# MULTICOMPONENT REACTIVE FLOWS
# (I) SYMMETRIZATION AND LOCAL EXISTENCE

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We consider the equations governing multicomponent reactive flows derived from the kinetic theory of dilute polyatomic reactive gas mixtures. Using an entropy function, we derive a symmetric conservative form of the system. In the framework of Kawashima's and Shizuta's theory, we recast the resulting system into a normal form, that is, in the form of a symmetric hyperbolic-parabolic composite system. We also characterize all normal forms for symmetric systems of conservation laws such that the nullspace associated with dissipation matrices is invariant. Using a result of Vol'pert and Hudjaev, we then prove the local existence of a bounded smooth solution to the Cauchy problem.

## 1. Introduction

In this paper, we investigate the system of equations modeling multicomponent reactive flows. We prove the well-posedness of the Cauchy problem locally in time with smooth initial conditions. In a subsequent paper, global existence and asymptotic stability around constant equilibrium states will be investigated [$\mathrm{GM96_{II}}$].

We first present the governing equations for multicomponent gaseous flows derived from the kinetic theory of dilute polyatomic reactive gas mixtures [EG94]. We express the conservation equations, the transport fluxes and the thermodynamic properties. An important point in these equations is that the transport fluxes have their natural symmetry properties. We use in particular the symmetric diffusion coefficients introduced by Waldmann and Trübenbacher [WT62] [Gi91]. On the contrary, Hirschfelder, Curtiss and Bird [HCB64] have artificially destroyed this symmetry [Gi91] [EG94]. The detailed form of the source term derived from the kinetic theory will not be needed in this first part of the paper and is postponed to the subsequent part of the paper [$\mathrm{GM96_{II}}$].

We then investigate the symmetrizability of the resulting system of conservation laws. This property is closely related to the existence of an entropy function as shown by Kawashima and Shizuta in the case of hyperbolic–parabolic systems [KS88], extending previous work on hyperbolic systems [Go62] [FL71] [Mo80]. We exhibit such an entropy function for the system of equations modeling multicomponent flows. This mathematical entropy is taken to be the opposite of the physical mixture entropy density per unit volume.

By using the corresponding entropic variable, we further derive a symmetric conservative formulation of the system. Chalot, Hughes and Shakib [CHS90] have carried out similar calculations in the case of flows in thermochemical nonequilibrium. However, they have used a multicomponent diffusion matrix which is not symmetric [HCB54] and which prohibits complete symmetrization. These authors have thus advocated Onsager's phenomenological coefficients in order to achieve symmetrization. On the contrary, by using the symmetric form of the transport fluxes and of the diffusion coefficients [WT62] [Gi91] [EG94], we have obtained a naturally symmetric conservative formulation.

Once a conservative symmetric form is obtained, we recast the system into a normal form, i.e., in the form of an hyperbolic–parabolic composite system. Normal forms for symmetric systems of conservation laws have been investigated by Kawashima and Shizuta [KS88]. These authors have shown in particular that symmetric systems of conservation laws such that the nullspace naturally associated with dissipation matrices is invariant can be cast into a normal form. Such symmetric systems of conservation laws are said to satisfy Condition N. In the framework of Kawashima and Shizuta's theory, we further characterize, in this paper, all normal forms for systems satisfying Condition N.

As an application, the symmetrized multicomponent reactive flows governing equations are shown to satisfy Condition N and are recast into two different normal forms. The first normal form has simpler matrix coefficients and generalizes previous results from Kawashima and Shizuta [KS88]. The second normal form has dissipative terms in conservative form and leaves unchanged the structure of the source term. Both forms can be used for the existence theorem obtained in this part of the paper and for the asymptotic stability of constant equilibrium states investigated in the subsequent part of the paper [$\mathrm{GM96}_{\mathrm{II}}$].

We next prove the existence of a unique solution to the Cauchy problem, locally in time, in a space of bounded smooth functions. Our method of proof relies on the results of Vol'pert and Hudjaev concerning the Cauchy problem for symmetric quasilinear hyperbolic-parabolic composite systems of partial differential equations [VH72], making use of a normal form of the system obtained in the previous sections. To the authors' knowledge, these results on multicomponent reactive flows, obtained in a mathematical framework deduced from the kinetic theory of gas mixtures [Gi91] [EG94], are new.

The governing equation for multicomponent reactive flows are presented in Section 2. Symmetrizability for systems of conservation laws, entropy functions and normal forms are investigated in Section 3. Finally, the Cauchy problem is investigated in Section 4.

## 2. Governing Equations

### 2.1. Conservation equations

The equations modeling multicomponent reactive flows express the conservation of species mass, momentum and energy. These equations can be written in the form [WT62] [EG94]

$$\partial_t U + \sum_{i \in C} \partial_i F_i + \sum_{i \in C} \partial_i \mathcal{F}_i = \Omega, \tag{2.1}$$

where $\partial_t$ is the time derivative operator, $U$ the conservative variable, $\partial_i$ the space derivative operator in the $i^{\text{th}}$ direction, $C = \{1, 2, 3\}$ the set of direction indices, $F_i$ the advective flux in the $i^{\text{th}}$ direction, $\mathcal{F}_i$ the dissipative flux in the $i^{\text{th}}$ direction and $\Omega$ the source term. The variable $U$ and the advective fluxes $F_i$, $i \in \mathcal{C}$, are given by

$$U = \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_{n_S} \\ \rho v_1 \\ \rho v_2 \\ \rho v_3 \\ \rho e^{\text{tot}} \end{pmatrix}, \qquad F_i = \begin{pmatrix} \rho_1 v_i \\ \vdots \\ \rho_{n_S} v_i \\ \rho v_1 v_i + \delta_{i1} p \\ \rho v_2 v_i + \delta_{i2} p \\ \rho v_3 v_i + \delta_{i3} p \\ \rho e^{\text{tot}} v_i + p v_i \end{pmatrix}, \tag{2.2}$$

where $\rho_k$ is the density of the $k^{\text{th}}$ species, $n_S$ the number of species, $S = [1, n_S]$ the set of species indices, $\rho = \sum_{k \in S} \rho_k$ the total density, $v_i$ the mass averaged flow velocity in the $i^{\text{th}}$ direction, $e^{\text{tot}}$ the total energy per unit mass of the mixture, and $p$ the thermodynamical pressure. For convenience, the dissipative flux $\mathcal{F}_i$ is splitted between the mass and heat diffusion flux $\mathcal{F}_i^{D\lambda}$ and the viscous flux $\mathcal{F}_i^{\kappa\eta}$ so that

$$\mathcal{F}_i = \mathcal{F}_i^{D\lambda} + \mathcal{F}_i^{\kappa\eta}. \tag{2.3}$$

The fluxes $\mathcal{F}_i^{D\lambda}$ and $\mathcal{F}_i^{\kappa\eta}$, and the source term $\Omega$, are given by

$$\mathcal{F}_i^{\kappa\eta} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \Pi_{i1} \\ \Pi_{i2} \\ \Pi_{i3} \\ \sum\limits_{j \in C} \Pi_{ij} v_j \end{pmatrix}, \qquad \mathcal{F}_i^{D\lambda} = \begin{pmatrix} \rho_1 \mathcal{V}_{1i} \\ \vdots \\ \rho_{n_S} \mathcal{V}_{n_S i} \\ 0 \\ 0 \\ 0 \\ q_i \end{pmatrix}, \qquad \Omega = \begin{pmatrix} m_1 \omega_1 \\ \vdots \\ m_{n_S} \omega_{n_S} \\ \rho g_1 \\ \rho g_2 \\ \rho g_3 \\ \rho g \cdot v \end{pmatrix}, \tag{2.4}$$

where $\Pi = (\Pi_{ij})_{i,j\in C}$ is the viscous stress tensor, $\mathcal{V}_k = (\mathcal{V}_{k1}, \mathcal{V}_{k2}, \mathcal{V}_{k3})^t$ the diffusion velocity of the $k^{\text{th}}$ species, $q = (q_1, q_2, q_3)^t$ the heat flux vector, $m_k$ the molar mass of the $k^{\text{th}}$ species, $\omega_k$ the molar production rate of the $k^{\text{th}}$ species, $g = (g_1, g_2, g_3)^t$ the external force per unit mass acting on the species, $v = (v_1, v_2, v_3)^t$ the velocity vector and $^t$ the transposition symbol.

These equations have to be completed by the relations expressing the transport fluxes $\Pi$, $\mathcal{V}_k$, $k \in S$, and $q$, the thermodynamic properties $p$ and $e^{\text{tot}}$, the chemical source terms $\omega_k$, $k \in S$, and the specific force $g$.

## 2.2. Transport fluxes

The expressions for the transport fluxes rigorously derived from the kinetic theory of dilute polyatomic gas mixture can be written as [WT62] [EG94]

$$\mathcal{V}_k = -\sum_{l\in S} D_{kl} d_l - \theta_k \partial_x \log T, \qquad k \in S, \tag{2.5}$$

$$\Pi = -(\kappa - \tfrac{2}{3}\eta)(\partial_x \cdot v)I - \eta\big(\partial_x v + (\partial_x v)^t\big), \tag{2.6}$$

$$q = -\lambda' \partial_x T - p \sum_{k\in S} \theta_k d_k + \sum_{k\in S} \rho_k h_k \mathcal{V}_k, \tag{2.7}$$

where $D = (D_{kl})_{k,l\in S}$ is the diffusion matrix, $d_k$ the diffusion driving force of the $k^{\text{th}}$ species, $\theta = (\theta_1, \dots, \theta_{n_S})^t$ the thermal diffusion vector, $\partial_x = (\partial_1, \partial_2, \partial_3)^t$ the usual differential operator, $T$ the absolute temperature, $\kappa$ the volume viscosity, $\eta$ the shear viscosity, $\lambda'$ the partial thermal conductivity and $h_k$ the enthalpy per unit mass of the $k^{\text{th}}$ species. The vectors $d_k$, $k \in S$, take into account the effects of various state variable gradients and are given by

$$d_k = \partial_x X_k + X_k \frac{\partial_x p}{p}, \tag{2.8}$$

where $X_k$ denotes the mole fraction of the $k^{\text{th}}$ species. Alternate expressions for the diffusion velocities and the heat flux vector are

$$\mathcal{V}_k = -\sum_{l\in S} D_{kl}(d_l + \chi_l \partial_x \log T), \qquad k \in S, \tag{2.9}$$

$$q = -\lambda \partial_x T + p \sum_{l\in S} \chi_l \mathcal{V}_l + \sum_{l\in S} \rho_l h_l \mathcal{V}_l, \tag{2.10}$$

where $\chi = (\chi_1, \dots, \chi_{n_S})^t$ is the thermal diffusion ratio vector and $\lambda$ the thermal conductivity. Both expressions (2.5) (2.9) for the diffusion velocities and (2.7) (2.10) for the heat flux vector will be used in the following.

These formulations of the pressure tensor (2.6), the diffusion velocities (2.5) (2.9), and the heat flux (2.7) (2.10) are due to Waldmann and Trübenbacher [WT62] [EG94]. In particular, the diffusion matrix $D$ associated with these fluxes is symmetric as specified in Section 2.7 where the properties of the various transport coefficients are expressed.

### 2.3. Thermodynamic properties

From the kinetic theory, the state law expressing the pressure $p$ is

$$p = \rho\, r\, T, \tag{2.11}$$

where

$$\rho\, r = R_{\mathrm{g}} \sum_{k \in S} \frac{\rho_k}{m_k}. \tag{2.12}$$

In these expressions, $\rho = \sum_{k \in S} \rho_k$ is the total mass density, $r$ the specific gas constant of the mixture, and $R_{\mathrm{g}}$ the universal gas constant. The specific total energy $e^{\mathrm{tot}}$ and the specific internal energy $e$ of the mixture are given by

$$e^{\mathrm{tot}} = e + \tfrac{1}{2} v \cdot v, \tag{2.13}$$

where

$$\rho e = \sum_{k \in S} \rho_k e_k. \tag{2.14}$$

The quantity $e_k$ is the internal energy per unit mass of the $k^{\mathrm{th}}$ species and can be written

$$e_k(T) = e_k^0 + \int_{T_0}^{T} c_{vk}(T')\ dT', \tag{2.15}$$

where $e_k^0$ is the energy of formation of the $k^{\mathrm{th}}$ species at the positive reference temperature $T_0$ and $c_{vk}$ is the specific heat at constant volume of the $k^{\mathrm{th}}$ species. The mixture specific heat at constant volume $c_v$ is also defined by

$$\rho c_v = \sum_{k \in S} \rho_k c_{vk}. \tag{2.16}$$

Similarly, the specific total enthalpy $h^{\mathrm{tot}}$ and specific enthalpy $h$ are written

$$h^{\mathrm{tot}} = h + \tfrac{1}{2} v \cdot v, \qquad \rho h = \sum_{k \in S} \rho_k h_k, \tag{2.17}$$

where $h_k$, the enthalpy per unit mass of the $k^{\mathrm{th}}$ species, reads

$$h_k(T) = e_k(T) + r_k T, \tag{2.18}$$

and where $r_k = R_{\mathrm{g}}/m_k$ is the specific gas constant of the $k^{\mathrm{th}}$ species. It is also convenient to denote by $e_k^{\mathrm{tot}}$ the total energy per unit mass of the $k^{\mathrm{th}}$ species and by $h_k^{\mathrm{tot}}$ the total enthalpy per unit mass of the $k^{\mathrm{th}}$ species

$$e_k^{\mathrm{tot}} = e_k + \tfrac{1}{2} v \cdot v, \qquad h_k^{\mathrm{tot}} = h_k + \tfrac{1}{2} v \cdot v. \tag{2.19}$$

The kinetic theory also yields the specific (physical) entropy of the $k^{\text{th}}$ species

$$s_k(\rho_k, T) = s_k^{00} + \int_{T_0}^{T} \frac{c_{vk}(T')}{T'}\, dT' - r_k \log\left(\frac{\rho_k}{\gamma_0 m_k}\right), \tag{2.20}$$

where $s_k^{00}$ is the standard formation entropy at the positive reference temperature $T_0$ and positive reference pressure $\gamma_0 R_{\mathrm{g}} T_0$. Finally, we will also need the expression of the chemical potential $\mu_k(\rho_k, T)$ of the $k^{\text{th}}$ species

$$\mu_k = e_k + r_k T - s_k T. \tag{2.21}$$

### 2.4. Source terms

The detailed description of the chemical source terms $\omega_k$, $k \in S$, is beyond the scope of these sections. It will only be needed for investigating global existence results and asymptotic stability of constant equilibrium states in the subsequent part of the paper [GM96$_{\mathrm{II}}$]. In the following sections, we only require that the chemical source terms $\omega_k$, $k \in S$, are functions of the natural variable $Y = (\rho_1, \dots, \rho_{n_S}, v_1, v_2, v_3, T)^t$

$$\omega_k = \omega_k(Y), \tag{2.22}$$

with a similar assumption for the specific force term $g$

$$g = g(Y). \tag{2.23}$$

**Remark 2.1.** In this paper, for sake of simplicity, we only consider a species independent specific force, as gravity for instance. When the specific forces are species dependent, the overall force term reads $g = \sum_{k \in S} \rho_k \mathfrak{g}_k$, and the diffusion driving force terms become $d'_k = d_k - (\rho_k/p)\mathfrak{g}_k$, where $\mathfrak{g}_k$ denotes the specific force acting on the $k^{\text{th}}$ species. In this situation, there are extra fluxes arising from the modified diffusion driving force terms, and symmetrization of these fluxes can only be achieved provided that special compatibility relations hold between the specific forces and the transport coefficients [Ma96].

