BOUNDARY LAYERS IN WEAK SOLUTIONS TO HYPERBOLIC CONSERVATION LAWS

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Abstract. This paper studies the boundary layers that generally arise in approximations of the entropy discontinuous solutions to the initial-boundary value problem associated with a nonlinear hyperbolic system of conservation laws. We consider the vanishing viscosity method and several finite difference schemes (Lax-Friedrichs type schemes, Godunov scheme). Assuming solely uniform L^∞ bounds and for entropy weak solutions, we derive several entropy inequalities satisfied by the boundary layers. Different approximation methods may generate different boundary layers, and so the boundary condition can be formulated only if an approximation scheme is selected.

We obtain several formulations for the boundary condition which in principle apply whether the boundary is characteristic or not. The formulations are based on families of sets of admissible boundary values, as we call them. Under some assumptions, the local structure of those sets together with the well-posedness of the corresponding initial-boundary value problems, is investigated. The results are illustrated with convex and non-convex conservation laws and examples from continuum mechanics

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1. Introduction.

This paper considers the initial-boundary value problem for an hyperbolic system of conservation laws

$$\partial_t u + \partial_x f(u) = 0, \qquad u(x,t) \in \mathcal{U} \subset \boldsymbol{R}^N, \quad x > 0,\, t > 0, \tag{1.1}$$

supplemented with

(1) an initial condition at time $t = 0$

$$u(x,0) = u_I(x), \qquad x > 0, \tag{1.2}$$

(2) the entropy inequality

$$\partial_t U(u) + \partial_x F(u) \leq 0, \tag{1.3}$$

(3) and a weak form of the following Dirichlet boundary condition at $x = 0$

$$u(0,t) = u_B(t), \qquad t > 0. \tag{1.4}$$

Indeed the hyperbolic problem (1.1)–(1.4) is usually not well-posed when the boundary data is required to be assumed in the (strong) sense (1.4), even when (1.1) is a linear system (cf. Kreiss [28]). It is the objective of this paper to provide a general framework which leads to (mathematically correct) formulations for the boundary condition. Following Dubois-LeFloch [15], our strategy is to reformulate (1.4) in the (weak) form

$$u(0+,t) \in \mathcal{E}(u_B(t)), \qquad t > 0, \tag{1.5}$$

where $\mathcal{E}(u_B(t)) \subset \mathcal{U}$ is a time-dependent set (the set of *admissible boundary values*) to be defined from the boundary data, and $u(0+,t)$ is the trace (its existence is discussed in this paper) of the solution u at the boundary. We are going to consider several methods of approximation for the problem (1.1)–(1.4), including the artificial vanishing viscosity method and a class of finite difference schemes, for which the boundary condition (1.4) can be easily implemented. As the approximation parameter goes to zero, a sharp transition layer generally develops near the boundary $\{x = 0\}$ and the limiting solution does not satisfy the boundary condition (1.4). Our aim in this paper is to provide some contribution to the following program: performe a rigorous analysis of the boundary layer for weak solutions, then derive several suitable definitions for the set in (1.5), and finally investigate the structure of the latter to decide whether the boundary-value problem is well-posed.

In (1.1), $\mathcal{U}$ is assumed to be a convex and open subset of $\boldsymbol{R}^N$, the flux-function $f : \mathcal{U} \to \boldsymbol{R}^N$ to be a smooth mapping, and the initial data u_I to belong to $L^\infty(\boldsymbol{R}_+,\mathcal{U})$. It will be convenient to assume that the boundary data u_B has bounded total variation on any interval $[0,T]$ for all $T > 0$. It is assumed that (1.1) admits at least one strictly convex entropy pair. By definition, a pair of functions $(U,F) : \mathcal{U} \to \boldsymbol{R} \times \boldsymbol{R}$ of class $\mathcal{C}^2$ is called a convex (or strictly convex) entropy pair iff $\nabla F^T = \nabla U^T \nabla f$ and the Hessian matrix $\nabla^2 U$ is non-negative (or positive definite). The existence of at least one strictly convex entropy pair implies that (1.1) is hyperbolic. For background on hyperbolic systems, we refer to Lax [29, 30, 31], Dafermos [11] and Smoller [44], concerning the theory of existence of entropy solutions to the pure Cauchy problem, to Glimm [21] and Liu [39] for initial data with small total variation, and DiPerna [12,13] for systems of two equations with L^∞ initial data.

This paper contributes to establishing a framework for the initial-boundary value problem for (1.1). It is intended to pursue the efforts initiated in recent years on this problem (Cf. review below). In particular we built upon the recent contributions in Gilscon-Serre [20] and Xin [48], who studied the boundary layers associated with the vanishing viscosity approximations assuming the solution to the hyperbolic problem be smooth. A formal asymptotic expansion is introduced in [20, 48] and the convergence including L^2 error estimates is proven for the boundary layer in the smooth regime.

One of the motivations here is to treat several approximation methods simultaneously and compare the results obtained with each of them. We consider the vanishing viscosity method, a class of Lax-Friedrichs type schemes, and the Godunov scheme.

In Section 2, we rigorously derive conditions satisfied by the boundary layer, which take the form of a family of *boundary entropy inequalities* and a *boundary layer equation*. The regularity of the relevant traces at the boundary are discussed. The whole analysis is performed by assuming only a uniform L^∞ bound on the approximate solutions; in particular no assumption is required on the regularity of the limiting solution to (1.1). Since high frequency oscillations in the approximate solutions can not be a priori excluded, the conditions above are formulated in terms of a boundary Young measure associated with the boundary layer. Note that, in the derivation of Section 2, the boundary is possibly characteristic, i.e. the eigenvalues of the matrix $\nabla f(u)$ may vanish for certain values of u.

Observe also that, in general, the equations and inequalities we derive depend upon the approximation method in use. Fundamentally the boundary condition can not be formulated from the mere knowledge of the function u_B, but depend upon the underlining "physical" regularization. This feature arises in weak solutions to many nonlinear hyperbolic problems.

In Section 3, we introduce several sets of admissible boundary values and investigate their local structure. When the boundary is non-characteristic, we establish that the sets based on the boundary layer equations are manifold with the "correct" dimension. That is, the corresponding initial-boundary value problem is well-posed, at least for constant boundary and initial data (a generalization to the Riemann problem). We also prove a similar (but stronger) result for the set based on the boundary layer equation derived by the Godunov scheme. Strictly speaking this scheme does not produce any boundary layer; however analyzing that scheme leads to a formulation of the boundary condition as it was first pointed out in [15, 16]. We recall that setting the boundary condition via an upwinding difference scheme is a classical idea in the computing literature.

Sections 4 and 5 are devoted to studying several examples of particular interest. It is expected that, in general, different approximation method for (1.1) leads to a different set in (1.5). However we prove in Section 4, for both convex and non-convex conservation laws, that this is not the case when $N = 1$. In other words the boundary layer for the scalar conservation laws is independent of the approximation method. The same is true of the linear hyperbolic systems; and we conjecture that this also holds for the nonlinear systems in the class with coinciding shock and rarefaction curves introduced by Temple [47]. In Section 5, we consider examples from continuum mechanics, i.e. the system of nonlinear elasticity and the system of gas dynamics. Additional analysis on systems will be provided in [26].

