On a model of two conservation laws arising in Magnetohydrodynamics

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Abstract

We study a 2×2 system of conservation derived through a sequence of approximations from the equations of Magnetohydrodynamics (MHD). We solve the Riemann problem for the inviscid system in a unique way and find explicit traveling wave solutions for the viscous system. It is established that the solutions must be searched in the space of bounded Radon measures.

1. Introduction

In this paper we study a system of conservation laws, denoted by Model I,

$$u_t + (u^2/2)_x = \epsilon_1 u_{xx}, \tag{1.1}$$

$$v_t + ((u-1)v)_x = \epsilon_2 v_{xx}, \tag{1.2}$$

where ϵ_1 and ϵ_2 are small viscosity coefficients, and with initial data $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$. Later in this introduction, we show how this system may be derived as an approximation to a model for magnetohydrodynamics.

An interesting feature of this model is that, as we will show, the solution to the Riemann initial value problem contains singular or delta-function shock waves. Due to the nonlinearity of the system, care must be taken to show that such shock waves actually satisfy the system, and in what sense they do so. To this end, we rely on the work of Dal Maso, LeFloch, and Murat [4].

Such shock waves, containing singularities in v, as well as v_x and v_t, have been studied for several cases. Korchinski [7] was perhaps the first to find these distributional shock waves as solutions to a Riemann problem. Recently, Keyfitz

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and Kranzer [6] studied the system

$$u_t + (u^2 - v)_x = 0, \tag{1.3}$$

$$v_t + (u^3/3 - u)_x = 0. \tag{1.4}$$

This system is both strictly hyperbolic and genuinely nonlinear, but its hugoniot locus is compact, so that there are regions in the (u, v) plane without classical solutions to the Riemann Problem. They introduce "singular shocks" and determine admissibility conditions for them. Their main result is that the Riemann problem for (1.3), (1.4) is well posed when the set of solutions is enlarged to include admissible singular shocks.

They further attempt to show that admissible singular shocks are actually limits of solutions to the Dafermos - DiPerna regularized system

$$u_t + (u^2 - v)_x = \epsilon t u_{xx}, \tag{1.5}$$

$$v_t + (u^3/3 - u)_x = \epsilon t v_{xx}. \tag{1.6}$$

Their system differs from Model I in several respects: the singular shocks they find are time *independent* (whereas ours have strengths which grow linearly in time) both characteristic families are genuinely nonlinear, their regularization (with a single viscosity) is different, and in addition, their system has no physical interpretation.

Much closer to Model I were the systems recently studied by Tan, Zhang, and Zheng [11]. Their first model was previously considered by Korchinski, under a rescaling of time.

$$u_t + (u^2)_x = 0, \tag{1.7}$$

$$v_t + (uv)_x = 0. \tag{1.8}$$

This model comes from dropping the "y" dependence in the 2-D Euler equations with constant density and pressure. The above system is nonstrictly hyperbolic, with one genuinely nonlinear field and a second linearly degenerate field. They construct solutions to the Riemann problem using "delta-shock" waves, which are standard shock waves plus Dirac delta functions. They introduce the following definition for a delta-shock, $x = x(t)$: (i) it is a shock wave for equation (1.7), (ii) $u|_{x(t)} = u_l + u_r$, (iii) the singular portion of v is in the form of a sum of measures, and (iv) all 4 characteristics enter the shock. They also show that all Riemann solutions are weak star limits of

$$u_t + (u^2)_x = \epsilon t u_{xx}, \tag{1.9}$$

$$v_t + (uv)_x = 0. \tag{1.10}$$

Furthermore, they show that Riemann solutions containing δ-shock waves are weak star limits of solutions to (1.9), (1.10) with *constant* viscosity. Finally, they solve the Riemann problem for a strictly hyperbolic system similar to Model I:

$$u_t + (u^2)_x = 0, \tag{1.11}$$

$$v_t + ((2u + 1)v)_x = 0. \tag{1.12}$$

While similar to Model I, this sytem is derived from a very different physical model. We will compare and contrast our findings with the behavior of the MHD model from which our system was derived. Also, they do not study a viscous version of (1.11), (1.12), and the regularization of their first model is different from Model I and its two viscosities. Finally, we utilize a somewhat different technique than Tan et al to solve the Riemann problem for Model I.

LeFloch [10] has studied a system of two conservation laws with a single characteristic speed:

$$u_t + f(u)_x = 0, \tag{1.13}$$

$$v_t + (a(u)v)_x = 0, \tag{1.14}$$

where f is a smooth, strictly convex function and $a(u) = f'(u)$. He proves an existence and uniqueness result of an entropy weak solution to the system. Provided that $u_0 \in BV(\mathbf{R})$ and $v_0 \in L^\infty(\mathbf{R}) \bigcap L^1(\mathbf{R})$, with u_0 satisfying an entropy condition, then there is exactly one solution in $L^\infty(\mathbf{R}_+, \mathrm{BV}(\mathbf{R})) \times L^\infty(\mathbf{R}_+, \mathrm{M}(\mathbf{R}))$, where v is not a function, but rather a Borel Measure.

This work of LeFloch follows the general theory of weak solutions to nonconservative nonlinear hyperbolic systems in Dal Maso, LeFloch, and Murat [4]. They propose a definition for the nonconservative product

$$g(u)\frac{dv}{dx}, \tag{1.15}$$

where $g : \mathbf{R}^p \to \mathbf{R}^p$ is a smooth function and $u, v : (a, b) \to \mathbf{R}^p$ are discontinuous functions of bounded variation. An example of where a product of the form (1.15) would occur is if both u and v are Heaviside functions. Then for $g(u) = u$, we would have the product $H \cdot \delta$, of a Heaviside function with a Dirac mass, where $u = H$ and $dv/dx = \delta$. Such a product cannot be defined in the sense of distributions, and their definition extends the work of Volpert [12]. We will encounter this type of nonconservative product in our solution to the Riemann problem for Model I.

