NON-CLASSICAL SHOCK WAVES
AND
KINETIC RELATIONS (I)

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Abstract. This paper analyzes the non-classical shock waves which arise as limits of certain diffusive-dispersive approximations to hyperbolic conservation laws. Such shocks occur for non-convex fluxes and connect regions of different convexity. They have negative entropy dissipation for a single convex entropy function, but not all convex entropies, and do not obey the classical Oleinik entropy criterion. We derive necessary conditions for the existence of non-classical shock waves, and construct them as limits of traveling wave solutions for several diffusive-dispersive approximations.

We introduce a "kinetic relation" to act as a selection principle for choosing a unique non-classical solution to the Riemann problem. The convergence to non-classical weak solutions for the Cauchy problem is investigated. Using numerical experiments, we demonstrate that, for the cubic flux-function, the Beam-Warming scheme produces non-classical shocks while no such shocks are observed with the Lax-Wendroff scheme. All of these results depend crucially on the sign of the dispersion coefficient.

Contents

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1. Introduction.

In this paper we study the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0, \qquad u(x,t) \in \boldsymbol{R},\, x \in \boldsymbol{R},\, t > 0, \tag{1.1}$$

where the flux-function $f : \boldsymbol{R} \to \boldsymbol{R}$ is non-convex; i.e. $f''(u)$ does not possess a definite sign. Our primary focus is on the prototypical case of the cubic flux $f(u) = u^3$. A function $U : \boldsymbol{R} \to \boldsymbol{R}$ is called an entropy for (1.1) with associated entropy flux $F : \boldsymbol{R} \to \boldsymbol{R}$ if

$$\partial_t U(u) + \partial_x F(u) = 0 \tag{1.2}$$

for all smooth solutions of (1.1). Equation (1.2) follows from (1.1) iff $F'(u) = U'(u)f'(u)$. In general, however, solutions to (1.1) do not remain smooth, even for smooth initial data. On the other hand, discontinuous or weak solutions of (1.1) are not uniquely determined by their initial data. A criterion must therefore be imposed to select the correct, physically meaningful solutions. When the flux f is convex and when the weak solution u is, say, a function of locally bounded variation, a single entropy inequality of the form [17]

$$\partial_t U(u) + \partial_x F(u) \le 0, \tag{1.3}$$

where U is a given strictly convex entropy, is sufficient to determine a unique solution to the Cauchy problem for (1.1). In this case the entropy inequality (1.3) is equivalent to the viscosity criterion, which states that $u(x,t) = \lim_{\epsilon \to 0} u(x,t;\epsilon)$, where the latter solves

$$\partial_t u + \partial_x f(u) = \epsilon\, \partial_{xx} u, \qquad \epsilon > 0. \tag{1.4}$$

When f is non-convex, the set of entropy inequalities (1.3) for all convex U also selects a unique discontinuous solution, i.e. the classical entropy weak solution, which can also be selected by the vanishing viscosity approach (1.4). When the solution u admits left and right traces $u_\pm = u(x\pm,t)$ at each point, it can be equivalently selected via the Oleinik entropy criterion

$$\frac{f(w) - f(u_-)}{w - u_-} \ge \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \tag{1.5}$$

for all w between u_- and u_+. The theory of classical entropy weak solutions is reviewed in Section 2 of this paper.

When f is non-convex, a single entropy inequality (1.3) is not sufficient to exclude some solutions disallowed by the viscosity criterion (1.4). In other words, there are many discontinuous solutions that satisfy (1.1) and (1.3) in the sense of distributions and assume the same initial data. On the other hand, there are shock waves which are limits of traveling waves for some regularizations of (1.1) with viscosity and dispersive effects kept in balance, but which do not satisfy (1.5). While these weak solutions do satisfy a single entropy inequality (1.3), they do not satisfy the whole set of inequalities (1.3). Moreover those solutions appear to be both stable under small perturbations and physically relevant in a number of situations. In this paper we refer to these waves as *non-classical shocks*, or *non-classical weak solutions.*

Let (U,F) be a strictly convex entropy pair for the equation (1.1). We give the following definition:

Definition 1.1. *A shock wave of the scalar conservation law* (1.1) *for a non-convex flux f is called a non-classical shock wave if it satisfies the single entropy inequality* (1.3)*, but not the Oleinik entropy criterion* (1.5). □

A non-classical shock is *undercompressive* in the sense (see, especially, Isaacson-Marchesin-Plohr [14], Liu [27], Liu-Zumbrun [28], and Shearer-Schaeffer-Marchesin-Paes-Leme [39]) that characteristics pass through it rather than focus on it. This leads to non-uniqueness of solutions for the Riemann problem.

Shock waves for the cubic scalar conservation law fall into three categories:

(1) u_l, u_r have the same sign.
(2) u_l, u_r have opposite signs, but the Oleinik entropy criterion (1.5) is satisfied.
(3) u_l, u_r have opposite signs, the single entropy inequality (1.3) holds, but the Oleinik criterion is not fulfilled.

Category (3) shocks are the non-classical ones. This work is aimed at developing a suitable framework for the study of the existence, uniqueness, and behavior of non-classical solutions to conservation laws. The present paper is a prelude to a study of non-classical shocks in systems of hyperbolic conservation laws which lack genuine nonlinearity or/and strict hyperbolicity. Such systems, arising for instance in magnetohydrodynamics and nonlinear elasticity theory, are endowed with one convex entropy pair, which can be used to formulate an entropy inequality like (1.3) (as well as an entropy balance like (1.8)-(1.9) below).

The principal objectives of this paper are:

(i) the existence and the behavior of non-classical shocks as limits of *diffusive-dispersive traveling waves,*

(ii) and their characterization via a suitable *selection principle,* which we call a *kinetic relation* by analogy with a similar strategy used in material science.

In Section 3, we study the existence and the behavior of non-classical shocks generated by several diffusive-dispersive regularizations for the cubic scalar equation. The "correct" shock waves associated with a given regularization are those that admit a diffusive-dispersive profile. The shocks obtained as limits of traveling waves are then the building blocks used to solve the Riemann problem for (1.1) in the class of solutions defined in Definition 1.1.

Following Wu [42] and Jacobs, McKinney, and Shearer [15], a special emphasis will be placed on the modified Korteweg-deVries-Burgers (MKdVB) equation:

$$\partial_t u + \partial_x u^3 = \epsilon\,\partial_{xx} u + \delta\,\partial_{xxx} u, \tag{1.6}$$

where the parameters ϵ and δ are kept in balance, say $\delta = A\,\epsilon^2$ where $A > 0$ is a constant. Indeed when $\delta = o(\epsilon^2)$ the dispersion effects are negligible and the solutions $u(x,t;\epsilon,\delta)$ to (1.6) converge to a classical weak solution as $\epsilon \to 0$. When $\epsilon^2 = o(\delta)$, the dispersion is dominant and, as $\delta \to 0$, the solutions become highly oscillatory and do not converge weakly to solutions of the cubic conservation law. The effects of dispersion (in the absence of diffusion) have been investigated for the quadratically nonlinear KdV equation by Lax and Levermore [20]. For a more general survey of oscillatory phenomena in both continuous and discrete settings, see the review paper of Lax, Levermore and Venakides [21], and the references contained therein.

In Section 3 we also study an equation augmented with a nonlinear diffusive term (motivated by von Neumann-Richtmyer's pseudo-viscosity [33]) and a linear dispersive term:

$$\partial_t u + \partial_x u^3 = \epsilon\,\partial_x\left(|\partial_x u|\partial_x u\right) + \delta\,\partial_{xxx} u, \tag{1.7}$$

we show that these effects are balanced for $\delta = A\,\epsilon^1$ and prove the existence of a family of TW solutions to (1.7). Using shock waves that admit such diffusive-dispersive profiles, we are able to solve the Riemann problem for (1.1) uniquely, in the class of non-classical solutions. The results are illustrated numerically by solving (1.7) directely for small diffusion and a fixed value of the ratio of diffusion-dispersion. It turns out that, compared to the linear diffusive case (1.6), also studied numerically by Wu [42] and Jacobs et al [15], the classical shocks for the nonlinear diffusion are significantly more oscillatory. It is checked that these oscillations reflect a fundamental feature of the equation (3.17) and are not merely generated by the numerics.

In Subsection 3c, we also prove the existence of some non-classical shocks for the equivalent equation associated with the Beam-Warming scheme, and the non-existence of such shocks for Lax-Wendroff equivalent equation. This result is closely related to the *sign* of the dispersion coefficient in those schemes.

The second issue addressed in this paper is the characterization of the non-classical weak solutions. In Section 4, we propose to use a *kinetic relation* as a selection principle to select the physically meaningful, non-classical shocks. Kinetic relations have been used in material science to control the rate at which phase transitions proceed. In that context they are prescribed only at subsonic phase transitions, not across shocks, and yield the rate of entropy dissipation across the phase boundary; cf. Abeyaratne-Knowles [1], LeFloch [23], and Truskinovsky [40].

For phase transition problems as well as for the non-convex conservation laws studied here, the entropy inequality (1.3) is inadequate to determine a unique solution. It seems natural to extract additional information by studying traveling waves for an augmented system, incorporating small effects (diffusion, dispersion, capillarity, etc). This approach has a long history in the mathematical theory of phase transitions (cf. for instance [2, 38, 40] and the references cited therein). Here we view the non-classical shocks, which span regions of different convexity in the cubic scalar equation, as "phase transitions" and adapt techniques that were proven successful in phase dynamics.

Let (U, F) be a strictly convex entropy pair for (1.1). Our approach in the present paper is to replace the entropy inequality (1.3) by an *entropy balance*, i.e. a stronger condition of the form

$$\partial_t U(u) + \partial_x F(u) = \mu_U, \tag{1.8}$$

where μ_U is a non-positive bounded Borel measure, called the entropy dissipation measure for the entropy U. This measure must be determined by an analysis of the traveling wave solutions associated with a given approximation method for (1.1). The measure μ_U has a non-zero mass only along the curves of discontinuity for the solution u. For non-classical shocks, we impose the dissipation measure to be equal to a given function $\phi_U(s)$ of the speed of propagation, s,

$$-s\big(U(u_+) - U(u_-)\big) + F(u_+) - F(u_-) = \mu_U\big\{(x,t)\big\} = \phi_U(s), \tag{1.9}$$

which we refer to as the *kinetic relation*. In general, different approximation methods generate different kinetic relations.

Given a strictly convex entropy U and a kinetic function $\phi_U(s)$, which is assumed to satisfy certain assumptions of regularity and consistency (cf. Section 4), we make the following definition.

Definition 1.2. *A weak solution to the scalar conservation law* (1.1) *is an admissible, non-classical, weak solution if*

(i) the entropy inequality (1.3) *holds,*

(ii) and every non-classical shock of Definition 1.1 satisfies the kinetic relation (1.9). □

As we demonstrate below, this definition provides us with a selection principle that picks up the unique non-classical solution in the case of the Riemann problem for the cubic flux $f(u) = u^3$. In order to discriminate between this non-classical solution, and the classical one, a *nucleation criterion* must also be imposed. For the Riemann problem, piecewise smooth solutions are sought, and the left and right traces of the solution exist and so (1.9) is well defined. For the general Cauchy problem, it would be interesting to determine the proper functional space in which to search for the solution u; see the discussion in Section 5.

Turning, in Section 4, to the Riemann problem for cubic flux we derive conditions on the function ϕ_U which ensure that the problem has a unique non-classical solution. Allowing non-classical shocks gives rise to a one-parameter family of solutions and the kinetic relation is then employed to select a unique solution. Furthermore, we construct the function $\phi_U(s)$ which is generated by the modified Korteweg-deVries-Burgers approximation (1.6) and its nonlinear version (1.7). This, in turn, allows us to recover the Riemann solution obtained by Jacobs-McKinney- Shearer [15] based on (1.6), and the Riemann solution we construct in Section 3, based on (1.7). In principle, a kinetic relation can also be derived for finite difference schemes such as the Beam-Warming scheme considered in Section 6.

The analogy between non-classical shocks and phase transitions is an imperfect one: for the Riemann problem with cubic flux, there may be either classical or non-classical shocks when $u_l u_r < 0$, and we must impose a "nucleation criterion" to determine which of the two possibilities actually occurs. Our nucleation is in a different sense than that of Abeyaratne and Knowles [1]. Their criterion concerned the nucleation, or opening up of a second phase, within a region of a single phase. We show that this latter form of nucleation can never occur for the cubic scalar equation.

Instead, we need a condition to say when a non-classical shock will form, even when the initial data lies on opposite sides of $u = 0$. While the nucleation criterion we introduce will, in general, depend on the specific regularization of (1.1), we find that there is a "universal nucleation criterion" which must be satisfied by any augmented equation, in order to observe non-classical shocks. As we show in Section 4, a more refined notion of what constitutes a different "phase" can substitute for this universal condition. For the equation (1.7), the universal criterion is shown to be sufficient: the equation always takes the non-classical solution if it exists, while for the equation (1.6), an additional condition, relating the initial data to a relevant parameter in the equation, must also be satisfied.

In Section 5, based on the compensated compactness method and after the important work by Schonbek [37], we study the vanishing diffusion-dispersion limit for the Cauchy problem for the cubic scalar equation. A main observation in Section 5 is that the regularizations (1.6), (1.7) generate non-classical solutions that do satisfy the entropy inequality (1.3) for $U(u) = u^2/2$, but, in general, do not satisfy (1.3) for non-quadratic entropies.

Finally, in Section 6, we present numerical evidence for non-classical shocks in finite difference schemes. The main observation of this section is that non-classical shocks are observed for the Beam-Warming scheme, while no such shocks are obtained with the Lax-Wendroff scheme. These results depend crucially on the sign of the dispersion coefficient and the type of flux-function under consideration. We recall that in their pioneering paper on the entropy consistency for difference schemes, Harten-Hyman-Lax [9] –for a flux function having two changes of convexity– numerically demonstrated the existence of stationary and stable discrete shocks that do not satisfy the Oleinik entropy criterion. Section 6 also points to the stability of the non-classical shocks. In addition, the design of numerical methods which compute non-classical shocks turns out to be very subtle. Section 6 is intended as a step towards the design of shock-capturing schemes for problems involving non-classical shocks. Currently, the most reliable and practical approach to numerical calculation of shocks is to resolve their inner structure. Other strategies include Glimm's random choice scheme, the front tracking methods, and the level set methods; cf. the discussions in LeFloch [23], Zhong-Hou-LeFloch [43], and Hou-LeFloch-Rosakis [13], respectively.

2. Classical Solutions to Conservation Laws.

In this section we review the classical weak solutions to a non-convex conservation law (1.1). For definiteness we consider the cubic flux-function $f(u) = u^3$, i.e.

$$\partial_t u + \partial_x u^3 = 0, \qquad u(x,t) \in \boldsymbol{R}, \quad x \in \boldsymbol{R},\, t > 0, \tag{2.1}$$

and we specify an initial condition

$$u(x,0) = u_0(x), \qquad x \in \boldsymbol{R}. \tag{2.2}$$

The standard approach to constructing the weak solutions is to add a vanishing viscosity of the form $\epsilon\, \partial_{xx} u$, $\epsilon > 0$, to the right-hand side of (2.1):

$$\partial_t u^\epsilon + \partial_x \big(u^\epsilon\big)^3 = \epsilon\, \partial_{xx} u^\epsilon, \qquad \epsilon > 0, \tag{2.3}$$

which yields an equation for the approximate solution $u^\epsilon(x,t)$.

When $u_0 \in L^2(\boldsymbol{R}) \cup L^\infty(\boldsymbol{R})$, the solutions u^ϵ to (2.3) satisfy the maximum principle and, in particular, for all $t \geq 0$

$$\|u^\epsilon(.,t)\|_{L^\infty(\boldsymbol{R})} \leq \|u_0\|_{L^\infty(\boldsymbol{R})}. \tag{2.4}$$

The functions u^ϵ converge, almost everywhere, to a function $u \in L^\infty(\boldsymbol{R} \times \boldsymbol{R}_+)$ which is a solution to (2.1) and satisfies *all* the entropy inequalities (1.3) in the sense of distributions. The derivation of (1.3) for the limiting solution u is based on the observation that the right-hand side of

$$\partial_t U(u^\epsilon) + \partial_x F(u^\epsilon) = \epsilon\, \partial_{xx} U(u^\epsilon) - \epsilon\, U''(u^\epsilon) |\partial_x u^\epsilon|^2 \tag{2.5}$$

is the sum of a term converging to zero in the Sobolev space $H^{-1}_{\rm loc}$ and a non-positive bounded Borel measure, provided U is convex.

Additional estimates on the solution u are known when more regularity on u_0 is available. For instance, if we further assume that u_0 belongs to the space $BV(\boldsymbol{R})$ of bounded functions of bounded variation, the approximate solutions u^ϵ are uniformly bounded in $L^\infty(\boldsymbol{R}_+, BV(\boldsymbol{R}))$ and so u belongs to the same space, with the additional property that

$$TV(u(.,t)) \leq TV(u_0), \qquad t \geq 0, \tag{2.6}$$

where we denote by TV the total variation of a function.

The properties (2.4)–(2.6), which are fundamental to the classical theory of shock waves, are violated by the non-classical weak solutions.

Turning now to the issue of uniqueness, it is known that the set of all entropy inequalities (1.3) allows one to characterize uniquely the solution obtained by the vanishing viscosity approximation. When $u_0 \in L^1(\boldsymbol{R}) \cap L^\infty(\boldsymbol{R})$, it is proved that u satisfies the L^1 contraction principle

$$\|u(.,t) - v(.,t)\|_{L^1(\boldsymbol{R})} \leq \|u_0 - v_0\|_{L^1(\boldsymbol{R})}, \tag{2.7}$$

where v is the limit associated with an initial data $v_0 \in L^1(\boldsymbol{R}) \cap L^\infty(\boldsymbol{R})$. The property (2.7) clearly implies the uniqueness of the solution for the Cauchy problem. Finally, for BV solutions, the entropy inequalities understood in the sense of distributions are equivalent to a pointwise formulation, i.e. the Oleinik entropy inequalities (1.5). This provides us with at least three approaches to the selection of the classical weak solutions.

To complete this brief review, we explicitly describe the Oleinik solution of the Riemann problem, which will be useful in the course of this paper. The following construction is closely related to the convex or concave hull of the cubic function. The initial data for the Riemann problem, by definition, consists of two constant states, u_l and u_r, with

$$u(x,0) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0. \end{cases} \tag{2.8}$$

To be specific, we shall fix $u_l > 0$ in the whole of this paper, the other case being completely analogous. We distinguish between three cases depending upon the value of u_r compared to the points $u = -u_l/2$ and $u = u_l$. For $u_r \geq u_l$, the solution to (2.1)-(2.8) is the rarefaction wave

$$u(x,t) = \begin{cases} u_l, & x < 3\,u_l^2\,t, \\ \sqrt{x/3t}, & 3u_l^2 t < x < 3\,u_r^2\,t, \\ u_r, & x > 3\,u_r^2\,t. \end{cases} \tag{2.9}$$

If $-u_l/2 \leq u_r < u_l$, the solution contains a single shock wave:

$$u(x,t) = \begin{cases} u_l, & x < st, \\ u_r, & x > st, \end{cases} \tag{2.10}$$

where the shock speed s is given by the Rankine-Hugoniot (RH) condition

$$s = u_l^2 + u_l u_r + u_r^2. \tag{2.11}$$

The significance of the limiting case $u_r = -u_l/2$ is that, at this point, the shock speed and the right-hand wave speed coincide; i.e. $s = f'(u_r)$. We see, therefore, that a single shock satisfying the Oleinik entropy criterion (1.5) cannot connect u_l to points $u_r < -u_l/2$. Instead, the entropy satisfying solution in this region consists of a shock to $-u_l/2$, immediately followed by a rarefaction wave from $-u_l/2$ to u_r; that is,

$$u(x,t) = \begin{cases} u_l, & x < (3/4)\,u_l^2\,t, \\ -\sqrt{x/3t}, & (3/4)\,u_l^2\,t < x < 3\,u_r^2\,t, \\ u_r, & x > 3\,u_r^2\,t. \end{cases} \tag{2.12}$$

We observe that the Riemann solution depends continuously upon its initial states: if $u_r \to u_l$ with u_l fixed, then all the values u taken by the solution converge to u_l as well. Furthermore each of the above discontinuities can be realized as the end points of a viscous profile, i.e. a traveling wave solution of the equation (2.3).