To complete this presentation, we give a short overview of the literature on the boundary conditions for (1.1). Most of the activity was restricted to scalar equations, i.e. $N = 1$. The pioneering work by Leroux [34] and Bardos-Leroux-Nedelec [4] based on the vanishing viscosity method provides a derivation of "the"correct formulation of the boundary condition for multidimensional *scalar* conservation laws. Specically, [4] shows that (1.4) should be replaced by the weaker statement:

$$\big(\mathrm{sgn}(u(0+,t)-k)-\mathrm{sgn}(u_B(t)-k)\big)\big(f(u(0+,t))-f(k)\big)\geq 0 \qquad \text{for all } k\in\boldsymbol{R}, \tag{1.6}$$

where $\mathrm{sgn}(a) = -1$ if $a < 0$, $\mathrm{sgn}(a) = 0$ if $a = 0$, and $\mathrm{sgn}(a) = 1$ if $a > 0$. The convergence of finite difference schemes, again for scalar equations, is established by Leroux in an unpublished work: it is remarkable that the finite difference scheme approach leads to the same formulation (1.6) of the boundary condition. The condition is used by LeFloch [32] in order to extend Lax's explicit formula [30] to the initial-boundary value problem. Joseph [24, 25] used the vanishing viscosity method and the Hopf-Cole transformation to extend Lax's formula for the inviscid Burgers equation. Another derivation is given by Joseph and Veerappa Gowda [27]; see also Gisclon [18] and LeFloch-Nedelec [33]. We also refer to the paper [46] by Szepessy for a very general result of existence and uniqueness.

The statement (1.6) is a special case (when applied to Kruzkov entropies) of a more general inequality:

$$F(u(0,t))-F(u_B(t))-\nabla U(u_B(t))\big(f(u(0,t))-f(u_B(t))\big)\leq 0, \tag{1.7}$$

which has to hold for every convex entropy pair (U, F). The latter was derived formally using the vanishing viscosity method in Dubois-LeFloch [15], who pointed out that (1.7) holds even when $N \geq 2$ and introduced the notion of set of admissible boundary values, cf. (1.5). These inequalities were obtained independently by Bourdel-Delorme-Mazet [8] based on an analysis of the characteristics of the system (1.1), and by Benabdallah [5] for a specific system. The first result of existence for the initial-boundary value problem for a system was given by Benabdallah-Serre [6, 7]: the vanishing viscosity method applied to the p-system of gas dynamics converges to a solution to (1.1) satisfying the set of inequalities (1.7).

The Glimm scheme with various type of boundary conditions was studied by Liu, for instance [36, 37, 38]. In the case that the boundary is assumed to be non-characteristic and the number of boundary conditions is equal to the number of positive eigenvalues of the matrix ∇f, Goodman proves the convergence of the Glimm scheme in his unpublished thesis [22]; cf. also Dubroca-Gallice cite17 and Sablé-Tougeron [41, 42]

More recently Amadori [1, 2] used the formulation in [15] and proved the convergence of a front tracking scheme in the characteristic case. In particular, Amadori establishes that a condition of the form (1.5) can be satisfied pointwise except at countably many times.

2. Boundary Layers in Weak Solutions.

In this section, we consider sequences of approximate solutions to the initial boundary value problem (1.1)–(1.4), and aim at characterizing their limiting behavior near the boundary. Here we rigorously derive entropy inequalities satisfied by the boundary layer. We deal with a sequence of L^∞ functions with uniformly bounded amplitude. As is well-known, for general systems of conservation laws, proving the strong convergence of a sequence of approximate solutions is an open problem. It seems therefore natural to formulate those entropy inequalities in terms of a Young measure (for instance Ball [3] for this concept) associated with the sequence of approximate solutions. Further analysis can be performed on a case by case basis only.

In the following, certain averages will be shown to belong to the space $BV(\boldsymbol{R}_+)$ of functions of locally bounded total variation, i.e. measurable and bounded functions $w : \boldsymbol{R}_+ \to \boldsymbol{R}$ whose distributional derivative is a bounded Borel measure on every interval $(0, T)$ for all $T > 0$. We denote by $TV_0^T(w)$ the total variation, and by $\|w\|_{BV(0,T)} = \|w\|_{L^\infty(0,T)} + TV_0^T(w)$ the norm, of a BV function w on an interval $(0, T)$. By convention, a BV function will be always normalized by selecting its right continuous representative.

2.1 Vanishing Viscosity Method. Let u^ϵ be the approximate solutions obtained by solving the following parabolic regularization of (1.1)-(1.4):

$$\partial_t u^\epsilon + \partial_x f(u^\epsilon) = \epsilon\, \partial_{xx}^2 u^\epsilon, \qquad x > 0,\, t > 0, \tag{2.1}$$

$$u^\epsilon(x,0) = u_I^\epsilon(x), \qquad x > 0, \tag{2.2}$$

$$u^\epsilon(0,t) = u_B^\epsilon(t), \qquad t > 0. \tag{2.3}$$

The smooth functions $u_I^\epsilon \in L^\infty(\boldsymbol{R}_+)$ and $u_B^\epsilon \in BV(\boldsymbol{R}_+)$ are chosen to be uniformly bounded and a.e. convergent approximations of the corresponding data u_I and u_B. We assume the existence of a (smooth enough) solution u^ϵ to the problem (2.1)–(2.3). Note that compatibility conditions at $(x,t) = (0,0)$, such as $u_I^\epsilon(0) = u_B^\epsilon(0)$, are implicitly required. We shall also assume that

$$u^\epsilon \text{ is uniformly bounded in } L^\infty(\boldsymbol{R}_+^2). \tag{2.4}$$

We introduce a new function v^ϵ by setting

$$v^\epsilon(y,t) = u^\epsilon(\epsilon y, t), \tag{2.5}$$

so that the system of equations (2.1) transforms into

$$\epsilon\, \partial_t v^\epsilon + \partial_y f(v^\epsilon) = \partial_{yy}^2 v^\epsilon. \tag{2.6}$$

It is expected that the $(\epsilon \to 0)$ limit of the v^ϵ's will give us a good description of the boundary layer at $x = 0$, at least under additional assumptions, although a different scaling may more adapted in certain cicumstances. Indeed the scaling used here will be justified on several examples of interest by the results in Sections 4 and 5.

By definition (e.g. Ball [3]), a Young measure associated with a sequence u^ϵ satisfying (2.4) is a weak-star measurable mapping ν from the (x,t) plane to the space $\mathrm{Prob}(\mathcal{U})$ of all probability measures (i.e. non-negative measures with mass one) with the property that for every continuous function $g : \mathcal{U} \to \boldsymbol{R}$

$$g(u^\epsilon) \to <\nu, g> \qquad \text{weakly}-\star \text{ in } L^\infty(\boldsymbol{R}_+^2). \tag{2.7}$$

In view of (2.4), the functions v^ϵ also are uniformly bounded in $L^\infty(\boldsymbol{R}_+^2))$. We denote by μ a Young measure associated with the functions v^ϵ.