### 2.5. Derivation from the kinetic theory

In this section we briefly discuss the derivation of the preceding equations from the semi-classical kinetic theory of reactive polyatomic gaz mixtures [WT62] [EG94]. This derivation requires that the chemical characteristic times are larger than the characteristic mean free times of the mixture [EG94].

More specifically, the Boltzmann equations describing reactive mixtures take the form

$$\mathfrak{D}_k(f_k) = \sum_{l \in S} \mathfrak{C}_{kl}(f_k, f_l) + \mathfrak{S}_k(f_1, \dots, f_{n_S}), \qquad k \in S.$$

where $f_k$ is the distribution function of the $k^{\rm th}$ species, $\mathfrak{D}_k$ the transport differential operator of the $k^{\rm th}$ species, $\mathfrak{C}_{kl}$ the non reactive collision operator for the species pair $(k,l)$, and $\mathfrak{S}_k$ the reactive collision operator for the $k^{\rm th}$ species. These operators have complex expressions and we refer to [EG94] for more details. The distribution function $f_k$ is a function of space and time, of the peculiar velocity $c_k$, and quantum state $\kappa$ of the molecule of the $k^{\rm th}$ species.

It is then possible to distinguish a number of different regimes for the reactive mixture, depending on the relative order of magnitude of the nonreactive and reactive collision terms. For this purpose, the Boltzmann equations are rewritten in the form

$$\mathfrak{D}_k(f_k) = \frac{1}{\epsilon}\sum_{l\in S}\mathfrak{C}_{kl}(f_k,f_l) + \epsilon^{\alpha}\ \mathfrak{S}_k(f_1,\dots,f_{n_S}), \qquad k\in S, \tag{2.24}$$

where $\epsilon$ is the formal expansion parameter associated with the Enskog procedure and where the parameter $\alpha$ depends on the regime under consideration. Accordingly, the species distribution functions are expanded in the form

$$f_k = f_k^0\big(1+\epsilon\phi_k + O(\epsilon^2)\big), \qquad k\in S,$$

where $f_k^0$ is the local Maxwellian distribution.

When the chemistry characteristic times are larger than the corresponding mean free times and the flow times, we then have $\alpha=1$ in (2.24). This regime is termed the "slow reaction regime". In this situation, the perturbed distribution functions $\phi_k=\phi_k^{\rm f}$, $k\in S$, are independent of the reactive collision terms. As a consequence, the transport fluxes are exactly those obtained as if there were no chemical reactions and are given by (2.5)–(2.10). In addition, the macroscopic source terms are evaluated from the Maxwellian reactive collision terms $\mathfrak{S}_k(f_1^0,\dots,f_{n_S}^0)$. In particular, these Maxwellian rates $\omega_k^0(\rho_1,\dots,\rho_{n_S},T)$, $k\in S$, are compatible with the law of mass action [EG94].

On the other hand, when the chemistry characteristic times are larger than the corresponding mean free times, but are of the same order of magnitude than the flow time, we then have $\alpha=0$ in (2.24). This is the "tempered reaction regime" discussed in [EG94]. In this situation, the species perturbed distribution functions $\phi_k$, $k\in S$, can be split into $\phi_k=\phi_k^{\rm f}+\phi_k^{\rm r}$, where $\phi_k^{\rm f}$ corresponds to the chemically frozen case, and $\phi_k^{\rm r}$ only depends on the reactive collision term. The fluxes due to macroscopic gradients are those associated with $\phi_k^{\rm f}$, $k\in S$, and are still given by (2.5)–(2.10). However, there is an additional chemical pressure correction of the pressure tensor $\Pi$, associated with the perturbed distribution functions $\phi_k^{\rm r}$, $k\in S$. Furthemore, the macroscopic source terms are also modified and are no longer compatible with the law of mass action [EG94]. These rates are a perturbation of the Maxwellian rates $\omega_k^0$, $k\in S$, and involve functionals associated with the linearization of the chemistry source terms $\mathfrak{S}_k(f_1,\dots,f_{n_S})$. They take the form of a regular function of the state variables $(\rho_1,\dots,\rho_{n_S},T)$ and of the divergence of the velocity $\partial_x{\cdot}v$ [EG94]. The description and analysis of the corresponding full set of equations is beyond the scope of the present paper. We note, however, that, by neglecting the chemical pressure term and the perturbations of the production rates, we recover the equations considered in this paper.

For all these regimes, the macroscopic quantities are then expressed in terms of the distribution functions with the relations

$$\rho_k = \sum_{\mathrm{K} \in \mathfrak{Q}_k} \int \mathfrak{m}_k f_k dc_k$$

$$\rho v = \sum_{\substack{k \in S \\ \mathrm{K} \in \mathfrak{Q}_k}} \int \mathfrak{m}_k c_k f_k dc_k$$

and

$$\tfrac{1}{2}\rho v \cdot v + \rho e = \sum_{\substack{k \in S \\ \mathrm{K} \in \mathfrak{Q}_k}} \int (\tfrac{1}{2}\mathfrak{m}_k c_k \cdot c_k + \mathfrak{e}_{k\mathrm{K}}) f_k dc_k,$$

where $\mathfrak{m}_k$ denotes the mass of the $k^{\text{th}}$ species, $\mathfrak{e}_{k\mathrm{K}}$ the energy of a molecule of type $k$ in state $\mathrm{K}$, and $\mathfrak{Q}_k$ the set of quantum state indices of the $k^{\text{th}}$ species [EG94]. Finally, the conservation equations are obtained by multiplying the Boltzmann equations by the collisional invariants, integrating with respect to the particle velocity and summing with respect to the quantum states of the particles [WT62] [EG94].

### 2.6. Mathematical assumptions

We introduce here the mathematical assumptions concerning the transport coefficients, the thermodynamic properties, the chemical source terms and the specific force term. We assume that the natural variable $Y = (\rho_1, \dots, \rho_{n_S}, v_1, v_2, v_3, T)^t$ takes its values in the open convex set $\mathcal{O}_Y$

$$\mathcal{O}_Y = (0,\infty)^{n_S} \times \mathbb{R}^3 \times (T_0, \infty), \tag{2.25}$$

where $T_0$ is positive and we assume the following dependence and regularity properties.

($H_1$) The transport coefficients $(D_{kl})_{k,l \in S}$, $(\theta_1, \dots, \theta_{n_S})^t$, $(\chi_1, \dots, \chi_{n_S})^t$, $\kappa$, $\eta$, $\lambda'$, and $\lambda$ are $C^\infty$ functions of $(\rho_1, \dots, \rho_{n_S}, T) \in (0,\infty)^{n_S} \times (T_0, \infty)$.

($H_2$) The specific heats $c_{vk}$, $k \in S$, are $C^\infty$ functions of $T \in [T_0, \infty)$. Moreover there exists a positive constant $a$ with $0 < a \le c_{vk}(\xi)$, for $\xi \ge T_0$ and $k \in S$.

($H_3$) The force term $g$ is a $C^\infty$ function of $Y \in \mathcal{O}_Y$.

($H_4$) The chemical production rate vector $\omega = (\omega_1, \dots, \omega_{n_S})^t$ is a $C^\infty$ function of $Y \in \mathcal{O}_Y$.

($H_5$) The shear viscosity $\eta$, the thermal conductivity $\lambda$, and the partial thermal conductivity $\lambda'$ are positive and the volume viscosity $\kappa$ is nonnegative.

$(H_6)$ The matrix $D = (D_{kl})_{k,l\in S}$ is symmetric positive semi-definite and its nullspace is spanned by $\varrho = (\rho_1, \ldots, \rho_{n_S})^t$. We have in particular the mass constraints

$$\sum_{k\in S} \rho_k D_{kl} = 0, \qquad l \in S.$$

$(H_7)$ The thermal diffusion vector $\theta = (\theta_1, \ldots, \theta_{n_S})^t$ satisfies the mass constraint

$$\sum_{k\in S} \rho_k \theta_k = 0.$$

$(H_8)$ The thermal diffusion ratio vector $\chi = (\chi_1, \ldots, \chi_{n_S})^t$ satisfies the relations

$$\sum_{l\in S} D_{kl}\chi_l = \theta_k, \quad k \in S, \qquad\qquad \sum_{k\in S} \chi_k = 0.$$

The partial thermal conductivity $\lambda'$ is given by

$$\lambda = \lambda' - \frac{p}{T} \sum_{l\in S} \chi_l \theta_l.$$

$(H_9)$ The chemical production rate vector $\omega = (\omega_1, \ldots, \omega_{n_S})^t$ satisfies the mass conservation relation

$$\sum_{k\in S} m_k \omega_k = 0.$$

We point out that all these assumptions are suggested by the semi-classical kinetic theory of dilute polyatomic reactive gas mixtures [Gi91] [EG94]. As previously mentioned, the diffusion coefficients considered here are symmetric and, therefore, are consistent with Onsager reciprocal relations. On the contrary, Hirschifelder, Curtiss and Bird have introduced an alternate definition [HCB54] and artiticially destroyed the symmetry of the diffusion process [Gi91] [EG94]. The mass constraints of the diffusion matrix and the thermal diffusion vector also imply the mass conservation relation $\sum_{k\in S} \rho_k \mathcal{V}_k = 0$. In addition, the positivity properties of the transport coefficients are associated with the positivity of the entropy production quadratic form [Gi91] [EG94].

Further note that the gas species specific heats—and therefore the energies and enthalpies—obtained from the kinetic theory, could also be extended—from a mathematical point of view—up to zero temperature, but not the gas entropy which explodes like $\log T$. However, since the basic assumptions of the kinetic theory of dilute gas mixtures are not valid at low temperatures, where the gases are ultimately transformed into liquids and then into solids, we have chosen to restrict the temperature domain to $[T_0, \infty)$, where $T_0$ is positive, for modeling gas mixtures.

### 2.7. The quasilinear form

By expressing the natural variable $Y$ in term of the conservative variable $U$, we now rewrite the system of conservation equations (2.1) in a quasilinear form. For this purpose, we first investigate the map $Y \to U$ and its range.

**Proposition 2.2.** The map $Y \longmapsto U$ is a $C^\infty$ diffeomorphism from the open set $\mathcal{O}_Y = (0,\infty)^{n_S} \times \mathbb{R}^3 \times (T_0,\infty)$ onto an open set $\mathcal{O}_U$. The open set $\mathcal{O}_U$ is convex and given by

$$\mathcal{O}_U = \{\ z \in \mathbb{R}^{n_S+4};\, z_i > 0,\ 1 \le i \le n_S, \quad z_{n_S+4} - \phi(z_1,\dots,z_{n_S+3}) > 0\ \}, \tag{2.26}$$

where

$$\phi(z_1,\dots,z_{n_S+3}) = \tfrac{1}{2}\frac{z_{n_S+1}^2 + z_{n_S+2}^2 + z_{n_S+3}^2}{\sum_{i\in S} z_i} + \sum_{i\in S} z_i e_i^0.$$

**Proof.** From the smoothness assumption $(H_2)$ and the expression (2.2) we first deduce that the map $Y \to U$ is $C^\infty$ over the domain $\mathcal{O}_Y$. The matrix $\partial_Y U$ is easily shown to be nonsingular over $\mathcal{O}_Y$ thanks to its triangular structure. Finally, it is straightforward to show that the map $Y \to U$ is one to one so that $Y \to U$ is a $C^\infty$ diffeomorphism onto an open set $\mathcal{O}_U$. From $(H_2)$, it is then easily established that $\mathcal{O}_U$ is given by (2.26). The convexity of $\mathcal{O}_U$ is then a direct consequence of the convexity of $\phi$ which is established by evaluating its second derivative. □

From (2.5)–(2.8) and Proposition 2.2, the dissipation fluxes can then be expressed as functions of the conservative variable gradients

$$\mathcal{F}_i = -\sum_{j\in C} G_{ij}(U)\partial_j U \tag{2.27}$$

where $G_{ij}(U)$, $i,j \in C$, are the dissipation matrices. These matrices are square matrices of dimension $n_S+4$, and, from (2.3), they admit the following decomposition

$$G_{ij} = G_{ij}^{\kappa\eta} + G_{ij}^{D\lambda}, \tag{2.28}$$

where $\mathcal{F}_i^{\kappa\eta} = -\sum_{j\in C} G_{ij}^{\kappa\eta}(U)\partial_j U$ and $\mathcal{F}_i^{D\lambda} = -\sum_{j\in C} G_{ij}^{D\lambda}(U)\partial_j U$. We may further introduce the jacobian matrices $A_i$, $i \in C$, of the advection fluxes $F_i$, $i \in C$,

$$A_i = \partial_U F_i, \tag{2.29}$$

and finally rewrite the system into the quasi-linear form

$$\partial_t U + \sum_{i\in C} A_i \partial_i U = \sum_{i,j\in C} \partial_i\big(G_{ij}\partial_j U\big) + \Omega(U), \tag{2.30}$$

where the matrix coefficients are defined on the open convex set $\mathcal{O}_U$. The detailed form of the coefficient matrices $A_i$, $i \in C$, and $G_{ij}(U)$, $i,j \in C$, will not be needed in the following, and, therefore, will not be given and we refer to [Ma96] for more details.

