MULTICOMPONENT REACTIVE FLOWS
(II) ASYMPTOTIC STABILITY OF EQUILIBRIUM STATES

Vincent GIOVANGIGLI and Marc MASSOT

Centre de Mathématiques Appliquées,
Ecole Polytechnique, 91128 Palaiseau Cedex, France

We consider the equations governing multicomponent reactive flows derived from the kinetic theory of dilute polyatomic reactive gas mixtures. We first discuss the structure of the chemical source term associated with Maxwellian distributions. We then investigate an abstract second order quasilinear system with a source term, around a constant equilibrium state. Assuming the existence a generalized entropy function, the invariance of the nullspace naturally associated with dissipation matrices, stability conditions for the source term, and a dissipative structure for the linearized equations, we establish global existence and asymptotic stability around the constant equilibrium state in all space dimensions and we obtain decay estimates. These results are then applied to multicomponent reactive flows using a normal form obtained in the previous part of the paper and the properties of Maxwellian chemical source terms.

1. Introduction

In this paper, we investigate the system of equations modeling multicomponent reactive flows. Global existence and asymptotic stability around constant equilibrium states are obtained for the Cauchy problem in all space dimensions.

The governing equations are first summarized, referring to the first part of the paper [$\mathrm{GM96_I}$] for more details. In particular, these equations admit an entropy function and can be written in symmetric form. We complete the presentation of these equations by investigating the structure of Maxwellian chemical source terms. These terms are compatible with the law of mass action and guarantee element and mass conservation. The space spanned by the reaction vectors is in particular included in the orthogonal complement of the space spanned by the element vectors.

We then consider an abstract second order quasilinear system with a source term and which admits an entropy function. We assume that the nullspace naturally associated with dissipation matrices of the symmetrized system is invariant. The latter system can then be cast into a normal form. Under stability conditions on the source term, and assuming conditions which guarantee the dissipative structure of the linearized normal system around the constant equilibrium state, we obtain global existence and asymptotic stability of the stationary state. As stability conditions, we assume that the chemical entropy production is nonnegative and that the source term lies in the range of its derivative at equilibrium. Our method of proof relies on the first part of this study [$\mathrm{GM96_I}$], on Kawashima's theory [Ka84], on a priori estimates provided by the entropy conservation law and on stability properties of the source term. Decay estimates towards the constant stationary state are also obtained in all space dimensions. This work extends previous results of Kawashima [Ka84] in space dimension $d \geq 3$ for general source term. It also extend results of Kawashima [Ka84] concerning entropic systems with no source term in space dimension $d \geq 1$.

We then apply these results to the system modeling multicomponent reactive flows. A detailed study of the Maxwellian chemical source term provided by the kinetic theory allows us to establish the existence of constant stationary states and to investigate the structure of the corresponding linearized equations. We then obtain global existence, asymptotic stability of the constant equilibrium state together with decay estimates, and, to the author's knowledge, these results are new.

In Section 2 we summarize the governing equations and discuss the chemical source term. In Section 3 we investigate an abstract hyperbolic-parabolic composite system of conservation laws with a source term. Finally, multicomponent reactive flows are considered in Section 4.

2. Governing Equations

2.1. Conservation equations

The equations modeling multicomponent reactive flows express the conservation of species mass, momentum and energy. These equations have been investigated in [$\mathrm{GM96_I}$] and can be written in the quasilinear form

$$\partial_t U + \sum_{i \in C} \partial_i F_i(U) = \sum_{i,j \in C} \partial_i \Big(G_{ij}(U) \partial_j U \Big) + \Omega(U), \tag{2.1}$$

where ∂_t is the time derivative operator, U the conservative variable, ∂_i the space derivative operator in the i^{th} direction, $C = \{1, 2, 3\}$ the set of direction indices, $F_i(U)$ the advective flux in the i^{th} direction, $G_{ij}(U)$, $i, j \in C$, the dissipation matrices, and Ω the source term. We will freely use, in the following, the notation and results established in the first part of the paper [$\mathrm{GM96_I}$].

In particular, we have $U = (\rho_1, \ldots, \rho_{n_S}, \rho v_1, \rho v_2, \rho v_3, \rho e^{\text{tot}})^t$ where ρ_k is the density of the k^{th} species, n_S the number of species, v_i the mass averaged flow velocity in the i^{th} direction, and e^{tot} the total energy per unit mass of the mixture. The dissipative fluxes are given by $\mathcal{F}_i = -\sum_{j \in C} G_{ij}(U) \partial_j U$, $i \in C$. The relations expressing the advective fluxes F_i, $i \in C$, and the dissipative fluxes $\mathcal{F}_i$, $i \in C$, are detailed in [$\mathrm{GM96_I}$]. The quasilinear system (2.1) can also be symmetrized and recast into various normal forms [$\mathrm{GM96_I}$].

In the first part of the paper [$\mathrm{GM96_I}$], in order to obtain local existence results, the source term was simply taken to be a regular function of the state variables, satisfying mass conservation relations. On the contrary, in order to investigate global existence around constant equilibrium states, we now have to complete the presentation of the governing equations by investigating the structure of the source term.

2.2. The chemical source term

In the following sections, the source term is taken to be

$$\Omega = (m_1\omega_1, \ldots, m_{n_S}\omega_{n_S}, 0, 0, 0, 0)^t, \tag{2.2}$$

where m_k is the molar mass of the $k^{\rm th}$ species and ω_k the molar production rate of the $k^{\rm th}$ species. In particular, we no longer consider force terms in Ω and only chemistry effects are included in the model. The molar production rates that we consider are the Maxwellian production rates obtained from the kinetic theory [EG94]. These rates are obtained in the kinetic framework of the "slow reaction regime", or in the "tempered reaction regime", when the chemistry characteristic times are larger than the mean free times of the molecules [EG94] [$\mathrm{GM96_I}$].

2.2.1. *Maxwellian production rates*

We consider a system of n_R reversible reactions for n_S species

$$\sum_{k\in S} \nu'_{ki}\mathcal{S}_k \;\rightleftharpoons\; \sum_{k\in S} \nu''_{ki}\mathcal{S}_k, \qquad i\in R, \tag{2.3}$$

where $\mathcal{S}_k$ is the chemical symbol of the $k^{\rm th}$ species, ν'_{ki} and ν''_{ki} the stoichiometric coefficients of the $k^{\rm th}$ species in the $i^{\rm th}$ reaction, and $R = [1, n_R]$ the set of reaction indices. The chemical species are assumed to be constituted by elements and we denote by $\mathcal{E}_{kl}$ the number of $l^{\rm th}$ element in the $k^{\rm th}$ species. We also denote by $E = [1, n_E]$ the set of element indices and by n_E the number of elements.

The molar production rate of the $k^{\rm th}$ species is then given by

$$\omega_k = \sum_{i\in R} \nu_{ki}\tau_i, \tag{2.4}$$

where

$$\nu_{ki} = \nu''_{ki} - \nu'_{ki}, \tag{2.5}$$

and where τ_i is the rate of progress of the $i^{\rm th}$ reaction. This rate τ_i is given by

$$\tau_i = K_{fi}\prod_{k\in S}\gamma_k^{\nu'_{ki}} - K_{ri}\prod_{k\in S}\gamma_k^{\nu''_{ki}}, \tag{2.6}$$

where $\gamma_k = \rho_k/m_k$ is the molar concentration of the k^{th} species, K_{fi} and K_{ri} are the direct and reverse rate constants of the i^{th} reaction. The quantities K_{fi} and K_{ri} are functions of the temperature and their ratio is the equilibrium constant K_{ei} of the i^{th} reaction

$$K_{ri}(T) = \frac{K_{fi}(T)}{K_{ei}(T)}. \tag{2.7}$$

The equilibrium constant $K_{ei}(T)$ is given by

$$\log K_{ei}(T) = \sum_{k\in S} \frac{\nu_{ki} m_k}{R_{\mathrm{g}}} \big(s_k(m_k, T) - \frac{h_k(T)}{T} \big), \tag{2.8}$$

where h_k is the enthalpy per unit mass of the k^{th} species , R_{g} the universal gas constant, and $s_k(m_k, T)$ is the specific entropy of the k^{th} species at unit concentration, that is at $\rho_k = m_k$.

Remark 2.1. The direct rate constant is usually approximated by using an Arrhenius empirical relation

$$K_{fi}(T) = \mathfrak{A}_i T^{\mathfrak{b}_i} \exp\Big(-\frac{\mathfrak{T}_i}{R_{\mathrm{g}} T} \Big),$$

where $\mathfrak{A}_i$ is the pre-exponential factor, $\mathfrak{b}_i$ the pre-exponential exponent and $\mathfrak{T}_i$ the activation energy of the i^{th} reaction, but the exact expression of $K_{fi}(T)$ will not be needed in the following.

2.2.2. *Mathematical assumptions*

The mathematical assumptions concerning the chemical productions rates are the following.

(H_{10}) The rate constants $\big(K_{fi}(T)\big)_{i\in R}$ are C^∞ positive bounded functions of $T \in [T_0, \infty)$ and the reverse constants are given by $K_{fi}(T)/K_{ei}(T)$ where $K_{ei}(T)$ is the equilibrium constant of the i^{th} reaction.

(H_{11}) The element vectors $\mathcal{E}_l$, $l \in E$, defined by $\mathcal{E}_l = (\mathcal{E}_{1l}, \dots, \mathcal{E}_{n_S l})^t$, and the reaction vectors ν_i, $i \in R$, defined by $\nu_i = (\nu_{1i}, \dots, \nu_{n_S i})^t$, satisfy the element conservation relations

$$\langle \nu_i, \mathcal{E}_l \rangle = 0, \qquad i \in R, \quad l \in E,$$

where $\langle x, y \rangle$ denotes the scalar product between vectors x and y. The space spanned by the reaction vectors is denoted by $\mathcal{R}$ and the space spanned by the element vectors is denoted by $\mathcal{E}$

$$\mathcal{R} = \mathrm{span}\{ \ \nu_i, \ l \in R \ \},$$

$$\mathcal{E} = \mathrm{span}\{ \ \mathcal{E}_l, \ l \in E \ \},$$

in such a way that

$$\mathcal{R} \subset \mathcal{E}^{\perp}, \qquad \mathcal{E} \subset \mathcal{R}^{\perp}.$$

(H_{12}) The vector of chemical production rates $\omega = (\omega_1, \ldots, \omega_{n_S})^t$ is given by (2.4) (2.5) which can be written in vector form

$$\omega = \sum_{i \in R} \tau_i \, \nu_i, \tag{2.9}$$

with the rate of progress τ_i given by (2.6). In particular, we always have $\omega(U) \in \mathcal{R}$ for $U \in \mathcal{O}_U$.

(H_{13}) The species molar mass m_k, $k \in S$, are related to the elemental masses a_l, $l \in E$, by the relations

$$m_k = \sum_{l \in E} a_l \, \mathcal{E}_{kl}.$$

Denoting by m the mass vector $m = (m_1, \ldots, m_{n_S})^t$, these relations can be written in vector form

$$m = \sum_{l \in E} a_l \, \mathcal{E}_l. \tag{2.10}$$

With respect to these assumptions, we make the following comments. We first note that Properties (H_{10}) and (H_{12}) imply the smoothness Property (H_4) of [GM96$_{\mathrm{I}}$]. Similarly, since the force term g is taken to be zero in this part of the paper, the smoothness Property (H_3) of [GM96$_{\mathrm{I}}$] is also trivially satisfied. Properties (H_{12}) and (H_{13}) also imply the mass conservation relation (H_9) of [GM96$_{\mathrm{I}}$] since

$$\sum_{k \in S} m_k \nu_{ki} = \langle m, \nu_i \rangle = \sum_{l \in E} a_l \langle \mathcal{E}_l, \nu_i \rangle = 0,$$

and therefore

$$\sum_{k \in S} m_k \omega_k = \langle m, \omega \rangle = \sum_{i \in R} \tau_i \langle m, \nu_i \rangle = 0.$$

The element vectors may taken to be the atomic elements provided that the corresponding vectors are independent. When this is not the case, it is first necessary to eliminate linearly dependent atomic elements. Finally, the fundamental relation between the direct and reverse rate constants $K_{ri}(T) = K_{fi}(T)/K_{ei}(T)$ is a direct consequence of the kinetic theory [EG94].