Theorem 2.1. *The following statements hold for all convex entropy pairs (U, F) associated with the system (1.1), all functions $\theta \in BV(\boldsymbol{R}_+)$, and all bounded interval (T_1, T_2).*

1) When $\theta(t) \geq 0$, the distribution

$$y \to \int_{T_1}^{T_2} < \mu_{y,t}, F > \theta(t)\,dt - \frac{d}{dy}\int_{T_1}^{T_2} < \mu_{y,t}, U > \theta(t)\,dt$$

is in fact a function of locally bounded variation and thus is defined pointwise as a right continuous function. There exists a Young measure $\mu_{0,t}$, such that the following limit exists and is given by $\mu_{0,t}$:

$$\lim_{y \to 0+} \int_{T_1}^{T_2} < \mu_{y,t}, U > \theta(t)\,dt = \int_{T_1}^{T_2} < \mu_{0,t}, U > \theta(t)\,dt.$$

When $\theta(t) \geq 0$, the function

$$x \to \int_{T_1}^{T_2} < \nu_{x,t}, F > \theta(t)\,dt$$

has locally bounded variation. There exists a Young measure $\nu_{0,t}$, the "trace" of $\nu_{x,t}$ at $x = 0$, such that the following limit exists and is given by $\nu_{0,t}$:

$$\lim_{x \to 0+} \int_{T_1}^{T_2} < \nu_{x,t}, F > \theta(t)\,dt = \int_{T_1}^{T_2} < \nu_{0,t}, F > \theta(t)\,dt.$$

When $(U, F) = (\mathrm{id}, f)$, all the results above still hold when the function θ has no specific sign.

2) For all $0 < y_1 < y_2$ and in the sense of distributions for $t \in \boldsymbol{R}_+$, one has

$$\begin{aligned} F(u_B) + \nabla U(u_B)\big(< \nu_{0,t}, f > - f(u_B)\big) &\geq < \mu_{y_1,t}, F > - \partial_y < \mu_{y_1,t}, U > \\ &\geq < \mu_{y_2,t}, F > - \partial_y < \mu_{y_2,t}, U > \\ &\geq < \nu_{0,t}, F > . \end{aligned} \tag{2.8}$$

3) Moreover one has

$$\mu_{0,t} = \delta_{u_B(t)} \qquad a.e.\ t \in \boldsymbol{R}_+ \tag{2.9}$$

and, when $\theta \geq 0$,

$$\lim_{y \to \infty} \Big(\int_{T_1}^{T_2} < \mu_{y,t}, F > \theta(t)\,dt - \frac{d}{dy}\int_{T_1}^{T_2} < \mu_{y,t}, U > \theta(t)\,dt \Big) \geq \int_{T_1}^{T_2} < \nu_{0,t}, F > \theta(t)\,dt. \tag{2.10}$$

□

A few remarks about the results in Theorem 2.1 are now in order. The inequalities (2.8) actually hold in the (stronger) sense:

$$\begin{aligned} &\int_{T_1}^{T_2} \Big(F(u_B(t)) + \nabla U(u_B(t))\big(< \nu_{0,t}, f > - f(u_B(t))\big) \Big)\,\theta(t)\,dt \\ &\geq \int_{T_1}^{T_2} < \mu_{y_1,t}, F > \theta(t)\,dt - \frac{d}{dy}\big(\int_{T_1}^{T_2} < \mu_{y,t}, U > \theta(t)\,dt\big)\big|_{y=y_1} \\ &\geq \int_{T_1}^{T_2} < \mu_{y_2,t}, F > \theta(t)\,dt - \frac{d}{dy}\big(\int_{T_1}^{T_2} < \mu_{y,t}, U > \theta(t)\,dt\big)\big|_{y=y_2} \\ &\geq \int_{T_1}^{T_2} < \nu_{0,t}, F > \theta(t)\,dt \end{aligned}$$

for all non-negative $\theta \in BV(\boldsymbol{R}_+)$ and all $0 < y_1 < y_2$. Observe that this is a stronger statement than the convergence in the sense of distributions since θ is a function of bounded total variation, not necessarily having compact support in (T_1, T_2), rather than a smooth function with compact support. All the formulas to be derived in this section hold in this sense. Note also that (2.10) is an immediate consequence of (2.8) by taking $y \to \infty$.

The following inequalities, rigorously derived in Theorem 2.1,

$$F(u_B) + \nabla U(u_B)\big(< \nu_0, f > - f(u_B)\big) \ \geq < \nu_0, f > . \tag{2.11}$$

will be referred to as the *boundary entropy inequalities*. They do not refer explicitly to the boundary layer itself but only to its limiting values.

The inequalities (2.8) also contain constraints for the boundary layer. In particular, using the trivial entropies $(U, F) = \pm(u, f(u))$ in (2.8) leads us to the equation:

$$< \mu, f > - \partial_y < \mu, \mathrm{id} > = < \nu_{0,t}, f >, \tag{2.12}$$

where the right hand side is independent of the variable y and only depends on t.

For scalar equations and when the method of compensated compactness due to Murat-Tartar applies (i.e., mainly, for systems of two conservation laws), it is known that ν is a Dirac mass concentrated at a point $u(x,t)$ which is an entropy weak solution. In those two situations, it is conceivable that the Young measure μ also would be a Dirac mass.

If one assumes that μ is a Dirac mass, say

$$\mu_{y,t} = \delta_{v(y,t)} \quad \text{for almost every } \ (y,t) \tag{2.13}$$

with $v \in L^\infty$, then the formulas in Theorem 2.1 take a much simpler form. Namely if (2.12) holds, then (2.12) becomes what will be referred to as *boundary layer equation*:

$$f(v) - \partial_y v = < \nu_0, f > . \tag{2.14}$$

This is nothing but the equation that would be obtained *formally* by plugging an asymptotic expansion of the form $u_\epsilon(x,t) = u(x,t) + v(x/\epsilon, t) + O(\epsilon)$ in the equations (2.1). More generally, if (2.12) holds, the inequalities (2.8) become

$$\begin{aligned} F(u_B) + \nabla U(u_B)\big(< \nu_0, f > - f(u_B)\big) &\geq \ F(v(y_1)) - \partial_y U(v)_{|y=y_1} \\ &\geq \ F(v(y_2)) - \partial_y U(v)_{|y=y_2} \\ &\geq < \nu_0, f > . \end{aligned}$$

When ν_0 also is a Dirac mass for a.e. t, say $\nu_{0,t} = \delta_{u_0(t)}$, for instance when u has bounded variation in x and so admits a trace at $x = 0$ in a classical sense, then the *boundary layer equation* (2.14) becomes

$$f(v) - \partial_y v = < \nu_0, f > . \tag{2.15}$$

and the *boundary entropy inequalities* (2.11) take the form

$$F(u_0) - F(u_B) - \nabla U(u_B)\big(f(u_0) - f(u_B)\big) \leq 0, \tag{2.16}$$

which was derived in Dubois-LeFloch [14, 15] by assuming a uniform BV bound on the u^ϵ.

Note finally that the behavior of $\mu_{y,t}$ as $y \to \infty$ is controled by the set of inequalities (2.10), only. If it is assumed that v has a limit in a classical sense and $\partial_y v(y,t) \to 0$ as $y \to \infty$, then we can set

$$v_\infty(t) \equiv \lim_{y \to \infty} v(y,t)$$

and (2.10) becomes

$$F(v_\infty) \geq F(u_0) \quad \text{ for all entropy flux } \ F \tag{2.17}$$

(the flux F must be associated with a convex entropy). In fact (2.17) need not imply

$$v_\infty(t) = u_0(t). \tag{2.17'}$$

However (2.17) does imply

$$f(v_\infty(t)) = f(u_0(t))$$

so, in the non-characteristic case i.e. when ∇f is invertible, (2.17) implies (2.17'). In the characteristic case, (2.17') may very well be violated. This difficulty is related to the choice of the scaling in the definition of the functions v^ϵ. Cf. the examples in Sections 4 and 5.

