol{a}.\boldsymbol{\nu})(\boldsymbol{x})=0\}.$$

Let g be a measure defined on Γ_- ; we look for a solution u of the convection problem

$$\begin{cases} \boldsymbol{a}.\boldsymbol{\nabla}u=0 \text{ in } \Omega, \\ \\ u=g \text{ on } \Gamma. \end{cases} \tag{1.1}$$

When $g\in L^1_{loc}(\Gamma_-)$ is a locally integrable function, one can easily solve the problem (1.1) by means of the method of characteristics. Indeed let us introduce the characteristic curves associated with the differential operator $\boldsymbol{a}.\boldsymbol{\nabla}$, i.e., the solutions $t\to\boldsymbol{X}(t;\boldsymbol{x})$ of the differential system

$$\begin{cases} \dfrac{d\boldsymbol{X}}{dt}=\boldsymbol{a}(\boldsymbol{X}) \\ \\ \boldsymbol{X}(0;\boldsymbol{x})=\boldsymbol{x},\boldsymbol{x}\in\overline{\Omega}. \end{cases} \tag{1.2}$$

For each $\boldsymbol{x}\in\Omega$, the system (1.2) has a maximal solution defined on the interval $]\tau_{in}(\boldsymbol{x})\ \tau_{out}(\boldsymbol{x})[$ where

$$\begin{cases} \tau_{in}(\boldsymbol{x})=sup\ \{t\le 0\ ;\ \boldsymbol{X}(t;\boldsymbol{x})\in\Gamma_-\cup\Gamma_0\}, \\ \\ \tau_{out}(\boldsymbol{x})=inf\ \{t\ge 0\ ;\ \boldsymbol{X}(t;\boldsymbol{x})\in\Gamma_+\cup\Gamma_0\}. \end{cases} \tag{1.3}$$

Then denoting by

$$\Omega_c=\{\boldsymbol{x}=X(t;\boldsymbol{\xi})\ ;\ \boldsymbol{\xi}\in\Gamma_-\ ,\ 0<t<\tau_{out}(\boldsymbol{\xi})\}$$

the subset of Ω generated by the characteristic curves issued from the inflow boundary Γ_-, the function u defined by

$$u(\boldsymbol{x})=\begin{cases} g(\boldsymbol{X}(\tau_{in}(\boldsymbol{x});\boldsymbol{x}))\ ,\ \boldsymbol{x}\in\Omega_c, \\ \\ 0,\boldsymbol{x}\in\Omega\backslash\Omega_c \end{cases} \tag{1.4}$$

is a weak solution of (1.1) , i.e., satisfies

$$\int_\Omega u\boldsymbol{a}.\boldsymbol{\nabla}\varphi d\boldsymbol{x}=\int_{\Gamma_-}g(\boldsymbol{a}.\boldsymbol{\nu})\varphi dS\qquad\forall\varphi\in C^1_0(\Omega\cup\Gamma_-). \tag{1.5}$$

When meas $(\Omega\backslash\Omega_c)$ is > 0, or equivalently when one cannot reach almost every point $\boldsymbol{x} \in \Omega$ by means of a characteristic curve issued from Γ_-, such a weak solution is not unique at least in general.

In this section, we want to generalize the solution (1.4) to the case where g is a measure. For $\boldsymbol{\xi} \in \Gamma_-$; we set

$$\tau(\boldsymbol{\xi}) = \tau_{out}(\boldsymbol{\xi}) \tag{1.6}$$

and we introduce the subsets of Γ_- and Ω respectively

$$\begin{cases} \tilde{\Gamma}_- = \{\boldsymbol{\xi} \in \Gamma_- ; \tau(\boldsymbol{\xi}) < +\infty \ , \ X(\tau(\boldsymbol{\xi}), \boldsymbol{\xi}) \in \Gamma_+\}, \\ \tilde{\Omega} = \{\boldsymbol{x} = \boldsymbol{X}(t; \boldsymbol{\xi}) \ ; \ \boldsymbol{\xi} \in \tilde{\Gamma}_-, \ 0 < t < \tau(\boldsymbol{\xi})\}. \end{cases} \tag{1.7}$$

In other words, $\tilde{\Omega}$ is the subset of Ω generated by the characteristic curves issued from Γ_- which leave Ω in a finite time and at a point of Γ_+ (we exclude those characteristics which leave Ω at a point of Γ_0). Let us study the topological properties of $\tilde{\Omega}$ and $\tilde{\Gamma}_-$.

Proposition 1. *Assume that*

$$\boldsymbol{a}(\boldsymbol{x}) \neq \boldsymbol{0} \qquad \forall \ \boldsymbol{x} \in \overline{\tilde{\Omega}} \tag{1.8}$$

Then $\tilde{\Omega}$ and $\tilde{\Gamma}_-$ are open subsets of Ω and Γ_- respectively.

Proof. We first check that $\tilde{\Gamma}_-$ is an open subset of Γ_-. Let $\boldsymbol{\xi} \in \tilde{\Gamma}_-$ and let $\boldsymbol{\eta} = \boldsymbol{X}(\tau(\boldsymbol{\xi}); \boldsymbol{\xi}) \in \Gamma_+$. Using (1.8) we have $\boldsymbol{a}(\boldsymbol{X}(t; \boldsymbol{\xi})) \neq 0$ for all $t \in [0, \tau(\boldsymbol{\xi})]$. Moreover Γ_- and Γ_+ are local sections of the field $\boldsymbol{a}$ at the points $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ respectively, i.e., Γ_- and Γ_+ are C^1 varieties of dimension $d-1$ which are transverse to the characteristic curves at $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ respectively. Then, by the Poincaré mapping theorem [3], there exist a neighborhood N_ξ of $\boldsymbol{\xi}$ in Γ_- , a neighborhood N_η of $\boldsymbol{\eta}$ in Γ_+ and a C^1_- diffeomorphism $\boldsymbol{h}$ from N_ξ onto N_η of the form

$$\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{X}(\tau(\boldsymbol{\xi}) + \theta(\boldsymbol{x}); \boldsymbol{x}).$$

If $\boldsymbol{x} \in N_\xi$, we have

$$\tau(\boldsymbol{x}) = \tau(\boldsymbol{\xi}) + \theta(\boldsymbol{\xi}) < \infty \quad , \ \boldsymbol{h}(\boldsymbol{\xi}) = \boldsymbol{X}(\tau(\boldsymbol{x}); \boldsymbol{x}) \in N_\eta.$$

Hence $N_\xi \subset \tilde{\Gamma}_-$ which proves that $\tilde{\Gamma}_-$ is open. Moreover, since $\boldsymbol{a}.\boldsymbol{\nu}$ is continuous on Γ, we may assume that $N_\eta \subset \Gamma_+$.