## 3. Symmetrization and Normal Forms

For hyperbolic systems of conservation laws, the existence of a conservative symmetric formulation has been shown to be equivalent to the existence of an entropy function [Go62] [FL71] [Mo80]. These results have been generalized to the case of second order quasilinear systems of equations by Kawashima and Shizuta [KS88], using Kawashima's definition of an entropy function [Ka84]. In this section, the results of Kawashima and Shizuta about conservative symmetrizability are restated and applied to the multicomponent reactive flows governing equations.

Following Kawashima and Shizuta [KS88], we then investigate normal forms of the system, i.e., symmetric hyperbolic-parabolic composite forms. Kawashima and Shizuta [KS88] have shown that, when the nullspace naturally associated with dissipation matrices is a fixed subspace, a symmetric system of conservation equations can be put into a normal form. In the framework of their theory, we further characterize all normal forms for such symmetric systems of conservation laws.

We then show that the equations governing multicomponent reactive flows satisfy the latter requirement and we recast the symmetrized system into two normal forms. The first normal form has simpler matrix coefficients and generalizes the normal form of the Navier-Stokes equations previously obtained by Kawashima and Shizuta [KS88]. This form also perturbs the structure of the source term. The second normal form, given in the Appendix, is more natural but also more complex, and has dissipative terms in conservative form. Both forms can be used for the existence theorem obtained in this part of the paper and for the asymptotic stability of constant equilibrium states investigated in the subsequent part of the paper [GM96$_{\mathrm{II}}$].

### 3.1. Entropy functions and symmetric conservative forms

We consider an abstract second order quasilinear system in the form

$$\partial_t U^* + \sum_{i\in C^*} \partial_i F_i^*(U^*) = \sum_{i,j\in C^*} \partial_i\big(G_{ij}^*(U^*)\partial_j U^*\big), \tag{3.1}$$

where $U^* \in \mathcal{O}_{U^*}$, $\mathcal{O}_{U^*}$ is an open convex set of $\mathbb{R}^n$, $C^* = \{1,\ldots,d\}$ is the set of direction indices of $\mathbb{R}^d$, and $F_i^*(U^*)$ and $G_{ij}^*(U^*)$ are smooth functions of $U^* \in \mathcal{O}_{U^*}$. Note that the superscript $^*$ is used in order to distinguish between the abstract second order system (3.1) of size $n$ in $\mathbb{R}^d$ and the particular multicomponent reactive flows system (2.30) of size $n_S + 4$ in $\mathbb{R}^3$. All quantities associated with the abstract system have the corresponding superscript $^*$, so that for instance the unknown vector is $U^*$.

The following definition of a symmetric (conservative) form, for the system (3.1), is due to Kawashima and Shizuta [KS88].

**Definition 3.1.** Assume that $V^* \to U^*$ is a diffeomorphism from $\mathcal{O}_{V^*}$ onto $\mathcal{O}_{U^*}$, and consider the system in the $V^*$ variable

$$\widetilde{A}_0^* \partial_t V^* + \sum_{i \in C^*} \widetilde{A}_i^* \partial_i V^* = \sum_{i,j \in C^*} \partial_i \Big( \widetilde{G}_{ij}^* \partial_j V^* \Big), \tag{3.2}$$

where

$$\begin{cases} \widetilde{A}_0^* = \partial_{V^*} U^*, & A_i^* = \partial_{U^*} F_i^*, \\ \widetilde{A}_i^* = A_i^* \widetilde{A}_0^*, & \widetilde{G}_{ij}^* = G_{ij}^* \widetilde{A}_0^*. \end{cases} \tag{3.3}$$

The system (3.2) is said to be of the symmetric form if the matrix coefficients satisfy the following properties.

- $(S_1)$ The matrix $\widetilde{A}_0^*(V^*)$ is symmetric and positive definite for $V^* \in \mathcal{O}_{V^*}$,
- $(S_2)$ The matrices $\widetilde{A}_i^*(V^*)$, $i \in C^*$, are symmetric for $V^* \in \mathcal{O}_{V^*}$,
- $(S_3)$ We have $\widetilde{G}_{ij}^*(V^*)^t = \widetilde{G}_{ji}^*(V^*)$ for $i, j \in C^*$ and $V^* \in \mathcal{O}_{V^*}$,
- $(S_4)$ The matrix $\widetilde{B}^*(V^*, w) = \sum_{i,j \in C^*} \widetilde{G}_{ij}^*(V^*) w_i w_j$ is symmetric and positive semi-definite, for $V^* \in \mathcal{O}_{V^*}$ and $w \in \mathcal{S}^{d-1}$, where $\mathcal{S}^{d-1}$ is the unit sphere in $d$ dimensions.

On the other hand, the following generalized definition of an entropy function has been given by Kawashima [Ka84] [KS88].

**Definition 3.2.** A real-valued smooth function $\mathcal{H}^*(U^*)$ defined on a convex set $\mathcal{O}_{U^*}$ is said to be an entropy function for the system (3.1) if the following properties hold

- $(E_1)$ The function $\mathcal{H}^*$ is a strictly convex function on $\mathcal{O}_{U^*}$ in the sense that the Hessian matrix is positive definite on $\mathcal{O}_{U^*}$,
- $(E_2)$ There exists real-valued smooth functions $\sigma_i^* = \sigma_i^*(U^*)$ such that

$$\left(\partial_{U^*} \mathcal{H}^*\right) A_i^* = \partial_{U^*} \sigma_i^*, \quad i \in C^*, \qquad U^* \in \mathcal{O}_{U^*}, \tag{3.4}$$

- $(E_3)$ We have the property

$$\left(\partial_{U^*}^2 \mathcal{H}^*(U^*)\right)^{-1} (G_{ij}^*)^t = G_{ji}^* \left(\partial_{U^*}^2 \mathcal{H}^*(U^*)\right)^{-1}, \quad i, j \in C^*, \quad U^* \in \mathcal{O}_{U^*}. \tag{3.5}$$

- $(E_4)$ The matrix $\widetilde{B}^*(V^*, w) = \sum_{i,j \in C^*} G_{ij}^*(U^*) \left(\partial_{U^*}^2 \mathcal{H}^*(U^*)\right)^{-1} w_i w_j$ is symmetric positive semi-definite for $U^* \in \mathcal{O}_{U^*}$ and $w \in \mathcal{S}^{d-1}$.

Kawashima and Shizuta have established the equivalence between conservative symmetrizability and the existence of an entropy function for the system [KS88].

**Theorem 3.3.** The system (3.1) can be symmetrized on the open convex set $\mathcal{O}_{U^*}$ if and only if the system admits an entropy function $\mathcal{H}^*$ on $\mathcal{O}_{U^*}$. In this situation, the symmetrizing variable $V^*$ can be expressed in terms of the gradient of the entropy function $\mathcal{H}^*$

$$V^* = \Big(\partial_{U^*}\mathcal{H}^*\Big)^t. \tag{3.6}$$

The mathematical entropy $\mathcal{H}^*$ is generally taken to be the opposite of the physical mixture entropy density per unit volume. The variable $V^*$ is usually termed the entropic variable associated with the variable $U^*$.

### 3.2. Application to multicomponent flows

We now consider the system (2.30) and apply the previous general results to its differential part, that is, to the time derivative, convective and dissipative terms. We first note that the smoothness of the matrix coefficients is a direct consequence of assumptions $(H_1)$–$(H_2)$. We next define the mathematical entropy function $\mathcal{H}$ as the opposite of the physical mixture entropy density per unit volume

$$\mathcal{H} = -\sum_{k\in S} \rho_k s_k. \tag{3.7}$$

We then consider the associated entropic variable

$$V = \Big(\partial_U \mathcal{H}\Big)^t, \tag{3.8}$$

for which a straightforward calculation yields

$$V = \frac{1}{T}\Big(\mu_1 - \tfrac{1}{2}v\cdot v,\ \dots,\ \mu_{n_S} - \tfrac{1}{2}v\cdot v,\ v_1,\ v_2,\ v_3,\ -1\Big), \tag{3.9}$$

where $\mu_k$ is the chemical potential of the $k^{\rm th}$ species [Ma96].

**Proposition 3.4.** The change of variable $U \longmapsto V$ from the open convex set $\mathcal{O}_U$ onto the open set $\mathcal{O}_V = \mathbb{R}^{n_S+3} \times (-\infty, -1/T_0)$ is a $C^\infty$ diffeomorphism.

**Proof.** From Proposition 2.2, $Y \to U$ is a $C^\infty$ diffeomorphism from $\mathcal{O}_Y$ onto $\mathcal{O}_U$, so that we only have to show that $Y \to V$ is a $C^\infty$ diffeomorphism from $\mathcal{O}_Y$ onto the open set $\mathcal{O}_V$. From the smoothness assumption $(H_2)$ and the expressions (2.15) (2.20) (2.21), we first deduce that the map $Y \to V$ is $C^\infty$ over the domain $\mathcal{O}_Y$. The matrix $\partial_Y V$ is then easily shown to be nonsingular over $\mathcal{O}_Y$ thanks to its triangular structure. Finally, it is straightforward to show that the map $Y \to V$ is one to one—since $-s_k$ is an increasing function of $\rho_k$ at fixed $T$—and that its range is $\mathcal{O}_V$, and the proof is complete. □

The conservative symmetric form is now investigated in the following theorem.

**Theorem 3.5.** The function $\mathcal{H}$ is an entropy for system (2.30), that is, $\mathcal{H}$ satisties Properties $(E_1)$–$(E_4)$ of Definition 3.2. The system associated with the entropic variable $V \in \mathcal{O}_V$ can then be written

$$\widetilde{A}_0 \partial_t V + \sum_{i\in C} \widetilde{A}_i \partial_i V = \sum_{i,j\in C} \partial_i \Big( \widetilde{G}_{ij} \partial_j V \Big) + \widetilde{\Omega}, \tag{3.10}$$

and is of the symmetric form in the sense that Properties $(S_1)$–$(S_4)$ of Definition 3.1 are satisfied. The matrix $\widetilde{A}_0$ is given by

$$\widetilde{A}_0 = \begin{pmatrix} \left( \dfrac{\rho_k}{r_k} \delta_{kl} \right)_{k,l\in S} & & Sym \\ \left( \dfrac{\rho_l}{r_l} v_i \right)_{i\in C, l\in S} & \left( \Sigma_\rho v_i v_j + \rho T \delta_{ij} \right)_{i,j\in C} & \\ \left( \dfrac{\rho_l}{r_l} e_l^{\mathrm{tot}} \right)_{l\in S} & \left( \Sigma_e v_j + \rho T v_j \right)_{j\in C} & \Upsilon_e \end{pmatrix} \tag{3.11}$$

where

$$\Sigma_\rho = \sum_{k\in S} \frac{\rho_k}{r_k}, \qquad \Sigma_e = \sum_{k\in S} \frac{\rho_k}{r_k} e_k^{\mathrm{tot}}, \qquad \Upsilon_e = \sum_{k\in S} \frac{\rho_k}{r_k} e_k^{\mathrm{tot}\,2} + \rho T \left( v\cdot v + c_v T \right).$$

Since this matrix is symmetric, we only give its block lower triangular part and write "$Sym$" in the upper triangular part. On the other hand, denoting by $\xi = (\xi_1, \xi_2, \xi_3)^t$ an arbitrary vector of $\mathbb{R}^3$, the matrices $\widetilde{A}_i$, $i \in C$, are given by

$$\sum_{i\in C} \widetilde{A}_i \xi_i =$$

$$\begin{pmatrix} \left( \delta_{kl} \dfrac{\rho_l}{r_l} v\cdot\xi \right)_{k,l\in S} & & Sym \\ \left( \rho_l T \xi_i + \dfrac{\rho_l}{r_l} v_i \, v\cdot\xi \right)_{i\in C, l\in S} & \left( \Sigma_\rho v_i v_j v\cdot\xi + \rho T (v_i\xi_j + v_j\xi_i + v\cdot\xi \delta_{ij}) \right)_{i,j\in C} & \\ \left( h_l^{\mathrm{tot}} \dfrac{\rho_l}{r_l} \, v\cdot\xi \right)_{l\in S} & \left( \Sigma_h v_j v\cdot\xi + \rho T v_j v\cdot\xi + \rho T h^{\mathrm{tot}} \xi_j \right)_{j\in C} & \Upsilon_h v\cdot\xi \end{pmatrix} \tag{3.12}$$

where

$$\Sigma_h = \sum_{k\in S} \frac{\rho_k}{r_k} h_k^{\mathrm{tot}}, \qquad \Upsilon_h = \sum_{k\in S} \frac{\rho_k}{r_k} h_k^{\mathrm{tot}\,2} + \rho T \left( v\cdot v + (c_v + r) T \right).$$