Proof of Theorem 2.1. We decompose the proof into several steps. For the whole of this proof, we denote by (U, F) a given convex entropy pair.
Step 1: Preliminaries.

We gather here several properties of ν and μ that are readily obtained. Let us multiply the equation (2.6) by the gradient of U and obtain

$$\begin{aligned}\epsilon\, \partial_t U(v^\epsilon) + \partial_y \big(F(v^\epsilon) - \partial_y U(v^\epsilon)\big) &= -\nabla^2 U(v^\epsilon) \cdot \big(\partial_y v^\epsilon, \partial_y v^\epsilon\big) \\ &\le 0.\end{aligned} \tag{2.18}$$

Using the definition of the Young measure μ, it is a simple matter to pass to the limit in the inequality (2.18). For any $\theta \in BV$ and uniformly in $y \in \boldsymbol{R}_+$, we have

$$\begin{aligned}\Big|\int_{T_1}^{T_2} \partial_t U(v^\epsilon)\, \theta\, dt\Big| &\le \Big|\int_{T_1}^{T_2} U(v^\epsilon)\, \partial_t \theta\, dt\Big| \; + \; \Big|\big[U(v^\epsilon)\, \theta\big]_{T_1}^{T_2}\Big| \\ &\le O(\epsilon)\, \|\theta\|_{BV}\, \|U(v^\epsilon)\|_{L^\infty} \;\to\; 0,\end{aligned}$$

so we obtain

$$\partial_y \Big(\int_{T_1}^{T_2} < \mu_{y,t}, F > \, \theta\, dt - \frac{d}{dy} \int_{T_1}^{T_2} < \mu_{y,t}, U > \, \theta\, dt \Big) \; \le \; 0, \tag{2.19}$$

which provides the second inequality in (2.8). Therefore time-averages of the function $< \mu_{y,t}, F > -\partial_y < \mu_{y,t}, U >$ are non-increasing, and so have bounded variation on any compact set. The limits as $y \to 0+$ or $y \to +\infty$ exist, although at this stage of the proof, we can not exclude that those limits could be $\pm\infty$. We shall see later that actually $< \mu_{y,t}, F > -\partial_y < \mu_{y,t}, U > \in L^\infty$. Moreover the function

$$\int_{T_1}^{T_2} < \mu_{y,t}, U > \theta(t)\, dt$$

has a trace at $y = 0$, which defines $< \mu_{0,t}, U >$. Note also that (2.19) with the choices $(U, F) = \pm(\mathrm{id}, f)$ leads us to

$$< \mu_{y,t}, f > -\partial_y < \mu_{y,t}, \mathrm{id} >= C_*(t), \tag{2.20}$$

where $C_*(t)$ has to be determined. In fact it will be immediate from the results in Step 5 below that

$$C_*(t) \; = < \nu_{0,t}, \mathrm{id} > \qquad \text{for a.e. } \; t > 0.$$

Similarly, following DiPerna [13] and using the Young measure $\nu_{x,t}$ associated with u^ϵ, one can pass to the limit in (2.1) and obtain the entropy inequality:

$$\partial_t < \nu_{x,t}, U > + \partial_x < \nu_{x,t}, F > \le 0. \tag{2.21}$$

From (2.21), we deduce first that, for any smooth function $\theta(t) \ge 0$,

$$\frac{d}{dx} \int_{T_1}^{T_2} < \nu_{x,t}, F > \theta(t)\, dt \le \int_{T_1}^{T_2} < \nu_{x,t}, U > \partial_t \theta(t)\, dt \le O(1)\, \|\theta\|_{BV}. \tag{2.22}$$

For θ fixed, the right hand side of (2.22) is a constant, thus its left hand side is a locally bounded Borel measure and the function

$$g_\theta(x) \; \equiv \; \int_{T_1}^{T_2} < \nu_{x,t}, F > \theta(t)\, dt$$

has bounded total variation. Therefore the trace $\nu_{0,t}$ introduced in Theorem 2.1 exists, at least on entropy fluxes. This gives a meaning to the last term in the right hand side of (2.8). In fact it is possible to establish the estimate

$$TV(g_\theta) \,\leq\, O(1)\, \|\theta\|_{BV}$$

for arbitrary fucntions $\theta \in BV$. (For such θ, (2.22) can be obtained directly from (2.1).) Thus the trace $\nu_{0,t}$ exists for $\theta \in BV$ as well.

Observe that the traces $\mu_{0,t}$ and $\nu_{0,t}$ are uniquely determined on entropies and entropy fluxes, respectively. They can be easily extended as Young measures defined on the whole set of continuous functions, in a non-unique way however. Namely, to construct $\mu_{0,t}$, take any sequence $y_k \to 0$ and consider a Young measure associated with the sequence of measures $\{\mu_{y_k,t}\}$.

This completes the proof of the part 1) in Theorem 2.1.

Step 2: A General Identity.

It remains to analyze the behavior of μ at the end point $y = 0$ which shall provide us with the desired boundary entropy inequality. We are going to use a general identity which immediatly follows from the Green formula applied to (2.6).

Let $\theta(t)$ and $\varphi(x)$ be smooth functions not necessarily having compact support. We multiply the equation (2.6) by $\nabla U(v^\epsilon)\,\theta\,\varphi$ and integrate over the domain $(y_1, y_2) \times (0,T)$. Integrating by parts and re-ordering the terms, we obtain the identity

$$E_I^\epsilon + E_{II}^\epsilon + E_{III}^\epsilon = E_{IV}^\epsilon \tag{2.23}$$

with

$$E_I^\epsilon \equiv -\epsilon \int_{T_1}^{T_2} \int_{y_1}^{y_2} U(v^\epsilon) \partial_t \theta \varphi \, dy dt + \epsilon\, \theta(T_2) \int_{y_1}^{y_2} U(v^\epsilon(T_2))\, \varphi\, dy - \epsilon\, \theta(T_1) \int_{y_1}^{y_2} U(v^\epsilon(T_1))\, \varphi\, dy, \tag{2.24.I}$$

$$\begin{aligned} E_{II}^\epsilon \equiv - \int_{T_1}^{T_2} \int_{y_1}^{y_2} F(v^\epsilon) \theta \partial_y \varphi \, dy dt &+ \varphi(y_2) \int_{T_1}^{T_2} \big(F(v^\epsilon(y_2)) - \partial_y U(v^\epsilon)_{|y=y_2} \big) \theta \, dt \\ &- \varphi(y_1) \int_{T_1}^{T_2} \big(F(v^\epsilon(y_1)) - \partial_y U(v^\epsilon)_{|y=y_1} \big) \theta \, dt, \end{aligned} \tag{2.24.II}$$

$$\begin{aligned} E_{III}^\epsilon \equiv - \int_{T_1}^{T_2} \int_{y_1}^{y_2} U(v^\epsilon)\, \theta\, \partial_{yy} \varphi \, dy dt &+ \partial_y \varphi(y_2) \int_{T_1}^{T_2} U(v^\epsilon(y_2))\, \theta \, dy dt \\ &- \partial_y \varphi(y_1) \int_{T_1}^{T_2} U(v^\epsilon(y_1))\, \theta \, dy dt, \end{aligned} \tag{2.24.III}$$

and

$$E_{IV}^\epsilon \equiv - \int_{T_1}^{T_2} \int_{y_1}^{y_2} \nabla U(v^\epsilon) \cdot \big(\partial_y v^\epsilon, \partial_y v^\epsilon \big) \theta \varphi \, dy dt. \tag{2.24.IV}$$

In case that $\theta \geq 0$ and $\varphi \geq 0$ and since U is assumed to convex, one has

$$E_{IV}^\epsilon \leq 0, \tag{2.25}$$

so we can focus attention on estimating the terms E_I^ϵ, E_{II}^ϵ and E_{III}^ϵ.