2.2.3. *Mass weighted stoichiometric coefficients*

In the following, we will have to manipulate the mass weighted production rates $m_k \omega_k$, $k \in S$. To this purpose, we introduce the mass weights matrix M, of order n_S, defined by

$$M = \mathrm{diag}(m_1, \ldots, m_{n_S}),$$

which acts on vectors of length n_S. The mass weighted stoichiometric coefficients are then the vectors $M\nu_i$, $i \in R$, and the specific elemental compositions are the vectors $M^{-1}\mathcal{E}_l$, $l \in E$. The corresponding spaces $M\mathcal{R}$ and $M^{-1}\mathcal{E}$, spanned by these mass weighted vectors, are also such that $M\mathcal{R} \subset (M^{-1}\mathcal{E})^\perp$ and $M^{-1}\mathcal{E} \subset (M\mathcal{R})^\perp$ in the composition space $\mathbb{R}^{n_S}$. In particular, we have $M\omega \in M\mathcal{R}$ and the mass relation (2.10) can also be written

$$\mathcal{U} = \sum_{l\in E} a_l M^{-1}\mathcal{E}_l,$$

where $\mathcal{U} \in \mathbb{R}^{n_S}$ is the unity vector defined by $\mathcal{U} = (1,\ldots,1)^t$, which implies that $\mathcal{U} \in (M\mathcal{R})^\perp$.

Various forms of the multicomponent reactive flow governing equations have been investigated in [GM96$_\mathrm{I}$] and will again be considered in this part of the paper. We present here some properties of the corresponding source terms. We first observe that the source term of the symmetric form $\widetilde{\Omega}$ is identical to that of the original formulation, that is, $\widetilde{\Omega} = \Omega$. As a consequence, we have

$$\Omega = \widetilde{\Omega} = \big(m_1\omega_1,\ldots,m_{n_S}\omega_{n_S},0,0,0,0\big)^t \in M\mathcal{R} \times 0_{\mathbb{R}^4}. \tag{2.11}$$

On the other hand, the source terms for the auxiliary variables $U' = P^tU$ and $V' = P^{-1}V$, where the constant matrix P is given by Eq. (3.36) of [GM96$_\mathrm{I}$], are given by $\Omega' = \widetilde{\Omega}' = P^t\Omega$, that is to say

$$\Omega' = \widetilde{\Omega}' = (0,m_2\omega_2,\ldots,m_{n_S}\omega_{n_S},0,0,0,0)^t. \tag{2.12}$$

In particular, Ω' is also in a fixed subspace of $\mathbb{R}^{n_S+4}$ of dimension $\dim(\mathcal{R})$ since

$$\Omega' \in P^t(M\mathcal{R} \times 0_{\mathbb{R}^4}). \tag{2.13}$$

In addition, the source terms $\overline{\Omega}$ corresponding to the variable W is given by

$$\overline{\Omega} = (0,\ m_2\omega_2,\ldots,\ m_{n_S}\omega_{n_S},\ 0,\ 0,\ 0,\ -\frac{1}{T^2}\sum_{k\in S} e_k m_k \omega_k)^t, \tag{2.14}$$

and the source term corresponding to the variable $\widehat{W}$ is given by $\widehat{\Omega} = \Omega'$.

3. Global Existence and Asymptotic Stability for an Abstract System

In this section, we consider an abstract quasilinear second order system of with a source term and a convex entropy function. We assume that the nullspace naturally associated with dissipation matrices of the symmetrized system is invariant. The latter system can then be recast into a normal form. Under stability conditions on the source term, and assuming conditions which guarantee the dissipative structure of the linearized normal system around the constant equilibrium state, we prove global existence, asymptotic stability and decay estimates for the Cauchy problem.

3.1. An abstract system

3.1.1. *Conservation equations*

We consider an abstract system of conservation laws in the form

$$\partial_t U^* + \sum_{i\in C^*} \partial_i F_i^*(U^*) = \sum_{i,j\in C^*} \partial_i \Big(G_{ij}^*(U^*)\partial_j U^* \Big) + \Omega^*(U^*), \tag{3.1}$$

where $U^* \in \mathcal{O}_{U^*}$, $\mathcal{O}_{U^*}$ is an open convex set of $\mathbb{R}^n$, and $C^* = [1, d]$ is the set of the spatial coordinates of $\mathbb{R}^d$. The superscript * is introduced in order to distinguish between the abstract system (3.1) and the particular system (2.1) modeling multicomponent reactive flows. We assume that the following fundamental properties hold for system (3.1).

- (A_1) The flux vectors F_i^*, $i \in C^*$, the dissipation matrices G_{ij}^*, $i, j \in C^*$, and the source term Ω^*, are smooth functions of the variable $U^* \in \mathcal{O}_{U^*}$, where $\mathcal{O}_{U^*}$ is a convex open set of $\mathbb{R}^n$.

- (A_2) The system (3.1) admits an entropy function $\mathcal{H}^*$ over the domain $\mathcal{O}_{U^*}$, in the sense of definition 3.2 of [GM96$_\mathrm{I}$].

3.1.2. *Conservative symmetric formulation*

By using the entropic variable, $V^* = \big(\partial_{U^*}\mathcal{H}^*\big)^t$, associated with U^*, the system also takes the form

$$\widetilde{A}_0^*\partial_t V^* + \sum_{i\in C^*} \widetilde{A}_i^* \partial_i V^* = \sum_{i,j\in C^*} \partial_i \left(\widetilde{G}_{ij}^* \partial_j V^* \right) + \widetilde{\Omega}^*, \tag{3.2}$$

where the matrix coefficients are defined according to Definition 3.2 of [GM96$_\mathrm{I}$], that is, where $\widetilde{A}_0^* = \partial_{V^*} U^*$, $\widetilde{A}_i^* = (\partial_{U^*} F_i^*)\,\widetilde{A}_0^*$, and $\widetilde{G}_{ij}^* = G_{ij}^* \widetilde{A}_0^*$. The system (3.2) then satisfies the following properties [KS88] [GM96$_\mathrm{I}$].

- (S_1) The matrix $\widetilde{A}_0^*(V^*)$ is symmetric and positive definite for $V^* \in \mathcal{O}_{V^*}$,
- (S_2) The matrices $\widetilde{A}_i^*(V^*)$, $i \in C^*$, are symmetric for $V^* \in \mathcal{O}_{V^*}$,
- (S_3) We have $\widetilde{G}_{ij}^*(V^*)^t = \widetilde{G}_{ji}^*(V^*)$ for $i, j \in C^*$ and $V^* \in \mathcal{O}_{V^*}$,
- (S_4) The matrix $\widetilde{B}^*(V^*, w) = \sum_{i,j\in C^*} \widetilde{G}_{ij}^*(V^*) w_i w_j$ is symmetric positive semi-definite, for $V^* \in \mathcal{O}_{V^*}$ and $w \in \mathcal{S}^{d-1}$.

We further assume that the symmetrized system (3.2) satisfies Condition N as in Definition 3.8 of [GM96$_\mathrm{I}$].

(A_3) The null space of the matrix

$$\widetilde{B}^*(V^*,w) = \sum_{i,j\in C^*} \widetilde{G}^*_{ij}(V^*)w_i w_j,$$

does not depend on $V^* \in \mathcal{O}_{V^*}$ and $w \in \mathcal{S}^{d-1}$ and we denote by n_0 its dimension $n_0 = \dim\big(N(\widetilde{B}^*)\big)$.

We then know [GM96$_\mathrm{I}$] that we can find a constant non-singular matrix P of dimension n such that, if we consider the entropic variable $V^{*\prime} = P^{-1}V^*$ associated with $U^{*\prime} = P^t U^*$, the corresponding dissipation matrices have the form

$$\widetilde{G}^{*\prime}_{ij} = \begin{pmatrix} 0_{n_0\times n_0} & 0_{n_0\times(n-n_0)} \\ 0_{(n-n_0)\times n_0} & \widetilde{G}^{*\prime\, II,II}_{ij} \end{pmatrix}, \qquad i,j\in C^*, \tag{3.3}$$

and $\sum_{i,j\in C^*} \widetilde{G}^{*\prime II,II}_{ij} w_i w_j$ is symmetric positive definite for $V^{*\prime}\in\mathcal{O}_{V^{*\prime}}$, $w\in\mathcal{S}^{d-1}$.

3.1.3. *A normal form*

Following Theorem 3.10 of [GM96$_\mathrm{I}$], we introduce a normal variable

$$W^*(V^*) = (W^*_{I}, W^*_{II})^t = \big(\psi^{I}(U^{*\prime}_{I}), \varphi^{II}(V^{*\prime}_{II})\big)^t, \tag{3.4}$$

where ψ^{I} and φ^{II} are diffeomorphisms of $\mathbb{R}^{n_0}$ and $\mathbb{R}^{n-n_0}$ respectively. The system obtained by multiplying (3.2) on the left side by the matrix $(\partial_{W^*}V^*)^t$, is then in normal form and reads

$$\overline{A}^*_0\partial_t W^* + \sum_{i\in C^*} \overline{A}^*_i \partial_i W^* = \sum_{i,j\in C^*} \partial_i\left(\overline{G}^*_{ij}\partial_j W^*\right) + \overline{H}^* + \overline{\Omega}^*, \tag{3.5}$$

where $\overline{A}^*_0 = (\partial_{W^*}V^*)^t\,\widetilde{A}^*_0\,(\partial_{W^*}V^*)$, $\overline{A}^*_i = (\partial_{W^*}V^*)^t\,\widetilde{A}^*_i\,(\partial_{W^*}V^*)$, $i\in C$, $\overline{G}^*_{ij} = (\partial_{W^*}V^*)^t\,\widetilde{G}^*_{ij}\,(\partial_{W^*}V^*)$, $i,j\in C$, $\overline{H}^* = -\sum_{i,j\in C^*}\partial_i(\partial_{W^*}V^*)^t\,\widetilde{G}^*_{ij}\,(\partial_{W^*}V^*)\partial_j W^*$, and where $\overline{\Omega}^* = (\partial_{W^*}V^*)^t\,\Omega^*$. In particular, the following properties are shown to be satisfied [KS88] [GM96$_\mathrm{I}$].

$(\overline{S}_1)$ The matrix $\overline{A}^*_0(W^*)$ is symmetric and positive definite for $W^*\in\mathcal{O}_{W^*}$,

$(\overline{S}_2)$ The matrices $\overline{A}^*_i(W^*)$, $i\in C^*$ are symmetric for $W^*\in\mathcal{O}_{W^*}$,

$(\overline{S}_3)$ We have $\overline{G}^*_{ij}(W^*)^t = \overline{G}^*_{ji}(W^*)$ for $i,j\in C^*$ and $W^*\in\mathcal{O}_{W^*}$,

$(\overline{S}_4)$ The matrix $\overline{B}^*(W^*,w) = \sum_{i,j\in C^*}\overline{G}^*_{ij}(W^*)w_i w_j$ is symmetric positive semi-definite, for $W^*\in\mathcal{O}_{W^*}$ and $w\in\mathcal{S}^{d-1}$.