For future reference, we state:

Proposition 2.1.

(1) *Given a state $u_l > 0$, the set $\mathbf{S}(u_l)$ consisting of all states u_r that can be achieved through a diffusive traveling wave (i.e. a solution to (2.3)), taking the values u_l and u_r at the left and the right ends respectively, is given by*

$$\mathbf{S}(u_l) = [-\frac{u_l}{2}, u_l).$$

(2) *Given a state $u_l > 0$, the Riemann problem with initial data u_l and u_r is then solved by*
   (i) a rarefaction wave, if $u_r \geq u_l$;
   (ii) a shock wave, if $u_r \in [-u_l/2, u_l)$;
   (iii) a shock wave with attached rarefaction, if $u_r < -u_l/2$.

□

3. Non-Classical Shocks as Limits of Traveling Waves.

In this section we study the existence of traveling wave (TW) solutions to several diffusive-dispersive regularizations for the cubic scalar equation. As the coefficients of diffusion and dispersion vanish, the TW converge to non-classical shocks. The latter are the building blocks to our construction of a non-classical solution to the Riemann problem for the conservation law (1.1). We consider three examples: the modified Korteweg-deVries-Burgers equation, a nonlinear diffusive-dispersive approximation, and the equivalent equation for the Lax-Wendroff and Beam-Warming schemes.

3a. Modified Korteweg-deVries-Burgers Equation.

This subsection reviews the recent construction of Jacobs, McKinney, and Shearer [15] for the MKdVB equation:

$$\partial_t u + \partial_x u^3 = \epsilon\, \partial_{xx} u + \delta\, \partial_{xxx} u, \tag{3.1}$$

where $\epsilon > 0$ and δ are real parameters, namely the coefficients of linear diffusion and dispersion, respectively. There are two markedly different behaviors depending on whether (i) $\delta \leq 0$ or (ii) $\delta > 0$. This is in contrast to the original (quadratically nonlinear) Korteweg-deVries-Burgers equation, for instance studied by Bona and Schonbek [3], for which a change of sign in δ can be "undone" by the transformation $(x, u) \to (-x, -u)$.

We search for the traveling waves connecting two states u_l and u_r. Let

$$v(\xi) = u(x,t), \qquad \xi = \frac{x - s\,t}{\sqrt{\delta}}, \tag{3.2}$$

where s is given by the Rankine-Hugoniot relation (2.11). The TW solutions of (3.1) are smooth solutions $u(x,t)$ that only depend on the variable ξ and satisfy the boundary conditions

$$\lim_{\xi \to -\infty} v = u_l, \qquad \lim_{\xi \to +\infty} v = u_r, \tag{3.3}$$

$$\lim_{\xi \to \pm\infty} v' = 0, \qquad \lim_{\xi \to \pm\infty} v'' = 0, \tag{3.4}$$

where we use the notation $v' = dv/d\xi$. We assume here that $u_r < 0 < u_l$, since we are interested in trajectories connecting different regions of convexity.

In the case $\delta \leq 0$, all admissible TW were shown in [15] to approach the (Oleinik) shock trajectories of the classical shock theory reviewed in Section 2, in the limit $\epsilon, \delta \to 0$.

We now discuss the second case, $\delta > 0$, in more detail. The equation (3.1) becomes

$$-s\, v' + (v^3)' = \frac{\epsilon}{\sqrt{\delta}}\, v'' + v'''. \tag{3.5}$$

Forming the new parameter

$$\mu = \frac{\epsilon}{\sqrt{\delta}} \tag{3.6}$$

and taking the limit $\epsilon, \delta \to 0$, with μ fixed, keeps in balance the diffusive and dispersive effects. Integrating (3.5) once and applying conditions (3.3)-(3.4), we arrive at the system

$$\begin{aligned} v' &= w, \\ w' &= -\mu\, w + p(v) \end{aligned} \tag{3.7}$$

with $p(v) = C - s\,v + v^3$ and $C = s\,u_l - u_l^3$. The equilibrium points of (3.7) are points $(v,w) = (\bar v, 0)$ with $p(\bar v) = 0$. Given $u_l > 0$ and $u_r \neq -u_l/2$, there always exists three equilibria, which we denote by $v_l \equiv u_l > v_m > v_r$. So u_r equals either v_m or v_r. The states v_r and v_m are the two roots of the polynomial

$$v^2 + v_l\,v + v_l^2 - s = 0. \tag{3.8}$$

We note for future reference that, since $p(v)$ has no quadratic term,

$$v_r + v_m + v_l = 0. \tag{3.9}$$

The eigenvalues of (3.7) at the equilibria are

$$\begin{aligned} \lambda_\pm(\bar v) &= \frac{1}{2}\left(-\mu \pm \sqrt{\mu^2 + 4(3\bar v^2 - s)}\right) \\ &= \frac{1}{2}\left(-\mu \pm \sqrt{\mu^2 + 4p'(\bar v)}\right). \end{aligned} \tag{3.10}$$

Since $p'(v) = 3v^2 - s$ is positive at the outer equilibria, v_r and v_l, and so $\lambda_-(\bar v) < 0$ and $\lambda_+(\bar v) > 0$, the states v_r and v_l are saddle points. Since $p'(v_m) < 0$, the state v_m is stable ($\mu \neq 0$ and the $\lambda_\pm$'s have a non-zero real part) and is either a node, if $\mu^2 + 4(3v_m^2 - s) \geq 0$ (two negative real eigenvalues), or a spiral if $\mu^2 + 4(3v_m^2 - s) < 0$ (two complex conjugate eigenvalues).

Jacobs, McKinney, and Shearer show that there exists a heteroclinic saddle connection of the form

$$w(\xi) = \frac{1}{\sqrt{2}}(v(\xi) - v_r)(v(\xi) - v_l). \tag{3.11}$$

Substituting (3.11) into (3.7) and setting $v = 0$ (which, by continuity, must be a value attained by $v(\xi)$ for some ξ) leads us to the condition

$$v_r + v_l = \sqrt{2}\,\mu + 2\,v_m. \tag{3.12}$$

Combining (3.9) and (3.12) shows that in order for there to be a saddle-saddle connection of this type, we must have

$$\begin{aligned} v_m &= -\frac{\sqrt{2}}{3}\,\mu < 0, \\ v_r &= -v_l + \frac{\sqrt{2}}{3}\,\mu. \end{aligned} \tag{3.13}$$

So the right state $u_r = v_r$ is uniquely determined from the left state u_l (and the parameter μ which should be considered as fixed here). Using v_m of (3.13) in (3.8) gives the shock speed as a function of v_l and μ:

$$s = v_l^2 - \frac{\sqrt{2}}{3}\,\mu\,v_l + \frac{2}{9}\,\mu^2. \tag{3.14}$$

Comparing the two expressions in (3.13), we observe that the trajectory (3.11) connects two saddle points only if we actually have $v_r < v_m$, that is

$$v_l > \frac{2\sqrt{2}}{3}\,\mu; \tag{3.15}$$

otherwise (3.11) is a connection between a saddle and a stable node/spiral. Jacobs et al then prove that (3.11) represents the *only* trajectory joining the two saddle points.

Since $3\,v^2 - s > 0$ at both $v = v_r$ and $v = v_l$, we see that the traveling wave speed is slower than the characteristic speeds on both sides. Thus in the limit of $\epsilon, \delta \to 0$ the saddle-saddle trajectory becomes an undercompressive shock, through which the characteristics pass. This shock does not satisfy the Lax entropy criterion and, therefore, the Oleinik entropy criterion also fails to hold.

From the above discussion and the results in [15], we can state an analogue of Proposition 2.1 but now for the diffusive-dispersive approximation.

Proposition 3.1.

(1) *Given a state* $u_l > 0$, *the set* $\mathbf{S}(u_l)$ *consisting of all states* u_r *that can be achieved through a diffusive-dispersive traveling wave, i.e. a solution to* (3.1), *taking the values* u_l *and* u_r *at the left and the right ends respectively, is given by*

$$\mathbf{S}(u_l) = \begin{cases} [-\frac{u_l}{2}, u_l) & \text{if } u_l \le \frac{2\sqrt{2}}{3}\mu, \\ \{-u_l + \frac{\sqrt{2}}{3}\mu\} \cup [-\frac{\sqrt{2}}{3}\mu, u_l), & \text{if } u_l > \frac{2\sqrt{2}}{3}\mu, \end{cases}$$

where the coefficient μ *is the constant given by* (3.6).

(2) *Given a state* $u_l > 0$, *the Riemann problem with initial data* u_l *and* u_r *is solved as follows using the shock curve derived in (1) above. If* $u_l \le \frac{2\sqrt{2}}{3}\mu$, *the solution is the classical Riemann solution described in Proposition 2.1. If* $u_l > \frac{2\sqrt{2}}{3}\mu$, *the solution consists of*

(i) a rarefaction wave, if $u_r \ge u_l$;

(ii) a classical shock wave, if $u_r \in [-\frac{\sqrt{2}}{3}\mu, u_l)$;

(iii) two shock waves, if $u_r \in (-u_l + \frac{\sqrt{2}}{3}\mu, -\frac{\sqrt{2}}{3}\mu)$; *that is: a (slow) non-classical shock from* u_l *to* $-u_l + \frac{\sqrt{2}}{3}\mu$ *followed by a (fast) classical shock connecting to* u_r;

(iv) a shock wave and a rarefaction wave, if $u_r \le -u_l + \frac{\sqrt{2}}{3}\mu$; *that is: a slow non-classical shock wave from* u_l *to* $-u_l + \frac{\sqrt{2}}{3}\mu$, *followed by a rarefaction wave connecting to* u_r.

□

Observe that when $u_l \le \frac{2\sqrt{2}}{3}\mu$, the shock curve is an interval, and when $u_l > \frac{2\sqrt{2}}{3}\mu$, it consists of the union of an isolated point and an interval.

We shall return to the Riemann problem in Section 4. We note here that the differential equation (3.11) for the heteroclinic saddle-saddle connection has the explicit solution

$$v(\xi) = \frac{v_l + v_r}{2} - \frac{v_l - v_r}{2} \tanh\left(\frac{v_l - v_r}{2\sqrt{2}}\xi\right). \tag{3.16}$$

In Section 5 we show that this trajectory does not in general have negative entropy dissipation, except for the quadratic entropy.

3b. Nonlinear Diffusive-Dispersive Approximation.

In this section we study traveling wave solutions to the cubic scalar conservation law, augmented by a vanishing nonlinear diffusion (or pseudo-viscosity) and a linear dispersion:

$$\partial_t u + \partial_x u^3 = \epsilon\, \partial_x\left(|\partial_x u| \partial_x u\right) + \delta\, \partial_{xxx} u. \tag{3.17}$$

Both ϵ and δ are positive constants in (3.17). Under a change of sign for δ, the equation (3.17) represents a special case of the nonlinear-diffusive dispersive equation

$$\partial_t u + \partial_x u^3 = \epsilon\, \partial_x\left(b(\partial_x u)\, \partial_x u\right) + \delta\, \partial_{xxx} u \tag{3.18}$$

studied for instance by LeFloch and Natalini [24]. The particular choice $b(\lambda) = |\lambda|$ in (3.18) gives (3.17). For $\delta = 0$, equation (3.18) has been studied in Marcati-Natalini [32] as a simplified model of the pseudo-viscosity proposed by von Neumann-Richtmyer [33]. Observe that [33] also proposed the pseudo-viscosity $b(u) = \left(\partial_x u\right)_-$, where $\alpha_- = -\alpha$ if $\alpha \le 0$ and $\alpha_- = 0$ if $\alpha \ge 0$; the analysis presented here could also be extended to this type of diffusion. The nonlinear viscosity in (3.17) was also studied in connection with numerical methods by Raviart [34] and Richtmyer-Morton [35].

We make the change of variables

$$u(x,t) = v(\xi), \qquad \xi = \frac{x - st}{\sqrt{\delta}}, \tag{3.19}$$

where the wave speed, s, is determined by the Rankine-Hugoniot condition. This leads to the ODE

$$-s\,v' + \left(v^3\right)' = \frac{\epsilon}{\delta}\,\left(|v'|v'\right)' + v'''. \tag{3.20}$$

The boundary conditions are the same as for MKdVB, namely

$$\lim_{\xi\to-\infty} v(\xi) = u_l, \qquad \lim_{\xi\to+\infty} v(\xi) = u_r, \qquad \lim_{|\xi|\to\infty} v^{(j)}(\xi) = 0, \tag{3.21}$$

where $j = 1, 2, 3$. For a TW connecting $u_l > 0$ to u_r, the Rankine-Hugoniot gives

$$s = \frac{u_l^3 - u_r^3}{u_l - u_r} = u_l^2 + u_l\,u_r + u_r^2. \tag{3.22}$$

We can immediately integrate (3.20) once to get

$$C - s\,v + v^3 = \frac{\gamma}{2}|v'|v' + v'', \tag{3.23}$$

where $C = s\,u_l - u_l^3$ is provided by the boundary condition on v as $\xi \to -\infty$, and where the parameter $\gamma > 0$ is defined by

$$\gamma = \frac{2\,\epsilon}{\delta}. \tag{3.24}$$

(In contrast to the parameter $\mu = \epsilon/\sqrt{\delta}$, which emerged in the study of TW solutions for MKdVB, the parameter γ has δ and ϵ appearing with the same exponent.)

Equation (3.23) may be written as a first order system; specifically

$$\begin{aligned} v' &= w, \\ w' &= p(v) - \frac{\gamma}{2}\,|w|\,w \end{aligned} \tag{3.25}$$

with $p(v) \equiv C - s\,v + v^3$. The equilibrium points of (3.25) are points $(v, w) = (\bar v, 0)$ where $p(\bar v) = 0$. For the case that the cubic $p(v)$ has three distinct zeros, denoted by $v_l > v_m > v_r$, the fact that $p(v)$ lacks a quadratic term implies

$$v_l + v_m + v_r = 0. \tag{3.26}$$

Whenever the meaning is clear, we abbreviate the equilibrium point $(\bar v, 0)$ by $\bar v$.

Consider in passing the case that $\delta = 0$ in (3.17) (i.e. no dispersion). In terms of the variable $\xi = x - s\,t$, the integrated TW equation

$$\epsilon\,|v'|\,v' = p(v),$$

$\epsilon > 0$, is now on the real line, so trajectories satisfying the boundary conditions (3.21) must join *adjacent* equilibrium points. Then since

$$v' = \begin{cases} \sqrt{p(v)/\epsilon}, & v' > 0, \quad p(v) > 0, \\ -\sqrt{-p(v)/\epsilon}, & v' < 0, \quad p(v) < 0, \end{cases}$$

and $p(v) > 0$ for all $u_r < v < u_m$, and $p(v) < 0$ for all $u_m < v < u_r$, we see that the only trajectories allowed are from u_r to u_m, and from u_l to u_m. So, by (3.26), the only TW solutions connecting u_- to u_+ satisfy $-1/2 \le u_+/u_- < 1$. As $\epsilon \to 0$, these tend to the classical shocks of Section 2. Henceforth, we assume that $\delta > 0$.

Linearizing (3.25) about the equilibrium points, we find that the eigenvalues are given by

$$\begin{aligned} \lambda_\pm(\bar v) &= \pm\sqrt{p'(\bar v)} \\ &= \pm\sqrt{3\,\bar v^2 - s}. \end{aligned} \tag{3.27}$$

As was the case for MKdVB, the outer equilibrium points, v_l and v_r, are both saddle points, since at those points $3\,\bar v^2 - s > 0$, so $\lambda_- < 0$ and $\lambda_+ > 0$. In contrast to MKdVB, however, the middle equilibrium, v_m, is now an elliptic equilibrium point or center since $\lambda_\pm(v_m)$ are purely imaginary. While this fact may seem to indicate the presence of periodic orbits or homoclinic connections for (3.25), the following proposition shows that this is *not* the case.

Proposition 3.2. *System* (3.25) *has no non-trivial homoclinic orbits, nor does it have any non-trivial periodic orbits. In addition, all heteroclinic orbits from u_l to u_r must satisfy $u_l^2 > u_r^2$.* □

Note that the condition $u_l^2 > u_r^2$ is slightly stronger than the condition we will arrive at in Section 4 via entropy dissipation for a quadratic entropy function, which is $u_l^2 \geq u_r^2$.

Proof of Proposition 3.2. If (v, w) satisfy (3.25), then v satisfies (3.23). We use an energy estimate: multiply (3.23) by $v'(\xi)$ and integrate from $\xi = -\infty$ to $\xi = +\infty$. This gives

$$\left[C\, v - \frac{s}{2}v^2 + \frac{1}{4}v^4\right]_{-\infty}^{+\infty} = \frac{\gamma}{2}\int_{-\infty}^{+\infty} |v'|(v')^2 d\xi + \left[\frac{1}{2}(v')^2\right]_{-\infty}^{+\infty}. \tag{3.28}$$

From the boundary conditions (3.21) on v', the second term on the right-hand side of (3.28) vanishes. Meanwhile, the integral must be positive, since $\gamma > 0$ and $v' \not\equiv 0$ (we exclude the trivial solution). This implies that

$$C\,(u_r - u_l) - \frac{s}{2}(u_r^2 - u_l^2) + \frac{1}{4}(u_r^4 - u_l^4) > 0.$$

A brief calculation using $C = s\, u_l - u_l^3$ leads to

$$\frac{1}{4}(u_r - u_l)^2(u_l^2 - u_r^2) > 0. \tag{3.29}$$

We therefore have (1) $u_l \neq u_r$, and (2) $u_l^2 > u_r^2$.

Thus homoclinic orbits are not allowed and all heteroclinic orbits must satisfy the second condition above.

Furthermore, suppose there were a periodic orbit with period $P > 0$, then $v(\xi + P) = v(\xi)$ and $w(\xi + P) = w(\xi)$. Multiplying the second equation of (3.25) by w and integrating over one period gives

$$\int_P w\, w' d\xi = \int_P p(v) w\, d\xi - \frac{\gamma}{2}\int_P w^2 |w|\, d\xi, \tag{3.30}$$

so that the integral on the left-hand side of (3.30) is immediately seen to vanish, while from the first equation of (3.25), the first integral on the right-hand side of (3.30) is also zero. Thus the only solution to (3.30) is the trivial one, $w \equiv 0$, so there are no non-trivial periodic orbits for (3.25) and Proposition 3.2 is proved. □

The following corollary describes which equilibrium points of (3.25) may possibly be joined by a traveling wave.

Corollary 3.3. *For the equilibrium points $v_l > v_m > v_r$ of* (3.25) *there is the following dichotomy:*

(1) *If $v_m \geq 0$, the only trajectory from v_l is the heteroclinic connection to v_m.*
(2) *If $v_m < 0$, the equilibrium v_l may be joined to either v_m or v_r, with the latter trajectory being a heteroclinic saddle connection.* □

Proof of Corollary 3.3. From (3.26), $v_r = -v_l - v_m$. Since $v_l > 0$, if $v_m > 0$, then $|v_r| > |v_l|$ and by Proposition 3.2 there cannot be an orbit between v_l and v_r. If $v_m < 0$, then Proposition 3.2 is not sufficient to rule out orbits from v_l to either of v_m or v_r. □

We note that when $v_m < 0$, the state v_r is restricted by (3.26) and the fact that $v_m > v_r$, to belong the interval $-v_l < v_r < -v_l/2$.

For a TW with endpoints u_l and u_r at $\xi = -\infty$ and $\xi = +\infty$, respectively, both of these values must correspond to equilibrium points of (3.25). We may identify $u_l = v_l$ and by Corollary 3.3, we have $u_r = v_m$, if u_r is positive. In this case the trajectory spirals into the point $(v_m, 0)$, so the TW profile contains oscillations about u_r, which decay to zero as $\xi \to +\infty$. When $u_r < 0$, we cannot yet say whether $u_r = v_r$ or $u_r = v_m$. In the argument that follows, we will show that if $v_m < 0$, then for each $\gamma > 0$, there is a value of v_r in the range $-v_l < v_r < -v_l/2$, which can be joined to v_l by a heteroclinic saddle connection; i.e. when $v_m < 0$, we always have $u_r = v_r$. To this end, we integrate system (3.25) as follows.