Furthermore, concerning the dissipation matrices, we have the usual decomposition

$$\widetilde{G}_{ij} = \widetilde{G}_{ij}^{\kappa\eta} + \widetilde{G}_{ij}^{D\lambda}. \tag{3.13}$$

The viscous matrices $\widetilde{G}_{ij}^{\kappa\eta}$, $i, j \in C$, have the following structure

$$\widetilde{G}_{ij}^{\kappa\eta} = \begin{pmatrix} 0_{n_S\times n_S} & 0_{n_S\times 4} \\ 0_{4\times n_S} & \widetilde{\mathcal{K}}_{ij} \end{pmatrix}, \tag{3.14}$$

so that we only need the expressions of $\widetilde{\mathcal{K}}_{ij}$, $i,j \in C$. For sake of brevity, we only express $\widetilde{\mathcal{K}}_{11}$ and $\widetilde{\mathcal{K}}_{12}$

$$\widetilde{\mathcal{K}}_{11} = T \begin{pmatrix} (\kappa + \frac{4}{3}\eta) & 0 & 0 & (\kappa + \frac{4}{3}\eta)v_1 \\ 0 & \eta & 0 & \eta v_2 \\ 0 & 0 & \eta & \eta v_3 \\ (\kappa + \frac{4}{3}\eta)v_1 & \eta v_2 & \eta v_3 & (\kappa + \frac{1}{3}\eta)v_1^2 + \eta v{\cdot}v \end{pmatrix},$$

$$\widetilde{\mathcal{K}}_{12} = T \begin{pmatrix} 0 & (\kappa - \frac{2}{3}\eta) & 0 & (\kappa - \frac{2}{3}\eta)v_2 \\ \eta & 0 & 0 & \eta v_1 \\ 0 & 0 & 0 & 0 \\ \eta v_2 & (\kappa - \frac{2}{3}\eta)v_1 & 0 & (\kappa + \frac{1}{3}\eta)v_1 v_2 \end{pmatrix},$$

the other matrices being obtained by circular permutation and using to the relations

$$\widetilde{\mathcal{K}}_{12} = \widetilde{\mathcal{K}}_{21}^t, \qquad \widetilde{\mathcal{K}}_{13} = \widetilde{\mathcal{K}}_{31}^t, \qquad \widetilde{\mathcal{K}}_{23} = \widetilde{\mathcal{K}}_{32}^t.$$

On the other hand, the heat and mass diffusion matrices $\widetilde{G}_{ij}^{D\lambda}$, $i,j \in C$, satisfy

$$\widetilde{G}_{11}^{D\lambda} = \widetilde{G}_{22}^{D\lambda} = \widetilde{G}_{33}^{D\lambda} = \widetilde{G}^{D\lambda}, \qquad \widetilde{G}_{ij}^{D\lambda} = 0, \quad i \neq j,$$

where

$$\widetilde{G}^{D\lambda} = \begin{pmatrix} (\mathcal{D}_{kl})_{k,l \in S} & & Sym \\ 0_{3\times n_S} & 0_{3\times 3} & \\ \Big(\sum\limits_{k\in S} \mathcal{D}_{kl} h_k + \rho_l \theta_l T\Big)_{l\in S} & 0_{1\times 3} & \lambda' T^2 + 2\sum\limits_{k\in S} \rho_k \theta_k h_k T + \sum\limits_{k,l\in S} \mathcal{D}_{kl} h_k h_l \end{pmatrix}, \tag{3.15}$$

and the symmetric matrix $\mathcal{D}$ has been defined from the multicomponent diffusion matrix $D$ by

$$\mathcal{D}_{kl} = \rho_k \rho_l D_{kl} / r\rho. \tag{3.16}$$

Finally, the source term $\widetilde{\Omega}$ is given by

$$\widetilde{\Omega} = \Omega. \tag{3.17}$$

**Proof.** The calculation of the matrices $\widetilde{A}_0$, $\widetilde{A}_i$, $i \in C$, and $\widetilde{G}_{ij}$, $i,j \in C$, is lengthy but straighforward and, therefore, is omitted [Ma96]. This calculation is easily conducted by using the natural variable $Y$ as an intermediate variable. The symmetry properties of $\widetilde{A}_0$, $\widetilde{A}_i$, $i \in C$, and $\widetilde{G}_{ij}$, $i,j \in C$, required in $(S_1)$–$(S_4)$, are then obtained. We also have the identity $\widetilde{\Omega} = \Omega$, since (3.10) is derived by a change of variable.

Consider then a vector $x \in \mathbb{R}^{n_S+4}$, with components $(x_1, \ldots, x_{n_S+4})$. After a little algebra, we obtain that

$$x^t \widetilde{A}_0 x = \rho T \left( (x_{n_S+1} + v_1 x_{n_S+4})^2 + (x_{n_S+2} + v_1 x_{n_S+4})^2 + (x_{n_S+3} + v_1 x_{n_S+4})^2 \right) +$$

$$\sum_{k \in S} \frac{\rho_k}{r_k} \left( x_k + v_1 x_{n_S+1} + v_2 x_{n_S+2} + v_3 x_{n_S+3} + e_k^{\mathrm{tot}} x_{n_S+4} \right)^2 + \rho c_v T^2 x_{n_S+4}^2,$$

so that from $(H_2)$ and the positivity of $\rho_k$, $k \in S$, and $T$, we deduce that $\widetilde{A}_0$ is positive definite.

On the other hand, by using $(H_6)$–$(H_8)$, one can establish that

$$x^t \widetilde{G}^{D\lambda} x = \sum_{k,l \in S} \mathcal{D}_{kl} \big( x_k + (h_k + \frac{p\ \chi_k}{\rho_k}) x_{n_S+4} \big) \big( x_l + (h_l + \frac{p\ \chi_l}{\rho_l}) x_{n_S+4} \big) + \lambda T^2 x_{n_S+4}^2, \quad (3.18)$$

which shows that $\widetilde{G}^{D\lambda}$ is positive semi-definite thanks to $(H_6)$.

Furthermore, a straightforward calculation leads to the following expression for the quadratic form associated with $\widetilde{B}(V, w)$

$$x^t \widetilde{B}(V, w) x = T(\kappa + \tfrac{1}{3}\eta)(o_1 w_1 + o_2 w_2 + o_3 w_3)^2 + T\eta(o_1^2 + o_2^2 + o_3^2) + x^t \widetilde{G}^{D\lambda} x, \quad (3.19)$$

where $o_i = x_{n_S+i} + v_i x_{n_S+4}$, $i = 1, 2, 3$, and where $w_1^2 + w_2^2 + w_3^2 = 1$. We thus obtain that the matrix $\widetilde{B}$ is symmetric, because it is the sum of symmetric matrices, and is positive semi-definite for $V \in \mathcal{O}_V$ and $w \in \mathcal{S}^2$, thanks to the positivity assumptions $(H_5)$ and $(H_6)$. Finally, $\mathcal{H}$ also satisfies $(E_1)$–$(E_4)$ as is easily checked and is strictly convex since $\widetilde{A}_0$ is positive definite over the open convex set $\mathcal{O}_U$. □

### 3.3. Normal forms

We again consider an abstract second order quasilinear system in the form (3.1) with an entropy function $\mathcal{H}^*$ as in Definition 3.2. Introducing the symmetrizing variable $V^{*t} = \partial_{U^*} \mathcal{H}^*$, the corresponding system

$$\widetilde{A}_0^* \partial_t V^* + \sum_{i \in C^*} \widetilde{A}_i^* \partial_i V^* = \sum_{i,j \in C^*} \partial_i \Big( \widetilde{G}_{ij}^* \partial_j V^* \Big), \quad (3.20)$$

is then symmetric in the sense of Definition 3.1. Depending on the range of the dissipation matrices $\widetilde{G}_{ij}^*$, this system lies between the two limit cases of a hyperbolic system and of a strongly parabolic one. In this section, we use a sufficient condition on the matrices $\widetilde{G}_{ij}^*$, $i, j \in C^*$, the Condition N introduced by Kawashima ans Shizuta [KS88], under which the system can be recast in the form of an hyperbolic-parabolic composite system, defined as a normal form of the system. We then characterize all normal forms for symmetric systems of conservation laws satisfying Condition N.

Introducing a new variable $W^*$, associated with a diffeomorphism from $\mathcal{O}_{W^*}$ onto $\mathcal{O}_{V^*}$, and multiplying the conservative symmetric form on the left side by the transpose of the matrix $\partial_{W^*} V^*$ we then get a new system in the variable $W^*$ and we have the following definition of a normal form [KS88].

**Definition 3.6.** Consider a system in symmetric form as in Definition 3.1 and a diffeomorphism $W^* \to V^*$ from $\mathcal{O}_{W^*}$ to $\mathcal{O}_{V^*}$. The system in the new variable $W^*$

$$\overline{A}_0^* \partial_t W^* + \sum_{i \in C^*} \overline{A}_i^* \partial_i W^* = \sum_{i,j \in C^*} \partial_i \Big( \overline{G}_{ij}^* \partial_j W^* \Big) + \overline{H}^* \left( W^*, \partial_x W^* \right), \tag{3.21}$$

where

$$\begin{cases} \overline{A}_0^* = (\partial_{W^*} V^*)^t \ \widetilde{A}_0^* \ (\partial_{W^*} V^*), & \overline{G}_{ij}^* = (\partial_{W^*} V^*)^t \ \widetilde{G}_{ij}^* \ (\partial_{W^*} V^*), \\ \overline{A}_i^* = (\partial_{W^*} V^*)^t \ \widetilde{A}_i^* \ (\partial_{W^*} V^*), & \overline{H}^* = - \displaystyle\sum_{i,j \in C^*} \partial_i (\partial_{W^*} V^*)^t \ \widetilde{G}_{ij}^* \ (\partial_{W^*} V^*) \partial_j W^*, \end{cases} \tag{3.22}$$

satisfies

($\overline{S}_1$) The matrix $\overline{A}_0^*(W^*)$ is symmetric and positive definite for $W^* \in \mathcal{O}_{W^*}$,

($\overline{S}_2$) The matrices $\overline{A}_i^*(W^*)$, $i \in C^*$, are symmetric for $W^* \in \mathcal{O}_{W^*}$,

($\overline{S}_3$) We have $\overline{G}_{ij}^*(W^*)^t = \overline{G}_{ji}^*(W^*)$ for $i, j \in C^*$ and $W^* \in \mathcal{O}_{W^*}$,

($\overline{S}_4$) The matrix $\overline{B}^*(W^*, w) = \sum_{i,j \in C^*} \overline{G}_{ij}^*(W^*) w_i w_j$ is symmetric positive semi-definite, for $W^* \in \mathcal{O}_{W^*}$ and $w \in \mathcal{S}^{d-1}$.

This system is then said to be of the normal form if there exists a partition of $\{1, \dots, n\}$ into $\mathrm{I} = \{1, \dots, n_0\}$ and $\mathrm{II} = \{n_0 + 1, \dots, n\}$, such that the following properties hold.

($N_1$) The matrices $\overline{A}_0^*$ and $\overline{G}_{ij}^*$ have the block structure

$$\overline{A}_0^* = \begin{pmatrix} \overline{A}_0^{*\mathrm{I},\mathrm{I}} & 0 \\ 0 & \overline{A}_0^{*\mathrm{II},\mathrm{II}} \end{pmatrix}, \qquad \overline{G}_{ij}^* = \begin{pmatrix} 0 & 0 \\ 0 & \overline{G}_{ij}^{*\mathrm{II},\mathrm{II}} \end{pmatrix},$$

($N_2$) The matrix $\overline{B}^{*\mathrm{II},\mathrm{II}}(W^*, w) = \sum_{i,j \in C^*} \overline{G}_{ij}^{*\mathrm{II},\mathrm{II}}(W^*) w_i w_j$ is positive definite, for $W^* \in \mathcal{O}_{W^*}$ and $w \in \mathcal{S}^{d-1}$,

($N_3$) We have $\overline{H}^* \left( W^*, \partial_x W^* \right) = \Big( \overline{H}_{\mathrm{I}}^* \left( W^*, \partial_x W_{\mathrm{II}}^* \right), \overline{H}_{\mathrm{II}}^* \left( W^*, \partial_x W^* \right) \Big)^t$,

where we have used the vector and matrix block structure induced by the partitioning of $\{1, \dots, n\}$ into $\mathrm{I} = \{1, \dots, n_0\}$ and $\mathrm{II} = \{n_0 + 1, \dots, n\}$, so that we have $W^* = (W_{\mathrm{I}}^*, W_{\mathrm{II}}^*)^t$ for instance.

**Remark 3.7.** Note that, for convenience, we have kept the dissipative terms $\overline{G}_{ij}^*$ in conservation form in Definition 3.6, which is a minor modification of [KS88].

A sufficient condition for the system (3.20) to be cast into a normal form is that the nullspace naturally associated with dissipation matrices is a fixed subspace of $\mathbb{R}^n$. This is Condition N introduced by Kawashima and Shizuta.

**Definition 3.8.** Condition N is satisfied if the nullspace $N\big(\widetilde{B}^*\big)$ of

$$\widetilde{B}^*(V^*,w) = \sum_{i,j\in C^*} \widetilde{G}^*_{ij}(V^*)w_iw_j$$

does not depend on $V^* \in \mathcal{O}_{V^*}$ and $w \in \mathcal{S}^{d-1}$.