(N_1) The matrices $\overline{A}^*_0$ and $\overline{G}^*_{ij}$ have the block structure

$$\overline{A}^*_0 = \begin{pmatrix} \overline{A}^{*I,I}_0 & 0 \\ 0 & \overline{A}^{*II,II}_0 \end{pmatrix}, \qquad \overline{G}^*_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & \overline{G}^{*II,II}_{ij} \end{pmatrix},$$

(N_2) The matrix $\overline{B}^{*II,II}(W^*,w) = \sum_{i,j\in C^*} \overline{G}_{ij}^{*II,II}(W^*)w_iw_j$ is positive definite, for $W^* \in \mathcal{O}_{W^*}$ and $w \in \mathcal{S}^{d-1}$,

(N_3) We have $\overline{H}^*(W^*,\partial_x W^*) = \Big(0, \overline{H}^*_{II}(W^*,\partial_x W^*)\Big)^t$,

We will often use, in the following, the block decomposition induced by the partitioning between hyperbolic and parabolic variables $W^* = (W_I^*, W_{II}^*)^t$.

3.1.4. *Equilibrium state*

We further assume that the system (3.1) possesses an equilibrium state.

(A_4) There exists a constant equilibrium state $U^{*e} \in \mathcal{O}_{U^*}$ such that

$$\Omega(U^{*e}) = 0.$$

The equilibrium states corresponding to the various variables are also denoted with the superscript ^{e}, so that the equilibrium states in the variables V^* and W^*, for instance, are denoted by V^{*e} and W^{*e}, respectively.

3.1.5. *Conservation of entropy*

In order to derive a priori estimates for the solution, we will need a conservation equation for the generalized entropy function $\mathcal{H}^*$. This equation is easily obtained by taking the scalar product of equation (3.2) with the vector V^* and reads

$$\partial_t\mathcal{H}^* + \sum_{i\in C^*} \partial_i\sigma_i^* = \sum_{i,j\in C^*} \partial_i\big\langle V^*, \widetilde{G}_{ij}^*\partial_j V^*\big\rangle - \sum_{i,j\in C^*} \big\langle \partial_i V^*, \widetilde{G}_{ij}^*\partial_j V^*\big\rangle + \big\langle V^*, \widetilde{\Omega}(V^*)\big\rangle \quad (3.6)$$

where σ_i^*, $i \in C^*$, are the entropy fluxes [GM96$_\mathrm{I}$]. We will also need the associated function $\mathfrak{h}(U^*, U^{*e})$ defined by

$$\mathfrak{h}(U^*, U^{*e}) = \mathcal{H}^*(U^*) - \mathcal{H}^*(U^{*e}) - \langle V^{*e}, U^* - U^{*e}\rangle, \qquad U^* \in \mathcal{O}_{U^*}. \quad (3.7)$$

Thanks to the strict convexity of $\mathcal{H}^*$, this function plays the role of a distance between U^* and the stationary state U^{*e} and the following proposition is easily established.

Proposition 3.1. The function $\mathfrak{h}$ is a positive, smooth, strictly convex function of $U^* \in \mathcal{O}_{U^*}$, satisfying $\mathfrak{h}(U^*, U^{*e}) = 0$ if and only if $U^* = U^{*e}$. There exists also a neighborhood $\mathfrak{O} = \{\, z \in \mathbb{R}^n;\ |z - W^{*e}| < r \,\}$ of W^{*e} with r small enough such that for any W^* in $\mathfrak{O}$ and $U^* = U^*(W^*)$

$$c|W^* - W^{*e}|^2 \le \mathfrak{h}(U^*, U^{*e}) \le C|W^* - W^{*e}|^2, \quad (3.8)$$

where c and C are positive constants. In addition, the function $\mathfrak{h}$ satisfies the conservation equation

$$\begin{aligned} &\partial_t\mathfrak{h} + \sum_{i\in C^*} \partial_i\Big(\sigma_i^*(U^*) - \sigma_i^*(U^{*e}) - \big\langle V^{*e}, F_i^*(U^*) - F_i^*(U^{*e})\big\rangle\Big) = \\ &\sum_{i,j\in C^*} \partial_i\big\langle V^* - V^{*e}, \widetilde{G}_{ij}^*\partial_j V^*\big\rangle - \sum_{i,j\in C^*} \big\langle \partial_i V^*, \widetilde{G}_{ij}^*\partial_j V^*\big\rangle + \big\langle V^* - V^{*e}, \widetilde{\Omega}^*(V^*)\big\rangle. \end{aligned} \quad (3.9)$$

3.2. Local dissipative structure

In order to establish global existence in time and asymptotic stability of a constant stationary state, decay estimates for the linearized equations are needed. A condition which guarantees decay properties for the linearized system is the local dissipative structure introduced by Kawashima [Ka84]. This dissipative structure is completed here by stability properties of the source term.

3.2.1. *Linearized equations*

If we linearize system (3.5) around the constant stationary state W^{*e}, we obtain the following linear system in terms of the variable $z = W^* - W^{*e}$

$$\overline{A}_0^*(W^{*e})\partial_t z + \sum_{i\in C^*} \overline{A}_i^*(W^{*e})\partial_i z = \sum_{i,j\in C^*} \overline{G}_{ij}^*(W^{*e})\partial_i\partial_j z - \overline{L}^*(W^{*e})z, \tag{3.10}$$

where $\overline{L}^*$ is defined by $\overline{L}^* = -\partial_{W^*}\overline{\Omega}^*$. We assume that this linearized system has a dissipative structure in the sense of Kawashima.

(D_1) The matrix $\overline{A}_0^*(W^{*e})$ is symmetric positive definite, the matrices $\overline{A}_i^*(W^{*e})$, $i \in C^*$ are symmetric, we have $\overline{G}_{ij}^*(W^{*e})^t = \overline{G}_{ji}^*(W^{*e})$ for $i, j \in C^*$ and the matrix $\overline{L}^*(W^{*e})$ is symmetric positive semi-definite,

(D_2) There exist compensating matrices K^j, $j \in C^*$, such that the products $K^j\overline{A}_0^*(W^{*e})$, $j \in C^*$, are skew-symmetric, and such that the matrix

$$\sum_{i,j\in C^*} \tfrac{1}{2}\Big(K^i\overline{A}_j^*(W^{*e}) + \big(K^i\overline{A}_j^*(W^{*e})\big)^t\Big)w_iw_j + \overline{B}^*(W^{*e}, w) + \overline{L}^*(W^{*e}),$$

is positive definite for $w \in \mathcal{S}^{d-1}$.

Remark 3.2. The existence of the compensating matrices K^j, $j \in C^*$, implies that the linearized normal form is strictly dissipative in the sense that the eigenvalues $\lambda^*(\zeta, w)$ of the problem

$$\lambda^*\overline{A}_0^*(W^{*e})\phi + [\zeta \sum_{i\in C^*} \overline{A}_i^*(W^{*e})w_i - \zeta^2\overline{B}^*(W^{*e}, w) + \overline{L}^*(W^{*e})]\phi = 0,$$

for $\zeta \in \mathrm{i}\,\mathbb{R}$ and $w \in \mathcal{S}^{d-1}$, have a negative real part [SK85]. However, the converse is not known to be true [SK85]. The latter property only implies the existence of a combined compensating matrix $K(w)$, for $w \in \mathcal{S}^{d-1}$, such that $w \to K(w)$ is C^∞ on $\mathcal{S}^{d-1}$, $K(w)$ is real, the product $K(w)\overline{A}_0^*(W^{*e})$ is skew-symmetric, and $K(-w) = -K(w)$, for $w \in \mathcal{S}^{d-1}$, and such that the matrix

$$\sum_{j\in C^*} \tfrac{1}{2}\Big(K(w)\overline{A}_j^*(W^{*e}) + \big(K(w)\overline{A}_j^*(W^{*e})\big)^t\Big)w_j + \overline{B}^*(W^{*e}, w) + \overline{L}^*(W^{*e}),$$

is positive definite for $w \in \mathcal{S}^{d-1}$. It is not known, however, if the matrix $K(w)$ is of the form $\sum_{j\in C^*} K^j w_j$ [SK85]. Nevertheless, all the results obtained in this paper can be proved without the existence of matrices K^j, $j \in C$, by only using the combined compensating matrix $K(w)$, $w \in \mathcal{S}^{d-1}$, that is, by only using the strict dissipativity of the system [Ma96]. Nevertheless, in practical applications, it is generally possible to obtain compensating matrices K^j, $j \in C^*$, and to set $K(w) = \sum_{j\in C} K^j w_j$.

3.2.2. *Locally stable source terms*

We have already assumed that the matrix $\overline{L}^*(W^{*e})$ is symmetric in (D_1). We now further introduce local stability assumptions concerning the source term.

(D_3) The smallest subspace containing the source term $\widetilde{\Omega}^*(V^*) = \Omega^*\big(U^*(V^*)\big)$, for all $V^* \in \mathcal{O}_{V^*}$, is included in the range of $\widetilde{L}^*(V^{*e}) = -(\partial_{V^*}\widetilde{\Omega}^*)(V^{*e})$.

(D_4) There exists a neighborhood of V^{*e} in $\mathcal{O}_{V^*}$ and a positive constant c such that for any V^* in this neighborhood, we have

$$c\,\big|\,\widetilde{\Omega}^*(V^*)\big|^2 \;\le\; -\big\langle V^* - V^{e*},\, \widetilde{\Omega}^*(V^*)\big\rangle.$$

We can choose r small enough such that this inequality holds with $V^* = V^*(W^*)$ and W^* in $\mathfrak{O} = \{\, z \in \mathbb{R}^n;\ |z - W^{*e}| < r\,\}$.

Note that Properties (D_3) and (D_4) only concern the source term $\widetilde{\Omega}^*$. Property (D_3) will be used for the asymptotic stability of constant stationary states whereas Property (D_4) will be needed for the existence theorem.

3.2.3. *Global dissipative structure*

The physical meaning of the entropy conservation equation (3.6) is that when

$$\sum_{i,j\in C^*} \langle \partial_i V^*, \widetilde{G}^*_{ij}\partial_j V^*\rangle \ge 0, \tag{3.11}$$

and

$$\langle V^*, \widetilde{\Omega}(V^*)\rangle \le 0 \tag{3.12}$$

then the integral $\int_{\mathbb{R}^d} \mathcal{H}^*\,dx$ is nonincreasing in time, which corresponds to the second principle of thermodynamics.

Note however, that Property (S_4) does not imply the stronger condition (3.11). Similarly, we have only assumed, with Property (D4), that the source term is locally stable, and it does not implies (3.12) globally. However, for multicomponent reactive flows, Properties (3.11) and (3.12) are globally satisfied, as they should for any physically reasonable model. This suggests the following definition of a strong entropy which could be used in order to obtain global estimates, not necessarily in the neighborhood of a constant state.

Definition 3.3. A function $\mathcal{H}^*$ is be said to be a strong entropy for the system (3.1) if it is an entropy according to Definition 3.2 of [GM96$_\mathrm{I}$] and if the inequalities (3.11) and (3.12) hold over $\mathcal{O}_{V^*}$.

3.3. A global existence theorem

Now that we have stated Assumptions (A_1)–(A_4) and Dissipative conditions (D_1)–(D_4), concerning the various forms of the governing equations, we investigate global existence of solutions around the stationary state W^{*e} and its asymptotic stability. Of course, Assumptions (A_1)–(A_4) imply the various properties (S_1)–(S_4), $(\overline{S}_1)$–$(\overline{S}_4)$, and (N_1)–(N_3). Asymptotic decay towards the equilibrium state W^{*e} will be investigated in Section 3.4.