For $w < 0$ (i.e. $v' < 0$, which will be the case for a heteroclinic orbit leaving v_l and connecting to $v_r < v_l$), the second equation in (3.25) becomes

$$\frac{dw}{d\xi} - \frac{\gamma}{2} w^2 = p(v). \tag{3.31}$$

Setting $y(\xi) = w^2(\xi)$ in (3.31) results in

$$\frac{dy}{dv} - \gamma y = 2p(v), \tag{3.32}$$

which has the homogeneous solution $De^{\gamma v}$, and a particular solution of the form

$$y_P(v) = av^3 + bv^2 + cv + d. \tag{3.33}$$

Substituting (3.33) into (3.32) we find that

$$y(v) = De^{\gamma v} - \frac{2}{\gamma}\left[v^3 + \frac{3}{\gamma}v^2 - \left(s - \frac{6}{\gamma^2}\right)v - \frac{s}{\gamma} + C + \frac{6}{\gamma^3}\right]. \tag{3.34}$$

The constant D is found by the condition $y(v_l) = 0$, which gives (after using $p(v_l) = 0$) that

$$D = \frac{2e^{-\gamma v_l}}{\gamma^4}\left((3v_l^2 - s)\gamma^2 + 6v_l\gamma + 6\right). \tag{3.35}$$

In summary, $w = -\sqrt{y}$, where y is given by (3.34) and (3.35) and this trajectory leaves v_l and decreases in v (recall that $v' = w < 0$) until it reaches a value $v = v_0$ such that $y(v_0^-) < 0$. At this point we can no longer invert y to find $w(v)$.

We now look for a particular trajectory joining v_l to the other saddle point v_r. This means $y(v_r) = 0$ in the equation (3.34), and we also make use of the fact that $p(v_r) = 0$. We end up with an equation connecting the three parameters v_l, v_r, and γ :

$$e^{\gamma(v_r - v_l)}\left((3v_l^2 - s)\gamma^2 + 6v_l\gamma + 6\right) = (3v_r^2 - s)\gamma^2 + 6v_r\gamma + 6, \tag{3.36}$$

where we have divided both sides by $2/\gamma^4$, so that the apparent solution $\gamma = 0$ is actually a singularity of the equation.

We claim that for $v_l > 0$ and $v_r \in (-v_l, -v_l/2)$, the equation (3.36) is solved by a unique value of $\gamma > 0$. We offer analytical proof of existence for v_r in a subinterval of $(-v_l, -v_l/2)$, as well as the numerical solution of (3.36) via Newton's method, to support our claim.

Proposition 3.4. *For $v_l > 0$ and for v_r in the interval $(1 - \sqrt{3})v_l \le v_r < -v_l/2$, there is a value*

$$\gamma_+ = (3v_r^2 - s)^{-1}\left(3|v_r| + \sqrt{3(2s - 3v_r^2)}\right) \tag{3.37}$$

with s given by the R-H relation (3.22), such that (3.36) has a solution in $\gamma_+ < \gamma < +\infty$. □

Proof of Proposition 3.4. Denoting $A = 3v_l^2 - s$ and $B = 3v_r^2 - s$, we have that $A > B > 0$ by (3.22) and Proposition 3.3, and we can rewrite the quadratics on the left- and right-hand sides of (3.36) as

$$\begin{aligned} Q_l(\gamma; v_l, v_r) &= A(\gamma + 3v_l A^{-1})^2 + 3(3v_l^2 - 2s)A^{-1}, \\ Q_r(\gamma; v_l, v_r) &= B(\gamma + 3v_r B^{-1})^2 + 3(3v_r^2 - 2s)B^{-1}. \end{aligned} \tag{3.38}$$

Using the value of s from (3.22), we see that $3v_l^2 - 2s$ is positive for

$$-\frac{1+\sqrt{3}}{2}v_l < v_r < \frac{1+\sqrt{3}}{2}v_l,$$

and so the minimum value of Q_l, which actually occurs when $\gamma = -3v_l A^{-1} < 0$, is greater than zero for all v_r in $(-v_l, -v_l/2)$. Thus Q_l is strictly positive in the range of interest. This, in turn, means that the left-hand side of (3.36) is always positive.

For Q_r, on the other hand, $3v_r^2 - 2s \le 0$ in (3.38) when $(1-\sqrt{3})v_l \le v_r \le (1+\sqrt{3})v_l$. At its minimum value, which occurs at $\gamma = 3|v_r|B^{-1} > 0$, Q_r is therefore negative. In particular, $Q_r < 0$ in the applicable subinterval

$$(1-\sqrt{3})v_l \le v_r \le -v_l/2. \tag{3.39}$$

In fact, when (3.39) holds, $Q_r(\gamma; v_l, v_r) < 0$ for $\gamma_- < \gamma < \gamma_+$, where

$$\gamma_\pm = B^{-1}\left(3|v_r| \pm \sqrt{3(2s - 3v_r^2)}\right).$$

Since the left-hand side of (3.36) is always positive and tends to zero as $\gamma \to +\infty$, and the right-hand side of (3.36) is zero at $\gamma = \gamma_+$ and tends to $+\infty$, as $\gamma \to +\infty$, there must be a solution of (3.36) for $\gamma \in (\gamma_+, +\infty)$. This proves Proposition 3.4. □

The numerical solution of (3.36) in the interval $-v_l < v_r < -v_l/2$ indicates that for $v_l > 0$, there is a value of $\gamma > 0$ which solves the equation; i.e. there is a heteroclinic saddle connection whenever $v_m < 0$. Interestingly, the value of γ that solves (3.36) for a given value of v_r/v_l scales with u_l. For if we introduce the quantities

$$\alpha = \frac{v_r}{v_l} \qquad \text{and} \qquad \Gamma = v_l\gamma, \tag{3.40}$$

then using (3.22) we can rewrite (3.36) as

$$e^{\Gamma(\alpha-1)}\left((2-\alpha-\alpha^2)\Gamma^2 + 6\Gamma + 6\right) = (2\alpha^2 - \alpha - 1)\Gamma^2 + 6\alpha\Gamma + 6, \tag{3.41}$$

so that Γ depends only on α, and γ can be recovered from (3.40) as Γ/v_l. In Figure 3.1, below, we plot the numerical solution to (3.41) as Γ versus α, for $\alpha \in (-1, -1/2)$.

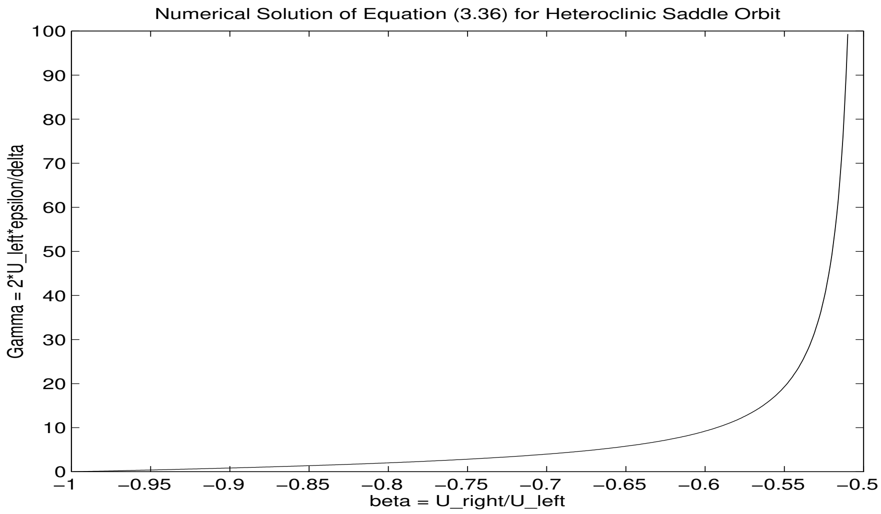

Figure 3.1

Based on the analytical and numerical evidence presented thus far, we have the following picture for solutions of the Riemann problem for (3.17) when ϵ and $\delta = 2\epsilon/\gamma$ are positive:

Consider the Riemann problem for the cubic equation with left- and right-hand states u_- and u_+, respectively. We construct here a solution of the Riemann problem by using shocks that have viscous profiles corresponding to the approximation (3.17). According to the just-mentioned scaling, we can take $u_- \equiv 1$. Then for a particular choice $\gamma = \gamma_0$ in (3.17), we have the scaled quantity Γ_0. For $u_+ > 1$, a centered rarefaction joins the two states. When $0 < u_+ < 1$, there is a classical shock connecting u_- to u_+. As u_+ is decreased below zero, the classical shock persists until

$$u_+ = -1 - \alpha(\Gamma_0),$$

where $\alpha(\Gamma_0)$ is the solution to (3.41). At this particular value of u_+, there is a shock from $u_- = 1$ to $\alpha(\Gamma_0)$, with speed $S_1 = 1 + \alpha + \alpha^2$. This is followed by a second shock from α to u_+, having speed $S_2 = \alpha^2 + u_+\alpha + u_+^2$. Note that $S_1 = S_2$ when $u_+ = -1 - \alpha$, so the pair of shocks first appears as an overshoot. As u_+ is further decreased, the slow shock is unchanged, but S_2 increases, so the gap between shocks widens for any fixed $t > 0$. When $u_+ = \alpha(\Gamma_0)$, there is a single (non-classical) shock, and when $u_+ < \alpha$, this shock is followed by a rarefaction to u_+. We mention the limiting cases:

$$\alpha(\Gamma_0) \to \begin{cases} -1/2, & \Gamma_0 \to +\infty, \\ -1, & \Gamma_0 \to 0. \end{cases}$$

We now consider the problem from another perspective: Again fix $u_- = 1$, and now fix $u_+ < 0$. We then ask what happens as Γ is decreased from infinity. There are three scenarios, depending on u_+ : When $-1/2 < u_+ < 0$, there is a value $\Gamma = \bar{\Gamma}$, satisfying (3.41) with $\alpha = -1 - u_+$, such that there is a classical shock for all $\Gamma > \bar{\Gamma}$. When $\Gamma = \bar{\Gamma}$, there is a non-classical shock to $-1 - u_+$, followed by a classical one to u_+. For $\Gamma < \bar{\Gamma}$, the intermediate state, $\alpha(\Gamma)$, decreases, approaching the value -1, as $\Gamma \to 0$.

If $-1 < u_+ < -1/2$, there is a value $\hat{\Gamma}$, which solves (3.41) with $\alpha = u_+$, and such that for $\Gamma > \hat{\Gamma}$, the solution is a non-classical shock followed by a rarefaction. This solution is almost indistinguishable from the classical shock-rarefaction for $\Gamma >> 1$. When $\Gamma = \hat{\Gamma}$, there is a single, non-classical shock to u_+, and for $\Gamma < \hat{\Gamma}$, there are two shocks, with the intermediate state decreasing to -1, as $\Gamma \to 0$.

Finally, when $u_+ < -1$, the solution again like the classical shock-rarefaction for $\Gamma \gg 1$. As Γ is decreased, an intermediate state opens up between the shock and rarefaction. The value of this intermediate state once again decreases with Γ, to -1. The value of the intermediate state is $\alpha(\Gamma)$.

For illustration, we have used a pseudospectral method to numerically solve the Riemann problem for (3.17). The results are plotted in Figures 3.2–3.4, and were generated by Daniel Mahoney, using his numerical scheme [29], which contains a pseudospectral decomposition in x, with time advancement carried out by an implicit, second-order Runge-Kutta technique. This decomposition is based on the method of Fornberg and Whitham [7]. We simulated Riemann initial data by taking a smooth, periodic function, which decreases very rapidly near $x = 0$. Periodicity was maintained by having a second region of more gradual increase. The spatial period was chosen sufficiently large compared with the wave speeds involved, so that noise from the boundaries could be kept away from the region of interest near $x = 0$. In the plots below, the initial data was taken to be

$$u(x, 0) = 1 + \frac{\bar{u} - 1}{2} \left(\tanh(\lambda x) - \tanh(\lambda x - 0.4\lambda x_m) \right),$$

where $\bar{u}$ is the value of the right-hand state, $\lambda = 2$ is the scaling in the data, and x_m is half the width (period) of the data.

In Figure 3.2, we plot the numerical solution for $u(x, t = 8)$, with $u_l = 1, u_r = -0.5$, and $\gamma = 1.0$. We used 2048 gridpoints over a spatial interval $x \in [-20\pi, 20\pi]$. The figure shows a monotone non-classical shock, to a value $u_m \approx -0.8849$, within $5(10^{-4})$ of the value found by Newton's method solution of equation (3.41). The time was taken sufficiently large so that the intermediate state was clearly distinct from the post-shock oscillations associated with the fast shock.

The slow, non-classical shock is monotone, and its speed agrees with that of RH, while the second shock is followed by oscillations; these are discussed following Figure 3.4, and represent a genuine feature of the TW solution. The oscillations have the effect of slowing down the second shock: It occurs at $x \approx 11.35$ in Figure 3.2, a value 3.8% lower than RH would give for this choice of u_m, u_r.

In Figure 3.3, we plot the numerical solution $u(x, t = 8)$, with 1024 gridpoints, for $\gamma = 2.01795, u_l \equiv 1$, and several values of u_r. The transition from classical shock, to non-classical shock, to (non-classical) shock-rarefaction can be seen, as u_r is decreased. The value of the intermediate state, $-.7995$, is again within $5(10^{-4})$ of the predicted value. The pulse near $x = 12$, for $u_r = -0.8$, appears to be a solitary wave; it cannot be one because of Proposition 3.2, but with the nonlinear diffusion, there appears to be great sensitivity to whether $u_m = u_r$. If they differ even slightly, as they do in this case, then the trajectory between the two equilibria can still execute a wide loop in the phase plane.

In Figure 3.4, are plotted three curves for $u(x, t = 8)$, also with 1024 gridpoints, corresponding to three values of γ, for $u_l = 1$ and $u_r = -0.25$. As γ is decreased, the oscillations are seen to increase.

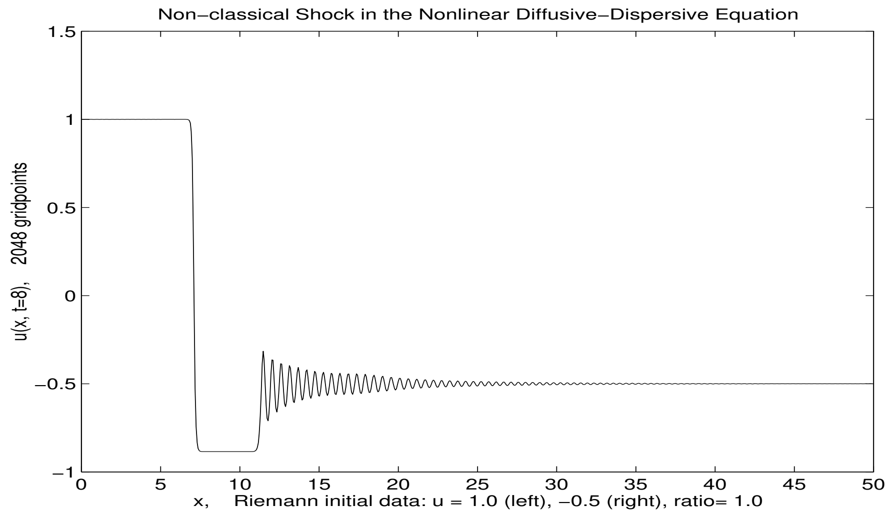

Figure 3.2

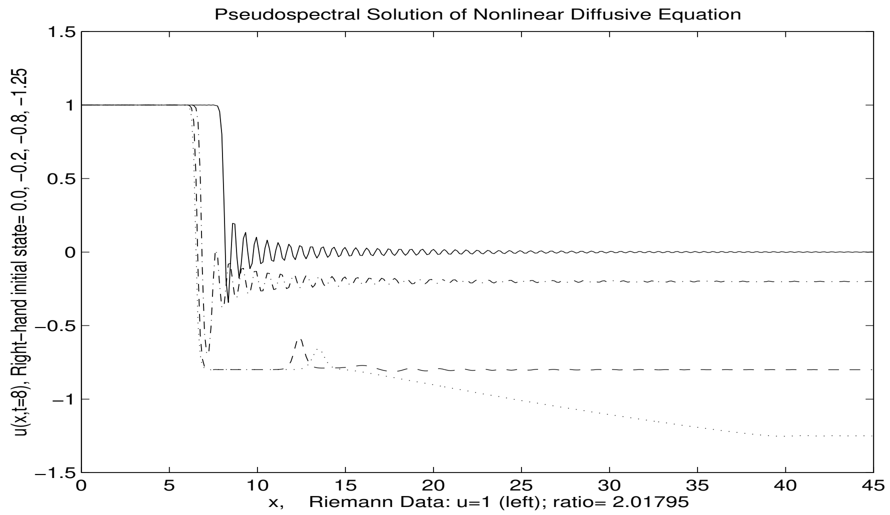

Figure 3.3

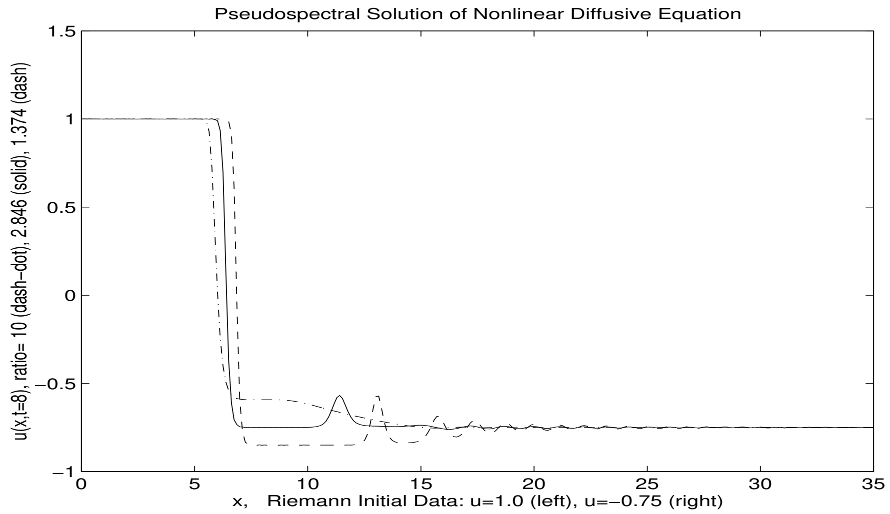

Figure 3.4

While the non-classical shocks displayed in the preceding plots are quite monotone, the classical shocks are followed by oscillatory envelopes. The envelopes decay more quickly as γ is increased or as the mesh size in the scheme is decreased. The oscillations are present in the diffusive-dispersive approximations but should not appear in the limiting solution to the hyperbolic problem. The computation therefore is quite delicate due to a subtle competition between two parameters: the ratio γ of diffusion to dispersion and the mesh size of the discretization scheme.

We also point out that, for $\gamma = O(1)$, the profiles for the non-linear diffusion are more oscillatory than those for the linear diffusion with its key parameter $\mu = O(1)$. This is expected since (3.17) contains a smaller amount of diffusion than (3.1) as first observed by von Neumann-Richtmyer [33]. To confirm that these oscillations are really due to the differential equation (3.17), and not a numerical artifact, we numerically integrated the TW equations for the case of a classical shock.

We chose $u_l = 1$, and $u_r = 0.5$, in the classical regime. We then selected several values of γ, and numerically integrated system (3.25), using (the slightly dissipative) fourth-order Runge-Kutta. In order the begin the calculation, it was necessary to move away from the equilibrium at $(v, w) = (1, 0)$. We did this by solving for the first intersection of the trajectory with the v-axis, using Newton's method on equations (3.34) and (3.35). From this point onward, the solution was found by Runge-Kutta. In Figure 3.5, we plot the integrated trajectories for $\gamma = 1, 5$, and 25. The resemblence to the oscillations in Figures 3.2–3.4 is immediately apparent.

For the linear diffusion (MKdVB), the classical shock is oscillatory only when $\mu^2 + 4p'(v_m) < 0$ in (3.10); otherwise, the middle equilibrium is a stable node and there are *no* oscillations. In the case of nonlinear diffusion (3.17), however, the middle equilibrium point - given by (3.27) - is an elliptic point, so that there will always be some oscillations associated with the classical shock. In both cases, TW analysis gives monotone profiles for the non-classical shock.