Step 3: Viscous Flux at the Boundary.

We prove here that the viscous flux at the boundary, i.e. the function $\partial_y v^\epsilon(0,t)$, is uniformly bounded in a certain sense and we determine its weak limit as $\epsilon \to 0$. We use the identity (2.23)-(2.24) with the following choice of parameters:

$$\text{supp}\ \theta \subset [T_1, T_2], \quad \text{supp}\ \varphi \subset [0,1), \quad y_1 = 0, \quad y_2 = 1, \quad (U,F) = (\text{id}, f).$$

For φ fixed, we obtain

$$|E_I^\epsilon| \le O(\epsilon)\,\|\theta\|_{BV},$$

$$\begin{aligned} E_{II}^\epsilon &= -\int_{T_1}^{T_2}\int_0^1 f(v^\epsilon)\,\theta\,\partial_y\varphi\,dydt - \varphi(0)\int_{T_1}^{T_2}\big(f(u_B^\epsilon) - \partial_y v^\epsilon(0,.)\big)\,\theta\,dt \\ &= O(1)\,\|\theta\|_{L^\infty} - \varphi(0)\int_{T_1}^{T_2}\big(f(u_B^\epsilon) - \partial_y v^\epsilon(0,.)\big)\,\theta\,dt, \end{aligned}$$

and

$$\begin{aligned} E_{III}^\epsilon &= -\int_{T_1}^{T_2}\int_0^1 v^\epsilon\theta\partial_{yy}\varphi\,dydt - \int_{T_1}^{T_2} u_B^\epsilon\theta\partial_y\varphi(0)\,dt \\ &= O(1)\,\|\theta\|_{L^\infty}. \end{aligned}$$

Since in this case $E_{IV}^\epsilon = 0$ and choosing φ so that $\varphi(0) \neq 0$, it follows

$$\Big|\int_{T_1}^{T_2}\big(f(u_B^\epsilon) - \partial_y v^\epsilon(0,.)\big)\theta\,dt\Big| \le O(1)\,\|\theta\|_{L^\infty} + O(\epsilon)\,\|\theta\|_{BV}. \tag{2.26}$$

More precisely we can pass to the limit in the identity (2.23) and get

$$\begin{aligned} &\varphi(0)\,\lim_{\epsilon\to 0}\int_{T_1}^{T_2}\big(f(u_B) - \partial_y v^\epsilon(0,t)\big)\theta\,dt \\ &\qquad = -\int_{T_1}^{T_2}\int_0^1 <\mu, f>\theta\partial_y\varphi\,dydt - \int_{T_1}^{T_2}\int_0^1 <\mu,\mathrm{id}>\theta\partial_{yy}\varphi\,dydt - \partial_y\varphi(0)\int_{T_1}^{T_2} u_B\theta\,dt. \end{aligned}$$

On the other hand, it has been observed in Step 1 that (2.20) holds and $<\mu,\mathrm{id}>$ has a trace at $y = 0$. Thus one has

$$\begin{aligned} &\int_{T_1}^{T_2}\int_0^1 <\mu, f>\theta\partial_y\varphi\,dydt + \int_{T_1}^{T_2}\int_0^1 <\mu,\mathrm{id}>\theta\partial_{yy}\varphi\,dydt \\ &\qquad = \int_{T_1}^{T_2}\int_0^1 C_*(t)\,\theta\partial_y\varphi\,dydt - \int_{T_1}^{T_2} <\mu_0,\mathrm{id}>\theta\partial_y\varphi(0)\,dt \\ &\qquad = -\int_{T_1}^{T_2} C_*(t)\,\theta\varphi(0)\,dt - \int_{T_1}^{T_2} <\mu_0,\mathrm{id}>\theta\partial_y\varphi(0)\,dt \end{aligned}$$

and therefore

$$\begin{aligned} &\varphi(0)\,\lim_{\epsilon\to 0}\int_{T_1}^{T_2}\big(f(u_B) - \partial_y v^\epsilon(0,t)\big)\theta\,dt \\ &\qquad = \varphi(0)\int_{T_1}^{T_2} C_*(t)\,\theta\,dt + \partial_y\varphi(0)\int_{T_1}^{T_2} <\mu_0,\mathrm{id}>\theta\,dt - \partial_y\varphi(0)\int_{T_1}^{T_2} u_B\theta\,dt. \end{aligned}$$

Choosing two test-functions φ, one such that $\varphi(0) = 0$ but $\partial_y\varphi(0) \neq 0$, and the other such that $\varphi(0) \neq 0$ but $\partial_y\varphi(0) = 0$, we deduce from the above formula that

$$\begin{aligned} \lim_{\epsilon\to 0}\int_{T_1}^{T_2}\big(f(u_B) - \partial_y v^\epsilon(0,t)\big)\theta\,dt &= \int_{T_1}^{T_2} C_*(t)\,\theta\,dt \\ \int_{T_1}^{T_2} <\mu_0,\mathrm{id}>\theta\,dt &= \int_{T_1}^{T_2} u_B\theta\,dt. \end{aligned} \tag{2.27}$$

The first statement in (2.27) is the desired convergence result. The second statement is a first step toward proving (2.9).

Step 4: Boundary Entropy Inequalities (I).

Using (2.27), we are now able to obtain the boundary entropy inequalities. We use the identity (2.23)-(2.24) with

$$\theta \geq 0, \quad \text{supp}\, \theta \subset [T_1, T_2], \quad \varphi \geq 0, \quad \text{supp}\, \varphi \subset [0, \infty), \quad y_1 = 0, \quad y_2 > 0,$$

and (U, F) arbitrary. We obtain

$$|E_I^\epsilon| \leq O(\epsilon)\, \|\theta\|_{BV},$$

$$\begin{aligned}
E_{II}^\epsilon &= -\int_{T_1}^{T_2}\int_0^{y_2} F(v^\epsilon)\,\theta\,\partial_y\varphi\,dydt - \varphi(0)\int_{T_1}^{T_2}\big(F(u_B^\epsilon) - \partial_y U(v^\epsilon)_{y=0}\big)\,\theta\,dt \\
&= -\int_{T_1}^{T_2}\int_0^{y_2} F(v^\epsilon)\,\theta\,\partial_y\varphi\,dydt - \varphi(0)\int_{T_1}^{T_2}\Big(F(u_B^\epsilon) - \nabla U(u_B^\epsilon)\,\partial_y v^\epsilon(0,.)\Big)\theta\,dt \\
&\to -\int_{T_1}^{T_2}\int_0^{y_2} <\mu, F>\,\theta\,\partial_y\varphi\,dydt - \varphi(0)\int_{T_1}^{T_2}\Big(F(u_B) - \nabla U(u_B)\big(f(u_B) - C_*(.)\big)\Big)\,\theta\,dt,
\end{aligned}$$

where we have used (2.27) and the fact that $u_B^\epsilon \in BV$ converges strongly to $u_B \in BV$, and

$$E_{III}^\epsilon = -\int_{T_1}^{T_2}\int_0^{y_2} U(v^\epsilon)\,\theta\,\partial_{yy}\varphi\,dydt - \int_{T_1}^{T_2} U(u_B^\epsilon)\,\theta\,\partial_y\varphi(0)\,dt.$$