3.3.1. *The main result*

Theorem 3.4. Consider the quasilinear system (3.1) and assume that (A_1)–(A_4) and (D_1)–(D_4) hold. Let $d \geq 1$, $l \geq [d/2]+2$, and let $W^{*0}(x)$ such that

$$W^{*0} - W^{*e} \in W_2^l(\mathbb{R}^d). \tag{3.13}$$

Then if $\|W^{*0} - W^{*e}\|_{l,2}$ is small enough, there exists a unique global solution to the Cauchy problem

$$\overline{A}_0^* \partial_t W^* + \sum_{i \in C^*} \overline{A}_i^* \partial_i W^* = \sum_{i,j \in C^*} \partial_i \Big(\overline{G}_{ij}^* \partial_j W^* \Big) + \overline{H}^* + \overline{\Omega}^*, \tag{3.14}$$

with initial conditions

$$W^*(0,x) = W^{*0}(x), \tag{3.15}$$

such that

$$\begin{cases} W_{I}^* - W_{I}^{*e} \in C^0\big([0,\infty); W_2^l(\mathbb{R}^d)\big) \cap C^1\big([0,\infty); W_2^{l-1}(\mathbb{R}^d)\big) \cap L^2(0,\infty; W_2^l(\mathbb{R}^d)), \\ W_{II}^* - W_{II}^{*e} \in C^0\big([0,\infty); W_2^l(\mathbb{R}^d)\big) \cap C^1\big([0,\infty); W_2^{l-2}(\mathbb{R}^d)\big) \cap L^2(0,\infty; W_2^{l+1}(\mathbb{R}^d)). \end{cases} \tag{3.16}$$

This solution W^* satisfies the following estimate

$$\begin{aligned} \|W^*(t) - W^{*e}\|_{l,2}^2 &+ \int_0^t \|\partial_x W_{I}^*(\tau)\|_{l-1,2}^2 \, d\tau \\ &+ \int_0^t \|\partial_x W_{II}^*(\tau)\|_{l,2}^2 \, d\tau \leq C \|W^{*0} - W^{*e}\|_{l,2}^2, \end{aligned} \tag{3.17}$$

and $\sup_{\mathbb{R}^d} |W^*(t) - W^{*e}|$ goes to zero as $t \to \infty$.

In order to establish this result, following Kawashima [Ka84], we restate a local existence theorem, we derive a priori estimates for the local solution of the Cauchy problem and we show that this local solution can be extended indefinitely.

3.3.2. *Local existence*

In this section we restate a local existence theorem due to Kawashima [Ka84]. Local existence is proved for an initial data near the stationary state with a control on the distance between the solution and the constant state.

Theorem 3.5. Assume that Properties (A_1)–(A_4) are satisfied. Let $d \geq 1$ and $l \geq [d/2]+2$ be integers. Let $\mathcal{O}_0$ be a bounded convex open set such that $\overline{\mathcal{O}}_0 \subset \mathcal{O}_{W^*}$, let $d_1 < d\big(\mathcal{O}_0, \partial\mathcal{O}_{W^*}\big)$, let b_0 be positive and let $\mathcal{O}_1 = \{\, z \in \mathbb{R}^n;\ d(z,\mathcal{O}_0) < d_1 \,\}$. Then there exists a positive constant T_1, depending only on $\mathcal{O}_0$, d_1 and b_0, such that for any $W^{*0}(x)$ with $W^{*0} - W^{*e} \in W_2^l(\mathbb{R}^d)$

$$\|W^{*0} - W^{*e}\|_{l,2} \leq b_0, \qquad\qquad W^{*0}(x) \in \mathcal{O}_0, \quad x \in \mathbb{R}^d, \tag{3.18}$$

the system (3.14) with initial condition W^{*0} has a unique solution W^* satisfying

$$\begin{cases} W_{\scriptscriptstyle I}^* - W_{\scriptscriptstyle I}^{*e} \in C^0\big([0,T_1]; W_2^l(\mathbb{R}^d)\big) \cap C^1\big([0,T_1]; W_2^{l-1}(\mathbb{R}^d)\big) \cap L^2\big(0,T_1; W_2^l(\mathbb{R}^d)\big), \\ W_{\scriptscriptstyle II}^* - W_{\scriptscriptstyle II}^{*e} \in C^0\big([0,T_1]; W_2^l(\mathbb{R}^d)\big) \cap C^1\big([0,T_1]; W_2^{l-2}(\mathbb{R}^d)\big) \cap L^2\big(0,T_1; W_2^{l+1}(\mathbb{R}^d)\big), \end{cases} \tag{3.19}$$

and $W^*(t,x) \in \mathcal{O}_1$, for $(t,x) \in [0,T_1] \times \mathbb{R}^d$. This solution also satisfies the estimate, for $t \in [0,T_1]$

$$N_l(t) \leq C_l \|W^{*0} - W^{*e}\|_{l,2}, \tag{3.20}$$

where $C_l > 1$ is a constant depending only on $\mathcal{O}_0$, d_1, and b_0, where we have defined $N_l(t) = N_l(0,t)$, and

$$N_l(t_1,t_2)^2 = \sup_{t_1 \leq \tau \leq t_2} \|W^*(\tau) - W^{*e}\|_{l,2}^2 + \int_{t_1}^{t_2} \Big(\|\partial_x W_{\scriptscriptstyle I}^*(\tau)\|_{l-1,2}^2 + \|\partial_x W_{\scriptscriptstyle II}^*(\tau)\|_{l,2}^2 \Big)\, d\tau. \tag{3.21}$$

This theorem is essentially obtained by first considering linear equations and establishing a priori estimates. Successive iterations are then shown to be convergent and we refer to Kawashima [Ka84] for more details.

3.3.3. *A priori estimates*

We first remark that the norm $N_l(T)$ can be used is order to control the solution in a given neighborhood of W^{*e} in $\mathcal{O}_{W^*}$.

Lemma 3.6. Let $l \geq [d/2]+1$ and let $\mathcal{B}$ be a bounded neighborhood of W^{*e} in $\mathcal{O}_{W^*}$. Then there exists a constant $\beta_0(\mathcal{B})$ such that if the solution W^* satisfies $N_l(T) \leq \beta_0(\mathcal{B})$, then $W^*(t,x) \in \mathcal{B}$ for any $(t,x) \in [0,T] \times \mathbb{R}^d$. In particular, when $N_l(T) \leq \beta_0(\mathfrak{O})$, then inequality (3.8) of Proposition 3.1 and the estimate of (D_4) hold.

We now want to estimate the quantity $N_l(T)$ when it satisfies a smallness assumption. We first restate a result of Kawashima concerning the norms of higher derivatives of $W_{\scriptscriptstyle I}^*$ and $W_{\scriptscriptstyle II}^*$. In the next lemmas, we complete this estimate by considering lower derivatives. These estimates involve the orthogonal projector $\mathcal{P}$ onto the range of $\overline{L}^*(W^{*e})$. The following result has been established by Kawashima [Ka84] and is given here without proof.

Lemma 3.7. Assume that Properties (A_1)–(A_4) and (D_1)–(D_2) hold. Let $d \geq 1$ and $l \geq [d/2]+2$ be integers and let the initial data satisfy $W^{*0} - W^{*e} \in W_2^l(\mathbb{R}^d)$. Let b be positive and W^* be a solution of (3.14) (3.15) satisfying

$$\begin{cases} W_{_I}^* - W_{_I}^{*e} \in C^0\big([0,T]; W_2^l(\mathbb{R}^d)\big) \cap C^1\big([0,T]; W_2^{l-1}(\mathbb{R}^d)\big) \cap L^2\big(0,T; W_2^l(\mathbb{R}^d)\big), \\ W_{_{II}}^* - W_{_{II}}^{*e} \in C^0\big([0,T]; W_2^l(\mathbb{R}^d)\big) \cap C^1\big([0,T]; W_2^{l-2}(\mathbb{R}^d)\big) \cap L^2\big(0,T; W_2^{l+1}(\mathbb{R}^d)\big), \end{cases}$$

and $N_l(T) \leq b$. Then there exists a constant $C_{g1} = C_{g1}(b)$ such that the following estimate holds for $t \in [0,T]$

$$\|\partial_x W^*(t)\|_{l-1,2}^2 + \int_0^t \Big(\|\mathcal{P}\partial_x W^*(\tau)\|_{l-1,2}^2 + \|\partial_x^2 W_{_{II}}^*(\tau)\|_{l-1,2}^2 + \|\partial_x^2 W_{_I}^*(\tau)\|_{l-2,2}^2 \Big)\, d\tau$$
$$\leq C_{g1}\Big(\|\partial_x W^{*0}\|_{l-1,2}^2 + N_l(T)^3 \Big). \tag{3.22}$$

We next estimate $\|W^* - Wseq\|_{0,2}$ and $\int_0^t (\|\partial_x W_{_{II}}^*(\tau)\|_{0,2}^2 + \|\overline{\Omega}^*(\tau)\|_{0,2}^2)\, d\tau$ by means of the modified entropy function $\mathfrak{h}$.

Proposition 3.8. Keep the hypotheses of Lemma 3.7 and assume also that (D_4) holds and that $b \leq \beta_0(\mathfrak{O})$. Then there exists a positive constant $C_{g2} = C_{g2}(b)$ such that the following estimate holds for $t \in [0,T]$

$$\|W^*(t) - W^{*e}\|_{0,2}^2 + \int_0^t \Big(\|\partial_x W_{_{II}}^*(\tau)\|_{0,2}^2 + \|\overline{\Omega}^*(\tau)\|_{0,2}^2 \Big)\, d\tau$$
$$\leq C_{g2}\Big(\|W^{*0} - W^{*e}\|_{0,2}^2 + N_l(T)^3 \Big). \tag{3.23}$$

Proof. Integrating the conservation equation (3.9) for $\mathfrak{h}$ over $[0,T]\times\mathbb{R}^d$ yields

$$\int_{\mathbb{R}^d} \mathfrak{h}(t)\, dx - \int_{\mathbb{R}^d} \mathfrak{h}(0)\, dx = -\sum_{i,j\in C^*} \int_0^t\!\!\int_{\mathbb{R}^d} \langle \partial_i V^*, \widetilde{G}_{ij}^* \partial_j V^* \rangle\, dx d\tau$$
$$+ \int_0^t\!\!\int_{\mathbb{R}^d} \langle V^* - V^{*e}, \widetilde{\Omega}^*(V^*) \rangle\, dx d\tau. \tag{3.24}$$

Using inequality (3.8) and (D_4) we then obtain

$$\|W^*(t) - W^{*e}\|_{0,2}^2 + \sum_{i,j\in C^*} \int_0^t\!\!\int_{\mathbb{R}^d} \langle \partial_i V^*, \widetilde{G}_{ij}^* \partial_j V^* \rangle\, dx d\tau$$
$$+ \int_0^t\!\!\int_{\mathbb{R}^d} \big| \widetilde{\Omega}^*(V^*) \big|^2\, dx d\tau \;\leq\; C\|W^{*0} - W^{*e}\|_{0,2}^2.$$

From Property (N_1) we have

$$\langle \partial_i V^*, \widetilde{G}^*_{ij} \partial_j V^* \rangle = \langle \partial_i V^{*\prime}_{II}, \widetilde{G}^{*\prime II,II}_{ij} \partial_j V^{*\prime}_{II} \rangle,$$

since $V^{*\prime} = P^{-1} V^*$ and $\widetilde{G}^{*\prime}_{ij} = P^t \widetilde{G}^*_{ij} P$, and there exists constants c and C such that

$$\sum_{i,j \in C^*} \int_0^t \!\! \int_{\mathbb{R}^d} \langle \partial_i V^{*\prime}_{II}, \widetilde{G}^{*\prime II,II}_{ij} \partial_j V^{*\prime}_{II} \rangle \, dx d\tau \;\geq\; c \int_0^t \!\! \int_{\mathbb{R}^d} |\partial_x V^{*\prime}_{II}|^2 \, dx d\tau \;-\; C\, N_l(t)^3,$$

since $\sum_{i,j \in C^*} \widetilde{G}^{*\prime II,II}_{ij}(W^{*e}) w_i w_j$ is positive definite. The proof is then complete since $W^*_{II} = \varphi_{II}(V^{*\prime}_{II})$ where φ_{II} is a diffeomorphism of $\mathbb{R}^{n-n_0}$ and $\overline{\Omega}^* = (\partial_{W^*} V^*)^t \, \widetilde{\Omega}^*$. $\square$

3.3.4. *More a priori estimates*

We now focus on a priori estimates concerning $\int_0^t \|\partial_x W^*_I(\tau)\|^2_{0,2} d\tau$.