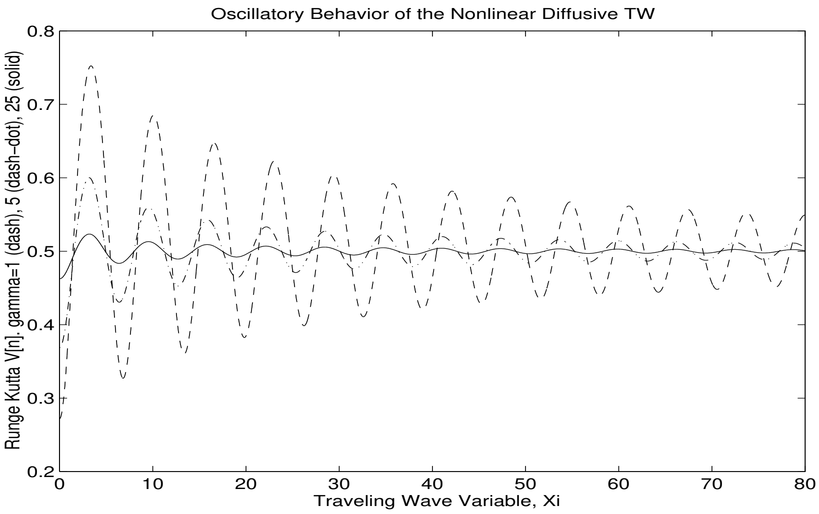

Figure 3.5

For the nonlinear diffusive-dispersive approximation (3.17), the shock curve is described as follows.

Proposition 3.5.

(1) *Given a state $u_l > 0$, the set $\mathbf{S}(u_l)$ consisting of all states u_r that can be achieved through a nonlinear diffusive-dispersive traveling wave (i.e. a solution to* (3.17)*), taking the values u_l and u_r at the left and the right ends respectively, is given by*

$$\mathbf{S}(u_l) = \big\{u_l\,\alpha(\gamma/u_l)\big\} \cup \big[-u_l\big(1+\alpha(\gamma/u_l)\big), u_l\big),$$

where the coefficient γ is the constant given by (3.24) *and the function $\alpha = \alpha(\Gamma)$ is given implicitly by* (3.41).

(2) *Given a state $u_l > 0$, the solution of the Riemann problem with initial data u_l and u_r is solved by*

(i) a rarefaction wave, if $u_r \geq u_l$;

(ii) a classical shock wave, if $u_r \in \big[-u_l\big(1+\alpha(\gamma/u_l)\big), u_l\big)$;

(iii) two shock waves, if $u_r \in \big(u_l\,\alpha(\gamma/u_l), -u_l\big(1+\alpha(\gamma/u_l)\big)\big)$; that is: a slow non-classical shock from u_l to $u_l\,\alpha(\gamma/u_l)$ followed by a fast classical shock connecting to u_r;

(iv) a shock wave and a rarefaction wave, if $u_r \leq u_l\,\alpha(\gamma/u_l)$; that is: a slow non-classical shock wave from u_l to $u_l\,\alpha(\gamma/u_l)$, followed by a rarefaction wave connecting to u_r.

□

3c. Equivalent Equations for the Lax-Wendroff and Beam-Warming Schemes.

In this subsection, we turn to traveling wave solutions of

$$\partial_t u + \partial_x u^3 = \delta\,\partial_{xxx} u^3 \tag{3.42}$$

and

$$\partial_t u + \partial_x u^3 = -\delta\,\partial_{xxx} u^3 \tag{3.43}$$

for $\delta > 0$. Equation (3.42) is the equivalent equation for the Beam-Warming numerical scheme for the cubic scalar equation, provided we identify $\delta = \Delta^2/3$ with Δ being the mesh spacing in x. Equation (3.43) is the equivalent equation for the Lax-Wendroff numerical scheme, for $\delta = \Delta^2/6$. In Section 6 we study the behavior of both numerical schemes on the cubic scalar equation, while equation (3.42) is derived in the Appendix.

In this section, therefore, we study traveling wave solutions of

$$\partial_t u + \partial_x u^3 = \kappa \frac{\delta}{3} \partial_{xxx} u^3, \qquad \kappa = \pm 1, \quad \delta > 0. \tag{3.44}$$

The sign of κ in (3.44) is paramount in determining whether there is a heteroclinic saddle connection, as we now show.

Proposition 3.6. *Given $u_l > 0$, there exists a unique traveling wave solution to* (3.42) *corresponding to an heteroclinic saddle connection in the phase plane, when $\kappa = +1$. This trajectory approaches a non-classical shock between u_l and $-u_l$, traveling with speed u_l^2, as $\delta \to 0$. When $\kappa = -1$, there is no saddle-saddle connection.* □

We mention that that Rosenau and Hyman [36] have also studied (3.44) when $\kappa = +1$, in connection with solitary waves having compact support.

Proof of Proposition 3.6. To find traveling waves, we again change variables to $v(\xi)$, where ξ is given by (3.19) and s by the Rankine-Hugoniot relation. The parameter δ is absorbed and v satisfies

$$-sv' + (v^3)' = \kappa \left(v^2 v'' + 2v(v')^2\right)' \tag{3.45}$$

with boundary conditions (3.21) enforced as $|\xi| \to +\infty$. Integrating (3.45) and using $\kappa^{-1} = \kappa$, gives

$$v^2 v'' + 2v(v')^2 = \kappa\, p(v), \tag{3.46}$$

where we once again define $p(v) = C - sv + v^3$, with $C = su_l - u_l^3$. Writing (3.46) as a first order system,

$$\begin{aligned} v' &= w, \\ w' &= v^{-2}\left(\kappa\, p(v) - 2vw^2\right), \end{aligned} \tag{3.47}$$

we find that the critical points of (3.47) match those of the previous two sections, see (3.26), and will be denoted by $u_l > u_m > u_r$. If $\kappa = +1$, u_l and u_r are once again saddle points, while u_m is a center. Alternatively, if $\kappa = -1$, then u_m is a saddle, with u_r and u_l being centers. Thus the only possible case where a heteroclinic saddle connection can exist is for $\kappa = +1$; we now show that there is indeed such a connection.

Setting $y = w^2$ reduces (3.47) to a linear equation in $y(v)$:

$$\frac{v^2}{2} \frac{dy}{dv} + 2v\, y = \kappa\, p(v). \tag{3.48}$$

Integrating (3.48) leads to

$$y(v) = \frac{\kappa}{6v^4}\, q(v), \tag{3.49}$$

where $q(v) = 2v^6 - 3sv^4 + 4Cv^3 + D$, and D is an integration constant. We therefore have

$$w(v) = \pm \frac{\sqrt{\kappa\, q(v)}}{\sqrt{6}v^2}. \tag{3.50}$$

We now enforce the conditions $w(u_l) = w(u_r) = 0$. The condition $q(u_l) = 0$ implies that $D = u_l^4(2u_l^2 - s)$, while setting $u_r = \alpha u_l$ and using the Rankine-Hugoniot relation with $q(u_r) = 0$ results in

$$(\alpha - 1)^3(\alpha + 1)(\alpha^2 + \alpha + 1) = 0, \tag{3.51}$$

whose only real zeros are $\alpha = \pm 1$. The choice $\alpha = -1$ corresponds to a heteroclinic saddle connection between $u = u_l$ and $u = -u_l$. We now confirm that $q(v) > 0$ for $-u_l < v < u_l$, so that $w(v)$ in (3.50) remains real. Setting $z = v/u_l$, we get (for $\alpha = -1$),

$$q(v) = \tilde{q}(z) = u_l^6 (z^2 - 1)^2 (2z^2 + 1),$$

so that $q(v)$ is, in fact, positive and so for $\kappa = 1$ (BW case) there is a real trajectory between u_l and $-u_l$.

The Rankine-Hugoniot condition gives $s = u_l^2$ for this trajectory, whereas the wave speed at both ends is $3u_l^2$. Thus the Lax entropy criterion is not satisfied. As $\delta \to 0$, the profile therefore converges to an undercompressive shock. □

Further analysis of the phase plane for (3.47) shows that TW solutions to (3.42) give, at best, an incomplete solution to the Riemann problem for the cubic scalar equation: when $u_r \neq -u_l$, it can be shown that there are only homoclinic orbits from u_l to itself. Thus, in general, when $u_r \neq u_l$, it is impossible to connect the two distinct states by a TW. Equation (3.42) apparently has too little structure to give rise to a large enough family of either classical or non-classical shocks for the purpose of this paper. Equation (3.42) is therefore not a good candidate to approximate weak solutions of the cubic scalar conservation law.

A better model, taking higher order effects into account, would be obtained from a higher-order equivalent equation for Beam-Warming. Indeed as described in Section 6, we numerically demonstrate the existence of non-classical shocks for the Beam-Warming scheme. It is heuristically expected that the same property should hold for the equivalent equation, although no rigorous connection can possibly exist.

4. Kinetic Relation and Selection of Non-Classical Shocks.

In Subsection 4.a below, we study multi-wave solutions of the Riemann problem for the equation

$$\partial_t u + \partial_x u^3 = 0. \tag{4.1}$$

Each shock wave should satisfy the Rankine-Hugoniot condition, cf. (2.11), and a single entropy inequality, cf. (1.3). First of all we describe a family of two-wave solutions, which can be composed of either two shock waves (one slow and one fast) or a (slow) shock wave followed by a (fast) rarefaction wave. For certain initial data in the Riemann problem, we deduce that, in addition to the classical solution, one can construct a one-parameter family of two-wave solutions. In order to select a unique non-classical solution, we introduce a kinetic relation for the slow shock. In order to select between this non-classical solution and the classical solution, an additional "nucleation" criterion must be imposed.

Next, in Subsection 4.b, we explain how the kinetic relation can be deduced from the traveling wave solutions associated with a given regularization of (4.1). Entropy dissipation across a shock, as a means of determining admissibility, goes back to the "entropy rate" admissibility criterion, introduced by Dafermos in [5].

4a. Non-classical solutions to the Riemann problem.

Our first objective in this section is to describe the allowable solutions to the Riemann problem for (4.1) in the class of non-classical weak solutions. For definiteness, we consider the quadratic entropy pair[1]

$$U(u) = \frac{u^2}{2}, \qquad F(u) = \frac{3u^4}{4}. \tag{4.2}$$

A shock from u_- to u_+ is said to be admissible if it satisfies the Rankine-Hugoniot relation and has negative entropy dissipation, i.e.

$$D(u_-, u_+) \equiv -s\big(U(u_+) - U(u_-)\big) + F(u_+) - F(u_-) \le 0, \tag{4.3}$$

where s is the shock speed.

Proposition 4.1. *Consider the Riemann problem with initial data $u_l > 0$ and u_r for the cubic scalar equation* (4.1) *in the class of non-classical weak solutions. Any admissible multi-wave solution of* (4.1) *is composed of either two shock waves (one of them possibly being trivial) or a shock wave followed by a rarefaction wave. In particular the Riemann solutions contain at most two waves.* □

[1] Our motivation in selecting a quadratic $U(u)$ is that (1.3) holds at the limit for the solutions constructed from (3.1) or (3.17). Based on our discussion in Section 5, see (5.17) and (5.18), similar -if not identical- results can be expected for the family of entropies $U(u) = u^{2n}/(2n)$.

Proof of Proposition 4.1. First of all, we observe that, across any admissible shock discontinuity (u_-, u_+), the Rankine-Hugoniot relation

$$s = u_+^2 + u_+u_- + u_-^2,$$

applied to (4.3), yields

$$\begin{aligned} D(u_-, u_+) &= \frac{3}{4}\left(u_+^4 - u_-^4\right) - \left(u_+^2 + u_+u_- + u_-^2\right)\left(\frac{u_+^2}{2} - \frac{u_-^2}{2}\right) \\ &= \frac{1}{4}(u_+ - u_-)^2(u_+^2 - u_-^2) \leq 0. \end{aligned}$$

Therefore, across each shock (u_-, u_+), we must have

$$u_+^2 \leq u_-^2. \tag{4.4}$$

We begin by searching for two-wave solutions. The classical shocks of Section 2 are special cases.

Step 1: Two Wave Solutions. In this case the solution is composed of three constant states: u_l, u_m, and u_r. We consider successively the following four combinations: shock-shock (S-S), shock-rarefaction (S-R), R-S, and R-R. To simplify the notation, we rescale each state, dividing by u_l, and set

$$\alpha \equiv \frac{u_m}{u_l}, \qquad \beta \equiv \frac{u_r}{u_l}. \tag{4.5}$$

The type of waves in the solution will ultimately depend on the quantities α and β.

Step 1a: Shock–Shock. Across each shock, (4.4) must hold, so we have

$$\alpha^2 \leq 1, \qquad \beta^2 \leq \alpha^2. \tag{4.6}$$

The shock speeds must satisfy $s_1 < s_2$ for the construction to be possible; thus, by Rankine-Hugoniot,

$$1 + \alpha + \alpha^2 < \alpha^2 + \alpha\beta + \beta^2.$$

Rearranging the terms, we get

$$(1 - \beta)(1 + \beta + \alpha) < 0. \tag{4.7}$$

Since $\beta > 1$ is excluded by (4.6), we must have $1 + \beta + \alpha < 0$ in (4.7). The allowable 2-shock region is a triangle in the (α, β)-plane:

$$\begin{aligned} &\alpha \geq -1, \\ &\beta < -\alpha - 1, \\ &\beta \geq \alpha. \end{aligned} \tag{4.8}$$

Observe that the single shock solution corresponds to the line segment $\alpha = \beta$, $-1/2 \leq \alpha < 1$.

Step 1b: Shock–Rarefaction. We now consider solutions composed of a shock from u_l to u_m, followed by a rarefaction from u_m to u_r. Using (4.5), the entropy inequality (4.4) gives

$$\alpha^2 \leq 1. \tag{4.9}$$

Since the wave speed must increase throughout the rarefaction, one has

$$\alpha^2 \leq \frac{u^2}{u_l^2} \leq \beta^2 \tag{4.10}$$

for all u inside the fan. Finally, the trailing edge of the fan must move faster than the shock:

$$\alpha^2 + \alpha + 1 < 3\alpha^2,$$

which, after rearranging, gives

$$(2\alpha + 1)(\alpha - 1) > 0. \tag{4.11}$$

By (4.9), $\alpha - 1 \leq 0$, so that (4.11) gives $\alpha < -1/2$. So the allowable S-R region lies inside $-1 \leq \alpha < -1/2$; in addition, careful consideration of (4.10) shows that we cannot have $\alpha < 0$ and $\beta > 0$, since then $u = 0$ inside the rarefaction, violating the inequality. Thus the allowable (α, β)-region is the unbounded trapezoid:

$$\begin{aligned} -1 \leq \alpha < -1/2, \\ \beta \leq \alpha. \end{aligned} \tag{4.12}$$

The right-hand boundary of the region, $\alpha = -1/2$, $\beta \leq \alpha$, corresponds to the classical shock–rarefaction.

Step 1c: Rarefaction–Shock. Next we discuss the case of a rarefaction joining u_l to u_m, followed by a shock from u_m to u_r. Using the notation (4.5), we get

$$1 \leq u^2 \leq \alpha^2 \tag{4.13}$$

for all u inside the rarefaction fan, and by (4.4) we have

$$\alpha^2 \geq \beta^2 \tag{4.14}$$

across the shock. The condition that the shock precedes the rarefaction reads

$$(2\alpha + \beta)(\alpha - \beta) < 0. \tag{4.15}$$

If $\alpha - \beta > 0$ in (4.15), then (4.14) gives $\alpha > 0$, so that $2\alpha + \beta > 0$, and (4.15) is not satisfied. A similar argument applies if $\alpha - \beta < 0$. Thus there is no allowable region in (α, β) for rarefaction-shock solutions.

Step 1d: Rarefaction–Rarefaction. Since both rarefactions are continuous, two consecutive rarefactions are equivalent to a single rarefaction joining u_l to u_r. This single wave solution was given in Section 2.

We plot the classical and two-wave solutions to the Riemann problem in Figure 4.1. The shaded regions for the two-shock and shock - rarefaction solutions correspond to the inqualities (4.8) and (4.12), respectively.

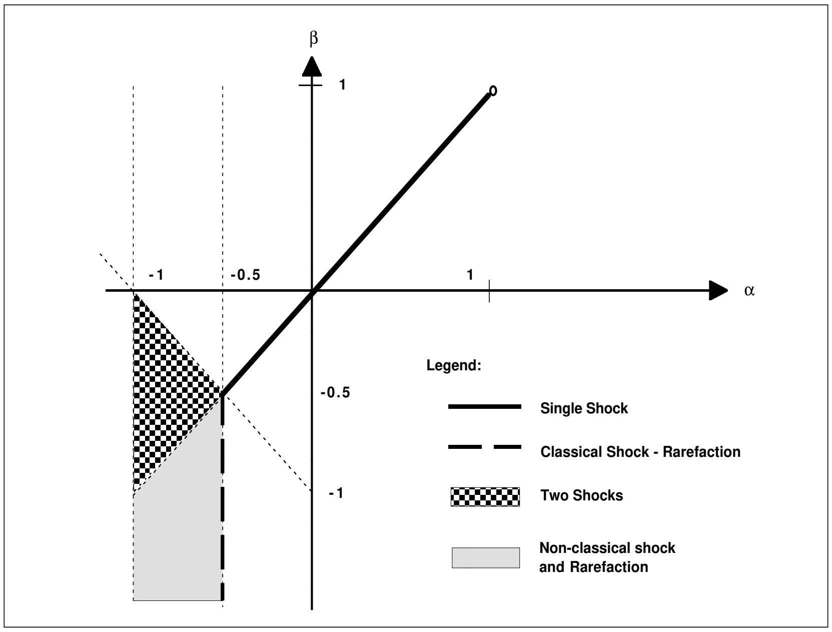

Figure 4.1: Allowable Regions for Classical and Non-classical Solutions

Step 2: Three Wave Solutions. Every multi-wave solution to the Riemann problem for (4.1) must begin on the left with either (i) S-S-S (ii) S-S-R (iii) S-R-S (iv) S-R-R (v) R-S-S (vi) R-S-R (vii) R-R-S (viii) R-R-R. Of these, (iv) is equivalent to the shock-rarefaction case previously discussed, (viii) is just the single rarefaction case, and (iii), (v), (vi) and (vii) are all disallowed because they contain a shock following a rarefaction; cf. Case 1c above. It remains to show that (i) and (ii) are also prohibited. Let the four states be u_l, u_{m_1}, u_{m_2} and u_r, with normalized values:

$$\alpha = \frac{u_{m_1}}{u_l}, \qquad \beta = \frac{u_{m_2}}{u_l}, \qquad \gamma = \frac{u_r}{u_l}. \tag{4.16}$$

Step 2a: Shock-Shock-Shock. The entropy inequality (4.4) gives

$$\alpha^2 \le 1, \qquad \beta^2 \le \alpha^2, \qquad \gamma^2 \le \beta^2, \tag{4.17a}$$

while the conditions $s_2 - s_1 > 0$ and $s_3 - s_2 > 0$ on the shock speeds give

$$(\beta - 1)(1 + \alpha + \beta) > 0, \tag{4.17b}$$

and

$$(\gamma - \alpha)(\gamma + \alpha + \beta) > 0, \tag{4.17c}$$

respectively. By (4.17a), we have $\beta^2 \le 1$, so that in (4.17b), we are compelled to take $1 + \alpha + \beta < 0$, or

$$\alpha + \beta < -1.$$

This, in turn, means that $\alpha < 0$. Then applying (4.17a) to equation (4.17c), we must have $\gamma - \alpha > 0$. We are now led to an impossible situation: we need $\gamma + \alpha + \beta > 0$ to satisfy (4.17c), but $\gamma + \alpha + \beta > \gamma - 1 > 0$ would imply $\gamma > 1$, in violation of (4.17a). We conclude that there are no 3-shock solutions to (4.1).