In order to characterize more easily normal forms for symmetric systems of conservation laws satisfying Condition N, we introduce the auxiliary variables $U^{*\prime}$ and $V^{*\prime}$, depending linearly on $U^*$ and $V^*$, respectively. The dissipation matrices corresponding to these auxiliary variables have nonzero coefficients only in the lower right block of size $n-n_0$, where $n_0 = \dim\big(N(\widetilde{B}^*)\big)$. Normal forms are then equivalently—and more easily—obtained from the $V^{*\prime}$ symmetric equation.

**Lemma 3.9.** Consider a system of conservation laws (3.2) which is symmetric in the sense of Definition 3.1. Denote by $\mathcal{H}^*$ the associated entropy function and by $V^* = (\partial_{U^*}\mathcal{H}^*)^t$ the symmetrizing variable, and assume that Condition N is satisfied over $\mathcal{O}_{V^*}$.

Further consider *any* constant nonsingular matrix $P$ of dimension $n$, such that its first $n_0$ columns span the nullspace $N(\widetilde{B}^*)$ associated with Condition N, where $n_0 = \dim\big(N(\widetilde{B}^*)\big)$. More specifically, assume that $P$ is such that

$$\mathrm{span}\Big\{\ (P_{1j},\ldots,P_{nj})^t;\ 1\le j\le n_0\ \Big\} = N(\widetilde{B}^*).$$

The auxiliary variable

$$U^{*\prime} = P^tU^*,$$

satisfies the equation

$$\partial_t U^{*\prime} + \sum_{i\in C^*} A^{*\prime}_i \partial_i U^{*\prime} = \sum_{i,j\in C^*} \partial_i\Big(G^{*\prime}_{ij}\partial_j U^{*\prime}\Big), \tag{3.23}$$

where $A^{*\prime}_i = P^tA^*_i(P^t)^{-1}$ and $G^{*\prime}_{ij} = P^tG^*_{ij}(P^t)^{-1}$. The corresponding entropy is then the functional $U^{*\prime} \to \mathcal{H}^*\big((P^t)^{-1}U^{*\prime}\big)$ and the associated entropic variable $V^{*\prime} = (\partial_{U^{*\prime}}\mathcal{H}^*)^t$ is given by

$$V^{*\prime} = P^{-1}V^*,$$

and satisfies the equation

$$\widetilde{A}^{*\prime}_0\partial_t V^{*\prime} + \sum_{i\in C^*} \widetilde{A}^{*\prime}_i \partial_i V^{*\prime} = \sum_{i,j\in C^*} \partial_i\Big(\widetilde{G}^{*\prime}_{ij}\partial_j V^{*\prime}\Big), \tag{3.24}$$

where $\widetilde{A}^{*\prime}_0 = P^t\widetilde{A}^*_0P$ , $\widetilde{A}^{*\prime}_i = P^t\widetilde{A}^*_iP$, and $\widetilde{G}^{*\prime}_{ij} = P^t\widetilde{G}^*_{ij}P$. In particular, $\widetilde{G}^{*\prime}_{ij}$ is in the form

$$\widetilde{G}^{*\prime}_{ij} = \begin{pmatrix} 0_{n_0\times n_0} & 0_{n_0\times(n-n_0)} \\ 0_{(n-n_0)\times n_0} & \widetilde{G}^{*\prime II,II}_{ij} \end{pmatrix}, \tag{3.25}$$

and $\widetilde{B}^{*\prime II,II}(V^{*\prime},w)=\sum_{i,j\in S}\widetilde{G}_{ij}^{*\prime II,II}w_iw_j$ is positive definite over $\mathcal{O}_{V^{*\prime}}\times\mathcal{S}^{d-1}$.

Finally, the normal form (3.21) is equivalently obtained by multiplying the $V^*$ equation (3.20) by $(\partial_{W^*}V^*)^t$ or the $V^{*\prime}$ equation (3.24) by $(\partial_{W^*}V^{*\prime})^t$.

**Proof.** Eq. (3.23) is easily established by multiplying Eq. (3.1) on the left by $P^t$. This also yields the relations $A_i^{*\prime}=P^tA_i^*(P^t)^{-1}$ and $G_{ij}^{*\prime}=P^tG_{ij}^*(P^t)^{-1}$. It is also easily checked that the functional $U^{*\prime}\to\mathcal{H}^*\big((P^t)^{-1}U^{*\prime}\big)$ is the corresponding entropy. From $V^{*\prime}=(\partial_{U^{*\prime}}\mathcal{H}^*)^t$ and the chain rule, we then get that $V^{*\prime}=P^{-1}V^*$ and Eq. (3.24) is obtained as in (3.2) (3.3). Since $\widetilde{B}^{*\prime}=P^t\widetilde{B}^*P$ and since the first $n_0$ columns of $P$ span $N(\widetilde{B}^*)$, we next deduce that $\widetilde{B}^{*\prime}$ is in the form

$$\widetilde{B}^{*\prime}=\begin{pmatrix}0_{n_0\times n_0} & 0_{n_0\times(n-n_0)}\\ 0_{(n-n_0)\times n_0} & \widetilde{B}^{*\prime II,II}\end{pmatrix},\tag{3.26}$$

so that all the matrices $\widetilde{G}_{ij}^{*\prime}$, $i,j\in C^*$, are also in the form (3.25). Moreover, the matrix $\widetilde{B}^{*\prime II,II}(V^{*\prime},w)$ is positive definite since the $n-n_0$ last columns of $P$ span a subspace complementary to $N(\widetilde{B}^*)$. □

Normal forms for symmetrizable systems of conservation laws satisfying Condition N are now completely characterized in the following theorem, in terms of the auxiliary variables $U^{*\prime}$ and $V^{*\prime}$.

**Theorem 3.10.** Keeping the assumptions and notation of Lemma 3.9, any normal form of the system (3.2) is given by a change of variable in the form

$$W^*=\big(\psi_I(U_I^{*\prime}),\varphi_{II}(V_{II}^{*\prime})\big)^t,\tag{3.27}$$

where $\psi_I$ and $\varphi_{II}$ are two diffeomorphisms of $\mathbb{R}^{n_0}$ and $\mathbb{R}^{n-n_0}$, respectively. The $n_0$ components of $U_I^{*\prime}$ can thus be termed hyperbolic components and the $n-n_0$ components of $V_{II}^{*\prime}$ parabolic components. Furthermore we have

$$\overline{H}^*(W^*,\partial_xW^*)=\big(0,\,\overline{H}_{II}^*(W^*,\partial_xW_{II}^*)\big)^t,$$

and when $\varphi_{II}$ is a constant linear mapping, we also have $\overline{H}_{II}^*=0$ and the dissipative terms are in conservative form.

**Proof.** The mapping $U^{*\prime}\longmapsto W^*$ is smooth from assumptions and a straightforward calculation yields that

$$\partial_{U^{*\prime}}W^*=\begin{pmatrix}\partial_{U_I^{*\prime}}\psi_I & 0_{(n-n_0)\times n}\\ \partial_{V_{II}^{*\prime}}\varphi_{II}\big(\widetilde{A}_0^{*\prime-1}\big)^{II,I} & \partial_{V_{II}^{*\prime}}\varphi_{II}\big(\widetilde{A}_0^{*\prime-1}\big)^{II,II}\end{pmatrix},$$

so that $\partial_{U^{*\prime}}W^*$ is invertible. On the other hand, the derivative of the mapping $U_{II}^{*\prime}\longmapsto V_{II}^{*\prime}(U_I^{*\prime},U_{II}^{*\prime})$, for a fixed value of $U_I^{*\prime}$, is given by $\partial_{U_{II}^{*\prime}}V_{II}^{*\prime}=\big(\widetilde{A}_0^{*\prime-1}\big)^{II,II}$ and is positive definite. Since $\mathcal{O}_{U^{*\prime}}$ is an open convex set, we deduce that $U^{*\prime}\longmapsto W^*$

is one to one. As a consequence, the mapping $U^{*\prime} \longmapsto W^*$ is a diffeomorphism from $\mathcal{O}_{U^{*\prime}} = P^t\mathcal{O}_{U^*}$ onto an open set denoted by $\mathcal{O}_{W^*}$ and, similarly, the mapping $V^{*\prime} \longmapsto W^*$ is a diffeomorphism from $\mathcal{O}_{V^{*\prime}} = P^{-1}\mathcal{O}_{V^*}$ onto $\mathcal{O}_{W^*}$.

Evaluating and inverting the matrix $\partial_{V^{*\prime}} W^*$, we next obtain the following expression for $\partial_{W^*} V^{*\prime}$

$$\partial_{W^*} V^{*\prime} = \begin{pmatrix} \big(\widetilde{A}_0^{*\prime\, I,I}\big)^{-1}\big(\partial_{U_I^{*\prime}}\psi_I\big)^{-1} & -\big(\widetilde{A}_0^{*\prime\, I,I}\big)^{-1}\widetilde{A}_0^{*\prime\, I,II}\big(\partial_{V_{II}^{*\prime}}\varphi_{II}\big)^{-1} \\ 0_{(n-n_0)\times n} & \big(\partial_{V_{II}^{*\prime}}\varphi_{II}\big)^{-1} \end{pmatrix}, \tag{3.28}$$

and a direct calculation using (3.24) then shows that properties $(N_1)$–$(N_3)$ are satisfied, keeping in mind that normal forms are equivalently obtained from the $V^{*\prime}$ equation (3.24) or the $V^*$ equation (3.20).

Consider a diffeomorphism $V^{*\prime} \longmapsto Z^* = \big(\varphi_I(V_I^{*\prime}, V_{II}^{*\prime}), \varphi_{II}(V_I^{*\prime}, V_{II}^{*\prime})\big)^t$ and assume that the system (3.21) in the variable $Z^*$ is of the normal form. By using the definition of $\overline{B}^*$ and the auxiliary variable $V^{*\prime}$, it is easily established that

$$\widetilde{B}^{*\prime} = \big(\partial_{V^{*\prime}} Z^*\big)^t\,\overline{B}^*\,\big(\partial_{V^{*\prime}} Z^*\big).$$

Using the block structure of $\overline{B}$ and $\widetilde{B}'$ then yields that

$$\begin{pmatrix} \big(\partial_{V_I^{*\prime}} Z_{II}^*\big)^t\,\overline{B}^{*II,II}\big(\partial_{V_I^{*\prime}} Z_{II}^*\big) & \big(\partial_{V_I^{*\prime}} Z_{II}^*\big)^t\,\overline{B}^{*II,II}\big(\partial_{V_{II}^{*\prime}} Z_{II}^*\big) \\ \big(\partial_{V_{II}^{*\prime}} Z_{II}^*\big)^t\,\overline{B}^{*II,II}\big(\partial_{V_I^{*\prime}} Z_{II}^*\big) & \big(\partial_{V_{II}^{*\prime}} Z_{II}^*\big)^t\,\overline{B}^{*II,II}\big(\partial_{V_{II}^{*\prime}} Z_{II}^*\big) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{B}^{*\prime II,II} \end{pmatrix}, \tag{3.29}$$

which implies that $\partial_{V_I^{*\prime}} Z_{II}^* = 0$ since $\widetilde{B}^{*II,II}$ is positive definite, and thus that

$$Z_{II}^* = \varphi_{II}(V_{II}^{*\prime}). \tag{3.30}$$

On the other hand, by using the definition of $\overline{A}_0^*$, it is easily establish that

$$\overline{A}_0^*\big(\partial_{U^{*\prime}} Z^*\big) = \big(\partial_{Z^*} V^{*\prime}\big)^t.$$

Using the block structure of $\overline{A}_0^*$ and the relation $\partial_{Z_I^*} V_{II}^{\prime *} = 0$, derived from $\partial_{V_I^{\prime *}} Z_{II}^* = 0$, we then obtain that $\partial_{U_{II}^{*\prime}} Z_I^* = 0$ and thus that

$$Z_I^* = \psi_I(U_I^{*\prime}), \tag{3.31}$$

where we have defined $\varphi_I(V_I^{*\prime}, V_{II}^{*\prime}) = \psi_I(U_I^{*\prime}, U_{II}^{*\prime})$. This shows that any diffeomorphism associated with a normal form is of type (3.27).

Moreover, a direct calculation yields

$$\overline{H}^* = \Big(0, -\sum_{i,j\in C^*} \partial_i\big((\partial_{V_{II}^{*\prime}}\varphi_{II})^{-1}\big)^t\,\widetilde{G}_{ij}^{*\prime II,II}\big(\partial_{V_{II}^{*\prime}}\varphi_{II}\big)^{-1}\partial_j W_{II}^*\Big)^t. \tag{3.32}$$

This shows that $\overline{H}_{I}^{*} = 0$, and that $\overline{H}_{II}^{*} = 0$ when $\varphi_{II}$ is constant linear mapping as was to be shown. □

**Remark 3.11.** Theorem 3.10 shows in particular that the general form (3.27) is independent of the choice of $P$. It is also possible, however, to check it directly. Consider indeed another matrix $Q$, as in Lemma 3.9, and define

$$U^{*\prime\prime} = Q^t U^*, \qquad V^{*\prime\prime} = Q^{-1} V^*.$$

Denoting by $\mathfrak{P}$ the matrix $\mathfrak{P} = P^{-1}Q$, we thus get

$$U^{*\prime\prime} = \mathfrak{P}^t U^{*\prime}, \qquad V^{*\prime\prime} = \mathfrak{P}^{-1} V^{*\prime}.$$

Since the nullspace $N(\widetilde{B})$ is spanned by the $n_0$ first columns of $P$ and $Q$, it is easily checked that $\mathfrak{P}^{II,I} = 0$. This implies that $\mathfrak{P}^{I,I}$ and $\mathfrak{P}^{II,II}$ are invertible and that

$$U_{I}^{*\prime\prime} = (\mathfrak{P}^{I,I})^t U_{I}^{*\prime}, \qquad V_{II}^{*\prime\prime} = (\mathfrak{P}^{II,II})^{-1} V_{II}^{*\prime}, \tag{3.33}$$

and we recover that the general form (3.27) is independent of $P$.