Lemma 3.9. Keep the assumptions of Lemma 3.7. Then there exists a positive constant $C_{g3} = C_{g3}(b)$ such that the following estimate holds for $t \in [0,T]$

$$\begin{aligned}
&\int_0^t \|\partial_x W^*_I(\tau)\|^2_{0,2} d\tau - C_{g3} \Big(\|W^*(t) - W^{*e}\|^2_{1,2} + \int_0^t \|\partial_x \mathcal{P} W^*(\tau)\|^2_{0,2} \, d\tau \\
&+ \int_0^t \|\partial_x W^*_{II}(\tau)\|^2_{1,2} \, d\tau + \int_0^t \|\overline{\Omega}^*(\tau)\|^2_{0,2} \, d\tau \Big) \leq C_{g3} \Big(\|W^{*0} - W^{*e}\|^2_{1,2} + N_l(T)^3 \Big).
\end{aligned} \tag{3.25}$$

Proof. We rewrite the system (3.14) in the form

$$\begin{aligned}
\overline{A}_0(W^{*e}) \partial_t W^* + \sum_{i \in C} \overline{A}^*_i(W^{*e}) \partial_i W^* - \sum_{i,j \in C} \overline{G}^*_{ij}(W^{*e}) \partial_i \partial_j W^* \\
- \overline{A}^*_0(W^{*e}) \big(\overline{A}^*_0(W^*)\big)^{-1} \overline{\Omega}^* = h,
\end{aligned} \tag{3.26}$$

where

$$\begin{aligned}
h = &\sum_{i \in C^*} \overline{A}^*_0(W^{*e}) \Big[\big(\overline{A}^*_0(W^{*e})\big)^{-1} \overline{A}_i(W^{*e}) - \big(\overline{A}^*_0(W^*)\big)^{-1} \overline{A}^*_i(W^*) \Big] \partial_i W^* \\
&- \sum_{i,j \in C^*} \overline{A}^*_0(W^{*e}) \Big[\big(\overline{A}^*_0(W^{*e})\big)^{-1} \overline{G}^*_{ij}(W^{*e}) - \big(\overline{A}^*_0(W^*)\big)^{-1} \overline{G}_{ij}(W^*) \Big] \partial_i \partial_j W^* \\
&+ \overline{A}^*_0(W^{*e}) \big(\overline{A}^*_0(W^*)\big)^{-1} \Big(\sum_{i,j \in C} \partial_{W^*} \overline{G}^*_{ij}(W^*) \partial_i W^* \partial_j W^* + \overline{H}^* \Big)
\end{aligned}$$

Multiplying this equation by K^j, taking the scalar product with $\partial_j W^*$, summing with respect to j, and integrating over $[0,t] \times \mathbb{R}^d$ yields the estimate

$$\int_0^t \|\partial_x W^*_I(\tau)\|^2_{0,2} d\tau - C_{g3} \Big(\|W^*(t) - W^{*e}\|^2_{1,2} + \int_0^t \|\partial_x \mathcal{P} W^*(\tau)\|^2_{0,2} \, d\tau$$

$$+\int_0^t \Big(\|\partial_x W_{II}^*(\tau)\|_{1,2}^2 + \|\overline{\Omega}^*(\tau)\|_{0,2}^2 \Big)\, d\tau \Big) \le C_{g3} \Big(\|W^{*0} - W^{*e}\|_{1,2}^2 + \int_0^t \mathcal{T}(\tau)\, d\tau \Big),$$

where

$$\mathcal{T}(t) = \sum_{i\in C^*} \int_{\mathbb{R}^d} \langle \partial_j W^*, K^j h \rangle \, dx,$$

thanks to the skew-symmetry of $K^j \overline{A}_0^*(W^{*e})$ which implies that

$$\Big\langle \partial_j W^*,\, K^j \overline{A}_0^*(W^{*e}) \partial_t W^* \Big\rangle = \tfrac{1}{2}\partial_t \Big\langle \partial_j W^*, K^j \overline{A}_0^*(W^{*e})(W^* - W^{*e}) \Big\rangle$$

$$+\ \tfrac{1}{2}\partial_j \Big\langle (W^* - W^{*e}), K^j \overline{A}_0^*(W^{*e}) \partial_t W^* \Big\rangle,$$

to Property (D_2) and the structure of $\overline{G}^*(W^{*e})_{ij}$, $i,j \in C^*$, and $\overline{L}^*(W^{*e})$ which yield that

$$\sum_{i,j\in C^*} \Big\langle \partial_j W^*, K^j \overline{A}_i^*(W^{*e}) \partial_i W^* \Big\rangle \ge c|\partial_x W^*|^2 - C\Big(|\partial_x W_{II}^*|^2 + |\partial_x \mathcal{P} W^*|^2 \Big),$$

to the estimates, for $j \in C$,

$$\int_0^t \!\! \int_{\mathbb{R}^d} \Big\langle \partial_j W^*, K^j \overline{A}_0^*(W^{*e}) \big(\overline{A}_0^*(W^*)\big)^{-1} \overline{\Omega}^* \Big\rangle^2 \, dx d\tau$$

$$\le \epsilon \int_0^t \|\partial_x W^*(\tau)\|_{0,2}^2 \, d\tau + (1/\epsilon) \int_0^t \|\overline{\Omega}^*(\tau)\|_{0,2}^2 \, d\tau,$$

and to similar estimates for the terms associated with second derivatives. We further deduce, from the expression of $\mathcal{T}$ that

$$\mathcal{T}(t) = O\Big(\int_{\mathbb{R}^d} \big(|\partial_x W^*|^3 + |W^* - W^{*e}|\, |\partial_x W^*|^2 + |W^* - W^{*e}|\, |\partial_x W^*|\, |\partial_x^2 W_{II}^*| \big)\, dx \Big),$$

and, therefore, we have

$$\int_0^t \mathcal{T}(t)\, dt \le C N_l(t)^3, \tag{3.27}$$

and the proof is complete. □

We can now combine the preceding lemmas in order to obtain a priori estimates for $N_l(T)$.

Proposition 3.10. Keep the hypotheses of Lemma 3.7 and assume also that (D_4) holds. Then there exists a positive constant $C_g = C_g(\beta_0(\mathfrak{O}))$, depending only on $\beta_0(\mathfrak{O})$, such that the following a priori estimate holds for $t \in [0, T]$

$$N_l(t) \le C_g \|W^{*0} - W^{*e}\|_{l,2}. \tag{3.28}$$

Proof. Letting $b = \beta_0(\mathfrak{O})$ in Lemma 3.7, Proposition 3.8 and Lemma 3.9 yields the inequality

$$N_l(T)^2 \le C_{g4}\left(\|W^{*0} - W^{*e}\|_{l,2}^2 + N_l(T)^3\right). \tag{3.29}$$

As a consequence, for $N_l(T) < 1/(2C_{g4})$, we obtain the estimate (3.28). □

3.3.5. *Global existence proof*

We now use repeatedly the local existence theorem and the a priori estimates [Ka84]. We apply Theorem 3.5 with $\mathcal{O}_0 = \mathfrak{O}$, $d_1 < d(\mathfrak{O}, \partial\mathcal{O}_{W^*})$, and $b_0 = \beta_0(\mathfrak{O})$, and we assume that W^{*0} satisfies

$$\|W^{*0} - W^{*e}\|_{l,2} \le b_1 = \min\left(\frac{\beta_0(\mathfrak{O})}{C_l}, \frac{\beta_0(\mathfrak{O})}{C_g(1+C_l^2)^{\frac{1}{2}}}\right). \tag{3.30}$$

The solution is then defined on $[0, T_1]$ and from (3.20) we obtain that $N_l(t) \le C_l b_1 \le \beta_0(\mathfrak{O})$ over $[0, T_1]$. As a consequence, the estimates obtained in Proposition 3.10 hold and $N_l(T_1) \le C_g b_1$. We then apply again Theorem 3.5 on $[T_1, 2T_1]$, with T_1 as a new initial time, still with $\mathcal{O}_0 = \mathfrak{O}$, and with the same d_1 and b_0. The solution on $[T_1, 2T_1]$ then satisfies the estimate $N_l(T_1, 2T_1) \le C_l N_l(T_1)$. As a consequence, we obtain

$$N_l(2T_1) \le \left(N_l(T_1)^2 + N_l(T_1, 2T_1)^2\right)^{\frac{1}{2}} \le (1 + C_l^2)^{\frac{1}{2}} N_l(T_1),$$

so that

$$N_l(2T_1) \le (1 + C_l^2)^{\frac{1}{2}} C_g b_1 \le \beta_0(\mathfrak{O}).$$

The estimates of Proposition 3.10 are therefore again valid on $[0, 2T_1]$ and we can again use Theorem 3.5 on $[2T_1, 3T_1]$. An easy induction then yields global existence and $N_l(t) \le C_g\|W^{*0} - W^{*e}\|_{l,2}$ for $t \ge 0$.

Defining then $\Phi(t) = \|\partial_x W^*(t)\|_{l-2,2}^2$, it follows from the a priori estimates and the relations (3.14) that

$$\int_0^\infty |\Phi(t)|\, dt + \int_0^\infty |\partial_t \Phi(t)|\, dt \;\le\; C\|W^{*0} - W^{*e}\|_{l,2}^2,$$

which shows that $\Phi(t) \to 0$ as $t \to \infty$. From the interpolation inequality

$$\sup_{\mathbb{R}^d} |W^*(t) - W^{*e}| \;\le\; C\,\|\partial_x^{l-1} W^*(t)\|_{0,2}^a\, \|W^*(t)\|_{0,2}^{1-a},$$

with $a = d/\big(2(l-1)\big) > 0$, we then deduce that $\sup_{\mathbb{R}^d} |W^*(t) - W^{*e}| \to 0$ as $t \to \infty$.

3.4. Asymptotic decay

In this section we establish decay estimates towards the equilibrium state.

Theorem 3.11. Assume that conditions (A_1)–(A_4) and (D_1)–(D_4) hold. Let $d \geq 1$ and $l \geq [d/2]+3$ be integers and let $W^{*0}(x)$ be such that

$$W^{*0} - W^{*e} \in W_2^l(\mathbb{R}^d). \tag{3.31}$$

Assume that $\|W^{*0} - W^{*e}\|_{l,2}$ is small enough so that there exists a unique global solution to the Cauchy problem from Theorem 3.5. Assume further that $W^{*0} - W^{*e} \in W_2^l(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, where $p=1$, if $d=1$, and $p \in [1,2)$, if $d \geq 2$. Defining for $\tau \leq l$

$$\|W^{*0} - W^{*e}\|_{\tau,2|p} = \|W^{*0} - W^{*e}\|_{\tau,2} + \|W^{*0} - W^{*e}\|_{0,p}, \tag{3.32}$$

then, if $\|W^{*0} - W^{*e}\|_{l,2|p}$ is small enough, the unique global solution to the Cauchy problem satisfies the decay estimate

$$\|W^*(t) - W^{*e}\|_{l-2,2} \leq C(1+t)^{-\gamma}\|W^{*0} - W^{*e}\|_{l-2,2|p}, \tag{3.33}$$

for $t \in [0,\infty)$, where C a positive constant and where $\gamma = d \times (1/2p - 1/4)$.