Since $E_{IV}^\epsilon \leq 0$ we pass to the limit in (2.23) and get

$$\begin{aligned}
&\varphi(0)\int_{T_1}^{T_2}\Big(F(u_B) - \nabla U(u_B)\big(f(u_B) - C_*(t)\big)\Big)\,\theta\,dt, \\
&\qquad \geq -\int_{T_1}^{T_2}\int_0^{y_2}\big(<\mu_{y,t}, F>\,\partial_y\varphi + <\mu_{y,t}, U>\,\partial_{yy}\varphi\big)\,\theta\,dydt \\
&\qquad\quad + \varphi(y_2)\int_{T_1}^{T_2}\big(<\mu_{y_2,t}, F> - \partial_y <\mu_{y,t}, U>_{y=y_2}\big)\,\theta\,dydt \\
&\qquad\quad + \partial_y\varphi(y_2)\int_{T_1}^{T_2} <\mu_{y_2,t}, U>\,\theta\,dt - \partial_y\varphi(0)\int_{T_1}^{T_2} U(u_B)\,\theta\,dt.
\end{aligned}$$

On one hand, using the test-function $\varphi(y) \equiv 1$, we deduce that

$$\int_{T_1}^{T_2}\Big(F(u_B) - -\nabla U(u_B)\big(f(u_B) - C_*(t)\big)\Big)\theta\,dt \geq \int_{T_1}^{T_2}\big(<\mu, F> + \partial_y <\mu, U>_{y=y_2}\big)\,\theta\,dt \tag{2.28}$$

which proves the first inequality in (2.8).

On the other hand, using the function $\varphi(y) = y$, we obtain

$$\begin{aligned}
0 \geq &-\int_{T_1}^{T_2}\int_0^{y_2} <\mu_{y,t}, F>\,\theta\,dydt + y_2\int_{T_1}^{T_2}\big(<\mu_{y_2,t}, F> - \partial_y <\mu_{y,t}, U>_{y=y_2}\big)\,\theta\,dydt \\
&+ \int_{T_1}^{T_2} <\mu_{y_2,t}, U>\,\theta\,dt - \int_{T_1}^{T_2} U(u_B)\,\theta\,dt,
\end{aligned}$$

which as $y_2 \to 0$ yields

$$\int_{T_1}^{T_2} U(u_B)\theta\,dt \geq \lim_{y\to 0+}\int_{T_1}^{T_2} <\mu_{y,t}, U>\theta\,dt. \tag{2.29}$$

In particular, plugging $(U,F)=(\mathrm{id},f)$ in (2.29), we recover the second statement in (2.27), which used together with (2.29) for any fixed, strictly convex entropy U gives:

$$\begin{aligned}
&\int_{T_1}^{T_2} < \mu_{0,t}, U - U(u_B) - \nabla U(u_B)(\mathrm{id}-u_B) > \theta\, dt \\
&\lim_{y\to 0+}\int_{T_1}^{T_2} < \mu_{y,t}, U - U(u_B) - \nabla U(u_B)(\mathrm{id}-u_B) > \theta\, dt \\
&\le \int_{T_1}^{T_2} U(u_B)\theta\, dt\ - \int_{T_1}^{T_2} U(u_B)\theta\, dt \\
&= 0.
\end{aligned}$$

But the function $u \to U(u) - U(u_B) - \nabla U(u_B)(u-u_B)$ is positive everywhere except at u_B where it achieves its global minimum value. It follows that $\mu_{0,t}$ is a Dirac mass concentrated at u_B. That proves (2.9).

Step 5: Boundary Entropy Inequalities (II).

We now establish the third inequalities in (2.8). We use once more the identity (2.23)-(2.24) with now

$$\theta \ge 0, \quad \mathrm{supp}\ \theta \subset [T_1,T_2], \quad \varphi \ge 0, \quad \mathrm{supp}\ \varphi \subset [y_1,\infty), \quad y_1>0, \quad y_2=\infty,$$

with a function φ depending on ϵ, that is

$$\varphi^\epsilon(y,t) \equiv \tilde\varphi(\epsilon y, t)$$

with $\tilde\varphi$ fixed. In that situation one can check that

$$\begin{aligned}
E_I^\epsilon &= -\int_{T_1}^{T_2}\int_{\epsilon y_1}^{\infty} U(u^\epsilon)\,\partial_t\theta\,\tilde\varphi\, dxdt \\
&\to -\int_{T_1}^{T_2}\int_0^\infty < \nu_{x,t}, U > \,\partial_t\theta\,\tilde\varphi\, dxdt,
\end{aligned}$$

$$\begin{aligned}
E_{II}^\epsilon &= -\int_{T_1}^{T_2}\int_{\epsilon y_1}^{\infty} F(u^\epsilon)\,\theta\,\tilde\partial_x\varphi\, dxdt \\
&\quad - \tilde\varphi(\epsilon y_1)\int_{T_1}^{T_2}\big(F(v^\epsilon) - \partial_y U(v^\epsilon)_{|y=y_1}\big)\theta\, dt \\
&\to -\int_{T_1}^{T_2}\int_0^\infty < \nu_{x,t}, F > \,\theta\,\partial_x\varphi\, dxdt - \tilde\varphi(0)\int_{T_1}^{T_2}\big(< \mu_{y_1,t}, F > - \partial_y < \mu, U >_{|y=y_1}\big)\,\theta\, dt,
\end{aligned}$$

and

$$\begin{aligned}
E_{III}^\epsilon &= -\epsilon\int_{T_1}^{T_2}\int_{\epsilon y_1}^{\infty} U(u^\epsilon)\,\theta\,\partial_{xx}\varphi\, dxdt\ -\ \partial_x\tilde\varphi(\epsilon y_1)\int_{T_1}^{T_2} U(v^\epsilon)_{|y=y_1}\,\theta\, dt \\
&\to 0.
\end{aligned}$$

Since $E_{IV}^\epsilon \le 0$ and

$$-\int_{T_1}^{T_2}\int_0^\infty < \nu_{x,t}, F > \,\theta\,\partial_x\tilde\varphi\, dxdt\ =\ \int_{T_1}^{T_2} < \nu_0, F > \,\theta\, dt\ +\ O(1)\,\|\tilde\varphi\|_{L^1},$$

we obtain an inequality of the form

$$\tilde\varphi(0)\int_{T_1}^{T_2}\big(< \mu_{y_1,t}, F > - \partial_y < \mu, U >_{|y=y_1}\big)\,\theta\, dt\ \ge\ \tilde\varphi(0)\int_{T_1}^{T_2} < \nu_0, F > \,\theta\, dt\ +\ O(1)\,\|\tilde\varphi\|_{L^1}, \tag{2.30}$$

which proves the third inequality in (2.8) by chosing $\tilde\varphi \ge 0$ such that $\|\tilde\varphi\|_{L^1} \to 0$ but $\tilde\varphi(0) > 0$.