### 3.4. Application to multicomponent flows

In this section we establish that system (3.10) satisfies Condition N and we investigate normal forms.

**Proposition 3.12.** The nullspace of the matrix $\widetilde{B}$ associated with system (3.10) is one dimensional and is given by

$$N(\widetilde{B}) = \mathrm{span}(1, \ldots, 1, 0, 0, 0, 0)^t. \tag{3.34}$$

**Proof.** According to Eq. (3.18) (3.19), the matrix $\widetilde{B}$ is positive semi-definite, so that its nullspace is constituted by the vectors $x$ of $\mathbb{R}^{n_S+4}$ such that $x^t\widetilde{B}x = 0$. On the other hand, we have

$$x^t\widetilde{B}(V,w)x = \left(\kappa + \tfrac{1}{3}\eta\right) T(o_1w_1 + o_2w_2 + o_3w_3)^2 + \eta T(o_1^2 + o_2^2 + o_3^2) +$$
$$\sum_{k,l\in S} \mathcal{D}_{kl}\left(x_k + (h_k + \frac{p\chi_k}{\rho_k})x_{n_S+4}\right)\left(x_l + (h_l + \frac{p\chi_l}{\rho_l})x_{n_S+4}\right) + \lambda T^2 x_{n_S+4}^2,$$

where $o_i = x_{n_S+i} + v_i x_{n_S+4}$, $i = 1, 2, 3$. As a consequence, $x^t\widetilde{B}x = 0$ implies that $x_{n_S+4} = 0$ and that $x_{n_S+i} = 0$, $i = 1, 2, 3$, thanks to $(H_5)$. Therefore, $x$ is in the nullspace of $\widetilde{B}(V,w)$ if and only if we have

$$\sum_{k,l\in S} \mathcal{D}_{kl} x_k x_l = 0. \tag{3.35}$$

Using $(H_6)$ and (3.16) we then obtain that the nullspace of $\widetilde{B}(V,w)$ is one dimensional and spanned by $(1,\dots,1,0,0,0,0)^t$ and is thus independent of $V \in \mathcal{O}_V$ and $w \in \mathcal{S}^2$. $\square$

Since the system of equations governing multicomponent reacting flows satisfies Condition N, we can now obtain from Lemma 3.9 the auxiliary variables $U'$ and $V'$. From Lemma 3.9 and Proposition 3.12 the matrix $P$ can be taken to be

$$P = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (3.36)$$

in such a way that

$$U' = P^t U = \Big(\rho,\ \rho_2, \dots,\ \rho_{n_S},\ \rho v_1,\ \rho v_2,\ \rho v_3,\ \rho e^{\mathrm{tot}}\Big)^t. \qquad (3.37)$$

The associated entropic variable is then $V' = P^{-1}V$ where $V$ is given by (3.9), and the corresponding symmetric system is easily obtained from (3.10).

**Proposition 3.13.** The system in the new dependent variable $V'$,

$$V' = \frac{1}{T}\Big(\mu_1 - \tfrac{1}{2}v\cdot v,\ \mu_2 - \mu_1, \dots,\ \mu_{n_S} - \mu_1,\ v_1,\ v_2,\ v_3,\ -1\Big)^t, \qquad (3.38)$$

can be written

$$\widetilde{A}_0'\partial_t V' + \sum_{i\in C} \widetilde{A}_i'\partial_i V' = \sum_{i,j\in C} \partial_i\Big(\widetilde{G}_{ij}'\partial_j V'\Big) + \widetilde{\Omega}', \qquad (3.39)$$

where $V' \in \mathcal{O}_{V'} = \mathbb{R}^{n_S+3} \times (-\infty, -1/T_0)$, $\widetilde{A}_0' = P^t\widetilde{A}_0P$, $\widetilde{A}_i' = P^t\widetilde{A}_iP$, $i = 1,2,3$, $\widetilde{G}_{ij}' = P^t\widetilde{G}_{ij}P$, $i,j = 1,2,3$, and where $\widetilde{\Omega}' = P^t\Omega = (0,\Omega_{II})^t$.

In particular, Properties $(S_1)$–$(S_4)$ of Definition 3.1 are satisfied and the dissipation matrices are given by

$$\widetilde{G}_{ij}' = \begin{pmatrix} 0 & 0_{1\times(n_S+3)} \\ 0_{(n_S+3)\times 1} & \widetilde{G}_{ij}^{II,II} \end{pmatrix}, \qquad (3.40)$$

where $\widetilde{G}_{ij}^{II,II}$ is the lower right block of size $n_S + 3$ of $\widetilde{G}$.

We now investigate normal forms for the system (3.10), or, equivalently, for the system (3.39). We first use the possibility of mixing parabolic components—the $V_{II}'$

components—established in Theorem 3.10, in order to simplify the analytic expression of the normal variable, and, consequently, of the matrix coefficients appearing in the normal form. More specifically, we consider the variable

$$W = \Big(\rho,\, \log(\rho_2^{r_2}/\rho_1^{r_1}), \dots,\, \log(\rho_{n_S}^{r_{n_S}}/\rho_1^{r_1}),\, v_1,\, v_2,\, v_3,\, T\Big)^t,$$

easily obtained by combining the $V'_{II}$ components, and derive the corresponding normal form of the governing equations. When there is only one gas, this normal form is identical with the one previously obtained by Kawashima and Shizuta in [KS88]. We also investigate, in the Appendix, the normal form associated with the "natural" normal variable $\widehat{W} = (U'_I, V'_{II})^t$ which guarantees a conservative form for the dissipative terms of the system, leaves invariant the source term $\Omega_{II}$, but has a more complex expression.

**Theorem 3.14.** The system in the variable $W = (W_I, W_{II})^t$, on the open convex set $\mathcal{O}_W = (0,\infty)\times\mathbb{R}^{n_S-1}\times\mathbb{R}^3\times(T_0,\infty)$, with hyperbolic variable $W_I = (\rho)$ and parabolic variable $W_{II} = \big(\log(\rho_2^{r_2}/\rho_1^{r_1}), \dots,\, \log(\rho_{n_S}^{r_{n_S}}/\rho_1^{r_1}),\, v_1,\, v_2,\, v_3,\, T\big)^t$, can be written

$$\overline{A}_0^{I,I}\,\partial_t W_I \;+ \textstyle\sum_{i\in C} \overline{A}_i^{I,I}\,\partial_i W_I + \sum_{i\in C} \overline{A}_i^{I,II}\,\partial_i W_{II} \;=\; 0, \tag{3.41}$$

$$\overline{A}_0^{II,II}\,\partial_t W_{II} + \textstyle\sum_{i\in C} \overline{A}_i^{II,I}\,\partial_i W_I + \sum_{i\in C} \overline{A}_i^{II,II}\,\partial_i W_{II} = \displaystyle\sum_{i,j\in C} \partial_i\Big(\overline{G}_{ij}^{II,II}\,\partial_j W_{II}\Big) + \overline{H}_{II} + \overline{\Omega}_{II}, \tag{3.42}$$

and is of the normal form. The matrix $\overline{A}_0$ is given by

$$\overline{A}_0 = \begin{pmatrix} \dfrac{1}{\Sigma_\rho} & & & 0 \\ & \mathcal{X} & & \\ & & \dfrac{\rho}{T} I_3 & \\ 0 & & & \dfrac{\rho c_v}{T^2} \end{pmatrix},$$

where $\mathcal{X}$ is a square matrix of dimension $n_S - 1$ given by

$$\mathcal{X}_{kl} = \delta_{kl}\frac{\rho_k}{r_k} - \frac{\rho_k\rho_l}{r_k r_l}\frac{1}{\Sigma_\rho}, \qquad k,l\in[2,n_S]. \tag{3.43}$$

Denoting by $\xi = (\xi_1,\xi_2,\xi_3)^t$ an arbitrary vector of $\mathbb{R}^3$, the matrices $\overline{A}_i$, $i = 1,2,3$, are given by

$$\sum_{i\in C} \overline{A}_i\xi_i = \begin{pmatrix} \dfrac{v\cdot\xi}{\Sigma_\rho} & & & Sym \\ 0_{(n_S-1)\times 1} & \mathcal{X}\, v\cdot\xi & & \\ \dfrac{\rho}{\Sigma_\rho}\xi & \xi\otimes\mathcal{Z} & \dfrac{\rho}{T} v\cdot\xi\, I_3 & \\ 0 & 0_{1\times(n_S-1)} & \dfrac{\rho r}{T}\xi^t & \dfrac{\rho c_v}{T^2} v\cdot\xi \end{pmatrix},$$

where $\mathcal{Z}$ is a vector of dimension $n_S - 1$ given by

$$\mathcal{Z}_l = \rho_l - \frac{\rho_l\,\rho}{r_l\,\Sigma_\rho}, \qquad l \in [2, n_S].$$

For the heat and mass diffusion matrices, $\overline{G}_{ij}^{D\lambda}$, $i, j = 1, 2, 3$, we have

$$\overline{G}_{11}^{D\lambda} = \overline{G}_{22}^{D\lambda} = \overline{G}_{33}^{D\lambda} = \overline{G}^{D\lambda} =$$

$$\begin{pmatrix} 0 & & & Sym \\ 0_{(n_S-1)\times 1} & (\mathcal{D}_{kl})_{k,l\geq 2} & & \\ 0_{3\times 1} & 0_{3\times(n_S-1)} & 0_{3\times 3} & \\ 0 & \frac{1}{T}\Big(\sum_{k\in S}\mathcal{D}_{kl} r_k + \rho_l\theta_l\Big)_{l\geq 2} & 0_{1\times 3} & \dfrac{\lambda'}{T^2} + 2\sum_{k\in S}\dfrac{\rho_k\theta_k r_k}{T^2} + \sum_{k,l\in S}\dfrac{\mathcal{D}_{kl} r_k r_l}{T^2} \end{pmatrix},$$

whereas the nondiagonal terms vanish $\overline{G}_{ij}^{D\lambda} = 0, \quad i \neq j$. The dissipation matrices due to the viscous effects $\overline{G}_{ij}^{\kappa\eta}$, $i, j = 1, 2, 3$, still have the structure (3.14) and the corresponding matrices $\overline{\mathcal{K}}_{ij}$, $i, j = 1, 2, 3$, are given by

$$\overline{\mathcal{K}}_{11} = \frac{1}{T}\begin{pmatrix} \kappa + \frac{4}{3}\eta & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \overline{\mathcal{K}}_{12} = \frac{1}{T}\begin{pmatrix} 0 & \kappa - \frac{2}{3}\eta & 0 & 0 \\ \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with the other ones deduced by circular permutation and from the relations

$$\overline{\mathcal{K}}_{12} = \overline{\mathcal{K}}_{21}^t, \qquad \overline{\mathcal{K}}_{13} = \overline{\mathcal{K}}_{31}^t, \qquad \overline{\mathcal{K}}_{23} = \overline{\mathcal{K}}_{32}^t.$$

Finally, the term $\overline{H}_{II}$ is easily computed from (3.32) whereas the corresponding source term $\overline{\Omega} = (\partial_{W^*} V^*)^t\,\Omega$ is given by

$$\overline{\Omega} = (0,\, m_2\omega_2, \ldots,\, m_{n_S}\omega_{n_S}, 0, 0, 0,\, -\frac{1}{T^2}\sum_{k\in S} e_k m_k \omega_k)^t.$$

**Proof.** The calculations are lenghty but straightforward and make use of Theorem 3.5, Proposition 3.13 and assumptions $(H_1)$–$(H_9)$. □

**Remark 3.15.** Note that when the source term $\Omega$ remains in a fixed subspace of $\mathbb{R}^{n_S} \times 0_{\mathbb{R}^4}$, the source term $\overline{\Omega}$ is no longer in a *fixed* subspace of $\mathbb{R}^{n_S+4}$ of the same dimension because of the coefficients $e_k/T^2$ in the term $\sum_{k\in S} e_k m_k \omega_k/T^2$ which introduce an explicit dependence on the state variables.

## 4. The Cauchy Problem

In this section we first restate an existence theorem of Vol'pert and Hudjaev concerning symmetric hyperbolic-parabolic composite systems. We only present a simplified quasilinear version of their existence result [VH72]. We then apply this existence result to the equations governing multicomponent reactive flows, using a normal form obtained in the previous sections.