Our primary motivation for considering (1.1), (1.2) is as an approximation to a model, due to Brio [1], which presumably retains the salient features of the fast and slow shock waves of Magnetohydrodynamics (MHD). By making some simplifying assumptions, he derives the 2×2 model, referred to as Model II,

$$u_t + ((u^2 + v^2)/2)_x = \epsilon_1 u_{xx}, \tag{1.16}$$

$$v_t + ((u - 1)v)_x = \epsilon_2 v_{xx}, \tag{1.17}$$

from the full MHD system. In particular, Brio and Rosenau [2] take the full one-dimensional ideal gas MHD equations, with dissipation and dispersive effects due to Hall current included, and make an asymptotic expansion. This expansion is carried out nearby a special constant state in which all three characteristic speeds of the linearized MHD equations coincide; such a state is referred to as a triple umbilic point. Expansion of the MHD equations up to second order in

the perturbation parameter leads to the 3×3 system:

$$\begin{aligned} u_t + ((cu^2 + |\mathbf{q}|^2)/2)_x &= \mu\, u_{xx}, \\ v_t + (v(u-1))_x &= \eta\, v_{xx} - \chi\, w_{xx}, \\ w_t + (w(u-1))_x &= \eta\, w_{xx} + \chi\, v_{xx}, \end{aligned}$$

where the thermodynamic variables (density, temperature, and pressure) are proportional to u, and the transverse components of the magnetic field are proportional to v and w, with $|\mathbf{q}|^2 = v^2 + w^2$. Also, μ is the viscosity coefficient, η is the ohmic resistivity, χ is the Hall parameter, and $c = \gamma + 1$, $\gamma =$ ratio of specific heats. The above system of three equations can be reduced to (1.16), (1.17) by taking $w = 0$, $\chi = 0$, and $c = 1$. We can then identify $\mu = \epsilon_1$ and $\eta = \epsilon_2$.

For (1.16), (1.17), Brio [1] (see also Cheng and Brio [3]) proves that the existence of traveling waves exhibits a global bifurcation. Setting $\alpha = \epsilon_1/\epsilon_2$, there exists α_0 such that for $\alpha > \alpha_0$, there are no traveling wave solutions which satisfy Liu's entropy condition. Alternatively, when $\alpha < \alpha_0$, there is a unique stable profile for such waves. Also distinct values of α provide distinct traveling wave profiles, which is at the origin of the non-uniqueness for the Riemann problem associated with the unviscid equations.

The goals of this study are to compare and contrast Model I and its solutions with the Brio's model for MHD (1.16), (1.17). We begin by comparing the systems I and II themselves and indicate why we believe that solutions of the simplified model are interesting with respect to Brio's model. In the third section, we solve the Riemann problem for the simplified model. The fourth section studies the existence and behavior of the traveling wave solutions to Model I.

2. Comparison of Models I and II

In this section, we compare the properties of nonlinearity of the two models under consideration. In particular, the properties of the shock curves are discussed. The inviscid (idealized) system is obtained by setting $\epsilon_1 = \epsilon_2 = 0$ in (1.1) and (1.2):

$$u_t + uu_x = 0, \tag{2.1}$$

$$v_t + ((u-1)v)_x = 0. \tag{2.2}$$

If we write (2.1), (2.2) in matrix form,

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} u & 0 \\ v & u-1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = 0,$$

we see that the eigenvalues are

$$\lambda_1 = u - 1 \quad \text{and} \quad \lambda_2 = u, \tag{2.3}$$

so that the system is strictly hyperbolic. The eigenvectors are given by

$$r_1 = (0,\, 1)^T \quad \text{and} \quad r_2 = (1,\, v)^T,$$

so that they form a basis in the (u,v)-plane. The first characteristic field is linearly degenerate, with $\nabla\lambda_1 \cdot r_1 = 0$, while the second field is genuinely nonlinear with $\nabla\lambda_2 \cdot r_2 = 1$. Thus the second field may contain shocks and rarefactions, while the first field can only support contact discontinuities.

If we consider a shock profile connecting two constant states, $(u,v) = (u_l, v_l)$ and $(u,v) = (u_r, v_r)$, then with $[u] \equiv u_r - u_l$, the Rankine-Hugoniot jump conditions for system (1.1), (1.2) are

$$-s[u] + \left[u^2/2\right] = 0, \tag{2.4}$$

$$-s[v] + [(u-1)v] = 0. \tag{2.5}$$

In addition, the shock speed from (2.4) must satisfy the entropy inequality

$$u_r < s < u_l. \tag{2.6}$$

When $[u] \neq 0$, the first of these equations gives the familiar Burgers equation shock speed

$$s = (u_l + u_r)/2 \equiv \bar{u}, \tag{2.7}$$

while the second equation yields a condition for v_r:

$$v_r = v_l \, \frac{2 + u_r - u_l}{2 - u_r + u_l}. \tag{2.8}$$

When $[u] = 0$ but $[v] \neq 0$, the equation (2.5) implies

$$s = u - 1. \tag{2.9}$$

For Brio's system, which we write in matrix form as

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} u & v \\ v & u-1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = 0,$$

the characteristic speeds are ($i = 1, 2$, $e_1 = -1$, $e_2 = 1$)

$$\lambda_i = u - 1/2 + e_i \sqrt{v^2 + 1/4},$$

with $\lambda_1 < \lambda_2$, so this system is also strictly hyperbolic. The right eigenvectors are given by

$$r_i = \begin{pmatrix} v \\ -1/2 + e_i \sqrt{1/4 + v^2} \end{pmatrix},$$

and so like the eigenvectors of our simplified model, these vectors are never parallel, and hence span the $(u,v)-$plane. Checking genuine nonlinearity for Brio's model, we find that

$$\nabla\lambda_i \cdot r_i = v(2 \ + e_i \frac{1}{\sqrt{4 + v^2}}). \tag{2.10}$$

Both of the quantities in (2.10) may be either positive, negative, or zero, depending upon the sign of v. Each of the two characteristic fields is therefore neither genuinely nonlinear nor linearly degenerate. However the two fields are genuinely nonlinear in each of the subdomains $\Omega_\pm = \{(u,v) \,/\, v \gtrless 0\}$. The situation here is analogous to that encountered in a cubic nonlinear scalar conservation law. If the initial data is such that v does not change signs, then Brio's model behaves exactly according to the theory for strictly hyperbolic and genuinely nonlinear systems. When v does change signs, however, we cannot utilize that theory.

The Rankine-Hugoniot jump relations for Brio's model may be written as

$$\begin{aligned}(-s+\bar u)[u]+\bar v[v]=0,\\ (-s+\bar u-1)[v]+\bar v[u]=0,\end{aligned}$$

where $\bar v = (v_l+v_r)/2$. If we solve for the shock speed in these equations, we get

$$s = \bar u - 1/2 \pm \sqrt{\bar v^2 + 1/4}.$$

In order to find the shock curves –the set of all points which can be connected to (u_l, v_l) on the right by a shock– we eliminate s from the jump relations and set $\mu \equiv [u]/[v]$. Then μ satisfies

$$\mu^2 - (1/\bar v)\,\mu - 1 = 0,$$

so that the shock curves are given by

$$S_{1,2}: \qquad u - u_l = \frac{v - v_l}{v + v_l}\left(1 \mp \sqrt{(v+v_l)^2+1}\right).$$

Denote by U_l the state (u_l, v_l) and by U_r the state (u_r, v_r). The 1-shocks (minus sign), which must have $\lambda_1(U_r) < s < \lambda_1(U_l)$, can be connected in the following manner: if $v_l > 0$, then we need $v > v_l$ in order for λ_1 to decrease. While if $v_l < 0$, the possible right-hand states must have $v < v_l$. For the 2-shocks (plus sign), $\lambda_2(U_r) < s < \lambda_2(U_l)$, we can connect those points with $-|v_l| < v < |v_l|$.