This complete the proof of Theorem 2.1. □

Remark 2.2. Additional uniform estimates and regularity can be obtained from the identity in Step 2 of the proof of Theorem 2.1. Let (U,F) be a non-negative entropy pair that is uniformly convex on $\mathcal{U}$. Use the identity (2.23)-(2.24) with

$$\theta \equiv 1, \quad T_1 = 0, \quad T_2 = T, \qquad \varphi \equiv 1, \quad y_1 = 0, \quad y_2 = \infty.$$

We assume addiitonally here that, for a fixed state u_∞ and for all t,

$$u^\epsilon(x,t) \to u_\infty, \quad u_x^\epsilon(x,t) \to 0 \qquad \text{as } x \to \infty.$$

The initial data u_I should also decay rapidly at infinity. We obtain the following identity

$$\epsilon \int_0^T U(v^\epsilon(T))\,dy - \epsilon \int_0^\infty U(v^\epsilon(y,0))\,dy + \int_0^T F(u_\infty)\,dt$$
$$- \int_0^T \big(F(u_B^\epsilon) - \nabla U(u_B^\epsilon)\partial_y v^\epsilon(0,.)\,dt + \int_0^T \int_0^\infty \nabla^2 U(v^\epsilon) \cdot \big(\partial_y v^\epsilon, \partial_y v^\epsilon\big)\,dydt \;=\; 0.$$

Since the following two terms are uniformly bounded

$$\big|\epsilon \int_0^\infty U(v^\epsilon(y,0))\,dy\big| \;=\; \big|\epsilon \int_0^\infty U(u_B^\epsilon)\,dx\,\big| \;\le O(1),$$

$$\big|\int_0^T \nabla U(u_B^\epsilon)\partial_y v^\epsilon(0,.)\,dt\big| \;\le\; O(1),$$

(Cf.(2.26) with $\theta \equiv 1$), we deduce the uniform bounds

$$\epsilon \int_0^T U(v^\epsilon(T))\,dy + \int_0^T \int_0^\infty \nabla^2 U(v^\epsilon) \cdot \big(\partial_y v^\epsilon, \partial_y v^\epsilon\big)\,dydt \;\le\; O(1). \tag{2.31}$$

For every Lipschitz continuous function g, it follows from (2.31) that the sequence $\partial_y g(v^\epsilon)$ is bounded in L^2, so converges weakly to a limit which is nothing but $\partial_y < \mu, g >$:

$$\partial_y g(v^\epsilon) \;\to\; \partial_y < \mu, g > \qquad \text{weak-}\star \text{ in } L^2(\boldsymbol{R}_+^2). \tag{2.32}$$

□

2.2. Finite Difference Schemes. We now extend the above analysis to several classes of finite difference schemes that are known to be consistent with the entropy inequality (1.3). Theorem 2.3 below deals with the entropy flux-splittings introduced by Chen-LeFloch [9], which also includes as a special case the Lax Friedrichs type schemes. We treat the Godunov scheme in Theorem 2.4.

We are given two mesh parameters τ and h with $\lambda \equiv \tau/h$ kept constant and small enough in order to garantee the stability of the scheme. We define the approximate solutions $u^h(x,t)$ by the scheme

$$u^h(x,t+\tau) = u^h(x,t+\tau) - \lambda g\big(u^h(x,t), u^h(x+h,t)\big) + \lambda g\big(u^h(x-h,t), u^h(x,t)\big) \tag{2.33}$$

and the initial and boundary conditions:

$$\begin{aligned} u^h(x,t) &= u_I(x) \qquad && \text{for all } t < \tau, \\ u^h(x,t) &= u_B(t) && \text{for all } x < h. \end{aligned} \tag{2.34}$$

By convention, the functions u^h are right continuous. For the Lax-Friedrichs type schemes, the numerical flux g is given by

$$g_{\text{Lax}}(v,w) = \frac{1}{2}(f(v) + f(w)) - \frac{Q}{\lambda}(w - v), \tag{2.35}$$

where $Q \in (0,1)$ is called the numerical coefficient of the scheme. (Symmetric positive definite matrices Q could also be dealt with.) For the flux-splitting schemes, g takes the form

$$g_{\text{split}}(v,w) = f^-(w) + f^+(v), \tag{2.36}$$

where $f = f^- + f^+$ is a given entropy flux-splitting for the system (1.1). By definition [9], the matrix $\nabla f^\pm$ have real eignevalues and a basis of eigenvectors and there exists a pair of functions $F_\pm$ such that $(U, F^\pm)$ is an entropy pair for the system associated with flux-functions $f^\pm$. Observe that (2.35) is a special case of (2.36) as was pointed out by Chen-LeFloch.

As in the analysis of Section 2.1, we assume a uniform L^∞ bound:

$$\|u^h\|_{L^\infty(\boldsymbol{R}^2_+)} \le O(1). \tag{2.37}$$

We rescale u^h and define the function $v^h : \boldsymbol{R}^2_+ \to \mathcal{U}$ by

$$v^h(y,t) = u^h(yh,t) \qquad y \ge 0,\, t \ge 0.$$

Let ν and μ be two Young measures associated with u^h and v^h, respectively.

The entropy flux-splitting schemes satisfy discrete entropy inequalities of the form

$$U(u^h(x,t+\tau)) - U(u^h(x,t+\tau)) + \lambda\Big(G(u^h(x,t),u^h(x+h,t)) - G(u^h(x-h,t),u^h(x,t))\Big) \le 0, \tag{2.38}$$

where G is called the numerical entropy flux. With obvious notation, we have

$$G_{\text{Lax}}(v,w) = \frac{1}{2}(F(v) + F(w)) - \frac{Q}{\lambda}(U(w) - U(v)) \tag{2.35bis}$$

and

$$G_{\text{split}}(v,w) = F^-(w) + F^+(v). \tag{2.36bis}$$

Note that (2.38) hold for (2.36)-(2.36bis) provided u takes its value in a sufficiently small neighborhood of a given state in $\mathcal{U}$. This is in constrast with the vanishing viscosity method where no such assumption was necessary.

Theorem 2.1 admits the following extension to the flux-splitting schemes. We omit the proof which follows the lines of the one of Theorem 2.1.

Theorem 2.3. *Assume that $\mathcal{U}$ is a small neighborhood of a constant state in $\boldsymbol{R}^N$. The measure $\mu_{y,t}$ is defined for all $y \ge 0$ and almost every t, and is constant for $y \in [k, k+1)$ for any integer k. For all convex entropy pairs (U,F), all $y \ge 0$, and in the sense of distributions in $t \in \boldsymbol{R}_+$, one has*

$$\begin{aligned} F^+(u_B) + <\mu_{1,t}, F^-> &\ge\ <\mu_{y,t}, F^+> + <\mu_{y+1,t}, F^-> \\ &\ge\ <\mu_{y+1,t}, F^+> + <\mu_{y+2,t}, F^-> \\ &\ge\ <\nu_{0,t}, F>, \end{aligned} \tag{2.39}$$

$$\mu_{0,t} = \delta_{u_B(t)} \qquad \textit{for a.e. } t > 0, \tag{2.40}$$

and

$$\lim_{y \to +\infty} <\mu_{y,t}, F^+> + <\mu_{y+1,t}, F^-> \ge <\nu_{0,t}, F>. \tag{2.41}$$