Let us next prove that $\tilde{\Omega}$ is an open subset of Ω. Let $\boldsymbol{x} \in \tilde{\Omega}$ and let $\sum_x$ be a local section of the field $\boldsymbol{a}$ at $\boldsymbol{x}$. Setting $\boldsymbol{\xi} = \boldsymbol{X}(\tau_{in}(\boldsymbol{x}); \boldsymbol{x}) \in \tilde{\Gamma}_-$, there exists again by the

Poincaré mapping theorem a C^1_- diffeomorphism from a neighborhood N_ξ of $\boldsymbol{\xi}$ in $\tilde{\Gamma}_-$ in a neighborhood N_x of $\boldsymbol{x}$ in $\sum_x$. Then the set

$$T_x = \{\boldsymbol{y} \in \Omega; \boldsymbol{y} = \boldsymbol{X}(t, \boldsymbol{z}), \boldsymbol{z} \in N_x, t \in] - \varepsilon, \varepsilon[\}$$

is contained in $\tilde{\Omega}$. Moreover since $\boldsymbol{a}$ is nonsingular in T_x, T_x is a neighborhood of $\boldsymbol{x}$ in Ω which proves our assertion. ■

Now, for any integer $k \geq 0$, we denote by $C_0^k(\Omega \cup \Gamma_-)$ the space of all C^k functions with compact support in $\Omega \cup \Gamma_-$ and by $\mathcal{M}^k(\Omega \cup \Gamma_-)$ its dual space. Note that $\mathcal{M}(\Omega \cup \Gamma_-) = \mathcal{M}^0(\Omega \cup \Gamma_-)$ is the space of measures on $\Omega \cup \Gamma_-$. We introduce also the space

$$T^k = \{\varphi \in C_0^k(\Omega \cup \Gamma_-); \boldsymbol{a}.\boldsymbol{\nabla}\varphi \in C_0^k(\Omega \cup \Gamma_-)\}$$

and we set $T = T^0$. Then $u \in \mathcal{M}(\Omega \cup \Gamma_-)$ *is a measure solution of the problem* (1.1) $\Omega \cup \tilde{\Gamma}_-$ *if it satisfies for all test function* $\varphi \in T$

$$(1.9) \qquad < u, \boldsymbol{a}.\boldsymbol{\nabla}\varphi > = < g, (\boldsymbol{a}.\boldsymbol{\nu})\varphi >$$

where $< \cdot, \cdot >$ denotes any duality pairing. We shall say that *such a measure solution is an admissible measure solution if in addition*

$$(1.10) \qquad supp\ (u) \subset \tilde{\Omega} \cup \tilde{\Gamma}_-.$$

Before proving the existence and uniqueness of an admissible measure solution of (1.1), we first establish a preliminary result. Let us introduce the subspace of T^k

$$\tilde{T}^k = \{\varphi \in T^k; supp(\varphi) \subset \tilde{\Omega} \cup \tilde{\Gamma}_-\}.$$

Then we can state

Proposition 2. *Let* $k = 0, 1$ *and let* $u \in \mathcal{M}^k(\Omega \cup \Gamma_-)$ *satisfy the support property* (1.10). *Then we have*

$$(1.11) \qquad < u, \boldsymbol{a}.\boldsymbol{\nabla}\varphi > = 0 \qquad \forall \varphi \in \tilde{T}^k \Longleftrightarrow u = 0.$$

Proof. Assume that u satisfies

$$< u, \boldsymbol{a}.\boldsymbol{\nabla}\varphi > = 0 \qquad \forall \varphi \in \tilde{T}^k.$$

If we denote by $\tilde{u}$ the restriction of u to $\tilde{\Omega} \cup \tilde{\Gamma}_-$, i.e.,

$$< \tilde{u}, \varphi > = < u, \varphi > \qquad \forall \varphi \in C_0^k(\tilde{\Omega} \cup \tilde{\Gamma}_-),$$

we obtain

$$< \tilde{u}, \boldsymbol{a}.\boldsymbol{\nabla}\varphi > = 0 \quad \forall \varphi \in \tilde{T}^k.$$

Let us check that this implies

$$< \tilde{u}, \psi >= 0 \quad \forall \psi \in C_0^k(\tilde{\Omega} \cup \tilde{\Gamma}_-)$$

and therefore $\tilde{u} = 0$. More precisely, given a function $\psi \in C_0^k(\tilde{\Omega} \cup \tilde{\Gamma}_-)$, we construct a function $\varphi \in C_0^k(\tilde{\Omega} \cup \tilde{\Gamma}_-)$ solution of the equation

$$\boldsymbol{a}.\boldsymbol{\nabla}\varphi = \psi. \tag{1.12}$$

Since

$$\frac{d}{dt}\varphi(\boldsymbol{X}(t;\boldsymbol{x})) = (\boldsymbol{a}.\boldsymbol{\nabla}\varphi)\ (\boldsymbol{X}(t;\boldsymbol{x})), \tag{1.13}$$

we obtain

$$\frac{d}{dt}\varphi(\boldsymbol{X}(t;\boldsymbol{x})) = \psi(\boldsymbol{X}(t;\boldsymbol{x}))$$

which yields

$$\varphi(\boldsymbol{X}(\tau_{out}(\boldsymbol{x});\boldsymbol{x}) - \varphi(\boldsymbol{x}) = \int_0^{\tau_{out}(\boldsymbol{x})} \psi(\boldsymbol{X}(t;\boldsymbol{x}))dt.$$

If we require that φ vanishes on Γ_-, we find the following solution of (1.12)

$$\varphi(\boldsymbol{x}) = -\int_0^{\tau_{out}(\boldsymbol{x})} \psi(\boldsymbol{X}(t;\boldsymbol{x}))dt\ ,\ \boldsymbol{x} \in \tilde{\Omega} \cup \tilde{\Gamma}_-$$

Next, we check that this function φ belongs to $C_0^k(\tilde{\Omega} \cup \tilde{\Gamma}_-)$. On one hand, using the Poincaré mapping theorem, we obtain that $\boldsymbol{x} \to \tau_{out}(\boldsymbol{x})$ is a C^1 mapping from $\tilde{\Omega} \cup \tilde{\Gamma}_-$ on $\mathbb{R}$ which proves that the function φ is of class C^k. On the other hand, in order to prove that the support of φ is contained in $\tilde{\Omega} \cup \tilde{\Gamma}_-$, we introduce the subset of $\tilde{\Gamma}_-$

$$V_\psi = \{\boldsymbol{\xi} \in \tilde{\Gamma}_-\ ;\ \exists\ t \in\ ]\ 0, \tau(\boldsymbol{\xi})\ [\ ,\ \boldsymbol{X}(t;\boldsymbol{\xi}) \in\ supp(\psi)\}$$

and, for $\boldsymbol{\xi} \in V_\psi$, we define

$$\tau^\psi(\boldsymbol{\xi}) = \tau_{out}^\psi(\boldsymbol{\xi}) = sup\ \{t \geq 0\ ;\ \boldsymbol{X}(t;\boldsymbol{\xi}) \in\ supp(\psi)\}.$$