The rarefaction or integral curves are found by their property that the derivatives of u and v along these curves are proportional to the u- and v-components of the eigenvectors $r_{1,2}$. We therefore have

$$\frac{du}{dv} = \frac{2v}{-1 \pm \sqrt{4v^2+1}.}$$

We can calculate these curves explicitly as

$$\begin{aligned}R_1: \qquad u &= -(1/2)\left(\sqrt{4v^2+1} - \log(1+\sqrt{4v^2+1})\right) + C_1,\\ R_2: \qquad u &= +(1/2)\left(\sqrt{4v^2+1} + \log(\sqrt{4v^2+1}-1)\right) + C_2,\end{aligned}$$

with constants C_1 and C_2 such that the curves pass through (u_l, v_l). Those states which can be connected to (u_l, v_l) by R_i must lie a direction in which λ_i is

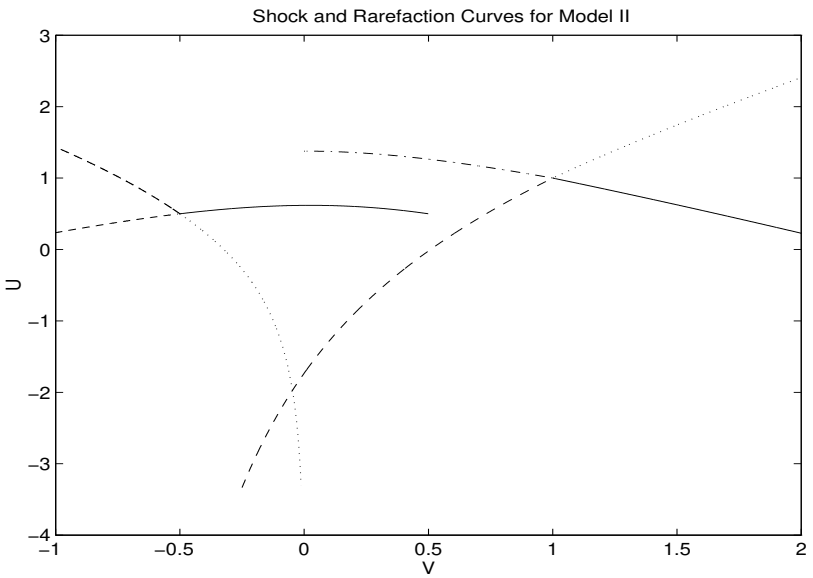

Figure 1: Shock and rarefaction curves for Model II. The left- and right-hand states are (1,1) and (0.5, -0.5), respectively. The curves are S_1 (solid), S_2 (dash), R_1 (dash-dot), and R_2 (dot). No classical Riemann solution exists for these states, since we must leave U_l via either $R_1(U_L)$ or $S_1(U_L)$ and neither of these curves intersects $R_2(U_R)$ or $S_2(U_R)$.

monotonically increasing. Thus our formulas for R_1 and R_2 are only valid until v changes sign; i.e. until genuine nonlinearity is lost. We can therefore connect the following states via rarefactions: if $v_l < 0$, R_1 is valid for $v_l < v < 0$, while R_2 can join points with $v < v_l$. When $v_l > 0$, R_1 connects points with $0 < v < v_l$, and R_2 links those with $v > v_l$. We sketch the rarefaction and shock curves for Model II in figure 1. We summarize this in the following table:

	Model I	*Model II*
strictly hyperbolic	yes	yes
1-field	Lin. Degen.	GNL in $\Omega_\pm$
2-field	GNL	GNL in $\Omega_\pm$

Table 1: Properties of Models I and II

For Model II, the "interesting" region of the (u, v) plane is the line $v = 0$. In each subdomains $\Omega_\pm$, the standard theory of hyperbolic conservation laws applies directly. For solutions crossing the line $v = 0$, non-uniqueness of Riemann solutions is anticipated. This has motivated us to neglect the v^2 term in the flux of the first equation of Model II to get Model I.

It is conceivable that solutions to the full model resemble the solutions to Model I, which we are going to derive in the coming sections of this paper. The study of Brio model however would certainly be much more involved. In fact Brio's model lacks a universal solution to the Riemann problem, as it can be checked that the (shock and rarefaction) wave curves described above do not intersect in a unique way when pair of states with v changing sign are considered.

Finally it is interesting to compare the mathematical entropies of systems I and II. We recall that a entropy pair $(U(u,v), F(u,v))$ is by definition a pair of Lipschitz continuous functions such that every solution to the system under consideration also satisfies the conservation law

$$U(u,v)_t + F(u,v)_x = 0. \tag{2.11}$$

For Model I, the entropies have the form:

$$U(u,v) = \varphi_1(u) + \int^v \varphi_2(u - \log w)\, dw, \tag{2.12}$$

where φ_1 and φ_2 are arbitrary functions. The entropies of Model II are given by solving a linear hyperbolic equation:

$$- vU_{uu} + U_{uv} + vU_{vv}0. \tag{2.13}$$

A particular solution to the latter is the so-called energy: $U(u,v) = (u^2+v^2)/2$. More generally any function $U(u,v) = \varphi_1(u) + \varphi_2(v)$ is an entropy to Model I. By contrast, all of the entropies to Model I are singular at $v = 0$.

3. Riemann Problem for the Inviscid System

In this section we study the Riemann problem for (2.1), (2.2) with initial data

$$u(x,0) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0. \end{cases} \qquad v(x,0) = \begin{cases} v_l, & x < 0, \\ v_r, & x > 0, \end{cases} \tag{3.1}$$

with constants u_l, u_r, v_l, and v_r. A unique solution can be derived in all cases; in one case, when $u_l \geqq u_r + 2$, the solution v happens to be a bounded measure, actually a combination of Heaviside-like functions and a propagating Dirac mass, instead of a function. Our results are contained in

THEOREM 3.1. *The system (2.1), (2.2), together with Riemann data (u_l, u_r) and (v_l, v_r) has the following solution:*

(i) When $u_r > u_l$, both u and v contain rarefaction fans, but v also has a jump discontinuity adjacent to its fan:

$$u(x,t) = \begin{cases} u_l, & x \leqq u_l t, \\ x/t, & u_l t < x < u_r t, \\ u_r, & u_r t \leqq x. \end{cases} \tag{3.2}$$

$$v(x,t) = \begin{cases} v_l, & x < (u_l - 1)t, \\ v_r e^{u_l - u_r}, & (u_l - 1)t \leqq x \leqq u_l t, \\ v_r e^{(x/t - u_r)}, & u_l t \leqq x \leqq u_r t, \\ v_r, & u_r t < x. \end{cases} \tag{3.3}$$