Proof. We consider equation (3.1) that we rewrite in the form

$$\begin{aligned}
\partial_t U^* + \sum_{i\in C^*} A_i^*(U^{*e})\partial_i U^* - \sum_{i,j\in C^*} G_{ij}^*(U^{*e})\partial_i\partial_j U^* - \partial_{U^*}\Omega^*(U^{*e})\,(U^* - U^{*e}) \\
= \sum_{i\in C^*} \partial_i\Big(F_i^*(U^{*e}) - F_i^*(U^*) + A_i^*(U^{*e})(U^* - U^{*e})\Big) \\
+ \sum_{i,j\in C^*} \partial_i\left(G_{ij}^*(U^*)\partial_j U^* - G_{ij}^*(U^{*e})\partial_j U^*\right) + \Omega^* - \partial_{U^*}\Omega^*(U^{*e})(U^* - U^{*e}).
\end{aligned} \tag{3.34}$$

Introducing the new variable

$$z(U^*) = \left(\partial_{U^*} W^*(U^{*e})\right)(U^* - U^{*e}), \tag{3.35}$$

we obtain from (3.34) that

$$\overline{A}_0^*(W^{*e})\partial_t z + \sum_{i\in C^*} \overline{A}_i^*(W^{*e})\partial_i z - \sum_{i,j\in C^*} \overline{G}_{ij}^*(W^{*e})\partial_i\partial_j z + \overline{L}^* z \;=\; h^1 + h^2, \tag{3.36}$$

and $z(0,x) = z(U^{*0})$. The first nonlinear term h^1 is in conservative form and reads

$$h^1 = \sum_{i\in C^*} \partial_i\,(b_i + b_i') \tag{3.37}$$

where

$$b_i = \left(\partial_{W^*} V^*(W^{*e})\right)^t \left(F_i^*(W^{*e}) - F_i^*(W^*)\right) + \overline{A}_i^*(W^{*e})\, z, \tag{3.38}$$

and

$$b_i' = \sum_{j\in C^*} \Big(\big(\partial_{W^*} V^*(W^{*e})\big)^t \, G_{ij}^*(U^*) \partial_{W^*} U^*(W^{*e}) - \overline{G}_{ij}^*(W^{*e}) \Big) \, \partial_j z. \tag{3.39}$$

On the other hand, the second term h^2 reads

$$h^2 = \big(\partial_{W^*} V^*(W^{*e})\big)^t \, \widetilde{\Omega}^* \; + \; \overline{L}^*(W^{*e}) \, z,$$

and lies in the subspace $R\big(\overline{L}^*(W^{*e})\big)$ thanks to (D_3), so that we have

$$(I - \mathcal{P}) \, h^2 = 0. \tag{3.40}$$

We now introduce the symbol associated with the linear part of (3.36)

$$\begin{aligned} \overline{S}(\xi) = \overline{A}_0^*(W^{*e})^{-1/2} \Big\{ \mathrm{i} \sum_{j\in C^*} \overline{A}_j^*(W^{*e}) \xi_j + \sum_{i,j\in C^*} \overline{G}_{ij}^*(W^{*e}) \xi_i \xi_j \\ + \; \overline{L}^*(W^{*e}) \Big\} \overline{A}_0^*(W^{*e})^{-1/2}, \end{aligned}$$

and we define for $\phi \in L^2(\mathbb{R}^d)$

$$\exp(-t\overline{S})\phi(x) \;=\; \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(\mathrm{i}x\cdot\xi) \, \exp\big(-t\overline{S}(\xi)\big) \, \hat{\phi}(\xi) \, d\xi,$$

where $\hat{\phi}$ is the Fourier transform of ϕ. We can then express the solution of system (3.34) by the relation [Ka84]

$$z(t) = \exp(-t\overline{S})z(0) + \int_0^t \exp(-(t-\tau)\overline{S}) h^1(\tau) \, d\tau + \int_0^t \exp(-(t-\tau)\overline{S}) h^2(\tau) \, d\tau, \tag{3.41}$$

and we estimate $\|z(t)\|_{l-2,2}$ in two steps. Defining $\mathfrak{z}(t)$ by

$$\mathfrak{z}(t) = \exp(-t\overline{S})z(0) + \int_0^t \exp(-(t-\tau)\overline{S}) h^2(\tau) d\tau, \tag{3.42}$$

we first deduce from Proposition 3.12 of [Ka84] the estimate

$$\|\mathfrak{z}(t)\|_{l-2,2} \leq C(1+t)^{-\gamma} \|z(0)\|_{l-2,2|p}, \qquad t \geq 0, \tag{3.43}$$

where $\gamma = d \times (1/2p - 1/4)$ and C denotes a positive constant, since the nonlinear term h^2 satisfies $(I - \mathcal{P}) \, h^2 = 0$.

On the other hand, by using the estimate 3.A.14 of [Ka84] or Theorem 1.2 of [SK85], we also obtain that

$$\| \int_0^t \exp(-(t-\tau)\overline{S}) h^1(\tau) \, d\tau \|_{l-2,2} \;\leq\; C \int_0^t \exp(-c(t-\tau)) \|h^1(\tau)\|_{l-2,2} \, d\tau$$

$$+C\int_0^t (1+t-\tau)^{-d/4-1/2} \| \sum_{i\in C^*} (b_i + b_i')\,(\tau)\|_{0,1}\, d\tau, \tag{3.44}$$

where c and C are positive constants. Since the initial data (3.15) is supposed to be sufficiently near W^{*e} in $W_2^l(\mathbb{R}^d)\cap L^p(X)$, the a priori estimates show that $\|z\|_{l,2}$ remains small for any time and we can write

$$b_i = O(|z|^2), \tag{3.45}$$

$$b_i' = (\partial_i z)\, O(|z|), \tag{3.46}$$

and we thus obtain that for $l \geq [d/2]+3$, and for $\|z\|_{l,2}$ sufficiently small, we have

$$\sum_{i\in C^*} \Big(\|b_i\|_{0,1} + \|b_i'\|_{0,1} \Big) \leq C\|z\|_{1,2}^2, \tag{3.47}$$

$$\sum_{i\in C^*} \Big(\|\partial_i b_i\|_{l-2,2} + \|\partial_i b_i'\|_{l-2,2} \Big) \leq C\|z\|_{l-2,2}\|z\|_{l,2}. \tag{3.48}$$

Combining inequalities (3.41) (3.43) (3.44) (3.47) and (3.48) finally yields

$$\begin{aligned} \|z(t)\|_{l-2,2} \;&\leq\; C(1+t)^{-\gamma}\|z(0)\|_{l-2,2|p} \\ &+ C\int_0^t \exp\big(-c(t-\tau)\big)\|z(\tau)\|_{l,2}\|z(\tau)\|_{l-2,2}\, d\tau + \\ &+ C\int_0^t (1+t-\tau)^{-d/4-1/2}\|z(\tau)\|_{1,2}^2\, d\tau. \end{aligned} \tag{3.49}$$

Defining then

$$|||z(t)|||_{l-2,\gamma} = \sup_{0\leq\tau\leq t} (1+\tau)^{\gamma}\|z(\tau)\|_{l-2,2}, \tag{3.50}$$

we obtain

$$|||z(t)|||_{l-2,\gamma} \leq C\|z(0)\|_{l-2,2|p} + C\mu_3(t)\|z(0)\|_{l,2}|||z(t)|||_{l-2,\gamma} + C\mu_4(t)|||z(t)|||_{l-2,\gamma}^2,$$

where

$$\mu_3(t) = \sup_{0\leq\tau\leq t} (1+\tau)^{\gamma} \int_0^{\tau} \exp\big(-c(\tau-\tau_1)\big)(1+\tau_1)^{-\gamma}\, d\tau_1,$$

and

$$\mu_4(t) = \sup_{0\leq\tau\leq t} (1+\tau)^{\gamma} \int_0^{\tau} (1+\tau-\tau_1)^{-d/4-1/2}(1+\tau_1)^{-2\gamma}\, d\tau_1.$$

Since $\mu_3(t)$ and $\mu_4(t)$ are uniformly bounded with respect to t, we obtain the desired estimate for $|||z(0)|||_{l-2,\gamma}$ small enough and this completes the proof. $\square$

4. Application to Multicomponent Reactive Flows

In this section we apply theorems 3.4 and Theorem 3.11 to the multicomponent reactive flow governing equations (2.1). To this purpose, we have to establish that Properties (A_1)–(A_4) and (D_1)–(D_4) are satisfied. These will be obtained by successively investigating normal forms of the governing equations, equilibrium points associated with Maxwellian chemistry and linearized equations around constant equilibrium states. We will use the normal variable W introduced in Theorem 3.14 of [GM96$_{\rm I}$] but the natural normal variable $\widehat{W}$ could be used as well.

4.1. Normal forms

Symmetrizability and normal forms associated with the multicomponent reactive flows governing equations have been investigated in [GM96$_{\rm I}$] and Properties (A_1), (A_2), and (A_3) have been established. More specifically, (A_1) and (A_2) are consequences of Theorem 3.5 of [GM96$_{\rm I}$] whereas (A_3) is a consequence of Proposition 3.12 of [GM96$_{\rm I}$].

The system governing reactive mixtures has also been recast into a normal form in Theorem 3.14 of [GM96$_{\rm I}$], by considering the variable

$$W = \big(\rho, \log(\rho_2^{r_2}/\rho_1^{r_1}), \dots, \log(\rho_{n_S}^{r_{n_S}}/\rho_1^{r_1}), v_1, v_2, v_3, T\big)^t,$$

on the open convex set $\mathcal{O}_W = (0,\infty) \times \mathbb{R}^{n_S-1} \times \mathbb{R}^3 \times (T_0,\infty)$. The corresponding equations are in the form

$$\overline{A}_0^{I,I}\partial_t W_I \quad + \textstyle\sum_{i\in C} \overline{A}_i^{I,I}\partial_i W_I + \sum_{i\in C} \overline{A}_i^{I,II}\partial_i W_{II} = 0, \tag{4.1}$$

$$\begin{aligned}\overline{A}_0^{II,II}\partial_t W_{II} + \textstyle\sum_{i\in C} \overline{A}_i^{II,I}\partial_i W_I + \sum_{i\in C} \overline{A}_i^{II,II}\partial_i W_{II} = \\ \sum_{i,j\in C} \partial_i\Big(\overline{G}_{ij}^{II,II}\partial_j W_{II}\Big) + \overline{H}_{II} + \overline{\Omega}_{II},\end{aligned} \tag{4.2}$$

where the matrix coefficients are detailed in [GM96$_{\rm I}$] and the corresponding source term is given in (2.14).

4.2. Equilibrium points

In this section we establish the existence of constant equilibrium states when the source terms are taken as in (H_{10})–(H_{13}). We restate existence and uniqueness of an equilibrium density vector at a fixed temperature in a given affine submanifold of element conservation, as well as detailed balance at equilibrium [Kr70] [SS65].