Then we have

$$supp(\psi) \subset S_\psi = \{\boldsymbol{x} = \boldsymbol{X}(t;\boldsymbol{\xi})\ ;\ 0 \leq t \leq \tau^\psi(\boldsymbol{\xi}), \boldsymbol{\xi} \in V_\psi\}$$

Since $supp(\psi)$ is bounded, V_ψ is a bounded subset of $\tilde{\Gamma}_-$ and therefore S_ψ is as bounded subset of $\overline{\tilde{\Omega}}$. Moreover, there exists a neighborhood $\mathcal{O}$ of $\partial\tilde{\Omega} \setminus \tilde{\Gamma}_-$ such that

$$\mathcal{O} \cap supp(\psi) = \emptyset.$$

Indeed, we can take

$$\mathcal{O} = \{\boldsymbol{x} = \boldsymbol{X}(t;\boldsymbol{\xi}) \ ; \ \boldsymbol{\xi} \in V_\psi, \ t \in [\frac{1}{2}(\tau(\boldsymbol{\xi}) + \tau^\psi(\boldsymbol{\xi})), \tau(\boldsymbol{\xi})[\}$$

$$\bigcup \{\boldsymbol{x} = \boldsymbol{X}(t;\boldsymbol{\xi}) \ ; \ \boldsymbol{\xi} \notin V_\psi \ , \ t \in [0, \tau(\boldsymbol{\xi})[\ \}.$$

Hence S^ψ is a compact set of $\tilde{\Omega} \cup \tilde{\Gamma}_-$. Clearly $supp(\varphi) \subset S_\psi$ which proves the support property of the function φ. Hence φ belongs to $\tilde{T}^k$ and we obtain $\tilde{u} = 0$. Together with the hypothesis (1.10), this yields $u = 0$. ∎

We can now state

Theorem 3. *Assume the hypothesis* (1.8). *Assume in addition the condition*

$$\sup_{\xi \in \tilde{\Gamma}_-} \tau(\boldsymbol{\xi}) < +\infty. \tag{1.14}$$

Then, for all measure $g \in \mathcal{M}(\Gamma_-)$ whose support in contained in $\tilde{\Gamma}_-$, the problem (1.1) *has a unique admissible measure solution defined by*

$$< u, \varphi > = - < g, \boldsymbol{\xi} \to (\boldsymbol{a}.\boldsymbol{\nu})(\boldsymbol{\xi}) \int_0^{\tau(\boldsymbol{\xi})} \varphi(\boldsymbol{X}(t;\boldsymbol{\xi}))dt > \quad \forall \varphi \in C_0^0(\Omega \cup \Gamma_-). \tag{1.15}$$

For simplicity, in all the sequel, we will denote by $\boldsymbol{a}.\boldsymbol{\nu} \int_0^\tau \varphi(\boldsymbol{X}(t))dt$ the function $\boldsymbol{\xi} \to (\boldsymbol{a}.\boldsymbol{\nu})(\boldsymbol{\xi}) \int_0^{\tau(\boldsymbol{\xi})} \varphi(\boldsymbol{X}(t;\boldsymbol{\xi}))dt$.

Proof. We have already observed that $\boldsymbol{\xi} \to \tau(\boldsymbol{\xi})$ is a C^1 function on $\tilde{\Gamma}_-$. Hence, for all $\varphi \in C_0^0(\Omega \cup \Gamma_-)$, the function $\boldsymbol{a}.\boldsymbol{\nu} \int_0^\tau (\boldsymbol{X}(t))dt$ is continuous on $\tilde{\Gamma}_-$. Since $supp(g) \subset \tilde{\Gamma}_-$, the formula (1.15) makes sense. Moreover, using the hypothesis (1.14), it is a simple matter to check that the linear form $\varphi \to < u, \varphi >$, defined by (1.15) is continuous on $C_0^0(\Omega \cup \Gamma_-)$. Next, we prove that the measure $u \in \mathcal{M}(\Omega \cup \Gamma_-)$ is indeed a measure solution of (1.1). Let $\varphi \in T$; we may write

$$< u, \boldsymbol{a}.\boldsymbol{\nabla}\varphi > = - < g, \boldsymbol{a}.\boldsymbol{\nu} \int_0^\tau (\boldsymbol{a}.\boldsymbol{\nabla}\varphi) \ (\boldsymbol{X}(t))dt > .$$

Since by (1.13)

$$\int_0^{\tau(\xi)} (\boldsymbol{a}.\boldsymbol{\nabla}\varphi) \ (\boldsymbol{X}(t;\boldsymbol{\xi}))dt = \varphi(\boldsymbol{X}(\tau(\boldsymbol{\xi});\boldsymbol{\xi})) - \varphi(\boldsymbol{X}(0;\boldsymbol{\xi})) = -\varphi(\boldsymbol{\xi}),$$

we obtain (1.9) which proves our assertion.

Let us then show that this solution is admissible. Given a function $\varphi \in C_0^0(\Omega \cup \Gamma_-)$ which vanishes in neighborhood of $\tilde{\Omega} \cup \tilde{\Gamma}_-$, we obtain for all $\boldsymbol{\xi} \in \tilde{\Gamma}_-$

$$\varphi(\boldsymbol{X}(t;\boldsymbol{\xi})) = 0 \quad \forall t \in [0, \tau(\boldsymbol{\xi})[$$

and therefore by (1.15)

$$< u, \varphi >= 0$$

which proves the support property (1.10).

Finally, the uniqueness of an admissible measure solution follows from Proposition 2 ($k = 0$). ■

Remark 1. Observe that we can slightly weaken the condition (1.14). Indeed, it is enough to assume that

$$\sup_{\xi \in \, supp(g)} \tau(\boldsymbol{\xi}) < +\infty. \blacksquare$$

2. An extension to first-order distribution solutions.

Assume now that the problem (1.1) depends on a parameter. For instance, in the case of free surface problems for the Vlasov-Poisson system considered in [2],[3], this parameter is the position of the free surface. Then a linearization of the weak formulation (1.2) of the problem (1.1) with respect to the parameter leads to consider the following problem : *Find a first order distribution* $u' \in \mathcal{M}^1(\Omega \cup \Gamma_-)$ *which satisfies for all test function* $\varphi \in T^1$

$$(2.1) \qquad < u', \boldsymbol{a}.\boldsymbol{\nabla} >= - < u, \boldsymbol{a}'.\boldsymbol{\nabla}\varphi > + < \mu, \varphi > .$$

In (2.1), u is the solution of (1.1) and μ is a distribution of $\mathcal{M}^1(\Gamma_-)$ whose support is contained in $\tilde{\Gamma}_-$ while u' and $\boldsymbol{a}' \in C^0(\bar{\Omega})^d$ are linearizations of u and the vector field $\boldsymbol{a}$ with respect to the parameter.