### 4.1. The mathematical framework

We consider the Cauchy problem for an abstract system of partial differential equations in the form

$$\begin{cases} \overline{A}_0^{*\,I,I}\partial_t W_I^* = -\sum_{i\in C^*} \overline{A}_i^{*\,I,I}\partial_i W_I^* + \overline{\Gamma}_I^*, \\ \overline{A}_0^{*\,II,II}\partial_t W_{II}^* = -\sum_{i\in C^*} \overline{A}_i^{*\,II,I}\partial_i W_I^* + \sum_{i,j\in C^*} \partial_i\big(\overline{G}_{ij}^{*\,II,II}\partial_j W_{II}^*\big) + \overline{\Gamma}_{II}^* \end{cases} \tag{4.1}$$

where $I = \{1,\dots,n_0\}$ and $II = \{n_0+1,\dots,n\}$ form a partition of $\{1,\dots,n\}$, $W^* = (W_I^*, W_{II}^*)$ is the corresponding decomposition of the unknown vector $W^*$, and where $C^* = \{1,\dots,d\}$ is the indexing set of spatial coordinates. These equations are considered in the strip $\overline{Q}_\Theta$ where $\Theta$ is positive and $Q_t = (0,t)\times\mathbb{R}^d$ for $t>0$. The matrix and vector coefficients are assumed to be in the form

$$\begin{cases} \overline{A}_0^{*\,I,I} = \overline{A}_0^{*\,I,I}(W_I^*, W_{II}^*), & \\ \overline{A}_0^{*\,II,II} = \overline{A}_0^{*\,II,II}(W_I^*, W_{II}^*), & \\ \overline{A}_i^{*\,I,I} = \overline{A}_i^{*\,I,I}(W_I^*, W_{II}^*, \partial_x W_{II}^*), & i\in C^*, \\ \overline{A}_i^{*\,II,I} = \overline{A}_i^{*\,II,I}(W_I^*, W_{II}^*, \partial_x W_{II}^*), & i\in C^*, \\ \overline{G}_{ij}^{*\,II,II} = \overline{G}_{ij}^{*\,II,II}(W_I^*, W_{II}^*), & i,j\in C^*, \\ \overline{\Gamma}_I^* = \overline{\Gamma}_I^*(W_I^*, W_{II}^*, \partial_x W_{II}^*), & \\ \overline{\Gamma}_{II}^* = \overline{\Gamma}_{II}^*(W_I^*, W_{II}^*, \partial_x W_{II}^*). & \end{cases} \tag{4.2}$$

We assume that the matrices $\overline{A}_0^{*\,I,I}(z)$, $\overline{A}_0^{*\,II,II}(z)$, and $\overline{G}_{ij}^{*\,II,II}(z)$, $i,j\in C^*$, are smooth functions of $z\in\mathcal{O}_{W^*}$, where $\mathcal{O}_{W^*}$ is a convex open set of $\mathbb{R}^n$. Similarly, we assume that the matrices $\overline{A}_i^{*\,I,I}(z,\zeta)$, $i\in C^*$, the matrices $\overline{A}_i^{*\,II,I}(z,\zeta)$, $i\in C^*$, and the vectors $\overline{\Gamma}_I(z,\zeta)$ and $\overline{\Gamma}_{II}(z,\zeta)$ are smooth functions of $z\in\mathcal{O}_{W^*}$, and $\zeta\in\mathbb{R}^{d\times(n-n_0)}$.

We will use the classical functional spaces $L^p(\mathbb{R}^d)$ with norm

$$\|\phi\|_{0,p} = |\phi|_{0,p} = \left(\int_{\mathbb{R}^d} |\phi(x)|^p dx\right)^{\frac{1}{p}}, \quad p \geq 1,$$

the Sobolev spaces $W_p^l(\mathbb{R}^d)$, $1 \leq p \leq \infty$, with norm

$$\|\phi\|_{l,p} = \sum_{k\in[0,l]} |\phi|_{k,p}, \quad |\phi|_{k,p} = \sum_{|\beta|=k} \|\partial^\beta \phi\|_{0,p}.$$

and the functional spaces $V_l(\mathbb{R}^d)$ with norm [VH72]

$$\|\phi\|_l = |\phi|_{0,\infty} + \sum_{k\in[1,l]} |\phi|_{k,2}.$$

We extend these definition to vector functions by using the euclidian norm of $\mathbb{R}^d$. According to the Sobolev inequalities, there is an imbedding of $W_2^l(\mathbb{R}^d)$ into $W_\infty^k(\mathbb{R}^d)$ for $l > d/2+k$, and an imbedding of $W_2^l(\mathbb{R}^d)$ into $V_l(\mathbb{R}^d)$ for $l > d/2$. In the following, $\mathcal{L}$ denotes an arbitrary fixed positive continuous convex function, on the open convex set $\mathcal{O}_{W^*}$, which grows without bound as any finite point of the boundary of $\mathcal{O}_{W^*}$ is approached.

### 4.2. An existence theorem in $V_l(\mathbb{R}^d)$

We consider the Cauchy problem for the system (4.1) (4.2), with smooth initial conditions

$$W^*(0,x) = W^{*0}(x). \tag{4.3}$$

The following theorem of Vol'pert and Hudjaev shows that, in a certain strip, there exists a solution which preserves the smoothness of the initial condition [VH72].

**Theorem 4.1.** Suppose that the system (4.1)–(4.3) satisfies the following assumptions where $l$ denotes an integer such that $l > d/2 + 3$.

($Ex_1$) The initial condition $W^{*0}$ satisfies $\sup_{x\in\mathbb{R}^d} \mathcal{L}\left(W^{*0}(x)\right) < +\infty$ and $W^{*0}$ is in the space $V_l(\mathbb{R}^d)$.

($Ex_2$) The matrix coefficients $\overline{A}_0^{*I,I}(z)$, $\overline{A}_0^{*II,II}(z)$, and $\overline{G}_{ij}^{*II,II}(z)$, $i,j \in C^*$, have continuous derivative of order $l > d/2+3$ with respect to $z \in \mathcal{O}_{W^*}$.

($Ex_3$) The matrix coefficients $\overline{A}_i^{*I,I}(z,\zeta)$ and $\overline{A}_i^{*II,I}(z,\zeta)$, $i \in C^*$, and the vectors $\overline{\Gamma}_I^*(z,\zeta)$ and $\overline{\Gamma}_{II}^*(z,\zeta)$, have continuous derivative of order $l > d/2+3$ with respect to $z \in \mathcal{O}_{W^*}$ and $\zeta \in \mathbb{R}^{d\times(n-n_0)}$.

($Ex_4$) The matrix coefficients $\overline{A}_0^{*I,I}$ and $\overline{A}_0^{*II,II}$ are symmetric and positive definite for $z \in \mathcal{O}_{W^*}$.

($Ex_5$) The matrix coefficients $\overline{A}_i^{*I,I}(z,\zeta)$, $i \in C^*$, are symmetric for $z \in \mathcal{O}_{W^*}$ and $\zeta \in \mathbb{R}^{d\times(n-n_0)}$.

($Ex_6$) The matrices $\overline{A}_0^{*I,I}$ and $\overline{A}_0^{*II,II}$ and the vectors $\overline{\Gamma}_I^*(z,0)$ and $\overline{\Gamma}_{II}^*(z,0)$, have continuous derivatives to order $l+3$ in $z$.

($Ex_7$) For any compact subset $K$ of $\mathcal{O}_{W^*}$, there exists $\alpha = \alpha(K)$ such that for any smooth function $z$ from $\mathbb{R}^d$ to $\mathbb{R}^n$ with values in $K$ we have

$$\int_{\mathbb{R}^d} \sum_{i,j\in C^*} \big(\partial_i\phi_{II}\big)^t\, \overline{G}_{ij}^{*II,II}\big(z(x)\big)\, \big(\partial_j\phi_{II}\big)\, dx \;\geq\; \alpha \int_{\mathbb{R}^d} \sum_{i\in C^*} \big(\partial_i\phi_{II}\big)^t \big(\partial_i\phi_{II}\big)\, dx, \tag{4.4}$$

where $\phi_{II}$ is any function in $W_2^1(\mathbb{R}^d)$ with $n-n_0$ components.

Then there exists $t_0$, $0 < t_0 \leq \Theta$, such that the Cauchy problem (4.1)–(4.2), admits a unique solution $W^* = (W_I^*, W_{II}^*)^t$ defined on $[0,t_0]\times\mathbb{R}^d$, which is continuous with its derivatives of first order in $t$ and second order in $x$, and for which the following quantities are finite

$$\sup_{0\leq t\leq t_0} \|W^*(t)\|_l, \qquad \sup_{\overline{Q}_{t_0}} \mathcal{L}(W^*), \tag{4.5}$$

$$\sup_{0\leq t\leq t_0} \|\partial_t W_I^*(t)\|_{l-1}, \qquad \int_0^{t_0} \Big(\|\partial_t W_{II}^*(\tau)\|_{l-1}^2 + \|W_{II}^*(\tau)\|_{l+1}^2\Big)\, d\tau. \tag{4.6}$$

Moreover, either $t_0 = \Theta$, or there exists $t_1$ such that the theorem is true for any $t_0 < t_1$ and such that for $t_0 \to t_1^-$, at least one of the quantities

$$\|W_I^*(t_0)\|_{1,\infty} + \|W_{II}^*(t_0)\|_{2,\infty}, \qquad \sup_{\overline{Q}_{t_0}} \mathcal{L}(W^*), \tag{4.7}$$

grows without bound, that is to say, the solution can be extended as long as quantities (4.7) remain finite.

**Remark 4.2.** Theorem 4.1 is a simplified quasilinear version of the results obtained by Vol'pert and Hudjaev. The dependence on $(t,x)$ of the various coefficients can be included as detailed in [VH72] [Ma96]. We have also kept the second order terms in divergence form for convenience.

### 4.3. Application to multicomponent flows

We now apply Theorem 4.1 to the system modeling multicomponent reactive flows in $\mathbb{R}^d$ with $d \in \{1,2,3\}$.

**Theorem 4.3.** Consider the Cauchy problem for the system (3.41) (3.42) in $\mathbb{R}^d$ with $d \in \{1,2,3\}$

$$\overline{A}_0\partial_t W + \sum_{i\in C} \overline{A}_i\partial_i W = \sum_{i,j\in C} \partial_i(\overline{G}_{ij}\partial_j W) + \overline{H} + \overline{\Omega}, \tag{4.8}$$

with initial conditions

$$W(0,x) = W^0(x), \tag{4.9}$$

where

$$W^0 \in V_l(\mathbb{R}^d), \qquad \inf_{\mathbb{R}^d} \rho^0(x) > 0, \qquad \inf_{\mathbb{R}^d} T^0(x) > T_0. \tag{4.10}$$

Then there exists $t_0$, $0 < t_0 < \infty$ such that (4.8) (4.9) admits a unique solution $W = (W_{I}, W_{II})^t$ with $W(t,x) \in \mathcal{O}_W$ in the strip $\overline{Q}_{t_0} = [0,t_0] \times \mathbb{R}^d$, continuous in $\overline{Q}_{t_0}$ with its derivatives of first order in $t$ and second order in $x$, and for which the following inequalities hold

$$\sup_{0\le t\le t_0} \Big( \|\rho(t)\|_l + \sum_{k\in[2,n_S]} \|\log(\rho_k^{r_k}/\rho_1^{r_1})(t)\|_l + \sum_{i\in C} \|v_i(t)\|_l + \|T(t)\|_l \Big) < +\infty,$$

$$\inf_{\overline{Q}_{t_0}} \rho(t,x) > 0, \qquad \inf_{\overline{Q}_{t_0}} T(t,x) > T_0,$$

$$\sup_{0\le t\le t_0} \|\partial_t \rho(t)\|_{l-1} < +\infty,$$

$$\int_0^{t_0} \Big( \sum_{k\in[2,n_S]} \|\partial_t \log(\rho_k^{r_k}/\rho_1^{r_1})(\tau)\|_{l-1}^2 + \sum_{i\in C} \|\partial_t v_i(\tau)\|_{l-1}^2 + \|\partial_t T(\tau)\|_{l-1}^2 +$$

$$\sum_{k\in[2,n_S]} \|\log(\rho_k^{r_k}/\rho_1^{r_1})(\tau)\|_{l+1}^2 + \sum_{i\in C} \|v_i(\tau)\|_{l+1}^2 + \|T(\tau)\|_{l+1}^2 \Big)\, d\tau < +\infty.$$

Moreover, either $t_0 = +\infty$, or there exists $t_1$ such that the theorem is true for any $t_0 < t_1$ and such that for $t_0 \to t_1^-$, either the following quantity

$$\|\rho(t_0)\|_{1,\infty} + \sum_{k\in[2,n_S]} \|\log(\rho_k^{r_k}/\rho_1^{r_1})(t_0)\|_{2,\infty} + \sum_{i\in C} \|v_i(t_0)\|_{2,\infty} + \|T(t_0)\|_{2,\infty}, \tag{4.11}$$

grows without bound or $\inf_{Q_{t_0}} T \to T_0$.

**Proof.** Assume first that $\kappa$ is a positive function and rewrite the system (3.41) (3.42) of Theorem 3.14 in the form (4.1) (4.2) with

$$\begin{cases} \overline{\Gamma}_{I} = \; -\displaystyle\sum_{j\in C} \overline{A}_j^{II,II} \partial_j W_{II}, \\[2ex] \overline{\Gamma}_{II} = \overline{H}_{II} + \overline{\Omega}_{II} - \displaystyle\sum_{j\in C} \overline{A}_j^{I,II} \partial_j W_{II}. \end{cases}$$

We can then apply Theorem 4.1 to the resulting system with $n = n_S + 1 + d$ and $n_0 = 1$, where $d \in \{1,2,3\}$ is the particular dimension of interest.