Let us introduce the vector ϱ of species densities

$$\varrho = (\rho_1, \dots, \rho_{n_S})^t, \tag{4.3}$$

and the vector $\mathcal{Y}$ of reduced chemical potential

$$\mathcal{Y} = \frac{1}{R_{\rm g}T}(\mu_1, \ldots, \mu_{n_S})^t, \tag{4.4}$$

which coincides with the first n_S components of V up to the scaling factor $R_{\rm g}$.

Proposition 4.1. The reduced entropy production due to chemical reactions

$$\sigma(\rho_1 \ldots, \rho_{n_S}, T) = \langle \mathcal{Y}, M\omega \rangle = \frac{1}{R_{\rm g}T} \sum_{k \in S} \mu_k m_k \omega_k,$$

defined on $(0,\infty)^{n_S} \times (T_0, \infty)$, is nonpositive and admits 0 as a maximum at equilibrium points. Any equilibrium point $(\rho_1^e, \ldots, \rho_{n_S}^e, T^e)^t$ in $(0,\infty)^{n_S} \times (T_0, \infty)$, that is, any point $(\rho_1^e, \ldots, \rho_{n_S}^e, T^e)^t$ where the source terms vanish

$$\omega_k(\rho_1^e, \ldots, \rho_{n_S}^e, T^e) = 0, \qquad k \in S, \tag{4.5}$$

is also such that the rates of progress of each reaction vanish

$$\tau_i(\rho_1^e, \ldots, \rho_{n_S}^e, T^e) = 0, \qquad i \in R, \tag{4.6}$$

which can also be written in the form

$$\Big\langle \mathcal{Y}(\rho_1^e, \ldots, \rho_{n_S}^e, T^e), M\nu_i \Big\rangle = 0, \qquad i \in R. \tag{4.7}$$

Proof. Some straightforward calculation yields that

$$\tau_i = \Delta_i \Big(1 - \exp\big\langle \mathcal{Y}, M\nu_i \big\rangle\Big), \tag{4.8}$$

where

$$\Delta_i = K_{fi} \prod_{k \in S} \gamma_k^{\nu'_{ki}}, \qquad i \in R, \tag{4.9}$$

thanks to the expression of the equilibrium constant K_{ei}. As a consequence, we have

$$\sigma = \sum_{i \in R} \langle \mathcal{Y}, M\nu_i \rangle\, \tau_i = \sum_{i \in R} \Delta_i \langle \mathcal{Y}, M\nu_i \rangle \Big(1 - \exp\big\langle \mathcal{Y}, M\nu_i \big\rangle\Big), \tag{4.10}$$

and, since Δ_i is positive, we deduce that $\sigma(\varrho, T) \le 0$ and that $\sigma(\varrho, T) = 0$ if and only if $\langle \mathcal{Y}, M\nu_i \rangle = 0$, $i \in R$, that is to say, if and only if $\tau_i = 0$, $i \in R$. In addition, $\tau_i = 0$, $i \in R$ if and only if $\omega_k = 0$, $k \in S$, from the expression of σ. $\square$

Proposition 4.2. Consider a temperature $T^e \in (T_0, \infty)$, a point ϱ_c of $(0,\infty)^{n_S}$ and the associated reaction simplex $(\varrho_c + M\mathcal{R}) \cap (0,\infty)^{n_S}$. Assume that Properties (H_{10})–(H_{13}) hold. Then there exists a unique equilibrium state ϱ^e in the reaction simplex $(\varrho_c + M\mathcal{R}) \cap (0,\infty)^{n_S}$ where the source term ω vanishes, and, therefore, where the reaction rates of progress also vanish. Finally, as a function of ϱ, the entropy production due to chemical reactions admits 0 as a strict maximum at ϱ^e over the reaction simplex $(\varrho_c + M\mathcal{R}) \cap (0,\infty)^{n_S}$.

Proof. We characterize the equilibrium point as the only extremum of the reduced Helmholtz free energy function [Kr70][SS65]

$$\mathfrak{H}(\varrho, T) = \langle \varrho, \mathcal{Y} - M^{-1}\mathcal{U} \rangle,$$

where $\mathcal{U} = (1,\dots,1)^t$. The function $\mathfrak{H}$ is a C^∞ strictly convex function of ϱ in $(0,\infty)^{n_S}$ at a fixed temperature $T = T^e$, and is given by

$$\mathfrak{H}(\varrho, T) = \frac{1}{R_{\mathrm{g}}} \sum_{k\in S} r_k \rho_k \big(\log \rho_k - 1 + a_k(T^e)\big),$$

where $a_k(T) = \mu_k(1,T)/r_kT$. The partial derivative with respect to the mass density vector ϱ is also given by

$$\partial_\varrho \mathfrak{H} = \mathcal{Y}.$$

The function $\mathfrak{H}$ is easily extended over the closure of $(\varrho_c + M\mathcal{R}) \cap (0,\infty)^{n_S}$ into a continuous function by using $0 \log 0 = 0$, and, therefore, $\mathfrak{H}$ admits a minimum on this convex compact set. Note that the boundedness of the reaction simplex is a direct consequence of $\mathcal{U} \in (M\mathcal{R})^\perp$, that is, of mass conservation. This minimum cannot be reached at the boundaries as easily checked by inspecting the sign of the normal derivative. As a consequence, it is reached in the interior, and, thanks to the strict convexity of $\mathfrak{H}$, this mimimum is unique. Since the minimum is reached in the interior of the reaction simplex, we also have

$$\mathcal{Y}^e = \mathcal{Y}(\varrho^e, T^e) \in (M\mathcal{R})^\perp. \tag{4.11}$$

As a consequence, $\sigma = \langle \mathcal{Y}, M\omega \rangle$ vanishes at (ϱ^e, T^e) which is therefore an equilibirum point. Conversely, from (4.10), any equilibrium point on the reaction simplex is such that the quantities $\langle \mathcal{Y}, M\nu_i \rangle$, $i \in R$, vanish so that the partial derivatives of $\mathfrak{H}$ along the reaction simplex are zero. Since $\mathfrak{H}$ is a strictly convex function, it reaches a minimum at this point. Therefore, this point coincides with the unique minimum of $\mathfrak{H}$ and the proof is complete. □

Noting that $\mathcal{Y}$ coincides with the first n_S components of V up to a positive scaling factor, we obtain, as a direct consequence of the preceding results, the following proposition, which implies Property (A_4).

Proposition 4.3. Given a temperature T^e and a mass density vector ϱ_c, and under assumptions (H_{10})–(H_{13}), there exists a unique constant stationary state $U^e = U(Y^e)$, with $Y^e = (\rho_1^e, \dots, \rho_{n_S}^e, 0, 0, 0, T^e)^t$, such that

$$\Omega(U^e) = 0, \tag{4.12}$$

and $\varrho^e \in (\varrho_c + M\mathcal{R}) \cap (0,\infty)^{n_S}$, $v^e = 0$, and $\mathcal{Y}^e \in (M\mathcal{R})^\perp$.

We will denote by V^e and W^e the equilibrium states in the variables V and W, respectively. In the following proposition, we establish Property (D_4).

Proposition 4.4. Let $U^e = U(Y^e)$, with $Y^e = (\rho_1^e, \ldots, \rho_{n_S}^e, 0, 0, 0, T^e)^t$, be a constant equilibrium state in $\mathcal{O}_U$. Then there exists a neighborhood $\mathfrak{V}$ of V^e such that

$$c\left|\widetilde{\Omega}(V)\right|^2 \;\leq\; -\langle V - V^e, \widetilde{\Omega}(V)\rangle, \qquad V \in \mathfrak{V}, \tag{4.13}$$

where c is a positive constant.

Proof. We first note that we have $V^e \in (M\mathcal{R})^\perp$ from Proposition 4.3. Combining this property with (2.11) we obtain that

$$\langle V - V^e, \widetilde{\Omega}(V)\rangle = \frac{1}{T}\sum_{k\in S}\mu_k m_k \omega_k = R_{\mathrm{g}}\langle \mathcal{Y}, M\omega\rangle. \tag{4.14}$$

From (4.10) we also have

$$\langle \mathcal{Y}, M\omega\rangle = \sum_{i\in R}\Delta_i\langle \mathcal{Y}, M\nu_i\rangle\Big(1 - \exp\langle \mathcal{Y}, M\nu_i\rangle\Big), \tag{4.15}$$

where $\Delta_i = K_{fi}\prod_{k\in S}\gamma_k^{\nu'_{ki}}$, $i \in R$, is a positive function. Denoting by $\mathfrak{V}$ any closed ball centered on V^e and included in $\mathcal{O}_V$, we deduce that for any V in $\mathfrak{V}$

$$c\sum_{i\in R}\langle \mathcal{Y}, M\nu_i\rangle^2 \;\leq\; -\langle \mathcal{Y}, M\omega\rangle \;\leq\; C\sum_{i\in R}\langle \mathcal{Y}, M\nu_i\rangle^2, \tag{4.16}$$

for positive constants c and C depending on $\mathfrak{V}$. On the other hand the relations $M\omega = \sum_{i\in R}\tau_i M\nu_i$ and (4.8) also imply that

$$|M\omega|^2 \;\leq\; C'\sum_{i\in R}\tau_i^2 \;\leq\; C''\sum_{i\in R}\langle \mathcal{Y}, M\nu_i\rangle^2, \tag{4.17}$$

where C' and C'' are positive constants, and combining (4.16) and (4.17) completes the proof. □

4.3. Linearized source term

In this section we describe the linearized chemical source term around an equilibrium state and we establish Properties (D_1) and (D_3).

Proposition 4.5. The linearized source term $\widetilde{L}(V^e) = -(\partial_V\widetilde{\Omega})(V^e)$ around the stationary state V^e is given by

$$\widetilde{L}(V^e) = \begin{pmatrix} (\mathcal{C}_{kl})_{k,l\in S} & 0_{n_S\times 4} \\ & \\ 0_{4\times n_S} & 0_{4\times 4} \end{pmatrix}, \tag{4.18}$$

where

$$\mathcal{C}_{kl} = \sum_{i\in R} \Lambda_i \; m_k \nu_{ki} \; m_l \nu_{li}, \tag{4.19}$$

and

$$\Lambda_i = \frac{K_{fi}(T^e)}{R_{\mathrm{g}}} \prod_{k\in S} (\gamma_k^e)^{\nu'_{ki}} = \frac{K_{ri}(T^e)}{R_{\mathrm{g}}} \prod_{k\in S} (\gamma_k^e)^{\nu''_{ki}}. \tag{4.20}$$

The matrix $\widetilde{L}(V^e)$ can also we written in the form

$$\widetilde{L}(V^e) = \sum_{i\in R} \Lambda_i \; \mathfrak{n}_i \otimes \mathfrak{n}_i,$$

where $\mathfrak{n}_i$ denotes the vector $\mathfrak{n}_i = (M\nu_i, 0_{\mathbb{R}^4})^t$. This matrix $\widetilde{L}(V^e)$ is symmetric positive semi-definite and satisfies

$$R\big(\widetilde{L}(V^e)\big) = M\mathcal{R} \times 0_{\mathbb{R}^4},$$

in such a way that we have $\Omega\big(U(V)\big) = \widetilde{\Omega}(V) \in R\big(\widetilde{L}(V^e)\big)$, for all $V \in \mathcal{O}_V$.