Note that u' is solution of the non homogeneous convection equation

$$\boldsymbol{a}.\boldsymbol{\nabla} u' = -\boldsymbol{\nabla}.(\boldsymbol{a}'.u) \quad in\ \Omega$$

Since $u \in \mathcal{M}(\Omega \cup \Gamma_-)$, it appears natural to look for a solution u' of (2.1) in the space $\mathcal{M}^1(\Omega \cup \Gamma_-)$.

As for u, we require for u' to have its support contained in $\tilde{\Omega} \cup \tilde{\Gamma}_-$: *a solution* $u' \in \mathcal{M}^1(\Omega \cup \Gamma_-)$ *of* (2.1) *is called admissible if in addition*

$$(2.2) \qquad supp(u') \subset \tilde{\Omega} \cup \tilde{\Gamma}_-.$$

Let us introduce the solution $t \to \boldsymbol{\beta}(t;\boldsymbol{\xi})$, $\boldsymbol{\xi} \in \tilde{\Gamma}_-$, of the linear differential system

$$(2.3) \qquad \begin{cases} \dfrac{d\boldsymbol{\beta}}{dt}(t;\boldsymbol{\xi}) = \boldsymbol{\nabla a}\ (\boldsymbol{X}(t;\boldsymbol{\xi})).\boldsymbol{\beta}\ (t;\boldsymbol{\xi}) + \boldsymbol{a}'(\boldsymbol{X}\ (t;\boldsymbol{\xi})), \\ \\ \boldsymbol{\beta}\ (0;\boldsymbol{\xi}) = \boldsymbol{0}. \end{cases}$$

Then we can state

Theorem 4. *Assume the hypotheses of Theorem 3. For all distribution $\mu \in \mathcal{M}^1(\Gamma_-)$ whose support is contained in $\tilde{\Gamma}_-$, the "linearized" problem* (2.1) *has a unique admissible solution $u' \in \mathcal{M}^1(\Omega \cup \Gamma_-)$ given by*

$$(2.4) \qquad \begin{cases} < u', \varphi > = - < \mu, \int_0^\tau \varphi(\boldsymbol{X}(t))dt > - \\ \qquad - < g, \boldsymbol{a}.\boldsymbol{\nu} \int_0^\tau \boldsymbol{\beta}(t).\boldsymbol{\nabla}\varphi(\boldsymbol{X}(t))dt > \qquad \forall \varphi \in C_0^1(\Omega \cup \Gamma_-). \end{cases}$$

In (2.4), $\int_0^\tau \varphi(\boldsymbol{X}(t))dt$ and $\boldsymbol{a}.\boldsymbol{\nu} \int_0^\tau \boldsymbol{\beta}(t).\boldsymbol{\nabla}\varphi(\boldsymbol{X}(t))dt$ denote the functions

$$\boldsymbol{\xi} \to \int_0^{\tau(\xi)} \varphi(\boldsymbol{X}(t;\boldsymbol{\xi}))dt$$

and

$$\boldsymbol{\xi} \to (\boldsymbol{a}.\boldsymbol{\nu})(\boldsymbol{\xi}) \int_0^{\tau(\xi)} \boldsymbol{\beta}(t;\boldsymbol{\xi}).\boldsymbol{\nabla}\varphi(\boldsymbol{X}(t;\boldsymbol{\xi}))dt$$

respectively.

Proof. Since $\boldsymbol{\xi} \to \tau(\boldsymbol{\xi})$ and $\boldsymbol{\xi} \to \beta(t;\boldsymbol{\xi})$ are respectively a C^1 function and a C^0 function on $\tilde{\Gamma}_-$, we first observe that

$$\int_0^\tau \varphi(\boldsymbol{X}(t))dt \in C_0^1(\tilde{\Gamma}_-) \ , \ \boldsymbol{a}.\boldsymbol{\nu} \int_0^\tau \boldsymbol{\beta}(t).\boldsymbol{\nabla}\varphi(\boldsymbol{X}(t))dt \in C_0^0(\tilde{\Gamma}_-).$$

Hence the formula (2.4) makes sense and again it is a simple matter to check that (2.4) defines an element $u' \in \mathcal{M}^1(\Omega \cup \Gamma_-)$. Let us next prove that u' is indeed a solution of (2.1). On one hand, using (1.15), we have for $\varphi \in T^1$

$$- < u, \boldsymbol{a}'.\boldsymbol{\nabla}\varphi > = < g, \boldsymbol{a}.\boldsymbol{\nu} \int_0^\tau (\boldsymbol{a}'.\boldsymbol{\nabla}\varphi)(\boldsymbol{X}(t))dt > .$$

On the other hand, by replacing u' by its expression (2.4), we find for $\varphi \in T^1$

$$\begin{aligned} < u', \boldsymbol{a}.\boldsymbol{\nabla}\varphi > = - \ < \mu, \int_0^\tau (\boldsymbol{a}.\boldsymbol{\nabla}\varphi)(\boldsymbol{X}(t))dt > \ - \\ - < g, \boldsymbol{a}.\boldsymbol{\nu} \int_0^\tau \boldsymbol{\beta}(t).(\boldsymbol{\nabla}(\boldsymbol{a}.\boldsymbol{\nabla}\varphi))(\boldsymbol{X}(t))dt > . \end{aligned}$$

Using again (1.13), we obtain

$$- < \mu, \int_0^\tau (\boldsymbol{a}.\boldsymbol{\nabla}\varphi)(\boldsymbol{X}(t))dt > = < \mu, \varphi > .$$

Hence, in order to prove that (2.1) holds, it remains only to check that for all $\varphi \in T^1$

$$(2.5) \qquad \begin{cases} - < g, \boldsymbol{a}.\boldsymbol{\nu} \int_0^\tau \boldsymbol{\beta}(t).(\boldsymbol{\nabla}(\boldsymbol{a}.\boldsymbol{\nabla}\varphi))(\boldsymbol{X}(t))dt > = \\ \qquad = < g, \boldsymbol{a}.\boldsymbol{\nu} \int_0^\tau (\boldsymbol{a}'.\boldsymbol{\nabla}\varphi) \ (\boldsymbol{X}(t))dt > . \end{cases}$$