Step 2b: Shock-Shock-Rarefaction. By (4.4) we have

$$\alpha^2 \le 1, \qquad \beta^2 \le \alpha^2, \tag{4.18}$$

while inside the rarefaction fan we must have

$$\beta^2 \le \frac{u^2}{u_l^2} \le \gamma^2. \tag{4.19}$$

Finally the speeds satisfy $s_1 < s_2 < 3\,u_{m_2}^2$, which leads to the inequalities

$$(\beta - 1)(1 + \alpha + \beta) > 0 \tag{4.20a}$$

and

$$(2\beta + \alpha)(\beta - \alpha) > 0, \tag{4.20b}$$

respectively. By (4.18), we see that (4.20a) is only satisfied if $\beta - 1 < 0$ and $1 + \alpha + \beta < 0$, so that

$$\alpha + \beta < -1, \tag{4.21}$$

and so, again using (4.18), we have $\alpha < 0$. This fact, together with $\alpha^2 \ge \beta^2$, means that we must take $\beta - \alpha > 0$ in (4.20b). This forces $2\beta + \alpha > 0$, which combined with (4.21), implies $\beta > 1$, contradicting (4.18). Consequently there are no S-S-R solutions of (4.1). The proof of Proposition 4.1 is complete. □

In what follows, we retain the notation, α and β, introduced in (4.5) and, in addition, define

$$V = \frac{s_1}{u_l^2}, \tag{4.22}$$

where s_1 is the speed of the first (slow) shock. By the Rankine-Hugoniot relation, one has

$$V = \alpha^2 + \alpha + 1. \tag{4.23}$$

The following corollary to Proposition 4.1 shows that the two-wave solutions to the cubic scalar equation comprise a one-parameter family. It is convenient to use V as a parameter.

Proposition 4.2. *For a given $u_l > 0$, the Riemann problem for (4.1) with data u_l and $u_r = \beta\, u_l$ admits the following one-parameter solutions.*

(1) *When $\beta \in (-1, 0)$, it admits a one-parameter family of non-classical, two-shock solutions where the parameter V describes the interval defined by*

$$\max\left(\frac{\beta^2+3}{4}, \frac{\beta^2+2\beta+4}{4}\right) \le V \le 1. \tag{4.24a}$$

Each solution consists of a slow non-classical shock with speed $u_l^2\, V$ connecting u_l to $u_m = \alpha\, u_l$, with

$$\alpha = \frac{-1-\sqrt{4V-3}}{2}, \tag{4.24b}$$

followed by a shock connecting u_m to $u_r = \beta\, u_l$. Here $\alpha < \beta$ and the solutions are non-monotone.

(2) *When $\beta \in (-\infty, -1/2)$, the Riemann problem also admits a one-parameter family of two-wave solutions consisting of a non-classical shock, again parametrized by speed V, describing the interval*

$$3/4 \le V \le \min\left(1, \frac{\beta^2+3}{4}\right), \tag{4.25}$$

and connecting u_l to $u_m = \alpha\, u_l$ with α given by (4.25) followed by a rarefaction connecting to $u_r = \beta\, u_l$. Here $\alpha \ge \beta$ and the solutions are monotonically decreasing.

□

We refer to Figure 4.1 for a graphical representation of the Riemann solution. We recall that the left state u_l is fixed in the whole of the discussion. It can be checked on Figure 4.1 that to a given value of β, i.e. to a given right state u_r, can be associated either one value of α (when $\beta > 0$), or the union of an interval and one point (when $\beta \in (-1/2, 0)$), or an interval (when $\beta \le -1/2$). When $\beta \in (-1, -1/2)$, the classical wave in the one-parameter family of solutions can be either a shock (Case (1)) or a rarefaction (Case (2)). Note the following limiting case in (2): when $\beta \in (-1, -1/2)$ and $\alpha = \beta$, the Riemann solution contains only a non-classical shock, and the rarefaction is degenerate.

Observe that when $\beta \in (-1/2, 0)$, the Riemann problem also admits the classical one-shock solution, which may actually be considered as a special case of the family (1) in which the two shock speeds coincide and therefore the state u_m cannot be "observed" in the physical space (x, t). Indeed, in the range $\alpha \in (-1, 0)$, the segments $\alpha = \beta$ and $\alpha + \beta + 1 = 0$ should be identified with each other, since they correspond to physically indistinguishable Riemann solutions.

Similarly, when $\beta \in (-\infty, -1/2,)$, the Riemann problem admits the classical shock-rarefaction solution as part of the family of solutions in (2). Indeed those solutions correspond to the choice $V = 3/4$.

Proof of Proposition 4.2. Consider for instance the family of two-shock solutions. From the discussion in the proof of Proposition 4.1, it is known that the intermediate state, $\alpha\, u_l$, lies in the region of the (α, β)-plane given by (4.8). Since the slow shock satisfies the Rankine-Hugoniot relation, we solve for α in (4.23) to get

$$\alpha = \frac{-1 \pm \sqrt{4V-3}}{2};$$

we must take the "-" sign to stay within the allowable region of (4.8). This establishes (4.24b).

From (4.8) we see that

$$\begin{aligned} -1 \le \alpha \le -\beta - 1, &\qquad \beta \ge -1/2, \\ -1 \le \alpha \le \beta, &\qquad \beta < -1/2. \end{aligned}$$

This leads to

$$-1 \le \alpha \le \min(\beta, |\beta| - 1).$$

From (4.25), α is clearly a decreasing function of V, so that, upon substituting the above relation into (4.23), we get the desired range of V in (4.24). Similar arguments apply to the family of shock-rarefaction solutions. The proof of Proposition 4.2 is completed. □

In order to select a unique solution in the one-parameter families of solutions, we introduce a *kinetic relation* based on a kinetic function, $\phi(s_1)$, depending on the wave speed, to be applied only to undercompressive shocks. Specifically, we require all non-classical shocks for (4.1) to satisfy

$$D(u_l, u_m) = \phi(s_1), \tag{4.26}$$

where

$$D(u_l, u_m) = \frac{1}{4}(u_l - u_m)^2 (u_m^2 - u_l^2). \tag{4.27}$$

It will be convenient to rescale the kinetic relation by

$$\phi(s_1) = u_l^4 \, \Phi(V). \tag{4.28}$$

For more generality, it could be assumed that the kinetics would depend upon, say, the left state u_l; however, in its application in Subsection 4.b below, we shall show that it is actually sufficient to consider a kinetic function of the variable V, only.

We show the following:

Proposition 4.3. *Assume that the kinetic function $\phi(s_1)$ is a smooth function defined for all $s_1 \in [0, \infty)$ and all $u_l > 0$, and satisifes the conditions*

$$\begin{aligned} &(a) \qquad \phi'(s_1) < 0, \qquad s_1 \in (0, \infty), \\ &(b) \qquad \phi(0) = 0, \\ &(c) \qquad \phi(s_1) \geq -(3/4)\, s_1^2, \qquad s_1 \in (0, \infty). \end{aligned} \tag{4.29}$$

Consider two initial states $u_l > 0$ and $u_r < 0$. Then, in the family of two-wave solutions derived in Proposition 4.2, there exists a unique solution consisting of a non-classical shock from u_l to an intermediate state denoted by $u_m(u_l) \in (-u_l, -u_l/2)$ with wave speed denoted by $s_1 \in (0, \infty)$ ans a classical wave from $u_m(u_l)$ to u_r such that the kinetic relation (4.26)- (4.28) *holds for the non-classical wave. The classical wave is a shock wave if $u_r > u_m(u_l)$ and a rarefaction wave if $u_r \leq u_m(u_l)$.* □

We recall that the classical solution is viewed here as a special case of a non-classical solution, in which the non-classical and the classical waves have that the same speed, so that the intermediate state cannot be observed physically. In general, Proposition 4.3 produces a non-classical solution. It does, however, select the classical solution in the special cases that either $u_r = -u_l - u_m(u_l) \in (-u_l/2, 0)$, or $V(u_l) = 3/4$ and $u_r \leq -u_l/2$.

In view of (4.29a), the function Φ is a monotone function of the variable V, and therefore the condition (4.26) is equivalent to saying that the speed of propagation for the non-classical shock is a function of the entropy dissipation, i.e.

$$s_1 \; = \; g\big(D\big) \quad \text{ with } \; D = D(u_l, u_m), \tag{4.30}$$

and g is a given monotone function of the variable D. Relations of the form (4.30) have been introduced in the litterature in material science in order to describe propagating boundaries between different phases of a material. In that context, the function g can be determined experimentally via a series of experiments, the entropy dissipation there being identified with a force acting on the interface, called a "driving traction".

Proof of Proposition 4.3. From Proposition 4.2, one knows that the admissible two-wave solutions to (4.1) form a one-parameter family, with the speed of the slow shock, normalized to V, serving as the parameter. We must now prove that the kinetic relation Φ, subject to the conditions (4.29), selects a unique value of V, everywhere in the range $3/4 \leq V \leq 1$. The latter inequalities are a consequence of (4.23) and (4.24), as we observed previously. This is accomplished by showing that, for u_l kept fixed, the functions $V \to D(u_l, u_m)$ and $V \to \phi(s_1, u_l)$ have exactly one intersection point for $V \in [3/4, 1]$.

Claim: $D(u_l, u_m)$ *is a strictly increasing function of* $V \in (3/4, 1)$.

Namely from (4.27) we have

$$E(\alpha) \equiv D(u_l, u_m) = \frac{u_l^4}{4}(\alpha - 1)^2(\alpha^2 - 1),$$

so that

$$\frac{d}{d\alpha}E(\alpha) = \frac{u_l^4}{4}(\alpha - 1)^2(2\alpha + 1).$$

Therefore $E'(\alpha) < 0$ for $\alpha < -1/2$, and, in particular, for $-1 \leq \alpha < -1/2$. From (4.25), α is monotonically decreasing with V, thus, in terms of V, the entropy dissipation function

$$E(\alpha(V)) = \frac{u_l^4}{4}\left(V^2 + V(3 + 2\sqrt{4V - 3}) - \frac{3}{2}\sqrt{4V - 3} - \frac{9}{2}\right) \tag{4.31}$$

is strictly increasing throughout the interval $3/4 < V \leq 1$, and the claim is proven.

Since E is increasing as a function of V, any decreasing function Φ over $V \in (3/4, 1)$ will cross the graph of E at exactly one place, provided that

$$\begin{aligned} u_l^4 \lim_{V \to 3/4+} \Phi(V, s_1) \;&\geq\; E(\alpha(3/4)) \;=\; -\,(3/4)^3\, u_l^4, \\ u_l^4 \lim_{V \to 1^-} \Phi(V, s_1) \;&\leq\; E(\alpha(1)) \;=\; 0. \end{aligned}$$

This proves the desired result. □

We conclude the discussion with the existence result for the Riemann problem. The notation $u_m(u_l)$ and $V(u_l)$ introduced in Proposition 4.3 is used herein. We observe that the state $-u_l - u_m(u_l)$ also satisfies the RH with u_l as the left state and $V(u_l)$ as the shock speed, and that

$$u_m(u_l) \;\leq\; u_l \;\leq\; -u_l - u_m(u_l) \;\leq\; 0.$$

The following result is now immediate in view of Propositions 4.1–4.3.

Theorem 4.4. *Let Φ be a fixed kinetic function satisfying the conditions* (4.29)*. Consider the Riemann problem with data $u_l > 0$ and u_r for the cubic scalar equation* (4.1) *in the class of non-classical weak solutions in which each non-classical shock satisfies the kinetic relation* (4.26)-(4.28)*. Then the Riemann problem can be solved*

(i) if $u_r \geq u_l$, by a rarefaction wave connecting monotonically u_l to u_r;

(ii) if $u_r \in [-u_l - u_m(u_l), u_l)$, by one classical shock connecting u_l to u_r.

(iii) if $u_r \in [u_m(u_l), -u_l - u_m(u_l))$, by one (slow) non-classical shock connecting u_l to $u_m(u_l)$, followed by a (fast) classical shock connecting $u_m(u_l)$ to u_r (the solution is non-monotone);

(iv) if $u_r < u_m(u_l)$, by a (slow) non-classical shock wave connecting u_l to $u_m(u_l)$, followed by (but not attached to) a rarefaction wave connecting $u_m(u_l)$ to u_r (the solution is monotone);

Furthermore, in the last two cases the classical solution is also available: when $u_r \in [-u_l/2, -u_l - u_m(u_l))$, it connects u_l to u_r by a classical shock, and when $u_r \leq -u_l/2$, it connects u_l to $-u_l/2$ by a classical shock attached to a rarefaction wave connecting, in turn, to u_r. □

No other solution exists in the class of self-similar solutions containing shock waves or rarefaction waves. Theorem 4.4 shows, in particular, that even after imposing the kinetic relation we are still left with two solutions. In the range of values $u_r \leq -u_l - u_m(u_l)$, an additional condition, which we shall call a *"nucleation criterion"*, is necessary to make the choice between a classical solution and a non-classical one.

The specific nucleation criterion must be determined on a case-by-case basis, utilizing information from the augmented equation. Nevertheless, we can make several general comments: Consider a regularization of (4.1), in which the coefficients of diffusion and dispersion are balanced, and whose ratio is denoted by the parameter τ. For initial data (u_l, u_r), the kinetic relation selects a unique non-classical shock, with intermediate state $u_m(u_l) = u(\tau, u_l)$.

(1) According to Theorem 4.4, we must have $u_r < -u_l - u(\tau, u_l)$, in order to have a non-classical shock. This result can be interpreted as a universal nucleation criterion. Alternatively, we can view the "phases" not as opposite signs of u, but rather as the following geometrical construct: draw the chord from u_l to $u(\tau, u_l)$, on the graph of $f(u) = u^3$; this chord intersects the graph at $u = -u_l - u(\tau, u_l)$, and all points, u, to the left of this intersection can then be interpreted as lying in a different phase from u_l. Even with this more refined notion of what constitutes a phase, an additional nucleation criterion may still be necessary, as we demonstrate in Section 4.b.

(2) From Proposition 3.5, we see that when the inequality in (1) holds, the non-classical solution is always selected for the nonlinear diffusive - dispersive regularization (1.7). For a discussion of nucleation in the Beam-Warming scheme, we refer to Section 6.

(3) There is no nucleation of new phases from a region having a single phase. This stands in contrast to the behavior of the 2×2 system studied by Abeyaratne and Knowles [1], [2]. In this latter case, the RH condition yields an equation for s^2, so that the system may have simultaneous right- and left-moving phase-transitions. In equation (4.1), however, the non-classical shock speed is determined uniquely by the left- and right-hand states; thus, from a region having a single phase, u_-, two successive non-classical shocks, the first to u_+ (in a different phase) and the second back to u_-, cannot serve to open up, i.e. "nucleate" a region of u_+-phase inside the u_--phase region. Furthermore, other possible constructions taking u_- to u_+, and back, such as a classical shock-rarefaction followed by a non-classical shock, are all excluded by the arguments of Proposition 4.1.

Still, when $u_r\, u_l < 0$ we found distinct solutions, with either classical or non-classical behavior, that are in the same spirit as those observed by Abeyaratne and Knowles.

4b. Kinetic relations derived from traveling waves.

By employing a specific kinetic relation, we now recover the solutions obtained as limits of vanishing viscosity-dispersion approximations. We recall first certain properties of the solution to the Riemann problem obtained by Jacobs, McKinney, and Shearer [15] using admissible traveling wave solutions to the MKdVB approximation. To restate their results in a format convenient for our analysis, we define a normalized parameter:

$$\eta \equiv \frac{\mu}{u_l} = \frac{\epsilon}{\sqrt{\delta} u_l}. \tag{4.32}$$

We recall that the traveling wave analysis performed in Section 3 yields a unique solution to the Riemann problem. If we are given Riemann initial data $u_l = v_l$ and u_r, along with a value of δ such that (3.15) holds, then the quantities $v_m(\delta)$ and $v_r(\delta, v_l)$ are determined by (3.13), while $s(\delta, v_l)$ is provided by (3.14). For a value of u_r just less than v_l, by standard results for vector fields there exists a TW connecting v_l to u_r, corresponding to a saddle-node connection and having speed given by the Rankine-Hugoniot relation:

$$\begin{aligned} s_1 &= v_l^2 + u_r v_l + u_r^2 \\ &= u_l^2 \left(1 - \frac{\sqrt{2}}{3}\eta + \frac{2}{9}\eta^2\right). \end{aligned} \tag{4.33}$$

As u_r, and along with it s_1, decreases, it eventually reaches $v_m(\delta)$, at which point there is no longer a saddle-node connection. Instead, there is a heteroclinic saddle orbit joining v_l to $v_r(\delta, v_l)$, with speed s of (3.14). This trajectory corresponds to an undercompressive shock wave. For $v_r(\delta, v_l) < u_r < v_m(\delta)$, there is a saddle-node connection between $v_r(\delta, v_l)$ and u_r, corresponding to an Oleinik-entropy satisfying shock; its speed is given by

$$s_2 = v_r^2 + v_r u_r + u_r^2. \tag{4.34}$$

This picture of slow and fast shocks persists as u_r is decreased to $v_r(\delta, v_l)$, whereupon the undercompressive shock still remains, but there is now a rarefaction fan from v_r to u_r. Note that since $s < 3v_r^2$, there is an intermediate state, u_m, between shock and rarefaction - they are no longer attached as in the classical case.

We now state our main theorem of this subsection, which reproduces the unique solution to the Riemann problem for MKdVB, through the use of a specific kinetic relation and nucleation criterion:

Theorem 4.5. *The unique solution to the Riemann problem for MKdVB (3.1) is equivalent to allowing non-classical shocks of Definition 1.1 for the cubic scalar equation (4.1), which satisfy the kinetic relation (4.26), for* $\Phi(V)$ *having the explicit dependence*

$$\begin{aligned} \phi(s_1) = u_l^4\Phi(V) &= E(\alpha(V)) \\ &= \psi\left(s_1(\eta, u_l)\right), \end{aligned} \tag{4.35}$$

where the function $E(\alpha(V))$ *is as in (4.31), and* $s_1(\eta, u_l)$ *is given by (4.33). The non-classical shocks must also satisfy the nucleation criteria:*

$$\begin{aligned} &(a) \qquad u_r < -\sqrt{2}\mu/3, \\ &(b) \qquad u_l > 2\sqrt{2}\mu/3. \end{aligned} \tag{4.36}$$

Proof of Theorem 4.5. In order to select the non-classical shocks, which are admissible TW solutions of MKdVB, we must utilize explicit information from the augmented equation (3.1), regarding the shock speed V, in the kinetic relation (4.26). In equation (4.31), we make the substitution

$$\begin{aligned} V &= u_l^{-2}s_1(\eta, u_l) \\ &= 1 - \sqrt{2}\eta/3 + 2\eta^2/9. \end{aligned} \tag{4.37}$$

We then find explicitly the right-hand side (4.35) of the kinetic relation, using (4.37). In the course of computing the function Ψ, we obtain $\sqrt{4V-3} = |2\sqrt{2}\eta/3 - 1| = 1 - 2\sqrt{2}\eta/3$, by equation (4.37) and the nucleation criterion (4.36b), respectively. A straightforward computation on (4.31) then gives

$$\psi(s_1(\eta, u_l)) = u_l^4\left(\frac{\eta^4}{81} - \frac{\sqrt{2}\,\eta^3}{9} + \frac{2\eta^2}{3} - \frac{2\sqrt{2}\,\eta}{3}\right). \tag{4.38}$$

The expression in (4.38) is also what the entropy dissipation,

$$D(u_l, u_m(u_l)) = D(u_l, -u_l + \sqrt{2}\mu/3)$$

for the non-classical MKdVB shock gives, after a brief calculation using equation (4.27). According to Proposition 4.3, therefore, the unique right-hand state selected by the kinetic relation will be $u_m(u_l) = -u_l + \sqrt{2}\mu/3$.

Turning to the nucleation criterion, we see that (4.36a) is just the universal requirement that $u_r < -u_l - u_m(u_l)$, for the specific value of u_m in MKdVB. The requirement that $u_l > 2\sqrt{2}\mu/3$ is quite specific to MKdVB; it can only be arrived at, as derived in (3.15), by an analysis of admissible TW solutions.