Indeed, Properties $(Ex_1)$–$(Ex_6)$ are easily checked from Theorem 3.14, the assumptions, and Properties $(H_1)$–$(H_9)$. On the other hand, in order to establish $(Ex_7)$, we consider separately the contributions of the matrices $\overline{G}_{ij}^{D\lambda}$, $i,j \in C$, and $\overline{G}_{ij}^{\kappa\eta}$,

$i,j \in C$. From Theorem 3.14, we have $\overline{G}^{D\lambda} = \overline{G}^{D\lambda}_{ii}$, $i \in C$, whereas $\overline{G}^{D\lambda}_{ij} = 0$ for $i \neq j$. After a little algebra, we obtain for $\xi_{II} = (\xi_2, \ldots, \xi_{n_S+4})^t \in \mathbb{R}^{n-n_0}$ that

$$\xi_{II}^t \overline{G}^{D\lambda\, II,II}(W)\xi_{II} = \sum_{k,l\in S} \mathcal{D}_{kl}\big(\xi_k + (\frac{r_k}{T} + \frac{p\chi_k}{T^2\rho_k})\xi_{n_S+4}\big)\big(\xi_l + (\frac{r_l}{T} + \frac{p\chi_l}{T^2\rho_l})\xi_{n_S+4}\big) + \frac{\lambda}{T^2}\xi_{n_S+4}^2,$$

where we have extended $\xi_{II}$ into $\xi = (0, \xi_{II})^t$. This identity shows that the matrix $\overline{G}^{D\lambda\, II,II}$ is positive definite on the subspace $\xi_1 = \xi_{n_S+1} = \xi_{n_S+2} = \xi_{n_S+3} = 0$ for any $W \in \mathcal{O}_W$. As a consequence, for any function $\phi_{II}$ in $W_2^1(\mathbb{R}^d)$ with $n - n_0$ components, denoted by $\phi_{II} = (\phi_2, \ldots, \phi_{n_S+4})$, we have

$$\partial_i \phi_{II}^t\, \overline{G}^{D\lambda\, II,II}_{ii}(W)\, \partial_i \phi_{II} \geq \alpha\big(\partial_i\phi_2^2 + \cdots + \partial_i\phi_{n_S}^2 + \partial_i\phi_{n_S+4}^2\big), \tag{4.12}$$

for $i \in C$, uniformly for $W$ in a given compact of $\mathcal{O}(W)$. On the other hand, after a little algebra, we also obtain from Theorem 3.14 that

$$\begin{aligned} \sum_{i,j\in C} \partial_i \phi_{II}^t\, \overline{G}^{\kappa\eta\, II,II}_{ij}(W)\, \partial_j \phi_{II} = \frac{\eta}{T} \sum_{i,j\in C} \Big(\partial_i \mathfrak{v}_j + \partial_j \mathfrak{v}_i - \tfrac{2}{3}(\sum_{k\in C} \partial_k \mathfrak{v}_k)\delta_{ij}\Big)^2 \\ + \frac{\kappa}{T}\Big(\sum_{k\in C} \partial_k \mathfrak{v}_k\Big)^2, \end{aligned} \tag{4.13}$$

where $\mathfrak{v}_i = \phi_{n_S+i}$, $i \in C$. Since $\eta$ and $\kappa$ are positive, we have

$$\sum_{i,j\in C} \partial_i \phi_{II}^t\, \overline{G}^{\kappa\eta\, II,II}_{ij}(W)\, \partial_j \phi_{II} \geq \alpha \sum_{i,j\in C} \big(\partial_i \mathfrak{v}_j + \partial_j \mathfrak{v}_i\big)^2, \tag{4.14}$$

where $\alpha$ is a positive constant, uniformly for $W$ in a compact of $\mathcal{O}_W$. Combining the estimates (4.12) and (4.14) with the identity

$$\int_{\mathbb{R}^d} \sum_{i,j\in C} \big(\partial_i \mathfrak{v}_j + \partial_j \mathfrak{v}_i\big)^2\, dx \;=\; \tfrac{1}{2}\int_{\mathbb{R}^d} \sum_{i,j\in C} \big(\partial_i \mathfrak{v}_j\big)^2\, dx \;+\; \tfrac{1}{2}\int_{\mathbb{R}^d} \Big(\sum_{k\in C} \partial_k \mathfrak{v}_k\Big)^2\, dx,$$

valid for $\mathfrak{v}_i \in W_2^1(\mathbb{R}^d)$, $i \in C$, we obtain $(Ex_7)$. Finally, we note that from the conservation of $\rho$

$$\rho(t,x) \geq \inf_{\mathbb{R}^d} \rho^0(x) \exp\left(-\int_0^t \|\partial_x \cdot v(s)\|_{0,\infty}\, ds\right),$$

and thus $\inf_{\mathbb{R}^d} \rho(t,x) > 0$ as long as (4.11) remains finite, so that only $T$ may reach the boundary of $\mathcal{O}_W$.

On the other hand, when $\kappa$ is only a nonnegative function, we rewrite the equation (3.42) in a different form. We first note that

$$\sum_{i,j\in C} \partial_i\Big(\overline{G}^{\kappa\eta\, II,II}_{ij} \partial_j W_{II}\Big) = \partial_x\cdot\Big(\frac{\eta}{T}\big(\partial_x v + (\partial_x v)^t\big) + \frac{\kappa - \frac{2}{3}\eta}{T}\partial_x\cdot v\, I\Big),$$

and we have the identity [VH72]

$$\partial_x\cdot\Big(\frac{\eta}{T}\big(\partial_x v+(\partial_x v)^t\big)+\frac{\kappa-\frac{2}{3}\eta}{T}\partial_x\cdot v\, I\Big)=\sum_{i\in C}\partial_i\Big(\frac{\eta}{T}\partial_i v\Big)+\partial_x\cdot\Big(\frac{\kappa+\frac{1}{3}\eta}{T}\partial_x\cdot v\, I\Big)$$
$$+\sum_{i\in C}\partial_i\Big(\frac{\eta}{T}\Big)\,\partial_x v_i-(\partial_x\cdot v)\,\partial_x\Big(\frac{\eta}{T}\Big).$$

We can thus rewrite equation (3.42) in the form

$$\overline{A}_0^{II,II}\,\partial_t W_{II}=-\sum_{i\in C}\overline{\mathcal{A}}_i^{II,I}\partial_i W_I+\sum_{i,j\in C}\partial_i\big(\overline{\mathcal{G}}_{ij}^{II,II}\,\partial_j W_{II}\big)+\overline{\Gamma}_{II},$$

with new coefficients $\overline{\mathcal{A}}_i^{II,I}$, $i\in C$, and $\overline{\mathcal{G}}_{ij}^{II,II}$, $i,j\in C$, such that

$$\sum_{i,j\in C}\partial_i\Big(\overline{\mathcal{G}}_{ij}^{\kappa\eta\;II,II}\,\partial_j W_{II}\Big)=\sum_{i\in C}\partial_i\Big(\frac{\eta}{T}\partial_i v\Big)+\partial_x\cdot\Big(\frac{\kappa+\frac{1}{3}\eta}{T}\partial_x\cdot v\, I\Big),$$

and with the lower order factors $\sum_{i\in C}\partial_i(\eta/T)\partial_x v_i$ and $-(\partial_x\cdot v)\partial_x(\eta/T)$ dispatched in the terms $\overline{\mathcal{A}}_i^{II,I}\partial_i W_I$, $i\in C$, and $\overline{\Gamma}_{II}$.

More specifically, we define $\overline{\mathcal{G}}_{ij}^{II,II}=\overline{\mathcal{G}}_{ij}^{D\lambda\;II,II}+\overline{\mathcal{G}}_{ij}^{\kappa\eta\;II,II}$, $i,j\in C$, where $\overline{\mathcal{G}}_{ij}^{D\lambda II,II}=\overline{G}_{ij}^{D\lambda II,II}$, $i,j\in C$, whereas $\overline{\mathcal{G}}_{ij}^{\kappa\eta II,II}$, $i,j\in C$, are given by

$$\overline{\mathcal{G}}_{ii}^{\kappa\eta II,II}=\overline{G}_{ii}^{\kappa\eta II,II},\quad i\in C$$

and

$$\mathcal{G}_{ij}^{\kappa\eta II,II}=(\kappa+\tfrac{1}{3}\eta)\ \big(\delta_{k(n_S+i)}\,\delta_{l(n_S+j)}\big)_{k,l\in[2,n_S+4]}\qquad i,j\in C\quad i\neq j.$$

Furthermore, we define $\overline{\mathcal{A}}_i^{II,I}$, $i\in C$, and $\overline{\Gamma}_{II}$ by

$$\overline{\mathcal{A}}_i^{II,I}=\overline{A}_i^{II,I}+(\partial_x\cdot v)\,\partial_\rho\Big(\frac{\eta}{T}\Big),$$

and

$$\overline{\Gamma}_{II}=\sum_{i\in C}\partial_i\Big(\frac{\eta}{T}\Big)\,\partial_x v_i-(\partial_x\cdot v)\Big(\partial_{W_{II}}\big(\frac{\eta}{T}\big)\Big)\,\partial_x W_{II}+\overline{H}_{II}+\overline{\Omega}_{II}-\sum_{j\in C}\overline{A}_j^{I,II}\partial_j W_{II},$$

and the end of the proof is similar to that of the preceding case. □

## Appendix

We consider here the normal form of the multicomponent reactive flows system of equations obtained with the "natural" normal variable $\widehat{W}=(U_I',V_{II}')^t$ suggested by

Theorem 3.10. This normal form has dissipative terms in conservative form and leaves unchanged the parabolic components of the source term $\Omega_{II}$. This form, however, has a more complex expression than the one obtained in Theorem 3.14. Theorem 4.1 can also be applied to this normal form of the system.

The system in the new variable

$$\widehat{W} = \frac{1}{T}\Big(\rho T,\ \mu_2 - \mu_1, \ldots,\ \mu_{n_S} - \mu_1,\ v_1,\ v_2,\ v_3,\ -1\Big)^t,$$

on the open convex set $\mathcal{O}_{\widehat{W}} = (0,\infty) \times \mathbb{R}^{n_S-1} \times \mathbb{R}^3 \times (-\infty, -1/T_0)$ can be written in the form

$$\widehat{A}_0^{I,I}\,\partial_t \widehat{W}_I \;+ \textstyle\sum_{i\in C} \widehat{A}_i^{I,I}\,\partial_i \widehat{W}_I \;+ \sum_{i\in C} \widehat{A}_i^{I,II}\,\partial_i \widehat{W}_{II} \;=\; 0, \tag{4.15}$$

$$\widehat{A}_0^{II,II}\,\partial_t \widehat{W}_{II} + \textstyle\sum_{i\in C} \widehat{A}_i^{II,I}\,\partial_i \widehat{W}_I + \sum_{i\in C} \widehat{A}_i^{II,II}\,\partial_i \widehat{W}_{II} = \displaystyle\sum_{i,j\in C} \partial_i\Big(\widehat{G}_{ij}^{II,II}\,\partial_j \widehat{W}_{II}\Big) + \widehat{\Omega}^{II}. \tag{4.16}$$

and is of the normal form. The matrix $\widehat{A}_0$ is given by

$$\widehat{A}_0 = \begin{pmatrix} \frac{1}{\Sigma_\rho} & & & Sym \\ 0 & \mathcal{X} & & \\ 0 & 0 & \rho T I_3 & \\ 0 & (\Xi_e)^t & \rho T v^t & \Upsilon'_e \end{pmatrix}, \tag{4.17}$$

where $\Xi_e$ is a vector given by given by

$$\Xi_{el} = \frac{\rho_l}{r_l}\left(e_l^{\rm tot} - \frac{\Sigma_e}{\Sigma_\rho}\right), \qquad l \in [2, n_S], \tag{4.18}$$

and where

$$\Upsilon'_e = \sum_{k\in S} \frac{\rho_k}{r_k} e_k^{{\rm tot}\,2} + \rho T\,(v\!\cdot\! v + c_v T) - \frac{\Sigma_e^2}{\Sigma_\rho}. \tag{4.19}$$

The matrices $\widehat{A}_i$, $i = 1, 2, 3$, are also given by

$$\sum_{i\in C} \widehat{A}_i \xi_i = \begin{pmatrix} \dfrac{v\!\cdot\!\xi}{\Sigma_\rho} & & & Sym \\ 0_{(n_S-1)\times 1} & \mathcal{X} v\!\cdot\!\xi & & \\ \dfrac{\rho T}{\Sigma_\rho}\xi & T\xi\otimes\mathcal{Z} & \rho T v\!\cdot\!\xi\, I_3 & \\ \dfrac{\rho T}{\Sigma_\rho} v\!\cdot\!\xi & (v\!\cdot\!\xi)\Xi_h^t & \rho T (v\!\cdot\!\xi) v^t + \rho T \Xi_{h0} \xi^t & \Upsilon'_h v\!\cdot\!\xi \end{pmatrix}, \tag{4.20}$$

where $\Xi_{h0}$ and $\Xi_{hl}$ are defined by

$$\Xi_{h0} = h^{\mathrm{tot}} - \frac{\Sigma_h}{\Sigma_\rho} + \frac{\rho T}{\Sigma_\rho}, \qquad \Xi_{hl} = \frac{\rho_l}{r_l}\left(h_l^{\mathrm{tot}} - \frac{\Sigma_h}{\Sigma_\rho}\right), \quad l \in [2, n_S],$$

and where

$$\Upsilon'_h = \sum_{k\in S} \frac{\rho_k}{r_k} {h_k^{\mathrm{tot}}}^2 + \rho T\,(v\cdot v + (c_v + r)T) - \frac{\Sigma_h^2}{\Sigma_\rho} + \frac{(\rho T)^2}{\Sigma_\rho}. \tag{4.21}$$

For the dissipative part, we also have

$$\widehat{G}_{ij}^{II,II} = \widetilde{G}_{ij}^{II,II}, \tag{4.22}$$

and an explicit expression for $\widetilde{G}_{ij}^{II,II}$ is given by (3.13)–(3.16). Finally, concerning the source term, we have

$$\widehat{\Omega}_I = \widetilde{\Omega}'_I = \Omega'_I = 0, \qquad \widehat{\Omega}_{II} = \widetilde{\Omega}'_{II} = \Omega'_{II} = \Omega_{II}. \tag{4.23}$$


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