We write
$$\boldsymbol{\nabla}(\boldsymbol{a}.\boldsymbol{\nabla}\varphi) = (\boldsymbol{a}\boldsymbol{\nabla})^T.\boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\boldsymbol{\nabla}\varphi.\boldsymbol{a}$$
and we notice that again by (1.13).
$$\int_0^\tau \boldsymbol{\beta}(t).(\boldsymbol{\nabla}\boldsymbol{\nabla}\varphi.\boldsymbol{a})(\boldsymbol{X}(t))dt = \int_0^\tau \boldsymbol{\beta}(t).\frac{d}{dt}\boldsymbol{\nabla}\varphi(\boldsymbol{X}(t))dt =$$
$$= -\int_0^\tau \frac{d\boldsymbol{\beta}}{dt}(t).\boldsymbol{\nabla}\varphi(\boldsymbol{X}(t))dt - \boldsymbol{\beta}(0).\boldsymbol{\nabla}\varphi(\boldsymbol{X}(0)).$$
Hence
$$\int_0^\tau \boldsymbol{\beta}(t).(\boldsymbol{\nabla}(\boldsymbol{a}.\boldsymbol{\nabla}\varphi))(\boldsymbol{X}(t))dt =$$
$$= -\int_0^\tau (\frac{d\boldsymbol{\beta}}{dt}(t) - \boldsymbol{\nabla}\boldsymbol{a}(\boldsymbol{X}(t)).\boldsymbol{\beta}(t)).\boldsymbol{\nabla}\varphi(\boldsymbol{X}(t))dt - \boldsymbol{\beta}(0).\boldsymbol{\nabla}\varphi(\boldsymbol{X}(0))$$
and (2.5) follows from (2.3).

Now, using the explicit form (2.4) of this solution $u' \in \mathcal{M}^1(\Omega \cup \Gamma_-)$, one can prove exactly as in the proof of Theorem 3 that u' satisfies the support property (2.2). This proves that u' is an admissible solution of the problem (2.1). The uniqueness of such an admissible solution follows from Proposition 2 with $k = 1$. ∎

3. Application to the Vlasov equation.

Let us apply the results of Sections 1 and 2 to Vlasov's equation. Let $\mathcal{O}$ be a *bounded* open subset of $\mathbb{R}^d$ with a C^1 boundary $\mathcal{C}$ and let $\boldsymbol{\nu}$ be the outward unit normal to $\mathcal{C}$. We introduce the position-velocity $(\boldsymbol{x}, \boldsymbol{v})$ phase-space $\mathbb{R}^d \times \mathbb{R}^d$ and the open subset $\Omega = \mathcal{O} \times \mathbb{R}^d$ of this phase-space. We denote by $\Gamma = \mathcal{C} \times \mathbb{R}^d$ the boundary of Ω and we set
$$\Gamma_- = \{(\boldsymbol{x}, \boldsymbol{v}) \in \Gamma_-; \boldsymbol{v}.\boldsymbol{\nu}(\boldsymbol{x}) < 0\}.$$
Next, we are given a vector field $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{v}) \in C^1(\overline{\Omega})^d$ which satisfies $\boldsymbol{\nabla}_{\boldsymbol{v}}.\boldsymbol{F} = 0$ and two functions n_e and $\boldsymbol{u}_e$ with the following hypotheses

(3.1)
$$n_e \in L^1(\mathcal{C}) \ , \ \boldsymbol{u}_e \in L^\infty(\mathcal{C})^d$$

(3.2)
$$(\boldsymbol{u}_e.\boldsymbol{\nu})(\boldsymbol{\xi}) < 0 \text{ for all } \boldsymbol{\xi} \in \mathcal{C}_e$$

where

(3.3)
$$\mathcal{C}_e = \{\boldsymbol{\xi} \in \mathcal{C}; n_e(\boldsymbol{\xi}) \neq 0\}.$$

Then we consider the following boundary problem for the Vlasov equation : *Find a measure* $f \in \mathcal{M}(\Omega \cup \Gamma_-)$ *solution of*

$$(3.4) \qquad \begin{cases} \boldsymbol{v}.\boldsymbol{\nabla}_{x} f + \dfrac{1}{m}\boldsymbol{F}.\boldsymbol{\nabla}_{v} f = 0 \ \ in \ \Omega, \\ \\ f = n_e \delta(\boldsymbol{v} - \boldsymbol{u}_e) \ \ on \ \Gamma_-. \end{cases}$$

The measure f is the distribution function of a population of particles with mass m and submitted to a force field $\boldsymbol{F}$. The boundary condition models a monokinetic emission of particles at the boundary $\mathcal{C}$ of the geometrical domain $\mathcal{O}$ characterized by a density $n_e \geq 0$ and a velocity $\boldsymbol{u}_e$. The subset $\mathcal{C}_e$ of is the emissive part of the boundary. In the applications that we have in mind (cf. [2],[3]), the particles posess an electric charge q and $\boldsymbol{F}$ is the Lorentz force $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$.

We observe that the framework of Section 1 indeed applies to the problem (3.4). The vector field $\boldsymbol{a}$ corresponds to the pair $(\boldsymbol{v}, \boldsymbol{F})$ and we have

$$\boldsymbol{\nabla}_{(x,v)}.(\boldsymbol{v}, \boldsymbol{F}) = \boldsymbol{\nabla}_{v}.\boldsymbol{F} = 0.$$

Then, we introduce the characteristic curves associated with the differential operator $\boldsymbol{v}.\boldsymbol{\nabla}_{x} + \dfrac{1}{m}\boldsymbol{F}.\boldsymbol{\nabla}_{v}$, i.e., the solutions $t \to (\boldsymbol{X}, \boldsymbol{V})(t; \boldsymbol{x}, \boldsymbol{v})$ of the differential system

$$(3.5) \qquad \begin{cases} \dfrac{d\boldsymbol{X}}{dt} = \boldsymbol{V} \quad , \quad m\dfrac{d\boldsymbol{V}}{dt} = \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{V}), \\ \\ \boldsymbol{X}(0; \boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{x} \ , \quad \boldsymbol{V}(0; \boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{v} \ , \quad (\boldsymbol{x}, \boldsymbol{v}) \in \overline{\Omega}. \end{cases}$$

The characteristic curve $(\boldsymbol{X}, \boldsymbol{V})(t; \boldsymbol{x}, \boldsymbol{v})$ is the trajectory in the phase-space of the particle passing trough the point $(\boldsymbol{x}, \boldsymbol{v})$.