Having selected $u_m(u_l) = -u_l + \sqrt{2}\mu/3$ by (4.35), we use this value in Theorem 4.4, which, combined with conditions (4.36a) and (4.36b), leads directly to the Riemann solution for MKdVB in Proposition 3.1. This proves Theorem 4.5. □

5. Remarks on the Convergence to Non-Classical Weak Solutions.

In this section we study the vanishing diffusive-dispersive limit for the Cauchy problem with diffusion and dispersion kept in balance. A significant open question concerns choosing the proper functional space, for which the limiting ($\epsilon \to 0$) Cauchy problem is well-posed. In this section we concentrate on the question of existence. We show that while the dispersive term is potent enough to modify the entropy criterion, it still is not an obstacle to strong convergence: as $\epsilon \to 0$, the solutions to the regularized equation do converge to a weak solution of the conservation law. Uniqueness for the Cauchy problem remains largely an open question, although our results in Section 4 provide an answer in the special case of the Riemann problem. In this section, we provide a detailed discussion of entropy inequalities, and comment particularly on their importance in providing a priori estimates on the solutions and in selecting non-classical solutions. In Theorem 5.1, we prove that all of the entropy inequalities –except for the one associated with the quadratic entropy– fail to hold in general for the non-classical solutions constructed by examination of admissible TW solutions.

Below we state our result on convergence of the MKdVB approximation.

Theorem 5.1. *Let $\{u^\epsilon\}_{\epsilon>0}$ be a family of smooth solutions to the Cauchy problem*

$$\begin{aligned}&\partial_t u + \partial_x u^3 = \epsilon\, \partial_{xx} u \;+\; \delta\, \partial_{xxx} u,\\ &u^\epsilon(x,0) = u_0^\epsilon(x),\end{aligned} \tag{5.1}$$

where $\epsilon > 0$ and $\delta = A\,\epsilon^2$ with $A > 0$ fixed, and with the initial data satisfying

$$\|u_0^\epsilon\|_{L^2(\boldsymbol{R})} + \|u_0^\epsilon\|_{L^4(\boldsymbol{R})} + \sqrt{\delta}\; \|\partial_x u_0^\epsilon\|_{L^2(\boldsymbol{R})} \;\le\; O(1).$$

(1) *Then the u^ϵ's remain uniformly bounded in $L^\infty(\boldsymbol{R}_+, L^2(\boldsymbol{R}) \cap L^4(\boldsymbol{R}))$ and converge in any $L^\infty_{\mathrm{loc}}(L^p_{\mathrm{loc}})$ $(2 \le p < 4)$ to a solution $u \in L^\infty(\boldsymbol{R}_+, L^2(\boldsymbol{R}) \cap L^4(\boldsymbol{R}))$ of*

$$\partial_t u + \partial_x u^3 = 0.$$

(2) *If we denote by $\mu = \lim_{\epsilon \to 0} (u^\epsilon)^4$, which (at least) belongs to the space of bounded Borel measures, we have*

$$\partial_t \frac{u^2}{2} + \partial_x \frac{3\,\mu}{4} \le 0. \tag{5.2}$$

If u^ϵ converges strongly in L^4, then $\mu = u^4$ and the function u satisfies the entropy inequality for the entropy $u^2/2$.

(3) *For an arbitrary (smooth and sub-quadratic) convex entropy U, the entropy inequality*

$$\partial_t U(u) + \partial_x F(u) \le 0 \tag{5.3}$$

is generally not *satisfied by the limiting solution.* □

The proof of Part (1) Theorem 5.1 is based on the compensated compactness method and follows from the seminal work by Schonbek [37]. In particular, the relevance of the space L^4 was first pointed out in [37]; a notion of L^p Young measure, necessary to apply the compensated compactness method here, was introduced therein. Parts (2)-(3) are based on the new observation that the *traveling wave solutions* associated with (5.1) satisfy the entropy inequality (5.3) with a quadratic entropy, but do not, in general, satisfy (5.3) with other entropies.

Lemma 5.2 below provides several a priori estimates generated by suitable entropy functions, that is $U(u) = u^2$ and $U(u) = u^4$. More precisely, we note that (5.1) reduces to the integrable, modified Korteweg-deVries equation when $\epsilon = 0$. In fact, the two main energy-like estimate below, cf. (5.4) and (5.5), are associated with the second and third time-invariants of MKdV: u^2 and $u^4 + 2\,\delta\,|\partial_x u|^2$. In the next lemma, Lemma 5.3, we estimate the entropy dissipation terms associated with an arbitrary sub-quadratic entropy. The proof of Theorem 5.1 then is a corollary of Lemmas 5.2–5.3.

We omit the subscript ϵ when there is no ambiguity. For later use in this section, it will be convenient to state Lemma 5.2 in the more general case of a nonlinear diffusion $b_m(w) = |w|^{m-1}$ for $m \ge 1$, i.e. Lemma 5.2 below concerns

$$\begin{aligned}&\partial_t u + \partial_x u^3 = \epsilon\, \partial_x\big(b_m(\partial_x u)\, \partial_x u\big) + \delta\, \partial_{xxx} u,\\ &u^\epsilon(x,0) = u_0^\epsilon(x),\end{aligned} \tag{5.4}$$

where it is natural to assume now $\epsilon > 0$ and $\delta = A\,\epsilon^{2/m}$ with $A > 0$ fixed.

Lemma 5.2. *For every $T > 0$, the solutions u^ϵ of (5.4) satisfy the uniform estimates*

$$\frac{1}{2}\int_{\boldsymbol{R}} u^2(T)\, dx + \epsilon \int_0^T \int_{\boldsymbol{R}} |\partial_x u|^{m+1}\, dx dt = \frac{1}{2}\int_{\boldsymbol{R}} u_0^2\, dx \;\le\; O(1) \tag{5.5}$$

and

$$\frac{1}{4}\int_{\boldsymbol{R}}\Big(u(T)^4+2\,\delta\,|\partial_x u(T)|^2\Big)\,dx+\int_0^T\int_{\boldsymbol{R}}\big(3\,\epsilon\,u^2\,|\partial_x u|^{m+1}+\epsilon\,\delta\,m\,|\partial_x u|^{m-1}\,|\partial_{xx}u|^2\big)\,dxdt$$
$$=\frac{1}{4}\int_{\boldsymbol{R}}\Big(u_0^4\,+2\,\delta\,|\partial_x u_0|^2\Big)\,dx\;\le\;O(1). \tag{5.6}$$

□

We mention that the estimate (5.5) is true regardless of the sign of δ. The estimate (5.6) is the crucial estimate in proving Theorem 5.1: when $\delta>0$ –which is the sign we are interested in– it provides a uniform control on u, $\partial_x u$ and, most importantly, $\partial_{xx}u$. This would not be the case if $\delta<0$.

Proof of Lemma 5.2. To derive the energy estimate (5.5), we multiply (5.4) by u and we integrate over x. Integrating by parts and applying the identities

$$u\,\partial_x\big(b_m(\partial_x u)\partial_x u\big)=\partial_x\big(u\,b_m(\partial_x u)\,\partial_x u\big)-|\partial_x u|^{m+1}\qquad\text{and}\qquad u\,\partial_{xxx}u=\partial_x\big(u\,\partial_{xx}u-|\partial_x u|^2/2\big),$$

we get

$$\frac{d}{dt}\int_{\boldsymbol{R}}\frac{u^2}{2}\,dx=-\epsilon\int_{\boldsymbol{R}}|\partial_x u|^{m+1}\,dx.$$

After integrating in time, we arrive at (5.4). Here and in the entirety of this section, we tacitly assume that the solutions u^ϵ and their derivatives decay sufficiently rapidly when $|x|\to\infty$.

We now turn to the higher order energy-like estimate (5.6), deriving it in two steps. We first multiply (5.1) by u^3 and integrate by parts:

$$\frac{d}{dt}\int_{\boldsymbol{R}}\frac{u^4}{4}\,dx=-\epsilon\int_{\boldsymbol{R}}3\,u^2\,|\partial_x u|^{m+1}\,dx-\delta\int_{\boldsymbol{R}}3\,u^2\,\partial_x u\,\partial_{xx}u\,dx.$$

Using the identity $2\,u^2\,\partial_x u\,\partial_{xx}u=\partial_x\big(u^2\,|\partial_x u|^2\big)-2\,u\,\big(\partial_x u\big)^3$ and integrating in time yields

$$\frac{1}{4}\int_{\boldsymbol{R}}u(T)^4\,dx+3\,\epsilon\int_0^T\int_{\boldsymbol{R}}u^2\,|\partial_x u|^{m+1}\,dxdt=\frac{1}{4}\int_{\boldsymbol{R}}u_0^4\,dx+3\,\delta\int_0^T\int_{\boldsymbol{R}}u\,\big(\partial_x u\big)^3\,dxdt. \tag{5.7}$$

By itself, (5.7) gives no uniform control since the sign of the last term of the right-hand side is indefinite. To circumvent this problem in finding (5.5), we employ a second step of multiplying (5.1) by $-\delta\,\partial_{xx}u$. After integrating by parts, we have

$$\frac{d}{dt}\left(\frac{\delta}{2}\int_{\boldsymbol{R}}|\partial_x u|^2\,dx\right)-3\,\delta\int_{\boldsymbol{R}}u^2\,\partial_x u\,\partial_{xx}u\,dx+\epsilon\,\delta\int_{\boldsymbol{R}}m\,b_m(\partial_x u)\,|\partial_{xx}u|^2\,dx=0,$$

since $d\big(b_m(w)\,w\big)/dw=m\,b_m(w)$. Integration in time leads to

$$\frac{\delta}{2}\int_{\boldsymbol{R}}|\partial_x u(T)|^2\,dx+\epsilon\,\delta\int_0^T\int_{\boldsymbol{R}}m\,b_m(\partial_x u)\,|\partial_{xx}u|^2\,dxdt$$
$$=\frac{\delta}{2}\int_{\boldsymbol{R}}|\partial_x u_0|^2\,dx-3\,\delta\int_0^T\int_{\boldsymbol{R}}u\,\big(\partial_x u\big)^3\,dxdt. \tag{5.8}$$

The terms without definite signs in (5.7) and (5.8) occur with opposite coefficients, so that by adding those two equations we arrive at the estimate (5.6). □

Lemma 5.3. *Consider the equation (5.1) with $\epsilon \to 0$, $\delta = A\,\epsilon^2$, and $A > 0$ fixed. Let (U, F) be an entropy pair with U having (at most) quadratic growth at infinity in the sense that*

$$|U''(u)| \le C_0 \qquad \textit{for all} \quad u, \tag{5.9}$$

where C_0 is a positive constant. Then we have

$$\partial_t U(u^\epsilon) + \partial_x F(u^\epsilon) = \mathcal{D}_1^\epsilon + \mathcal{D}_2^\epsilon, \tag{5.10a}$$

where the distributions $\mathcal{D}_1^\epsilon$ and $\mathcal{D}_2^\epsilon$ satisfy

$$\begin{aligned} &\mathcal{D}_1^\epsilon \longrightarrow 0 \quad \textit{in the Sobolev space } W^{-1,2}_{\text{loc}}(\boldsymbol{R} \times \boldsymbol{R}_+), \\ &\mathcal{D}_2^\epsilon \quad \textit{is a bounded Borel measure.} \end{aligned} \tag{5.10b}$$

□

Proof of Lemma 5.3. To bound U' in our estimates below we shall use the following consequence of (5.9):

$$|U'(u)| \le C_1 + C_0\,|u| \qquad \text{for all} \quad u,$$

with $C_1 = U'(0)$, say. From (5.1) and an integration by parts, we have

$$\begin{aligned} \partial_t U(u^\epsilon) + \partial_x F(u^\epsilon) &= U'(u^\epsilon)\,\epsilon\,\partial_{xx} u^\epsilon) + U'(u^\epsilon)\,\delta\,\partial_{xxx} u^\epsilon \\ &= \epsilon\,\partial_x\left(U'(u^\epsilon)\,\partial_x u^\epsilon\right) - \epsilon\,U''(u^\epsilon)\,|\partial_x u^\epsilon|^2 + \delta\,\partial_x\left(U'(u^\epsilon)\partial_{xx} u^\epsilon\right) - \delta\,U''(u^\epsilon)\,\partial_x u^\epsilon\,\partial_{xx} u^\epsilon \\ &\equiv T_1^\epsilon + T_2^\epsilon + T_3^\epsilon + T_4^\epsilon. \end{aligned} \tag{5.11}$$

We now examine each of the terms T_i^ϵ in the identity (5.11).

The terms T_1^ϵ tends to zero in the Sobolev space $W^{-1,2}_{\text{loc}}(\boldsymbol{R} \times \boldsymbol{R}_+)$. Namely, for each $\theta \in W^{1,2}(\boldsymbol{R} \times \boldsymbol{R}_+)$ having a compact support, one can use the Cauchy-Schwartz inequality and get

$$\begin{aligned} |< T_1^\epsilon, \theta >| &\equiv \left| \iint_{\boldsymbol{R}\times\boldsymbol{R}_+} \epsilon\,\partial_x\big(U'(u^\epsilon)\,\partial_x u^\epsilon\big)\,\theta\,dxdt \right| \\ &\le O(1) \iint_{\boldsymbol{R}\times\boldsymbol{R}_+} \epsilon\,(1 + |u^\epsilon|)\,|\partial_x u^\epsilon|\,|\partial_x \theta|\,dxdt \\ &\le O(\epsilon)\,\|(1 + |u^\epsilon|)\partial_x u^\epsilon\|_{L^2_{\text{loc}}(\boldsymbol{R}\times\boldsymbol{R}_+)} \|\partial_x \theta\|_{L^2(\boldsymbol{R}\times\boldsymbol{R}_+)} \\ &\le O(1)\,\epsilon^{1/2}\,\|\theta\|_{W^{1,2}(\boldsymbol{R}\times\boldsymbol{R}_+)} \qquad \longrightarrow 0, \end{aligned}$$

where we have used both energy estimates (5.5)-(5.6) and the bound (5.9) on U.

The term T_2^ϵ is immediately found to be bounded in the space of bounded Borel measures thanks to the entropy dissipation estimate in (5.5) and the bound (5.9).

The term T_3^ϵ converges to zero in $W^{-1,2}_{\text{loc}}(\boldsymbol{R} \times \boldsymbol{R}_+)$. We estimate it by the Cauchy-Schwartz inequality and the higher-order estimate (5.6), as follows:

$$\begin{aligned} |< T_3^\epsilon, \theta >| &\le \delta\,\|(1 + |u^\epsilon|)\partial_{xx} u^\epsilon\|_{L^2(\boldsymbol{R}\times\boldsymbol{R}_+)}\,\|\theta\|_{W^{1,2}(\boldsymbol{R}\times\boldsymbol{R}_+)} \\ &\le \delta\,\big(\|\partial_{xx} u^\epsilon\|_{L^2(\boldsymbol{R}\times\boldsymbol{R}_+)} + \sup_{t\in\boldsymbol{R}_+} \|\|u^\epsilon(t)\|_{L^2(\boldsymbol{R})} \|\partial_{xx} u^\epsilon\|_{L^2(\boldsymbol{R}\times\boldsymbol{R}_+)}\big)\,\|\theta\|_{W^{1,2}(\boldsymbol{R}\times\boldsymbol{R}_+)} \\ &\le O(1)\,\delta\,\left(\epsilon\,\delta\right)^{-1/2}\,\|\theta\|_{W^{1,2}(\boldsymbol{R}\times\boldsymbol{R}_+)} \\ &\le O(\epsilon^{1/2}\,)\,\|\theta\|_{W^{1,2}(\boldsymbol{R}\times\boldsymbol{R}_+)} \qquad \longrightarrow 0. \end{aligned}$$

We finally turn to the term T_4^ϵ. For every continuous function θ with compact support, we use the Cauchy-Schwarz inequality and (5.5)-(5.6) to obtain

$$\begin{aligned}|< T_4^\epsilon, \theta >| &\equiv \left| \iint_{\boldsymbol{R}\times\boldsymbol{R}_+} \delta\, U''(u^\epsilon)\partial_x u^\epsilon \partial_{xx} u^\epsilon\, \theta\, dxdt \right| \\ &\leq O(\delta)\, \|\partial_x u^\epsilon\|_{L^\infty(\boldsymbol{R}_+, L^2(\boldsymbol{R}))} \|\partial_{xx} u^\epsilon\|_{L^2(\boldsymbol{R}\times\boldsymbol{R}_+)} \|\theta\|_{L^\infty(\boldsymbol{R}\times\boldsymbol{R}_+)} \\ &\leq O(\delta)\, \epsilon^{-1/2} \left(\epsilon\, \delta\right)^{-1/2} \|\theta\|_{L^\infty(\boldsymbol{R}\times\boldsymbol{R}_+)} \\ &= O(1)\, \|\theta\|_{L^\infty(\boldsymbol{R}\times\boldsymbol{R}_+)},\end{aligned}$$

if $\delta = O(\epsilon^2)$, which proves that T_4^ϵ uniformly bounded as a sequence of bounded Borel measures.

Therefore (5.10) holds true if we set $\mathcal{D}_1^\epsilon = T_1^\epsilon + T_3^\epsilon$ and $\mathcal{D}_2^\epsilon = T_2^\epsilon + T_4^\epsilon$. The proof of Lemma 5.4 is complete. □

Proof of Theorem 5.1. Lemma 5.3 allows one to apply the compensated compactness method and derive Tartar's commutation relation for the Young measure associated with the sequence $\{u^\epsilon\}$. Compared with Tartar's proof, the major difference here is that the relevant space is L^4 rather than L^∞, and so L^4 Young measures are to be used. The convergence framework in Schonbek [37] applies here in a straightforward manner. We observe that, at this stage of the proof of Theorem 5.1, we only need Lemma 5.3 for entropies with compact support. The result is that the solutions u^ϵ converge strongly in any $L^\infty(L^p)$ space locally, $2 \leq p < 4$, to a solution $u \in L^\infty(\boldsymbol{R}_+, L^2(\boldsymbol{R}) \cap L^4(\boldsymbol{R}))$ of the cubic conservation law.

We now focus on the derivation of the entropy inequality, which is again based on Lemma 5.3. Applying this lemma with $U(u) = u^2/2$, we see that

$$\partial_t \left(\frac{1}{2} \left(u^\epsilon\right)^2 \right) + \partial_x \left(\frac{3}{4} \left(u^\epsilon\right)^4 \right) = \mathcal{D}_1^\epsilon + \mathcal{D}_2^\epsilon. \tag{5.12}$$

The first term $\mathcal{D}_1^\epsilon$ converges to zero in $W^{-1,2}$, whereas $\mathcal{D}_2^\epsilon$ is a (generally non-trivial) bounded measure and so it admits a weak-$\star$ limit, denoted below by $\mathcal{D}_2^\star$.

In order to pass to the limit in (5.12) as $\epsilon \to 0$, we observe that the sequence $\left(u^\epsilon\right)^2$ converges strongly to u^2, but that $\left(u^\epsilon\right)^4$ is merely bounded in L^1 in view of our estimate (5.6) and therefore converges to a measure which we denote by μ. Passing to the limit in (5.12) yields

$$\partial_t \frac{u^2}{2} + \partial_x \frac{3\,\mu}{4} = \mathcal{D}_2^\star.$$

Lastly we check that the measure $\mathcal{D}_2^\star$ is actually non-negative. Indeed $\mathcal{D}_2^\epsilon$ is given by the general formula

$$\mathcal{D}_2^\epsilon = -\epsilon\, U''(u^\epsilon)\, |\partial_x u^\epsilon|^2 + \delta\, U'''(u^\epsilon) \left(\partial_x u^\epsilon\right)^3, \tag{5.13}$$

so for $U(u) = u^2/2$, we simply obtain

$$\mathcal{D}^\epsilon = -\epsilon\, U''(u^\epsilon)\, |\partial_x u^\epsilon|^2$$

which is non-positive. This proves that $\mathcal{D}_2^\star \leq 0$ in the sense of bounded measures, and (5.13) implies (5.2).

We now consider the entropy inequalities for arbitrary entropies. To establish the claim (3) in Theorem 5.1, it is enough to treat traveling wave solutions to (5.1) and their limits. For an arbitrary entropy pair (U, F) and $\delta = A\epsilon^2$, we consider $\mathcal{D}_2^\epsilon$ defined by (5.13) and show that $\mathcal{D}_2^\epsilon$ need not converge to a non-positive measure when $U(u) \neq u^2$. According to the analysis made in Sections 3 and 4 on the traveling waves, a non-classical shock connects a left state $u_l > 2\sqrt{2}\mu/3$ to the particular state $u_m = -u_l + \sqrt{2}\mu/3$. For a traveling wave solutions with wave speed s, it is not hard to see that the limiting measure $\mathcal{D}_2^\star$ generated by the sequence $\mathcal{D}_2^\epsilon$ has its support on the line $x = s\, t$ and that the mass of the measure on this line is exactly

$$D_U(u_l; \mu) \equiv -\, s\left(U(-u_l + \sqrt{2}\mu/3) - U(u_l)\right) + F(-u_l + \sqrt{2}\mu/3) - F(u_l), \tag{5.14}$$

with $s = u_l^2 - \sqrt{2}\,\mu u_l/3 + 2\mu^2/9$.