(iia) When $u_r < u_l < u_r + 2$, then u has a single jump discontinuity, while v contains two such jumps:

$$u(x,t) = \begin{cases} u_l, & x < \bar{u}t, \\ u_r, & x > \bar{u}t, \end{cases} \qquad \bar{u} \equiv \frac{u_l + u_r}{2}. \tag{3.4}$$

$$v = \begin{cases} v_l, & x < (u_l - 1)t, \\ v^*, & (u_l - 1)t \leqq x < \bar{u}t, \\ v_r, & \bar{u}t < x, \end{cases} \quad v^* \equiv v_r \left(\frac{2 + u_l - u_r}{2 + u_r - u_l} \right). \tag{3.5}$$

(iib) For $u_l \geqq u_r + 2$, $u(x,t)$ is still given by (3.4), while

$$v(x,t) = \begin{cases} v_l, & x < \bar{u}t, \\ v_r, & x > \bar{u}t, \end{cases} \tag{3.6}$$

but has a Dirac mass propagating with the shock, whose strength is

$$\int_{\bar{u}t^-}^{\bar{u}t^+} v(x,t)dx = (t/2)\left[v_r(u_l - u_r + 2) - v_l(u_r - u_l + 2) \right]. \tag{3.7}$$

Proof: We introduce the integrated variable $V(x,t)$ defined by

$$V(x,t) = \int_{-C}^{x} v(y,t)dy, \qquad \text{so that} \qquad \frac{\partial V}{\partial x} = v, \tag{3.8}$$

where C is an arbitrary positive number. The system (2.1), (2.2) then becomes

$$u_t + uu_x = 0, \tag{3.9}$$

$$V_t + (u - 1)V_x = 0, \tag{3.10}$$

with Riemann initial data (u_l, u_r) in u, but the initial data for V is now piecewise linear (and continuous) and is given by

$$V(\xi, 0) \equiv V_0(\xi) = \begin{cases} v_l(C + \xi), & \xi \leqq 0, \\ v_l C + v_r \xi, & \xi > 0. \end{cases} \tag{3.11}$$

The procedure we follow is to solve (3.9) using the explicit solution for the Hopf equation, then substitute this formula for $u(x,t)$ into (3.10), which now becomes a linear equation with a variable coefficient. This coefficient may, however, be discontinuous and, if the jump in u is great enough (see case iib) it can lead to the formation of discontinuities in V.

The explicit solution to (3.9) is well-known and is given by

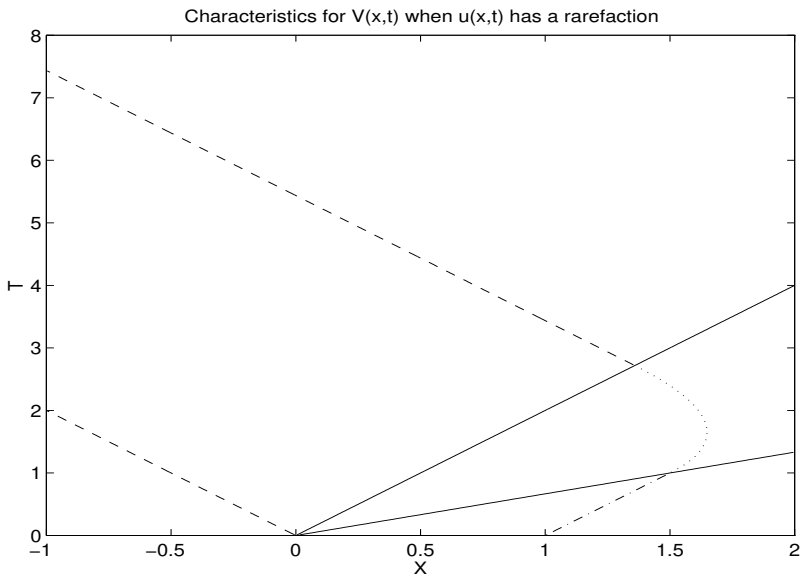

Figure 2: Characteristics for V, in the case $u_l < u_r$. The solid lines are the boundaries of the rarefaction fan in u; namely, $x = u_l t$ and $x = u_r t$.

(i) If $u_l < u_r$, the solution is the rarefaction fan of equation (3.2).

(ii) If $u_l > u_r$, the solution to (3.9) contains a jump discontinuity or shock traveling with a speed equal to the average of u_l and u_r. This solution is given in (3.4).

We can re-write (3.10) in characteristic form as V = constant, along curves satisfying

$$dx/dt = u(x,t) - 1. \tag{3.12}$$

That is, $V(x,t) = V_0(\xi)$, for the characteristic originating at $x = \xi$ and passing through the point (x,t). These characteristics can be continued until they reach a discontinuity in u. The solutions (3.2) and (3.4) can then be used to integrate equation (3.12) and find the characteristics.

The characteristics for case (i) are shown in figure 2. Those curves starting at points $\xi > 0$ impinge upon the rarefaction fan, are refracted through the fan, and emerge as straight lines with slope $u_l - 1$ - parallel to those curves beginning at $\xi < 0$.

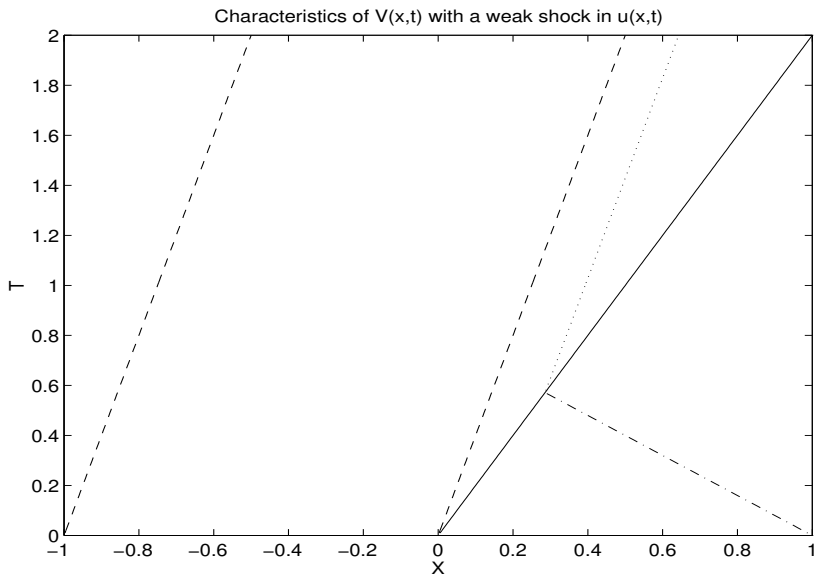

Figure 3: Characteristics for V in the case $u_r < u_l < u_r + 2$. The solid line is the shock in u, given by $x = \bar{u}t$. Characteristics intersecting the shock from the right (dash-dot line) emerge parallel (dotted line) to those originating on the left (dashed line).