Note that here

$$(3.6) \qquad \begin{cases} \tilde{\Gamma}_- = \{(\boldsymbol{x}, \boldsymbol{v}) \in \Gamma_-; \tau(\boldsymbol{x}, \boldsymbol{v}) < +\infty, \\ \\ \boldsymbol{V}(\tau(\boldsymbol{x}, \boldsymbol{v}); \boldsymbol{x}, \boldsymbol{v}).\boldsymbol{\nu}(\boldsymbol{X}(\tau(\boldsymbol{x}, \boldsymbol{v}); \boldsymbol{x}, \boldsymbol{v})) > 0\} \\ \\ \tilde{\Omega} = \{(\boldsymbol{X}, \boldsymbol{V})(t; \boldsymbol{x}, \boldsymbol{v}); (\boldsymbol{x}, \boldsymbol{v}) \in \tilde{\Gamma}_- \ , \ 0 < t < \tau(\boldsymbol{x}, \boldsymbol{v})\} \end{cases}$$

where, for $(\boldsymbol{x}, \boldsymbol{v}) \in \Gamma_-$, $\tau(\boldsymbol{x}, \boldsymbol{v})$ denotes the transit time in $\mathcal{O}$ of the particle emitted at the point $\boldsymbol{x}$ of the boundary $\mathcal{C}$ with the velocity $\boldsymbol{v}$. Thus the characterisitic curves which span $\tilde{\Omega}$ are the trajectories of the particles emitted from $\mathcal{C}$ which leave the domain $\mathcal{O}$ in finite time. Therefore, by restricting ourselves to admissible solutions of (3.4), we exclude the existence of trapped particles in $\mathcal{O}$ and also that of infinite charge accumulation points (which are clearly unphysical).

Now, we set for simplicity

$$(\boldsymbol{X},\boldsymbol{V})(t;\boldsymbol{\xi}) = (\boldsymbol{X},\boldsymbol{V})(t;\boldsymbol{\xi},\boldsymbol{u}(\boldsymbol{\xi}))\ ,\ \tau(\boldsymbol{\xi}) = \tau(\boldsymbol{\xi};\boldsymbol{u}(\boldsymbol{\xi})). \tag{3.7}$$

Theorem 5. *Assume that* $\boldsymbol{F}$ *has no critical point in* $\tilde{\Omega}$. *Assume in addition that the functions* n *and* $\boldsymbol{u}$ *satisfy the hypotheses* (3.1),(3.2) *and*

$$\sup_{\xi\in\mathcal{C}_e} \tau(\boldsymbol{\xi}) < \infty, \tag{3.8}$$

$$\boldsymbol{V}(\tau(\boldsymbol{\xi});\boldsymbol{\xi}).\boldsymbol{\nu}(\boldsymbol{X}(\tau(\boldsymbol{\xi});\boldsymbol{\xi})) > 0 \qquad \forall\boldsymbol{\xi}\in\mathcal{C}_e. \tag{3.9}$$

Then the problem (3.4) *has a unique admissible measure solution* $f\in\mathcal{M}(\Omega\cup\Gamma_-)$ *given by*

$$f(\boldsymbol{x},\boldsymbol{v}) = -\int_{\mathcal{C}_e}(n_e\boldsymbol{u}_e.\boldsymbol{\nu})(\boldsymbol{\xi})\int_0^{\tau(\xi)}\delta(\boldsymbol{x}-\boldsymbol{X}(t;\boldsymbol{\xi}))\delta(\boldsymbol{v}-\boldsymbol{V}(t;\boldsymbol{\xi}))dtdS(\boldsymbol{\xi}) \tag{3.10}$$

Proof. It follows from the hypotheses (3.1),(3.2) that $n_e\delta(\boldsymbol{v}-\boldsymbol{u}_e)$ is indeed a measure of $\mathcal{M}(\Gamma_-)$. The conditions (3.8) and (3.9) ensure that the support of this measure is contained in $\tilde{\Gamma}_-$. Using Remark 1, we may apply Theorem 3 which implies the existence of a unique admissible measure solution $f\in\mathcal{M}(\Omega\cup\Gamma_-)$ defined by

$$< f,\varphi > = - < n_e\delta(\boldsymbol{v}-\boldsymbol{u}_e),(\boldsymbol{\xi},\boldsymbol{v})\to\boldsymbol{v}.\boldsymbol{\nu}(\boldsymbol{\xi})\int_0^{\tau(\xi,v)}\varphi(\boldsymbol{X}(t;\boldsymbol{\xi},\boldsymbol{v}),\boldsymbol{V}(t;\boldsymbol{\xi},\boldsymbol{v}))dt > =$$

$$= -\int_{\mathcal{C}}(n_e\boldsymbol{u}_e.\boldsymbol{\nu})(\boldsymbol{\xi})\int_0^{\tau(\xi)}\varphi(\boldsymbol{X}(t;\boldsymbol{\xi}),\boldsymbol{V}(t;\boldsymbol{\xi}))dt\ dS(\boldsymbol{\xi}) \qquad \forall\varphi\in C_0^0(\Omega\cup\Gamma_-)$$

which yields (3.10). ∎

Remark 2. The formula (3.10) defines a so-called *laminar solution* of the Vlasov equation (3.4). In fact, one can easily check that the associated number density $n=n(\boldsymbol{x})$ defined by

$$n = \int f d\boldsymbol{v} = -\int_{\mathcal{C}_e}(n_e\boldsymbol{u}_e.\boldsymbol{\nu})(\boldsymbol{\xi})\int_0^{\tau(\xi)}\delta(\boldsymbol{x}-\boldsymbol{X}(t;\boldsymbol{\xi}))dt\ dS(\boldsymbol{\xi}) \tag{3.11}$$

is a function of $L^1(\mathcal{O})$. ∎

Let us next introduce the "linearized" Vlasov equation : *Find* $f'\in\mathcal{M}^1(\Omega\cup\Gamma_-)$ *solution of*

$$(3.12)\qquad \begin{cases} < f', \boldsymbol{v}.\boldsymbol{\nabla}_{x}\varphi + \dfrac{1}{m}\boldsymbol{F}.\boldsymbol{\nabla}_{v}\varphi > = - < f, \dfrac{1}{m}\boldsymbol{F}'.\boldsymbol{\nabla}_{v}\varphi > + \\ \qquad\qquad + < \mu, \varphi > \qquad \forall \varphi \in T^1. \end{cases}$$