That $D_U(u_l;\mu)$ is non-positive for $U(u) = u^2/2$ can be checked directly from (5.14); one finds

$$D_U(u_l;\mu) = -\mu\frac{2\sqrt{2}}{3}\,\big(u_l - \frac{\mu}{3\sqrt{2}}\big)^3, \tag{5.15}$$

which is always negative since $u_l > 2\sqrt{2}\mu/3$. Of course this result is also a direct consequence of Part (2) of the theorem.

First of all, we claim that given any non-classical shock, i.e. any $u_l > 2\sqrt{2}\mu/3$, there exists at least one entropy for which $D_U(u_l;\mu)$ is positive. To show this, we use the Kruzkov entropy

$$U(u) = |u-k|, \qquad F(u) = \mathrm{sgn}(u-k)(u^3-k^3)$$

with the choice $k = -u_l/2$. We then have

$$\begin{aligned} D_U(u_l;\mu) &= \frac{3}{4}\,u_l^3 - \frac{\sqrt{2}\mu}{3}\,u_l^2 + \frac{2\mu^2}{9}\,u_l \\ &\geq \frac{\sqrt{2}\,\mu}{6}\,u_l^2 + \frac{2\mu^2}{9}\,u_l \\ &> 0. \end{aligned}$$

The first inequality above is obtained by substituting (3.15) for one factor of u_l in the u_l^3 term.

It seems natural that, given any non-quadratic entropy U, there should exist a non-classical shock having positive entropy dissipation for that entropy. This is actually false, without certain restrictions on the entropy. We verify this by using the ODE (3.11) for the traveling wave. Consider the TW $u^\epsilon(x,t) = v(\xi)$ associated with a non-classical shock. Multiplying the equation (3.5) by $U'(v)$, we obtain an equivalent expression for the mass on the discontinuity of the entropy dissipation measure:

$$D_U(u_l;\mu) \equiv \int_{\boldsymbol{R}} \big(-\mu\, U''(v)\,(v')^2 + U'''(v)\,(v')^3\big)\,d\xi. \tag{5.16}$$

This term need not be negative for general traveling waves and general convex entropies. It is clear, however, that $D_U(u_l;\mu)$ is actually non-positive if U''' is not "too" large, i.e. if, for instance,

$$U''(u) > 0, \quad \text{and} \quad |U'''(u)|\,\sup_{\xi\in\boldsymbol{R}} |v'(\xi)| \leq \mu\, U''(u), \qquad \text{for all } u,$$

because this condition ensures that the *integrand* of (5.16) is non-positive.

We now consider the particular family of entropies:

$$U(u) = u^{2n}/(2n) \tag{5.17}$$

with $n \geq 2$. For this family, we show that $D_U(u_l;\mu)$ is indeed negative. Substituting (5.17) into (5.16) gives

$$\begin{aligned} D_U(u_l;\mu) &= -\int_{\boldsymbol{R}} (v')^2\big(\mu\, U''(v) - U'''(v)\,v'\big)\,d\xi \\ &= -(2n-1)\int_{\boldsymbol{R}} v^{2n-4}\,(v')^2\,\big(\mu\, v^2 - (2n-2)\,v\,v'\big)\,d\xi. \end{aligned} \tag{5.18}$$

We observe that the first term inside the integral, with μv^2, is always positive. Furthermore, when we use the explicit, tanh, form (3.16) of the non-classical MKdVB shock, we see that because $u_l > |u_r|$, we have that v is an odd function plus a positive constant; in addition, v' is a negative even function: $-\,\mathrm{sech}^2$. Therefore, $-v\,v'$

integrated over the real line is also positive. The negative sign in front of the integral gives the overall negative sign for the integrand of $D_U(u_l;\mu)$.

When the convex entropy is an even function, as in (5.15), $D_U(u_l;\mu)$ appears to have the "correct" (negative) sign, but when this symmetry is broken, as in the case of the Kruzkov entropy, above, or (as can be shown) by using $U(v) = e^{-v}$ in (5.14), the dissipation measure is, in general, positive.

This completes the proof of Theorem 5.1. $\square$

We note that Part (3) of Theorem 5.1 can be generalized to equations containing a nonlinear diffusion, especially (5.4), or to the regularization based on the Beam-Warming equivalent equation studied in Subsection 3c. For the equation (5.4), the preferred entropy is again the quadratic function. While for the Beam-Warming equivalent equation, we should instead use the quartic entropy function $U(u) = u^4/4$. Indeed if u solves (3.44), we get formally

$$\begin{aligned}\partial_t u^4/4 + \partial_x 3\, u^7/7 =& \kappa\, \tilde{\delta}\, u^3\, \partial_{xxx} u^3 \\ =& \kappa\, \tilde{\delta}\, \partial_x \big(u^3\, \partial_{xx} u^3 - \frac{1}{2} \left(\partial_x (u^3)\right)^2\big),\end{aligned} \tag{5.19}$$

where $\tilde{\delta} = \delta/3$. The right-hand side of (5.19) is in conservation form and should not, therefore, contribute in the limit $\delta \to 0$.

We conclude by commenting that the solutions u^ϵ constructed by (5.1) do not satisfy the maximum principle or the total variation diminishing property, as was the case of the classical weak solutions. There is generally no uniform bound, nor is a uniform bound for the amplitude available. However an estimate for the growth of $\|u^\epsilon\|_{L^\infty}$ can be obtained as follows:

$$\begin{aligned}\|u^\epsilon\|^2_{L^\infty(\boldsymbol{R}\times\boldsymbol{R}_+)} &\le 2 \sup_{x,t} \int_0^x |u^\epsilon(y,t)|\, |\partial_x u^\epsilon(y,t)|\, dy \\ &\le O(1) \sup_t \|u^\epsilon(t)\|_{L^2}\, \|\partial_x u^\epsilon(t)\|_{L^2} \\ &\le O(\delta^{-1/2}),\end{aligned}$$

thus

$$\|u^\epsilon\|_{L^\infty(\boldsymbol{R}\times\boldsymbol{R}_+)} \le O(\delta^{-1/4}).$$

6. Non-Classical Shocks in the Beam-Warming Scheme.

The purpose of this section is to demonstrate numerically that finite difference schemes may generate non-classical shocks. In principle, those waves should also be described in the framework introduced in Section 4, although studying discrete shock profiles needed to determine the kinetic relation in Section 4 is more delicate than studying continuous TW.

We focus here on two well-known, finite difference approximations to the cubic scalar conservation law (4.1); namely, the Lax-Wendroff (LW) and Beam-Warming (BW) schemes. We discretize space and time using the mesh points $x = k\,\Delta x$ and $t = n\,\Delta t$, and as $\Delta x, \Delta t \to 0$, we expect the numerical solution, u_k^n, to approximate $u(k\,\Delta x, n\,\Delta t)$, where the function u solves (4.1). For both schemes below we use the notation: $f_k^n = (u_k^n)^3$.

Both LW and BW are second order accurate schemes and are not total variation diminishing. The latter feature seems to be a necessary requirement for a conservative scheme to admit non-classical shocks. We recall that that monotone, first-order conservative schemes have their total variation decreasing in time. On the other hand, the two-shock non-classical solution represents an *increase* in total variation, compared with the Riemann initial data.

Being second order accurate is not, however, a sufficient (nor necessary) condition for a numerical scheme (having a nonconvex flux) to admit non-classical shocks. Indeed one of the observations in this section is that solutions to the Riemann problem for LW *do converge* to the classical, Oleinik solution (cf. Section 2) as $\Delta x, \Delta t \to 0$. Meanwhile, for a range of initial states, u_l and u_r, solutions to BW converge to non-classical shocks.

We conjecture that this disparity in behavior for BW and LW is due to a difference in the sign of the numerically induced dispersion, as explained below. It would be interesting to rigourously establish that the Lax-Wendroff schemes converges to the classical entropy solution. On the other hand we conjecture that any TVD scheme applied to the cubic conservation law and a smooth initial data, and satisfying one entropy inequality of the form (1.3) converges to the classical, entropy solution. Indeed the results in Section 4 show that any Riemann solution that does not satisfy the Oleinik entropy criterion is necessarily non-monotone, but such a behavior is excluded for a TVD scheme.

The archetypal second order finite difference scheme is the Lax-Wendroff (LW) scheme, a conservative form of which (cf. for instance LeVeque [26]) is

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right) + \frac{\lambda^2}{2}A_{j+1/2}\left(f(u_{j+1}^n) - f(u_j^n)\right) - \frac{\lambda^2}{2}A_{j-1/2}\left(f(u_j^n) - f(u_{j-1}^n)\right)\,), \tag{6.1}$$

where

$$\lambda = \frac{\Delta t}{\Delta x}, \tag{6.2}$$

is the ratio of timestep to gridspace, and $A_{j\pm1/2} = f'\big((u_j^n + u_{j\pm1}^n)/2\big)$, is the derivative of the flux, evaluated midway between consecutive gridpoints. We performed numerical experiments on the cubic scalar equation with this scheme, but observed only classical behavior, modulo some pre-shock oscillations - see Figure 6.3, for example.

Using another second-order finite difference scheme, due to Warming and Beam [41], we *were* able to detect non-classical behavior. While LW uses both left- and right-differences, Beam-Warming (BW) is a second order upwinding scheme:

$$v_j^n = u_j^n - \lambda(f_j^n - f_{j-1}^n), \tag{6.3a}$$

$$u_j^{n+1} = \frac{1}{2}\left(u_j^n + v_j^n\right) - \frac{\lambda}{2}\left(g_j^n - g_{j-1}^n\right) - \frac{\lambda}{2}\left(f_j^n - 2\,f_{j-1}^n + f_{j-2}^n\right), \tag{6.3b}$$

where $g_j^n = f(v_j^n)$ is the flux evaluated at the intermediate point.

Harten, Hyman and Lax [9] discovered an entropy violating shock for the LW solution to (1.1) with the non-convex flux $f(u) = u - \alpha\, u^2(1-u)^2$, $\alpha > 0$, and also one for a convex flux function with $f(1) = f(-1)$. Both of these non-classical shocks are stationary. Majda and Osher [31] point out that all examples of entropy violating shocks for LW are steady solutions of (6.1). By contrast, the non-classical shock we observe in BW is a traveling wave, whose speed depends, through RH, on the left- and right-hand states.

We now motivate our conjecture, that the sign of the numerical dispersion is crucial, by considering the case of a linear flux.

Analysis of BW and LW schemes for Linear Equations

For the linear advection equation

$$u_t + Au_x = 0, \tag{6.4}$$

with A constant, the Lax-Wendroff scheme reduces to

$$u_j^{n+1} = u_j^n - \frac{\lambda\, A}{2}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{\lambda^2\, A^2}{2}\left(u_{j+1}^n - 2\,u_j^n + u_{j-1}^n\right), \tag{6.5}$$

while the BW scheme becomes

$$u_j^{n+1} = u_j^n - \frac{\lambda\, A}{2}\left(3\,u_j^n - 4\,u_{j-1}^n + u_{j-2}^n\right) + \frac{\lambda^2\, A^2}{2}\left(u_j^n - 2\,u_{j-1}^n + u_{j-2}^n\right). \tag{6.6}$$

An important distinction between the modified equations for Lax-Wendroff and Beam-Warming, in the linear case, is in the sign of their dispersion coefficients. Taking $A > 0$ (which we must do in order to satisfy the CFL condition for the upwind BW, and which will be true for our nonlinear scheme, when $f'(u) = 3u^2 \geq 0$), the modified equations for both linear schemes (6.5) and (6.6) are

$$u_t + Au_x = \mu u_{xxx}. \tag{6.7}$$

For second-order approximations to a linear equation, there is no u_{xx} term, as there was in MKdVB, but the modified equations for the full nonlinear schemes on (4.1) do contain both second- and third derivative terms. For Lax-Wendroff,

$$\mu = (1/6)(\Delta x)^2 A\left((\lambda A)^2 - 1\right) \leq 0, \tag{6.8}$$

with the inequality coming from the CFL stability condition[2] for (6.5) is $|\lambda A| \leq 1$. While for BW, the dispersion coefficient is

$$\mu = (1/6)(\Delta x)^2 A(2 - \lambda A)(1 - \lambda A). \tag{6.9}$$

The CFL condition for (6.6) is $0 \leq \lambda A \leq 2$, and so in the equation (6.9), $\mu > 0$ for $0 < \lambda A < 1$. The usual manifestation of the sign difference is that pre-shock oscillations occur in LW, while BW has post-shock oscillations. In the context of this paper, we note that for $0 < \lambda A < 1$, we can achieve the "interesting" - positive - sign of the dispersion, with respect to the TW solutions in Section 3, for this equivalent equation.

A similar sign discrepancy is also present in the equivalent equations of LW and BW for the cubic conservation law (4.1). In Section 3c, we showed that this sign difference allows non-classical solutions in the equivalent equation for BW, while disallowing them in the equivalent equation for LW. In addition, Example 4 of this section will show that the intermediate state, following the non-classical shock, becomes unstable at values of λ well below the CFL limit, near $\lambda = 1/A$, where μ changes signs in (6.9).

Numerical Experiments with Cubic Flux

In all of the ensuing numerical experiments, we deal with Riemann initial data for LW (6.1) and BW (6.3a,b):

$$u_j^0 = \begin{cases} u_l, & j \leq 0, \\ u_r, & j > 0. \end{cases}$$

Example 1: In Figure 6.1, we plot the solution to BW, (6.3), for the choice $f_j^n = \left(u_j^n\right)^3$ and with $u_l = 1, u_r = -0.4$. The classical solution would be a shock from $u = 1$ to $u = -0.4$, traveling with speed $s = 0.76$, but the profile we observe is non-monotone, with an intermediate state, $u_m \approx -0.855\,u_l$, having formed as the single shock split in two: a slow, undercompressive shock and a fast, Oleinik shock. The Rankine-Hugoniot condition is satisfied numerically across both shocks, as it must be according to the Lax-Wendroff Theorem,[3] see [22]. Also note the small oscillations following both shocks; one must look closely near the undercompressive (slow) shock, however, in order to see them.

Then in Figure 6.2, we plot the numerical solution corresponding to the non-classical shock/rarefaction, with $u_l = 1, u_r = -1.2$ Instead of having a classical shock from u_l to $-0.5\,u_l$, followed *immediately* by a rarefaction to u_r, the BW scheme's profile jumps to an intermediate state $u_m \approx -0.853\,u_l$ and remains at that level until its right-hand boundary, where $x = 3\,u_m^2 t$. Oscillations behind the rarefaction are visible, while those behind the shock are invisible at the scale of this plot.[4]

Next, in Figure 6.3, we plot the solution of the Lax-Wendroff scheme for cubic flux, with initial data $u_l = 1, u_r = -0.4$. Although there are some pre-shock oscillations, there is only one wave speed. We took $\lambda = 0.2$ for Figure 6.3; but similar results were obtained for other values of λ, as well as for other choices of $u_r < 0$, which all led to classical solutions.

[2] The Leapfrog scheme also has $\mu < 0$; we therefore expect its behavior to be similar to that of LW.

[3] They proved that if a difference scheme is in conservation form (as is BW), then if it converges boundedly to some function as $\Delta x, \Delta t \to 0$, that function must be a weak solution.

[4] The slight discrepancy between the values of u_m in Figures 6.1 and 6.2 may be due to the small change in λ between the two figures; see Example 4 in this regard.

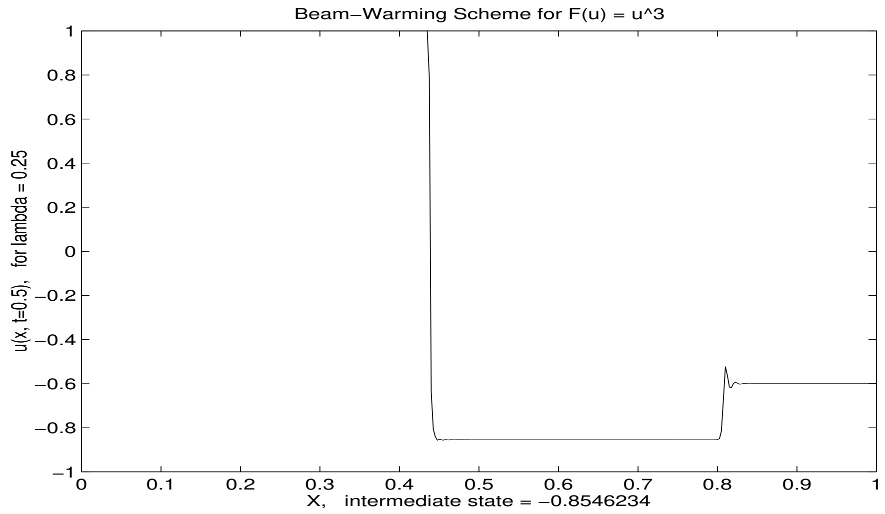

Figure 6.1

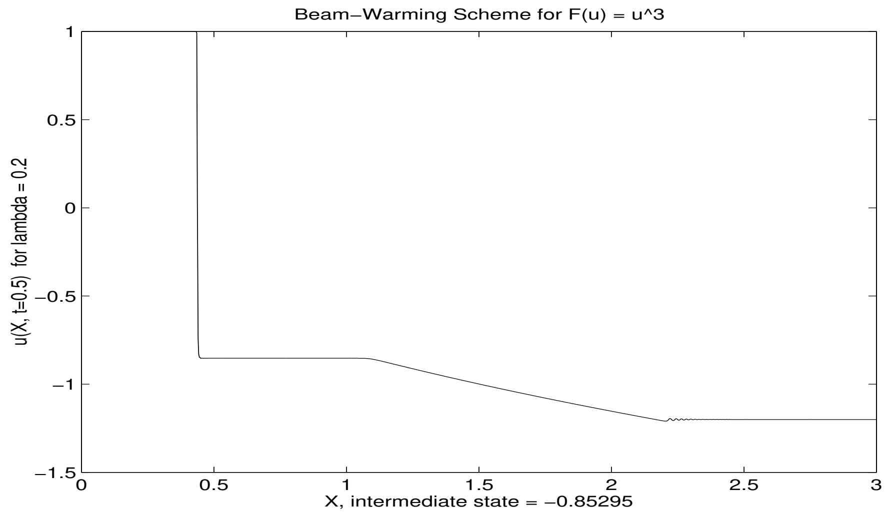

Figure 6.2

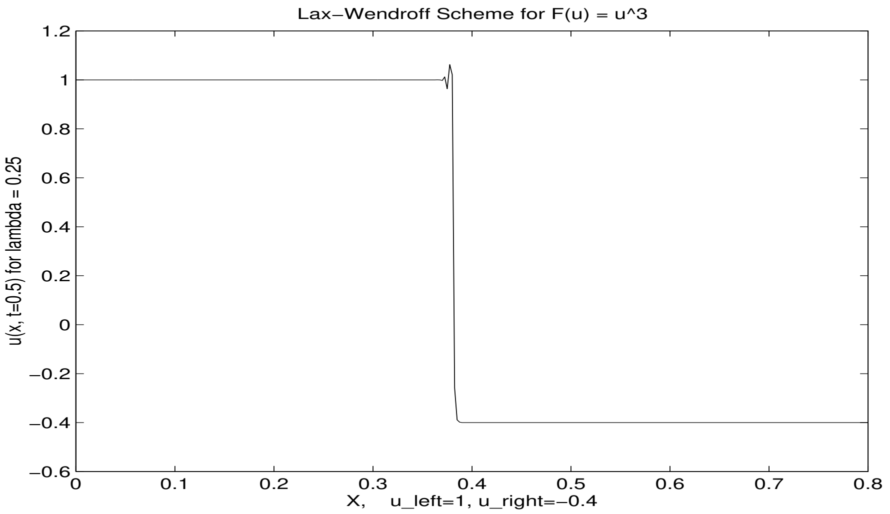

Figure 6.3

Example 2: Figure 6.4 shows the transition to undercompressive shock as u_r is decreased, while u_l is fixed at 1. From Figure 6.4, it appears that the intermediate state grows out of a post-shock overshoot. For $u_r = 0$, there is a relatively small overshoot, which grows as u_r is decreased, until it saturates at approximately -0.855, for $-0.15 < u_r < -0.18$. Further decreasing u_r does not change the value of the intermediate level. This behavior mimics that of MKdVB, where we need $u_r < v_m(\mu)$, with this latter quantity given by (3.13), in order to observe non-classical behavior. Here there is an effective value of μ for BW, determined by the scheme and the choices of λ and u_l.