Solving (3.12) with u given by (3.2), we find equations for the characteristics:

$$x = \begin{cases} (u_r - 1)t + \xi, & t < \xi, \\ t(u_r + \ln \xi - \ln t), & \xi \leqq t \leqq \xi e^{u_r - u_l}, \\ (u_l - 1)t + \xi e^{u_r - u_l}, & t > \xi e^{u_r - u_l}, \\ (u_l - 1)t + \eta, & \eta < 0. \end{cases} \qquad \xi > 0, \tag{3.13}$$

Then by (3.11) we have the following solution for V:

$$V(x,t) = \begin{cases} v_l(x - (u_l - 1)t) + Cv_l, & x < (u_l - 1)t, \\ v_r(x - (u_l - 1)t)e^{u_l - u_r} + Cv_l, & (u_l - 1)t \leqq x \leqq u_l t, \\ v_r t e^{(x/t - u_r)} + Cv_l, & u_l t \leqq x \leqq u_r t, \\ v_r(x - (u_l - 1)t) + Cv_l, & x > u_r t. \end{cases} \tag{3.14}$$

Taking the x-partial derivative of (3.14) gives the solution $v(x,t)$ from (3.3). We now turn to case (ii), where u has a discontinuity. In this case, $V(x,t)$ - and hence v - has two qualitatively different behaviors depending on the magnitude of $u_l - u_r$. Recalling that the shock speed is $\bar{u} = (u_l + u_r)/2$, we see that the characteristics for V starting at $\xi > 0$, straight lines with slope $u_r - 1$, always impinge on the shock. On the other hand, those characteristics beginning at

$\xi < 0$ will only intercept the shock if the condition

$$u_l - 1 > (u_l + u_r)/2, \tag{3.15}$$

is met. Or, equivalently, if $u_l - u_r > 2$.

In the case of "weak" shocks, with $u_l - u_r < 2$, the characteristics hitting the shock from the right have their slopes deflected from $u_r - 1$ to $u_l - 1$. They continue to propagate their initial values along straight lines, as they fill in the gap created by the characteristics with $\xi < 0$. The characteristics have the formula

$$x = \begin{cases} (u_r - 1)t + \xi, & \xi > t^*, \\ (u_l - 1)t + \left(\frac{2+u_r-u_l}{2+u_l-u_r}\right)\xi, & 0 < \xi < t^*, \\ (u_l - 1)t + \xi, & \xi < 0, \end{cases} \tag{3.16}$$

where $t^* \equiv (t/2)(2 + u_l - u_r)$. The characteristics for V are plotted in figure 3. Solving for ξ and applying (3.11) in the above formula yields

$$V(x,t) = \begin{cases} v_r(x - (u_l - 1)t) + Cv_l, & x > \bar{u}t, \\ v^*(x - (u_l - 1)t) + Cv_l, & (u_l - 1)t < x < \bar{u}t, \\ v_l(x - (u_l - 1)t) + Cv_l, & x \leqq (u_l - 1)t, \end{cases} \tag{3.17}$$

with v^* as defined in (3.5). Differentiating (3.17) with respect to x leads to the solution (3.5) for $v(x,t)$.

When $u_l - u_r > 2$, characteristics impinge on the shock, $x = \bar{u}t$, from both sides. In this case the characteristics are:

$$x = \begin{cases} (u_l - 1)t + \xi, & \xi < -(u_l - \bar{u} - 1)t, \\ (u_r - 1)t + \xi, & \xi > (\bar{u} - u_r + 1)t. \end{cases} \tag{3.18}$$

These characteristic curves are plotted in figure 4.

Since the two approaching characteristics propagate different values of V, there will be a jump in this variable, as well as in u, across the shock. Denoting by V_L and V_R the left- and right-hand limits, respectively of $V(x,t)$ at $x = \bar{u}t$, we have from (3.18) and (3.11) that

$$\begin{aligned} V_L &= v_l\left((\bar{u} - u_l + 1)t + C\right), \\ V_R &= v_r(\bar{u} - u_r + 1)t + Cv_l. \end{aligned}$$

So that the jump in V across the discontinuity is

$$[V] = V_L - V_R = (t/2)\left[v_l(u_r - u_l + 2) - v_r(u_l - u_r + 2)\right]. \tag{3.19}$$

The solution for $V(x,t)$ is

$$V(x,t) = \begin{cases} v_l(x - (u_l - 1)t) + Cv_l, & x < \bar{u}t, \\ v_r(x - (u_r - 1)t) + Cv_l, & x > \bar{u}t, \end{cases} \tag{3.20}$$

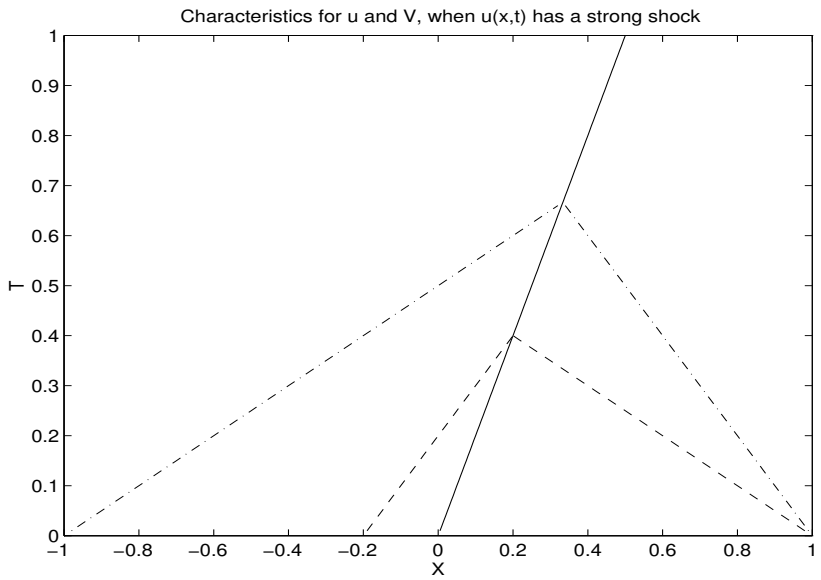

Figure 4: Characteristic curves for the case $u_l > u_r + 2$. Notice that both of the V characteristics (dashed lines), as well as those of u (dash-dot) impinge on the shock (solid line), given by $x(t) = \bar{u}t$.