In (3.12), $\mu \in \mathcal{M}^1(\Gamma_-)$ is a given first-order distribution while f' and $\boldsymbol{F}' \in C^0(\overline{\Omega})^d$ are suitable linearizations of f and $\boldsymbol{F}$ respectively. In the sequel, we will restrict ourselves to the case (cf. [3])

$$(3.13)\qquad \mu = \boldsymbol{g}_e.\boldsymbol{\nabla}\delta(\boldsymbol{v} - \boldsymbol{u}_e)$$

where the function $\boldsymbol{g}_e$ satisfies

$$(3.14)\qquad \boldsymbol{g}_e \in L^1(\mathcal{C})^d\ , \boldsymbol{g}_e = \mathbf{0} \text{ on } \mathcal{C}\backslash\mathcal{C}_e.$$

We denote by $t \to (\boldsymbol{\beta}(t;\boldsymbol{\xi})), \boldsymbol{\xi} \in \mathcal{C}_e$, the solution of the linear differential system

$$(3.15)\begin{cases} \dfrac{d\boldsymbol{\beta}}{dt}(t;\boldsymbol{\xi}) = \gamma(t;\boldsymbol{\xi}), \\ m\ \dfrac{d\gamma}{dt}(t;\boldsymbol{\xi}) = \boldsymbol{\nabla}_{x}\boldsymbol{F}(\boldsymbol{X}(t;\boldsymbol{\xi}), \boldsymbol{V}(t;\boldsymbol{\xi})).\boldsymbol{\beta}(t;\boldsymbol{\xi}) + \\ \qquad + \boldsymbol{\nabla}_{v}\boldsymbol{F}(\boldsymbol{X}(t;\boldsymbol{\xi}), \boldsymbol{V}(t;\boldsymbol{\xi})).\gamma(t;\boldsymbol{\xi}) + (n_e\boldsymbol{u}_e.\boldsymbol{\nu})(\boldsymbol{\xi})\boldsymbol{F}'(\boldsymbol{X}(t;\boldsymbol{\xi}), \boldsymbol{V}(t;\boldsymbol{\xi})), \\ \boldsymbol{\beta}(0;\boldsymbol{\xi}) = \mathbf{0}, \quad \gamma(0;\boldsymbol{\xi}) = -\boldsymbol{g}_e(\boldsymbol{\xi}). \end{cases}$$

Theorem 6. *Assume the hypotheses of Theorem* 5 *together with* (3.13) *and* (3.14). *Then, the problem* (3.12) *has a unique admissible solution* $f' \in \mathcal{M}^1(\Omega \cup \Gamma_-)$ *given by*

$$(3.16)\qquad \begin{cases} f'(\boldsymbol{x}, \boldsymbol{v}) = \displaystyle\int_{\mathcal{C}_e}\int_0^{\tau(\xi)} \{\boldsymbol{\beta}(t;\boldsymbol{\xi}).\boldsymbol{\nabla}\delta(\boldsymbol{x} - \boldsymbol{X}(t;\boldsymbol{\xi}))\delta(\boldsymbol{v} - \boldsymbol{V}(t;\boldsymbol{\xi})) + \\ \qquad + \delta(\boldsymbol{x} - \boldsymbol{X}(t;\boldsymbol{\xi}))\gamma(t;\boldsymbol{\xi}).\boldsymbol{\nabla}\delta(\boldsymbol{v} - \boldsymbol{V}(t;\boldsymbol{\xi}))\}dt\ dS(\boldsymbol{\xi}) \end{cases}$$

Proof. First, setting

$$\boldsymbol{a} = (\boldsymbol{v}, \frac{1}{m}\boldsymbol{F})^T,$$

we have

$$\boldsymbol{\nabla}\boldsymbol{a} = \begin{pmatrix} \mathbf{0} & \boldsymbol{I} \\ \dfrac{1}{m}\boldsymbol{\nabla}_{x}\boldsymbol{F} & \dfrac{1}{m}\boldsymbol{\nabla}_{v}\boldsymbol{F} \end{pmatrix}.$$

Hence, replacing in (2.3) $\boldsymbol{\beta}, \boldsymbol{a}, \boldsymbol{a}'$ by $(\boldsymbol{\beta}_1, \boldsymbol{\gamma}_1)^T, (\boldsymbol{v}, \frac{1}{m}\boldsymbol{F})^T$ and $\left(\boldsymbol{0}, \frac{1}{m}\boldsymbol{F}'\right)^T$ respectively gives the differential system

$$\begin{cases} \dfrac{d\boldsymbol{\beta}_1}{dt}(t;\boldsymbol{\xi}) = \boldsymbol{\gamma}_1(t;\boldsymbol{\xi}), \\ m\dfrac{d\boldsymbol{\gamma}_1}{dt}(t;\boldsymbol{\xi}) = \nabla_x \boldsymbol{F}(\boldsymbol{X}(t;\boldsymbol{\xi}), \boldsymbol{V}(t;\boldsymbol{\xi})).\boldsymbol{\beta}_1(t;\boldsymbol{\xi}) + \\ \qquad\qquad + \nabla_v \boldsymbol{F}(\boldsymbol{X}(t;\boldsymbol{\xi}), \boldsymbol{V}(t;\boldsymbol{\xi})).\boldsymbol{\gamma}_1(t;\boldsymbol{\xi}) + \boldsymbol{F}'(\boldsymbol{X}(t;\boldsymbol{\xi}), \boldsymbol{V}(t;\boldsymbol{\xi})), \\ \boldsymbol{\beta}_1(0;\boldsymbol{\xi}) = \boldsymbol{\gamma}_1(0;\boldsymbol{\xi}) = \boldsymbol{0}. \end{cases}$$

Next, applying Theorem 4, we obtain the existence and uniqueness of an admissible distribution $f' \in \mathcal{M}^1(\Omega \cup \Gamma_-)$ which satisfies for all $\varphi \in C_0^1(\Omega \cup \Gamma_-)$

$$\begin{aligned} < f', \varphi > = &- \int_{\mathcal{C}_e} (n_e \boldsymbol{u}_e.\boldsymbol{\nu})(\boldsymbol{\xi}) \int_0^{\tau(\xi)} \{\boldsymbol{\beta}_1(t;\boldsymbol{\xi}).\nabla_x \varphi(\boldsymbol{X}(t;\boldsymbol{\xi}), \boldsymbol{V}(t;\boldsymbol{\xi})) + \\ &+ \boldsymbol{\gamma}_1(t;\boldsymbol{\xi}).\nabla_v \varphi(\boldsymbol{X}(t;\boldsymbol{\xi}), \boldsymbol{V}(t;\boldsymbol{\xi}))\} dt\ dS(\boldsymbol{\xi}) - \\ &- < \mu, (\boldsymbol{\xi}, \boldsymbol{v}) \to \int_0^{\tau(\xi,v)} \varphi(\boldsymbol{X}(t;\boldsymbol{\xi},\boldsymbol{v}), \boldsymbol{V}(t;\boldsymbol{\xi},\boldsymbol{v})) dt > . \end{aligned}$$