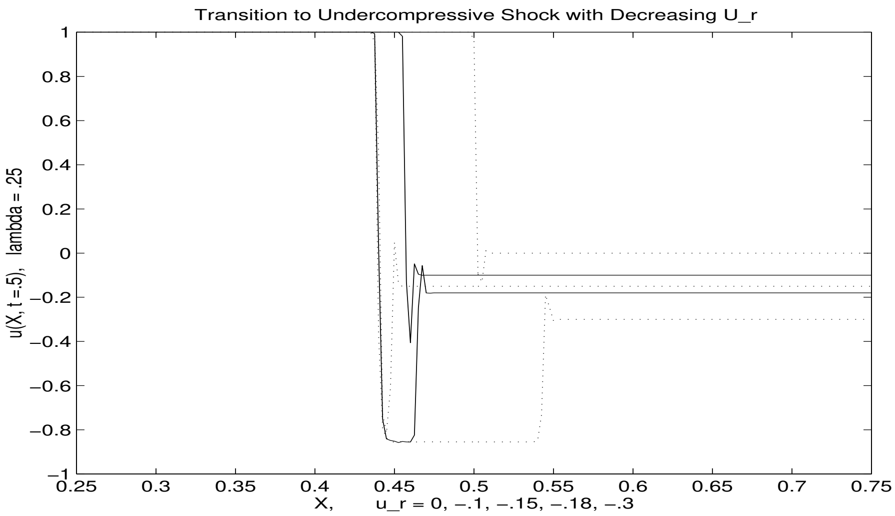

Figure 6.4

Example 3: In this experiment, we observe that in analogy with the condition (3.15) for MKdVB, u_l needs to be sufficiently large, compared to the effective μ for BW, in order to see the formation of an intermediate state. In Figure 6.5, we fix the ratio of $u_r/u_l = -1/2$, and we increase u_l from 0.1 to 0.4. From the plotted curves, it appears that the threshold of formation for an intermediate state is $u_l \approx 0.3$; when $u_l > 0.3$, we observe an intermediate state, with value $u_m \approx -0.85 u_l$.

Examples 2 and 3 are evidence of a nucleation criterion for the Beam-Warming scheme to have non-classical shocks. Based on the findings of these examples, the nucleation criterion here should be similar to that of MKdVB in Theorem 4.5.

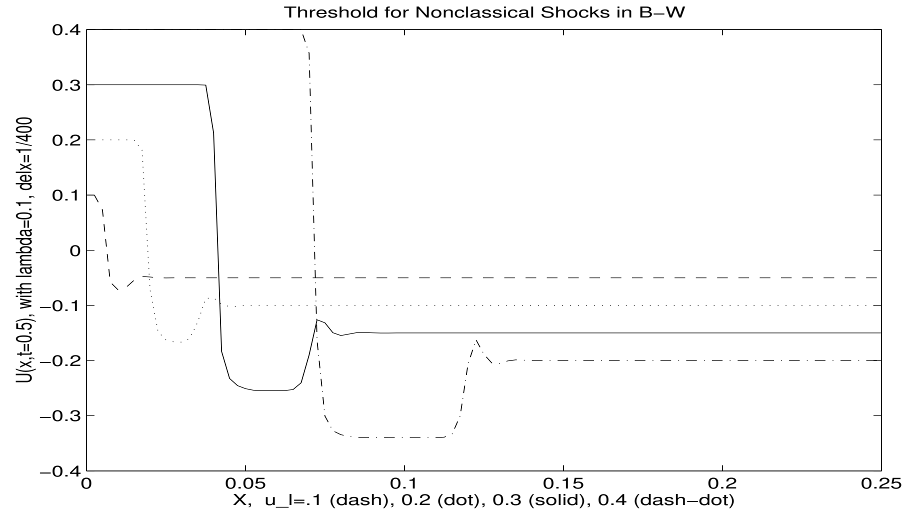

Figure 6.5

Example 4: Figure 6.6 displays the effect of variations in $\lambda = \Delta t/\Delta x$ on the profile. With $u_l = 1$ and $u_r = -0.4$, the wave speeds satisfy, $3u^2 \leq 3$. Thus the CFL condition for BW should be $\lambda < 2/3$, and, indeed, we observe this numerically. Well before this value of λ is reached, however, large-amplitude, medium-wavelength oscillations appear in the intermediate state (see the $\lambda = .5$ curve, for example). We believe this to be connected with the change in sign of the dispersion coefficient for the linear scheme (6.6) at $\lambda = 1/3$.

We further observe that the profiles appear to be converging as $\lambda \to 0$. This fact may be useful in future analysis of non-classical behavior for BW.

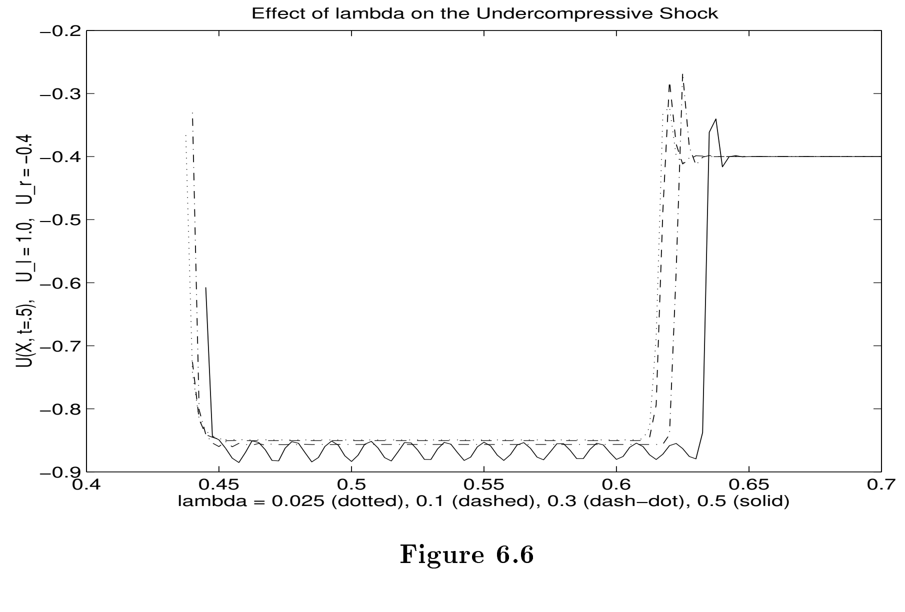

Figure 6.6

7. Concluding Remarks.

This paper presents a new approach towards defining shock waves in non-convex scalar conservation laws. Those non-classical shocks are generated by approximation methods for (1.1) that contain both diffusion and dispersion, e.g. the equations of the form (3.18) with diffusion and dispersion kept in balance, or the non-TVD numerical schemes such as the second order schemes in Section 6. It should be noted that the Lax-Wendroff and Beam-Warming schemes automatically balance diffusion and dispersion. Similar effects could presumably be obtained with singular source terms (relaxation, etc) added to the right hand side of the conservation law.

Understanding non-classical shocks may be based on the following strategy: Studying the existence and the behavior of the traveling wave solutions leads to the class of admissible shock waves for the conservation law. In certain cases, the class of TW solutions is rich enough so that a solution to the Riemann problem may be determined (Subsections 3.a and 3.b). It may not be be unique, however, and an additional criterion is then necessary. A nucleation criterion –specifying when new phase boundaries can arise– is required for some phase transition models [1] in materials science.

To any system derived from conservation principles in physics, is associated one entropy function and a corresponding entropy inequality. The more stringent requirement that the entropy dissipation should be known for each undercompressive shock can be used to select a unique solution, the "correct" non-classical solution, to the Riemann problem; cf. Section 4. It seems convenient to express this condition in the form of a kinetic relation. To complete the description, it is necessary to derive this kinetic relation from the properties of the TW solutions for the specific regularization which has been employed. The above analysis should also provide valuable insight towards deriving a priori estimates for the Cauchy problem.

This paper presented an instance where shock waves are sensitive to the form of regularization used to approach them. Such behavior has been pointed out recently for non-strictly hyperbolic systems (for instance [4, 14, 27, 42] and the references contained therein) and phase transition problems [2, 38, 40]. For other hyperbolic problems see also [6, 12, 15, 18, 19, 25]. Similar sensitivity has been observed in other contexts, e.g. the vortex sheet problem in incompressible fluid dynamics; cf. [16] and the references listed therein.

We now mention some of the outstanding problems in this area:

It is should be possible to prove an analogue of Theorem 5.1 for the Beam-Warming scheme. The existence of two special entropies, having favorable properties for dissipation, was crucial in the analysis of Section 5. When $\delta < 0$, we conjecture that the diffusive-dispersive approximations to the cubic conservation law converge to the classical, entropy weak solution. On the other hand, deriving a criterion for the solutions to the Cauchy problem, i.e. an extension to the kinetic relation introduced for the Riemann problem in Section 4, is one of the main challenges. Another approach to the characterization of limiting solutions may be necessary.

The stability of the traveling waves associated with non-classical shocks is also an open question. The present paper focused on the cubic conservation law; treatment of arbitrary non-convex fluxes is an open problem. It would be interesting to see whether our Definition 1.2 is sufficient to ensure uniqueness for a non-cubic flux. Systems are considered in Hayes-LeFloch [11].

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Note added:. After this paper was completed, Lev Truskinovsky pointed out to us that A. Kulikovsky, in Dokl. Acad. Nauk SSR 275 (6) (1984), 1349–1352 (in Russian), had introduced a kinetic relation for scalar conservation laws in a similar fashion as we do in Section 4 of the present paper.

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Appendix: Derivation of Beam-Warming Equivalent Equation.

In this appendix, we derive the equivalent equation associated with the Beam-Warming scheme when $F(u_j^n) = \big(u_j^n\big)^3$. Explicitly writing out the two-step scheme (6.3), we get

$$\begin{aligned} u_j^{n+1} = u_j - \frac{\lambda}{2}(u_j^3 - u_{j-1}^3) - \frac{\lambda}{2}\big(\big(u_j - \lambda(u_j^3 - u_{j-1}^3)\big)^3 \\ - \big(u_{j-1} - \lambda(u_{j-1}^3 - u_{j-2}^3)\big)^3\big) - \frac{\lambda}{2}(u_j^3 - 2u_{j-1}^3 + u_{j-2}^3), \end{aligned} \tag{A.1}$$

where $u_j \equiv u_j^n$. Writing (A.1) in ascending powers of λ leads to

$$\begin{aligned} u_j^{n+1} &= u_j - \frac{\lambda}{2}(3u_j^3 - 4u_{j-1}^3 + u_{j-2}^3) + \frac{3\lambda^2}{2}\left(u_j^2(u_j^3 - u_{j-1}^3) - u_{j-1}^2(u_{j-1}^3 - u_{j-2}^3)\right) \\ &+ \frac{3\lambda^3}{2}\left(-u_j(u_j^3 - u_{j-1}^3)^2 + u_{j-1}(u_{j-1}^3 - u_{j-2}^3)^2\right) + \frac{\lambda^4}{2}\left((u_j^3 - u_{j-1}^3)^3 - (u_{j-1}^3 - u_{j-2}^3)^3\right). \end{aligned} \tag{A.2}$$

To calculate the modified equation, we insert the Taylor series for quantities like

$$u_{j-1} = u - \Delta u_x + \frac{\Delta^2}{2}u_{xx} - \frac{\Delta^3}{6}u_{xxx} + \cdots,$$

where Δ is the spatial mesh size, into equation (A.2). Defining the three quantities:

$$\begin{aligned} \alpha &= 3u^2 u_x, \\ \beta &= -\frac{3}{2}u^2 u_{xx} - 3uu_x^2, \\ \gamma &= \frac{1}{2}u^2 u_{xxx} + 3uu_x u_{xx} + u_x^3, \end{aligned}$$

we can compactly write the Taylor expansions for expressions from (A.2):

$$\begin{aligned} u_j^3 - u_{j-1}^3 &= \alpha\,\Delta + \beta\Delta^2 + \gamma\Delta^3 + O(\Delta^4), \\ u_{j-1}^3 - u_{j-2}^3 &= \alpha\,\Delta + 3\beta\Delta^2 + 7\gamma\Delta^3 + O(\Delta^4), \end{aligned}$$

$$\begin{aligned} (u_j^3 - u_{j-1}^3)^2 &= \alpha^2\,\Delta^2 + 2\alpha\,\beta\Delta^3 + O(\Delta^4), \\ (u_{j-1}^3 - u_{j-2}^3)^2 &= \alpha^2\,\Delta^2 + 6\alpha\,\beta\Delta^3 + O(\Delta^4), \end{aligned}$$

$$\begin{aligned} (u_j^3 - u_{j-1}^3)^3 &= \alpha^3\,\Delta^3 + O(\Delta^4), \\ (u_{j-1}^3 - u_{j-2}^3)^3 &= \alpha^3\Delta^3 + O(\Delta^4). \end{aligned}$$

We therefore have

$$3u_j^3 - 4u_{j-1}^3 + u_{j-2}^3 = 2\alpha\Delta - 4\gamma\Delta^3 + O(\Delta^4), \tag{A.3}$$

$$u_j^2(u_j^3 - u_{j-1}^3) - u_{j-1}^2(u_{j-1}^3 - u_{j-2}^3) = (2\alpha uu_x - 2\beta u^2)\Delta^2 + (-\alpha u_x^2 - \alpha uu_{xx} + 6\beta uu_x - 6\gamma u^2)\Delta^3 + O(\Delta^4), \tag{A.4}$$

$$u_{j-1}(u_{j-1}^3 - u_{j-2}^3)^2 - u_j(u_j^3 - u_{j-1}^3)^2 = (4\alpha\beta u - \alpha^2 u_x)\Delta^3 + O(\Delta^4), \tag{A.5}$$

$$(u_j^3 - u_{j-1}^3)^3 - (u_{j-1}^3 - u_{j-2}^3)^3 = O(\Delta^4). \tag{A.6}$$

Inserting (A.3)-(A.6) into the right-hand side of (A.2) gives

$$u_j^{n+1} = u - \alpha\delta + 2\gamma\Delta^2\delta + O(\Delta^3\delta) + \frac{3}{2}(2\alpha u u_x - 2\beta u^2)\delta^2 + \frac{3}{2}(-\alpha u_x^2 - \alpha u u_{xx} + 6\beta u u_x - 6\gamma u^2)\Delta\delta^2 + O(\Delta^2\delta^2)$$
$$+ \frac{3}{2}(4\alpha\beta u - \alpha^2 u_x)\delta^3 + O(\Delta\delta^3) + O(\delta^4), \tag{A.7}$$

where $\delta \equiv \lambda\Delta$ is the timestep. Rearrangement of (A.7) yields

$$\begin{aligned} u_j^{n+1} &= u - 3u^2u_x\delta + \frac{1}{2}(9u^4u_{xx} + 36u^3u_x^2)\delta^2 + 2\gamma\Delta^2\delta + \frac{3}{2}(-\alpha u_x^2 - \alpha u u_{xx} + 6\beta u u_x - 6\gamma u^2)\Delta\delta^2 \\ &\quad + \frac{3}{2}(4\alpha\beta u - \alpha^2 u_x)\delta^3 + O(\Delta^4) \end{aligned} \tag{A.8}$$

Note that we have substituted the expressions for α and β in the two lowest-order terms of (A.8). We now expand the left-hand side of (A.8):

$$u_j^{n+1} = u + \delta u_t + \frac{\delta^2}{2}u_{tt} + \frac{\delta^3}{6}u_{ttt} + \cdots .$$

After substituting this into (A.8), cancelling the $O(1)$ terms and dividing by δ, we get

$$\begin{aligned} u_t + \frac{\delta}{2}u_{tt} + \frac{\delta^2}{6}u_{ttt} + O(\delta^3) &= -3u^2u_x + \frac{1}{2}(9u^4u_{xx} + 36u^3u_x^2)\delta + 2\gamma\Delta^2 \\ &\quad + \frac{3}{2}(-\alpha u_x^2 - \alpha u u_{xx} + 6\beta u u_x - 6\gamma u^2)\Delta\delta + \frac{3}{2}(4\alpha\beta u - \alpha^2 u_x)\delta^2 + O\Big(\frac{\Delta^4}{\delta}\Big). \end{aligned} \tag{A.9}$$

To compute the higher time-derivatives of u, we utilize the relation

$$u_t = -3u^2u_x + O(\Delta^2), \tag{A.10}$$

so that

$$\begin{aligned} u_{tt} &= -3(u^2u_x)_t + O(\Delta^2) \\ &= -6uu_xu_t - 3u^3(u_t)_x + O(\Delta^2) \\ &= 9u^4u_{xx} + 36u^3u_x^2 + O(\Delta^2). \end{aligned} \tag{A.11}$$

Similarly, we find that

$$u_{ttt} = -36\alpha u^3u_{xx} - 54u^4\gamma - 36\alpha^2u_x + 48\alpha\beta u + O(\Delta^2). \tag{A.12}$$

Note that (A.10) may be justified *a posteriori* by inserting (A.11) into (A.9). Expanding the $O(\Delta\delta)$ term from (A.9) yields

$$-\alpha u_x^2 - \alpha u u_{xx} + 6\beta u u_x - 6\gamma u^2 = -30u^3u_xu_{xx} - 27u^2u_x^3 - 3u^4u_{xxx}. \tag{A.13}$$

Combining (A.9)-(A.13) and retaining terms up to second order gives

$$\begin{aligned} u_t + 3u^2u_x &= (u^2u_{xxx} + 6uu_xu_{xx} + 2u_x^3)\Delta^2 - \frac{9}{2}(10u^3u_xu_{xx} + 9u^2u_x^3 + u^4u_{xxx})\Delta\delta \\ &\quad + \frac{9}{2}(12u^5u_xu_{xx} + 15u^4u_x^3 + u^6u_{xxx})\delta^2. \end{aligned} \tag{A.14}$$

The $O(\Delta^2)$ term may be written in conservation form as

$$\begin{aligned} u^2u_{xxx} + 6uu_xu_{xx} + 2u_x^3 &= \big(u^2u_{xx} + 2uu_x^2\big)_x \\ &= \frac{1}{3}(u^3)_{xxx}, \end{aligned} \tag{A.15}$$

The $O(\Delta\delta)$ term can also be expressed as an exact derivative:

$$\begin{aligned} 10u^3u_xu_{xx} + 9u^2u_x^3 + u^4u_{xxx} &= \left(3u^3u_x^2 + u^4u_{xx}\right)_x \\ &= \frac{1}{4}\left(u(u^4)_{xx}\right)_x. \end{aligned} \tag{A.16}$$

Similarly for the $O(\delta^2)$ term, we have

$$\begin{aligned} 12u^5u_xu_{xx} + 15u^4u_x^3 + u^6u_{xxx} &= \left(3u^5u_x^2 + u^6u_{xx}\right)_x \\ &= \frac{1}{4}\left(u^3(u^4)_{xx}\right)_x. \end{aligned} \tag{A.17}$$

Using (A.15)-(A.17) we can write (A.12) more compactly as

$$u_t + (u^3)_x = \Delta^2\left\{\frac{1}{3}(u^3)_{xxx} - \frac{9}{8}\lambda(u(u^4)_{xx})_x + \frac{9}{8}\lambda^2(u^3(u^4)_{xx})_x\right\}. \tag{A.18}$$

If we restrict λ to be small enough that the second and third terms on the right may be neglected in comparison to $O(\Delta^2)$, then the result is equation (3.44), with $\kappa = 1$.

The derivation of the modified equation for Lax-Wendroff is similar, and may be found in detail in Majda and Osher [30]. The modified equation for LW is

$$u_t + (u^3)_x = -\frac{\Delta^2}{6}(u^3)_{xxx} + O(\delta^2), \tag{A.19}$$

so that in equation (3.44), we actually have $\kappa = -1/2$ for LW, but this factor of 1/2 can be absorbed into Δ.