We now need to prove that our solution, containing a propagating Dirac mass is a solution to

$$V_t + (u-1)V_x = 0, \tag{3.21}$$

with $u, V \in \mathrm{BV}$. We use the definition of DalMaso et al [4] to give meaning to the product $(u-1)V_x$. We introduce the Borel measure

$$\mu = \left[(u-1)\frac{dV}{dx}\right]_\phi, \tag{3.22}$$

and provide the following definition: Where u and V are continuous, it is just the usual integral

$$\mu(B) \equiv \int_B (u(x)-1)\,\frac{dV}{dx}(x), \tag{3.23}$$

with $B \subset (a,b)$ a Borel set and the integration with respect to the Borel measure dV/dx. If x_0 is a point of discontinuity in (a,b),

$$\mu(\{x_0\}) \equiv \int_0^1 (\phi_1(s;u_l,u_r)-1)\,\frac{\partial \phi_2}{\partial s}(s;V_L,V_R)ds, \tag{3.24}$$

where ϕ_1 and ϕ_2 are Lipschitz continuous paths connecting u_l to u_r and V_L to V_R, respectively.

If we choose the canonical straight line path for V, we have

$$\phi_2: \quad V(s) = V_L + s(V_R - V_L), \qquad s \in [0,1]. \tag{3.25}$$

Integrating (3.21) gives

$$-\bar{u}(V_R - V_L) + \left[(u-1)\frac{dV}{dx}\right]_\phi = 0. \tag{3.26}$$

We must show that there exists a path $\phi_1(s;u_l,u_r) = u(s)$ such that $u(0) = u_l$ and $u(1) = u_r$ and so that (3.26) holds. From (3.24),

$$\begin{aligned}[(u-1)V_x]_\phi &= \int_0^1 (u(s)-1)\,V'(s)ds \\ &= (V_R - V_L)\int_0^1 (u(s)-1)\,ds,\end{aligned}$$

where we have used (3.25). We therefore need to show that

$$\bar{u} = \int_0^1 u(s)ds - 1, \tag{3.27}$$

for some path $u(s)$.

For the case we are considering, $u_r - 1 < \bar{u} < u_l - 1$. Furthermore,

$$u_r \leqq \int_0^1 u(s)ds \leqq u_l.$$

Putting these facts together, we see that there must be some path $u(s)$ which exactly satisfies (3.27).

This completes the proof of the Theorem.

We believe that solutions to Model II resemble the solutions to Model I obtained at least in Case (i) and (iia). This is because v remains close to 0 in the solution if v is close to zero at time $t = 0$. The approximation "v close to 0" was the basis to derive I from II. In case (iib), this approximation is clearly violated however. It is an open question whether the solution to the Riemann problem for Brio's model contains a Dirac mass.

4. Traveling Wave Profiles for the Viscous System

In this section we study traveling shock profiles for (1.1), (1.2) which connect the constant states (u_l, v_l) and (u_r, v_r). We prove that at least for certain values of the ratio ϵ_1/ϵ_2, there exist traveling wave solutions for Model I. This established the on-uniqueness of traveling wave profiles.

We show the following result.

THEOREM 4.1. *There exist (unique) traveling wave solutions to (1.1), (1.2) when $0 < u_l - u_r < 2$ and $\epsilon_1/\epsilon_2 = 2/n$, for n an integer. In particular, the profile exists in the limit of $\epsilon_2/\epsilon_1 \to 0$.*

Proof: By the Rankine-Hugoniot jump condition (2.4), these profiles must move with speed $\bar{u}$. Thus we look for solutions of the form $u(x,t) = U(\xi)$ and $v(x,t) = V(\xi)$, where $\xi = x - \bar{u}t$ is a coordinate moving with the shock speed. In addition, the entropy condition for (1.1) requires $u_l > u_r$. Substituting this ansatz into (1.1) and (1.2) gives

$$-\bar{u}U' + UU' = \epsilon_1 U'' \tag{4.1}$$

$$-\bar{u}V' + ((U-1)V)' = \epsilon_2 V''. \tag{4.2}$$

This system has boundary conditions $(U,V) \to (u_l, v_l)$, as $\xi \to -\infty$, $(U,V) \to (u_r, v_r)$, as $\xi \to \infty$, and $U', V' \to 0$, as $\xi \to \pm\infty$. The ODE (4.1), derived from Burgers equation, has the well-known hyperbolic tangent profile

$$U(\xi) = \bar{u} - \delta \tanh\left(\frac{\delta}{2\epsilon_1}\xi\right), \tag{4.3}$$

where $\delta = (u_l - u_r)/2$ is a measure of the shock strength.

Meanwhile, (4.2) can be immediately integrated once to give

$$(U - 1 - \bar{u})V + (\bar{u} + 1 - u_l)v_l = \epsilon_2 V'. \tag{4.4}$$

We now make a change of variables from ξ to η, with $\epsilon_2 \eta = \xi$ and substitute the explicit formula for $U(\eta)$ into equation (4.4) to get

$$\frac{dV}{d\eta} + (1 + \delta \tanh(\lambda\eta))V = C, \tag{4.5}$$

where $\lambda = \delta\epsilon_2/(2\epsilon_1)$ and $C = (v_l/2)(u_r - u_l + 2) = v_l(1-\delta)$. This equation can be integrated with the integrating factor

$$\begin{aligned} I(\eta) &= \exp\left(\int_\eta (1 + \delta \tanh \lambda\eta')d\eta'\right) \\ &= (\cosh(\lambda\eta))^{\delta/\lambda}\, e^{\eta}. \end{aligned}$$

The solution $V(\eta)$ is then found by

$$V(\eta) = [I(\eta)]^{-1}\, C \int_\eta I(\bar\eta)d\bar\eta. \tag{4.6}$$

This result can be computed explicitly when $\delta/\lambda = n$, an integer. In other words, when the viscosities are related by

$$\epsilon_2/\epsilon_1 = 2/n. \tag{4.7}$$

In this case,

$$V(\eta) = C\, \frac{\displaystyle\sum_{k=0}^{n} \binom{n}{k} \frac{e^{((2k-n)\lambda+1)\eta}}{(2k-n)\lambda+1} + D}{\displaystyle\sum_{k=0}^{n} \binom{n}{k} e^{((2k-n)\lambda+1)\eta}}, \tag{4.8}$$

where D is a constant of integration. Note that there are no singularities in (4.8): if $(2k-n)\eta + 1 = 0$, for $k = K$, then that term in the summation is simply

$$\binom{n}{K}\eta.$$

This situation can only occur in the case where $\delta > 1$.