It remains to evaluate the second term in the right hand side of the above expression. Since φ vanishes on $\Gamma_+ \cup \Gamma_0$, we notice that

$$\begin{aligned} &\nabla_v \int_0^{\tau(\xi,v)} \varphi(\boldsymbol{X}(t;\boldsymbol{\xi},\boldsymbol{v}), \boldsymbol{V}(t;\boldsymbol{\xi},\boldsymbol{v})) dt = \\ &= \int_0^{\tau(\xi,v)} (\nabla_v \boldsymbol{X}^T.\nabla_x \varphi(\boldsymbol{X}, \boldsymbol{V}) + \nabla_v \boldsymbol{V}^T.\nabla_v \varphi(\boldsymbol{X}, \boldsymbol{V}))(t;\boldsymbol{\xi},\boldsymbol{v}) dt. \end{aligned}$$

Hence, using (3.13),(3.14), we have

$$\begin{aligned} &- < \mu, (\boldsymbol{\xi}, \boldsymbol{v}) \to \int_0^{\tau(\xi,v)} \varphi(\boldsymbol{X}(t;\boldsymbol{\xi},\boldsymbol{v}), \boldsymbol{V}(t;\boldsymbol{\xi},\boldsymbol{v})) dt >= \\ &= \int_{\mathcal{C}_e} \boldsymbol{g}_e(\boldsymbol{\xi}). \int_0^{\tau(\xi)} (\nabla_v \boldsymbol{X}^T.\nabla_x \varphi(\boldsymbol{X}, \boldsymbol{V}) + \nabla_v \boldsymbol{V}^T.\nabla_v \varphi(\boldsymbol{X}, \boldsymbol{V}))(t;\boldsymbol{\xi},\boldsymbol{u}_e(\boldsymbol{\xi})) dt\ dS(\boldsymbol{\xi}) = \\ &= \int_{\mathcal{C}_e} \int_0^{\tau(\xi)} \{\boldsymbol{\beta}_2(t;\boldsymbol{\xi}).\nabla_x \varphi(\boldsymbol{X}(t;\boldsymbol{\xi}), \boldsymbol{V}(t;\boldsymbol{\xi})) + \boldsymbol{\gamma}_2(t;\boldsymbol{\xi}).\nabla_v \varphi(\boldsymbol{X}(t;\boldsymbol{\xi}), \boldsymbol{V}(t;\boldsymbol{\xi})) dt\ dS(\boldsymbol{\xi}). \end{aligned}$$

where we have set

$$\boldsymbol{\beta}_2(t;\boldsymbol{\xi}) = \nabla_v \boldsymbol{X}(t;\boldsymbol{\xi},\boldsymbol{u}_e(\boldsymbol{\xi})).\boldsymbol{g}_e(\boldsymbol{\xi}),\ \ \boldsymbol{\gamma}_2(t;\boldsymbol{\xi}) = \nabla_v \boldsymbol{V}(t;\boldsymbol{\xi},\boldsymbol{u}_e(\boldsymbol{\xi})).\boldsymbol{g}_e(\boldsymbol{\xi}).$$

On the other hand, it follows from (3.5) that $t \to (\boldsymbol{\beta}_2(t;\boldsymbol{\xi}), \gamma_2(t;\boldsymbol{\xi}))$ is the solution of the differential system

$$\begin{cases} \dfrac{d\boldsymbol{\beta}_2}{dt}(t;\boldsymbol{\xi}) = \gamma_2(t;\boldsymbol{\xi}), \\ m\dfrac{d\gamma_2}{dt}(t;\boldsymbol{\xi}) = \nabla_x \boldsymbol{F}(\boldsymbol{X}(t;\boldsymbol{\xi}), \boldsymbol{V}(t;\boldsymbol{\xi})).\boldsymbol{\beta}_2(t;\boldsymbol{\xi}) + \\ \qquad\qquad + \nabla_v \boldsymbol{F}(\boldsymbol{X}(t;\boldsymbol{\xi}), \boldsymbol{V}(t;\boldsymbol{\xi})).\gamma_2(t;\boldsymbol{\xi}), \\ \boldsymbol{\beta}_2(0;\boldsymbol{\xi}) = \mathbf{0} \quad , \quad \gamma_2(0;\boldsymbol{\xi}) = \boldsymbol{g}_2(\boldsymbol{\xi}). \end{cases}$$

Therefore, we obtain

$$< f', \varphi > = -\int_{\mathcal{C}_e}\int_0^{\tau(\xi)} \{\boldsymbol{\beta}(t;\boldsymbol{\xi}).\nabla_x \varphi(\boldsymbol{X}(t;\boldsymbol{\xi}), \boldsymbol{V}(t;\boldsymbol{\xi})) + \\ + \gamma(t;\boldsymbol{\xi}).\nabla_v \varphi(\boldsymbol{X}(t;\boldsymbol{\xi}), \boldsymbol{V}(t;\boldsymbol{\xi}))\} dt\ dS(\boldsymbol{\xi})$$

where

$$\begin{cases} \boldsymbol{\beta}(t;\boldsymbol{\xi}) = (n_e \boldsymbol{u}_e.\boldsymbol{\nu})(\boldsymbol{\xi})\ \boldsymbol{\beta}_1(t;\boldsymbol{\xi}) - \boldsymbol{\beta}_2(t;\boldsymbol{\xi}) \\ \gamma(t;\boldsymbol{\xi}) = (n_e \boldsymbol{u}_e.\boldsymbol{\nu})(\boldsymbol{\xi})\ \gamma_1(t;\boldsymbol{\xi}) - \gamma_2(t;\boldsymbol{\xi}) \end{cases}$$

is the solution of (3.15). This yields (3.16). ■

Remark 3. As an alternate proof Theorem 6, we could have checked directly that (3.15) and (3.16) indeed define a solution of the problem (3.12). ■

Remark 4. It follows from (3.16) that $n' = \int f' d\boldsymbol{v}$ is the first order distribution

$$n' = \int_{\mathcal{C}_e}\int_0^{\tau(\xi)} \boldsymbol{\beta}(t;\boldsymbol{\xi}).\nabla\delta(\boldsymbol{x} - X(t;\boldsymbol{\xi}))dt\ dS(\boldsymbol{\xi}). \tag{3.17}$$

Note that in applications to free boundary problems for the Vlasov- Poisson equations, we make use only of the expressions (3.11) and (3.17) of the density n and the linearized density n' and not of the expressions of f and f'. Again, we refer to [2],[3] for details. ■

References.

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