Proof. Evaluating the matrix $\widetilde{L}(V^e)$ is straightforward from (H_{10})–(H_{13}). The expresion (4.19) shows that $\widetilde{L}(V^e)$ is symmetric and yields

$$\big\langle x, \widetilde{L}(V^e)x \big\rangle = \sum_{i\in R} \Lambda_i \big\langle \mathfrak{n}_i, \, x \big\rangle^2 = \sum_{i\in R} \Lambda_i \Big(\sum_{k\in S} x_k m_k \nu_{ki} \Big)^2, \tag{4.21}$$

where $x = (x_1, \dots, x_{n_S+4})^t$, so that $\widetilde{L}(V^e)$ is positive semi-definite. Furthermore, the nullspace of $\widetilde{L}(V^e)$ is also constituted by the vectors orthogonal to the vectors $\mathfrak{n}_i$, $i \in R$, and we thus conclude that $R\big(\widetilde{L}(V^e)\big) = N\big(\widetilde{L}(V^e)\big)^\perp = M\mathcal{R} \times 0_{\mathbb{R}^4}$. $\square$

Property (D_3) is now a direct consequence of Proposition 4.6. In addition, the relation $\overline{\Omega} = (\partial_W V)^t \widetilde{\Omega}$ also yields by linearization that

$$\overline{L}(W^e) = \big(\partial_W V\,(W^e)\big)^t \;\; \widetilde{L}(V^e) \;\; \partial_W V(W^e), \tag{4.22}$$

taking into account that $\widetilde{\Omega}(V^e) = 0$, and, therefore, $\overline{L}(W^e)$ is also symmetric positive semi-definite. Taking into account that (4.1) (4.2) is a normal form, we thus obtain that Property (D_1) is satisfied.

Remark 4.6. The fact that the rank of $\widetilde{L}$ is maximal at V^e shows that the equilibrium point is regular in the sense of Krambeck [Kr70].

4.4. The linearized normal form

If we linearize the symmetric hyperbolic-parabolic system (4.1) (4.2) around this constant stationary state W^e, we obtain the linear symmetric system

$$\overline{A}_0(W^e)\partial_t z + \sum_{i\in C} \overline{A}_i(W^e)\partial_i z = \sum_{i,j\in C} \overline{G}_{ij}(W^e)\partial_i\partial_j z - \overline{L}(W^e)z. \tag{4.23}$$

In this section, we investigate the existence of compensating matrices K^j, $j \in C$.

Proposition 4.7. For a sufficiently small positive α, the matrices K^j, $j \in C$, defined by

$$\sum_{j\in C} \xi_j K^j = \alpha \begin{pmatrix} 0 & 0 & \xi^t & 0 \\ 0 & 0 & 0 & 0 \\ -\xi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \left(\overline{A}_0(W^e)\right)^{-1}, \tag{4.24}$$

where $\xi = (\xi_1, \xi_2, \xi_3)^t$, are compensating matrices. In particular, the products $K^j \overline{A}_0$ are skew-symmetric and the matrix

$$\sum_{i,j\in C} \tfrac{1}{2}\Big(K^i \overline{A}_j(W^e) + \big(K^i \overline{A}_j(W^e)\big)^t\Big) w_i w_j + \overline{B}(W^e, w) + \overline{L}(W^e),$$

is positive definite for $w \in \mathcal{S}^2$.

Proof. It is obvious by construction that the products $K^j \overline{A}_0$, $j \in C$, are skew-symmetric. On the other hand, a direct calculation yields

$$\sum_{i,j\in C} \xi_i K^i \overline{A}_j(W^e) \xi_j = \begin{pmatrix} \dfrac{\alpha T^e}{\Sigma_\rho^e} |\xi|^2 & \dfrac{\alpha T^e}{\Sigma_\rho^e} \mathcal{Z}^t |\xi|^2 & 0 & \alpha r^e |\xi|^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha \rho^e T^e \xi \otimes \xi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.25}$$

where $\mathcal{Z}$ is a vector of $\mathbb{R}^{n_S - 1}$ given by

$$\mathcal{Z}_l = \rho_l^e - \frac{\rho_l^e \, \rho^e}{r_l \, \Sigma_\rho^e}, \qquad l \in [2, n_S],$$

and where the superscript ^{e} indicates that the corresponding quantity is evaluated at W^e. As a consequence, for $\xi \in \mathcal{S}^2$, we have $|\xi| = 1$, and there exists $C \in (0, \infty)$ such that

$$\langle x^t, \sum_{ij\in C} \xi_i K^i \overline{A}_j(W^e) \xi_j x \rangle \;\geq\; \frac{\alpha T^e}{2\Sigma_\rho^e} \Big(x_1^2 - C \sum_{l\in[2, n_S+4]} x_l^2 \Big). \tag{4.26}$$

As a consequence, using Property (N_2), the matrix

$$\sum_{ij\in C} \tfrac{1}{2}\Big(K^i \overline{A}_j(W^e) + \big(K^i \overline{A}_j(W^e)\big)^t\Big) w_i w_j + \overline{B}(W^e, w),$$

is positive definite for $w \in \mathcal{S}^2$ and α sufficiently small. $\square$

Remark 4.8. The normal form in the W variable has particular properties. More specifically, we have the relations

$$K^j \overline{A}_0(W^e) \left(\overline{A}_0(W)\right)^{-1} \overline{\Omega} = 0, \qquad j \in C, \tag{4.27}$$

which can be established by a direct calculation. Indeed, from (2.14) and the expression of $\overline{A}_0$ in Theorem 3.14 of [GM96$_{\rm I}$] we easily get that

$$\big(\overline{A}_0(W)\big)^{-1}\,\overline{\Omega} = R_{\rm g}\Big(0, (\frac{\omega_2}{\rho_2} - \frac{\omega_1}{\rho_1}), \ldots, (\frac{\omega_{n_S}}{\rho_{n_S}} - \frac{\omega_1}{\rho_1}), 0, 0, 0, -\frac{1}{\rho c_v}\sum_{k\in S} e_k m_k \omega_k\Big)^t,$$

and from the sparse structure of $\sum_{j\in C}\xi_j K^j \overline{A}_0(W^e)$ we immediately obtain (4.27). As a consequence, for this particular formulation, the term involving $\|\overline{\Omega}\|_{0,2}$ is not needed in the estimate (3.25).

Remark 4.9. Similarly, with the natural $\widehat{W}$ formulation, one can establish the particular estimate

$$c|\mathcal{P}(\widehat{W} - \widehat{W}^e)| \le |\widehat{\Omega}| \le C|\mathcal{P}(\widehat{W} - \widehat{W}^e)|,$$

where c and C are positive constants. As a consequence, the terms involving the norm $\|\widehat{\Omega}\|_{0,2}$ in a priori estimates can be replaced by the corresponding terms involving the norm $\|\mathcal{P}(\widehat{W} - \widehat{W}^e)\|_{0,2}$.

4.5. Global existence and asymptotic stability for reactive flows

In the previous sections, we have established that Properties (A_1)–(A_4) and (D_1)–(D_4) are satisfied. Therefore, Theorem 3.4 and Theorem 3.11 can be applied to the system (4.1) (4.2) governing multicomponent reactive flows, written in the $W = (W_{I}, W_{II})^t$ variable, with the hyperbolic variable $W_{I} = (\rho)$ and the parabolic variable $W_{II} = \big(\log(\rho_2^{r_2}/\rho_1^{r_1}), \ldots, \log(\rho_{n_S}^{r_{n_S}}/\rho_1^{r_1}), v_1, v_2, v_3, T\big)^t$.

Theorem 4.10. Consider system (4.1) (4.2), let $d \in \{1, 2, 3\}$, $l \ge [d/2] + 2$ and let $W^0(x)$ such that

$$W^0 - W^e \in W_2^l(\mathbb{R}^d).$$

Then, if $\|W^0 - W^e\|_{l,2}$ is small enough, there exists a unique global solution to the the Cauchy problem

$$\overline{A}_0\partial_t W + \sum_{i\in C}\overline{A}_i\partial_i W = \sum_{i,j\in C}\partial_i\Big(\overline{G}_{ij}\partial_j W\Big) + \overline{H} + \overline{\Omega},$$

with initial condition

$$W(0, x) = W^0(x),$$

such that

$$\rho - \rho^e \in C^0\big([0,\infty); W_2^l(\mathbb{R}^d)\big) \cap C^1\big([0,\infty); W_2^{l-1}(\mathbb{R}^d)\big) \cap L^2\big(0,\infty; W_2^l(\mathbb{R}^d)\big),$$

$$W_{II} - W_{II}^e \in C^0\big([0,\infty); W_2^l(\mathbb{R}^d)\big) \cap C^1\big([0,\infty); W_2^{l-2}(\mathbb{R}^d)\big) \cap L^2\big(0,\infty; W_2^{l+1}(\mathbb{R}^d)\big).$$

Furthermore, W satisfies the estimate

$$\|W(t) - W^e\|_{l,2}^2 + \int_0^t \Big(\|\partial_x \rho(\tau)\|_{l-1,2}^2 + \sum_{k\geq 2} \|\partial_x \log(\rho_k^{r_k}/\rho_1^{r_1})(\tau)\|_{l,2}^2 + \|\partial_x v(\tau)\|_{l,2}^2 + \|\partial_x T(\tau)\|_{l,2}^2 \Big)\, d\tau \leq C\|W^0 - W^e\|_{l,2}^2,$$

and $\sup_{x\in\mathbb{R}^d} |W(t) - W^e|$ goes to zero as $t \to \infty$.

In addition, assume that $l \geq [d/2] + 3$ and $W^0 - W^e \in W_2^l(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, with $p = 1$, if $d = 1$, and $p \in [1, 2)$, if $d \geq 2$. Then, if $\|W^0 - W^e\|_{l,2|p}$ is small enough, the unique global solution to Cauchy problem also satisfies the decay estimate

$$\|W(t) - W^e\|_{l-2,2} \leq C(1+t)^{-\gamma}\|W^0 - W^e\|_{l-2,2|p}, \qquad t \in [0,\infty),$$

where C is a positive constant and $\gamma = d \times (1/2p - 1/4)$.

References

[EG94] A. Ern et V. Giovangigli, *Multicomponent transport algorythms*, In **Lectures Notes in Physics**, Springer Verlag, **Vol. m24** (1994).

[GM96$_{\mathrm{I}}$] V. Giovangigli and M. Massot, *Multicomponent Reactive Flows, (I) Symmetrization and Local Existence,* (1996) Submitted for publication.

[Ka84] S. Kawashima, *Systems of hyperbolic-parabolic composite type, with application to the equations of magnetohydrodynamics*, *Doctoral Thesis Kyoto University*, (1984).

[KS88] S. Kawashima and Y. Shizuta, *On the Normal Form of the Symmetric Hyperbolic-Parabolic Systems associated with the Conservation Laws*, *Tôhoku Math. J.*, **40** (1988) 449-464.

[Kr70] F. J. Krambeck, *The mathematical structure of chemical kinetics*, *Arch. Rational Mech. Anal.*, **38** (1970) 317-347.

[Ma96] M. Massot, *Modélisation mathématique et numérique de la combustion des mélanges gazeux*, *Doctoral Thesis Ecole Polytechnique*, in preparation (1996).

[SK85] Y. Shizuta and S. Kawashima, *Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation*, *Hokkaido Math. J.*, **14** (1985) 249-275.

[SS65] N. Z. Shapiro and L. S. Shapley, *Mass Action Law and the Gibbs Free Energy Function*, *SIAM J. Appl. Math.*, **13** (1965) 353-375.

[VH72] A.I. Vol'pert and S.I. Hudjaev, *On the Cauchy Problem for Composite Systems of Nonlinear Differential Equations*, *Math USSR Sbornik*, **Vol.16 No 4** (1972) 517-544.

[WT62] L. Waldmann und E. Trübenbacher, *Formale Kinetische Theorie von Gasgemischen aus anregbaren Molekülen*, *Zeitschr. Naturforschg.*, **17a** (1962) 363-376.