The exponents in both the numerator and denominator of (4.8) range from $(1-\delta)\eta$ to $(1+\delta)\eta$. From the entropy condition for Burgers equation, $\delta > 0$. We need therefore to consider two cases, according to whether δ is greater than or less than one:

1. When $\delta < 1$, all of the exponentials in (4.8) have the form $e^{p\eta}$, for $p > 0$. As $\eta \to -\infty$, all of these exponentials tend to zero. Consequently, we must take $D = 0$ in (4.8) to obtain a bounded solution as $\eta \to -\infty$. Thus we can write

$$V(\eta) = v_l(1-\delta)(2\cosh(\lambda\eta))^{-n} \sum_{k=0}^{n} \binom{n}{k} \frac{e^{(2k-n)\lambda\eta}}{(2k-n)\lambda+1}. \tag{4.9}$$

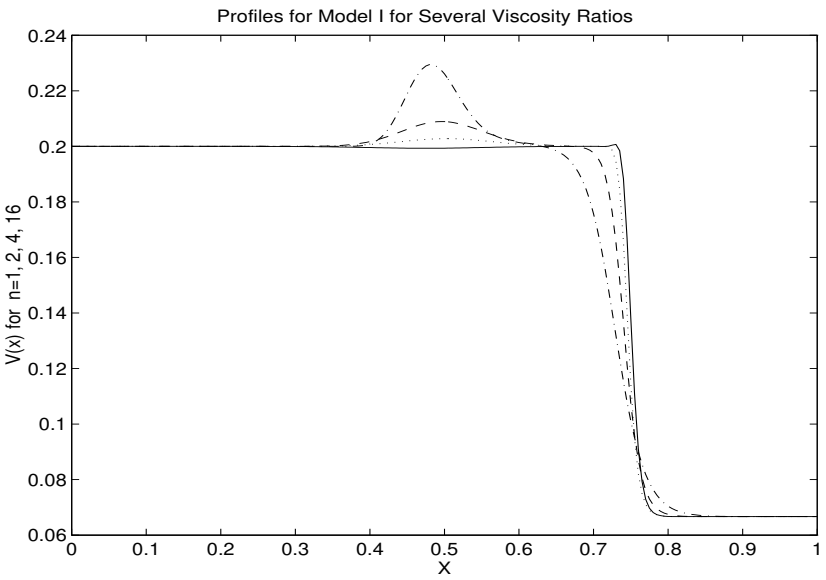

Figure 5: $V(x, t = 0.5)$ for Model I with $n = 1$(solid), 2 (dot), 4 (dash), and 16 (dash-dot).

As $\eta \to -\infty$, the dominant exponentials in (4.9) will be $e^{-\delta\eta}$. That is, the $k = 0$ term. Therefore,

$$V(\eta) \to v_l(1-\delta)\frac{1}{1-\delta} = v_l, \quad \text{as} \quad \eta \to -\infty. \tag{4.10}$$

Similarly, as $\eta \to \infty$, the dominant exponent will be $e^{\delta\eta}$, i.e., the $k = n$ term. Thus,

$$V(\eta) \to v_l(1-\delta)\frac{1}{(1+\delta)} = v_r, \quad \text{as} \quad \eta \to \infty. \tag{4.11}$$

In figure 5, we display the numerical solution to the Riemann problem for Model I, when $\delta < 1$. The left- and right-hand values of u and v were chosen to satisfy the jump condition (2.8). The numerical solution was computed by solving system (2.1), (2.2) using a modified Lax-Friedrichs scheme. The solutions are displayed for several values of $n = 2\epsilon_1/\epsilon_2$.
Then in figure 6 the solution is displayed at times $t = 0.5$ and 1.0 for $n = 16$. Notice that this solution is composed of a traveling wave and a "hump". This latter feature is moving with a different speed and spreading out in time. In figure 7, we plot the numerical solution to the Brio model (1.16), (1.17), with the same initial data. This solution is almost identical to that of Model I.

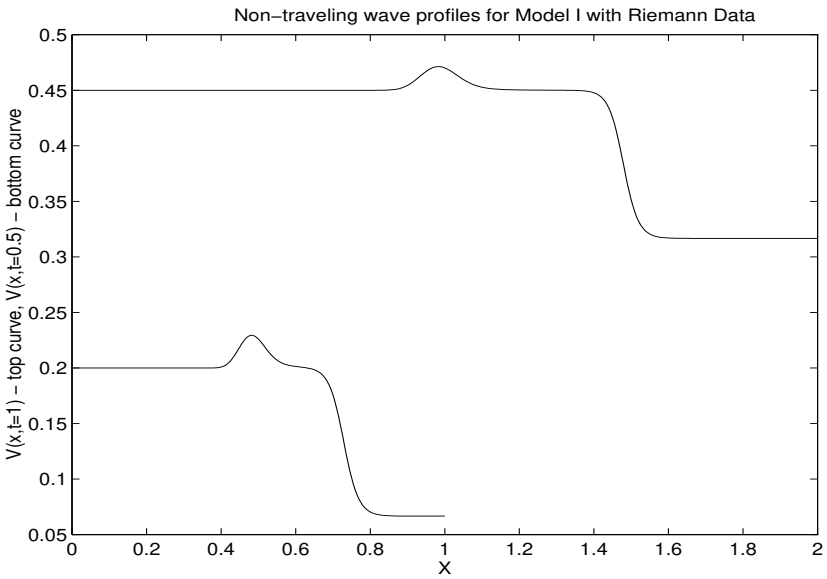

Figure 6: Non- traveling wave nature of solution to Model I with Riemann initial data. The values of V in the top figure were shifted up by 0.25.

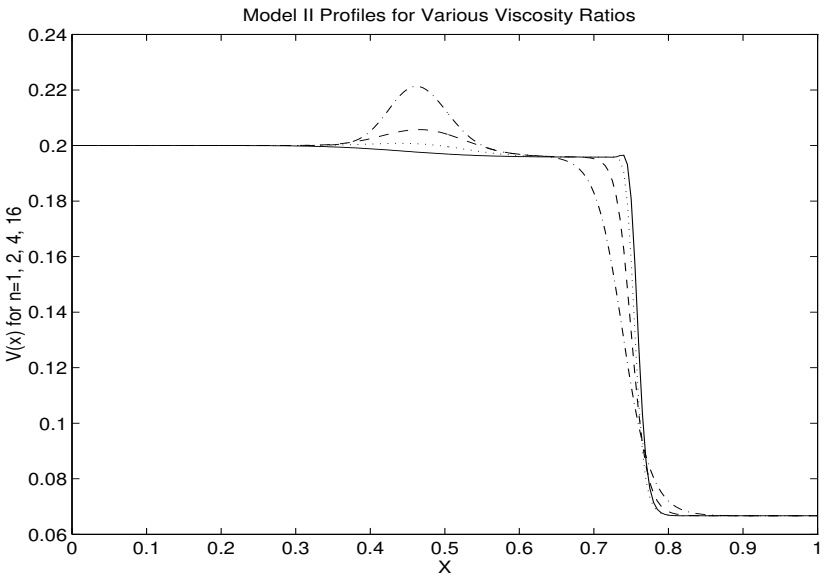

Figure 7: $V(x, 0.5)$ for Model II with $n = 1,\ 2,\ 4,$ and 16.

To examine the nature of this solution, we consider the case where $\epsilon_1 = \epsilon_2$; i.e. $n = 2$. In this case the hump is visible in figure 5, but less pronounced than the $n = 16$ case. From equation (4.9), we have

$$V(\eta) = v_l(1-\delta)(2\cosh(\lambda\eta))^{-2}\left[\frac{e^{-2\lambda\eta}}{1-2\lambda} + 2 + \frac{e^{2\lambda\eta}}{1+2\lambda}\right]. \tag{4.12}$$

After some manipulation and use of the jump condition (2.8) connecting v_l and v_r, we may rewrite the solution as

$$V(\eta) = v_r + \lambda v_r \operatorname{sech}^2(\lambda\eta)\left(1 + \frac{e^{-2\lambda\eta}}{1-2\lambda}\right) \tag{4.13}$$

$$= v_l - \lambda \operatorname{sech}^2(\lambda\eta)\left(v_l + v_r\frac{e^{2\lambda\eta}}{1-2\lambda}\right). \tag{4.14}$$

We will use the first formula for $\eta > 0$, and the second for $\eta < 0$. We now define the Heaviside function

$$\tilde{H}(\eta) = \begin{cases} v_l, & \eta < 0, \\ v_r, & \eta > 0. \end{cases} \tag{4.15}$$

This represents the Riemann initial data moving with the TW speed $\bar{u}$. If we now compute the integral of the TW profile minus this Heaviside function and use (4.13) and (4.14), we obtain

$$\begin{aligned} \int_{-\infty}^{\infty}\left(V(\eta) - \tilde{H}(\eta)\right)d\eta &= v_r\int_0^{\infty}\operatorname{sech}^2(\lambda\eta)\lambda d\eta - v_l\int_{-\infty}^{0}\operatorname{sech}^2(\lambda\eta)\lambda d\eta \\ &= v_r - v_l < 0. \end{aligned}$$

Thus there is a time-independent "mass deficiency" in the TW profile, compared with the Heaviside function moving with an identical speed.
As the hyperbolic tangent profile for u moves off to the right with speed $\bar{u}$, the second equation of Model I becomes approximately,

$$v_t + (u_l - 1)v_x = \epsilon_2 v_{xx}, \tag{4.16}$$

which is just the linear advection-diffusion equation. The initial data for this equation, $v(x,0)$, is a is a pulse (like a Dirac mass) centered at $x = 0$, and having strength $v_l - v_r$. In time, the pulse evolves into a Gaussian which propagates with speed $u_l - 1$, as it spreads out. A similar mass deficiency and resulting pulse appears to occur for larger values of n as well.

2. When $\delta > 1$, the exponentials in (4.8) have the form $e^{p\eta}$, but now p has mixed signs. So as $\eta \to \pm\infty$, some exponentials will always dominate and we therefore cannot determine D from the argument of case (i). We

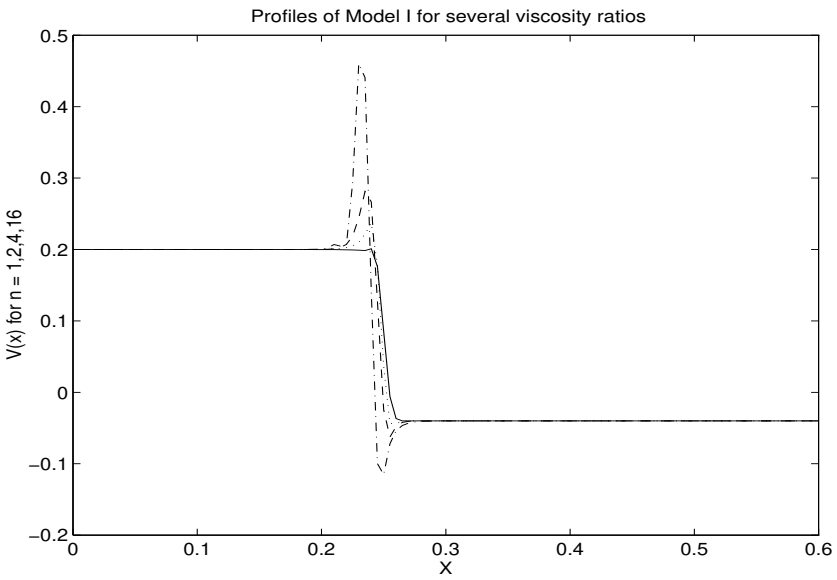

Figure 8: Traveling wave profiles for V in Model I with $\delta > 1$ and $n = 1, 2, 4$, and 16.

conjecture that D, which depends on u_l, u_r, v_l, and v_r, is determined by the integral (conservation) condition

$$\int_{-\infty}^{\infty} \Big(V(\eta) - \tilde{H}(\eta) \Big)\, d\eta = 0. \tag{4.17}$$

This surmise is based on the fact that for the inviscid problem with $\delta > 1$, both characteristic families propagate with the shock speed $\bar{u}$. Numerical experiments also indicate that (4.17) holds. In figure 8, we plot these numerical solutions for $\delta > 1$ and a range of ϵ_1/ϵ_2. Finally, in figure 9, we plot the numerical solution to the Brio model for the same initial data.

Remark : It would be interesting to show that (3.9) is monotone in η for all n. This is indeed true at least in several cases, i.e. when $n = 1, 2$ and also when $V'(0)$ is proportional to $-|v_l|$ for all n.

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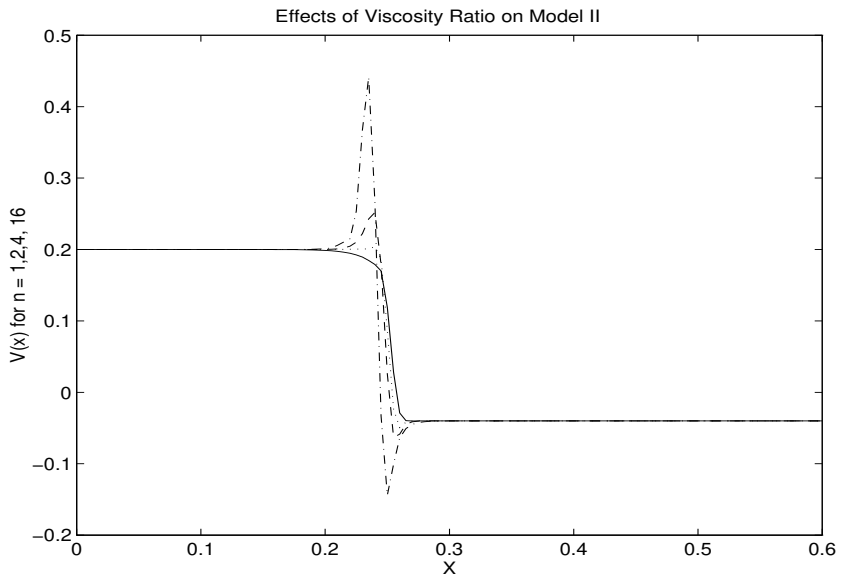

Figure 9: Traveling wave profiles for V in Model II with $\delta > 1$ and $n = 1, 2, 4,$ and 16.

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