□

Consider next the Godunov scheme corresponding to the flux g given by

$$g_{\text{Godunov}}(v,w) = f(R(v,w)), \tag{2.42}$$

where we denote by $R(v,w)$ the value at $x/t = 0+$ of the solution to the Riemann problem with v and w as left and right initial data, respectively. The entropy flux is

$$G_{\text{Godunov}}(v,w) = F(R(v,w)), \tag{2.42bis}$$

Here it is more convenient to consider the values $R(u^h(x,t), u^h(x+h,t))$ and define a function w^h

$$w^h(y,t) = R(u^h(yh,t), u^h(yh+h,t)) \tag{2.43}$$

for all $y \ge 0$. We denote by π a Young measure associated with w^h and by ν a Young measure for u^h. It is not difficult to extend Theorem 2.3 as follows:

Theorem 2.4. *The measure $\pi_{y,t}$ is defined for all $y \geq 1/2$ and almost every t, and is constant in y for $y \in [k-1/2, k+1/2)$ for any integer $k \geq 1$. For all convex entropy pairs (U, F), all $y \geq 1/2$, and in the sense of distributions in $t \in \boldsymbol{R}_+$, one has*

$$\begin{aligned} < \pi_{1/2,t}, F > &\geq \; < \pi_{y,t}, F > \\ &\geq \; < \pi_{y+1,t}, F > \\ &\geq \; < \nu_{0,t}, F >, \end{aligned} \tag{2.44}$$

and, at $y = 1/2$ and $y = \infty$, π satisfies

$$< \pi_{1/2,t}, F >= \lim_{h \to 0} R(u_B, v^h(1,t)), \tag{2.45}$$

and

$$\lim_{y \to \infty} < \pi_{y,t}, F > \geq < \nu_{0,t}, F > . \tag{2.46}$$

□

We conclude this section by giving the main conditions satisfied by the discrete boundary layer, which will be studied in the rest of this paper.

Assuming in the results of Theorem 2.3 that μ is a Dirac mass, say $\mu = \delta_v$, the *discrete boundary layer equation* associated with the scheme (2.33) takes the form:

$$\begin{aligned} &g(v(y-1), v(y)) - g(v(y), v(y+1)) = 0 \qquad \text{for all } \; y \geq 1, \\ &v(y) = u_B, \qquad y \in [0,1), \end{aligned} \tag{2.47}$$

while the *discrete boundary entropy inequality* is

$$G(u_B, v_1) \geq F(u_0), \tag{2.48}$$

where v_1 plays the role of a parameter. Formally, Theorem 2.4 leads to the same equations (2.47)-(2.48) with flux and entropy-fluxes given by (2.42).

3. Sets of Admissible Boundary Values.

Based on the results in Section 2, we introduce in this section several sets which can be used to formulate the boundary condition. For every method of approximation considered in Section 2, we introduce two different sets of admissible boundary values:

(1) One is based on the entropy inequalities, $\mathcal{E}^{\text{entropy}}(u_B)$ and yields a boundary condition of the form (1.5). This boundary condition is rigorously satisfied by the limiting function generated by a sequence of approximate solution. as was proven in Section 2. For arbitrary systems having few or even just one entropy, the set $\mathcal{E}^{\text{entropy}}(u_B)$ may be too large to lead to a well-posed problem;
(2) Another set, $\mathcal{E}^{\text{layer}}(u_B)$, is based on the boundary layer equation, which was obtained formally after the analysis in Section 2. This set is more adapted to deal with general systems and lead to a well-posed problem when the boundary is non characteristic.

In this section, we study the local structure of those sets; under certain assumptions, we can prove that the sets $\mathcal{E}^{\text{layer}}(u_B)$ are manifolds with dimension equal to the number of negative wave speeds of the system (1.1). This ensures that the initial-boundary value problem is well posed if, for instance, the data are constant states (boundary Riemann problem) as can be seen by applying the theory in [35]. We recall that (1.1) is assumed to be strictly hyperbolic throughout this section and we denote by $\lambda_j(u)$ the N real and distinct eigenvalues of the matrix $\nabla f(u)$ and by $\ell_j(u)$ and $r_j(u)$ corresponding basis of left and right eigenvectors.

3.1 Vanishing Viscosity Method. For the sake of generality, we consider

$$\partial_t u^\epsilon + \partial_x f(u^\epsilon) = \epsilon\, \partial_x \big(B(u^\epsilon) \partial_x u^\epsilon\big), \qquad x > 0,\, t > 0. \tag{3.1}$$

Theorem 2.1 could be partially extended to this case. We assume that the viscosity matrix $B(u)$ depends smoothly upon its argument u and is positive. We consider entropies U that are *B-convex* in the sense that $\nabla^2 U(u) B(u) > 0$ for all u under consideration. The boundary layer equation here takes the form

$$\partial_y f(v) = \partial_y \big(B(v)\partial_y v\big) \tag{3.2}$$

and the boundary entropy inequalities have the same form (2.16) but now U must be B-convex.

Following Dubois-LeFloch [15], we introduce a set based on the boundary entropy inequalities. From now on, the time-dependence may be omitted.

Definition 3.1. *Given $u_B \in \mathcal{U}$, the set of admissible boundary values based on the entropy inequalities associated with the vanishing viscosity method* (3.1) *is*

$$\mathcal{E}^{\text{entropy}}_{\text{viscosity}}(u_B) = \big\{u_0 \in \mathcal{U};\ \textit{for all } B\textit{-convex } (U,F),\ F(u_B) + \nabla U(u_B)\big(f(u_0) - f(u_B)\big) \geq F(u_0)\big\}. \tag{3.3}$$

□

It is obvious that this set may be quite large when the system (1.1) only admits few entropies. For most systems ($N \geq 3$), this set is too large to be used to formulate the boundary condition. In any case, it is difficult to get information on its local structure at u_B. For general systems, the following observation is immediate.

Proposition 3.2. *Fix a state $u_B \in \mathcal{U}$ and suppose that for some p one has*

$$\lambda_p(u_B) < 0 < \lambda_{p+1}(u_B) \tag{3.4}$$

and the basis $r_j(u)$ is a family of eigenvectors for $B(u)$. Then the set obtained by formally plugging the expansion

$$\begin{aligned} f(u_0) &\approx f(u_B) + \nabla f(u_B)(u_0 - u_B) + \nabla^2 f(u_B)\cdot\big(u_0 - u_B, u_0 - u_B\big),\\ F(u_0) &\approx F(u_B) + \nabla F(u_B)(u_0 - u_B) + \nabla^2 F(u_B)\cdot\big(u_0 - u_B, u_0 - u_B\big)\end{aligned} \tag{3.5}$$

in the definition of $\mathcal{E}^{\text{entropy}}_{\text{viscosity}}(u_B)$ is an affine manifold of dimension p containing u_B and spanned by the vectors $r_j(u_B)$, $j = 1, 2, \cdots, p$. □

Proof of Proposition 3.2. The inequality under consideration in (3.3) then becomes

$$\nabla^2 U(u_B)\,\nabla f(u_B)\big(u_0 - u_B, u_0 - u_B\big) \ \leq\ 0.$$

Since U is an entropy and the system is strictly hyperbolic, the matrix $\nabla^2 U(u_B)\cdot\big(r_j(u_B), r_j(u_B)\big)$ is a diagonal matrix. On the other hand, $\nabla^2 U(u_B)\,B(u_B)$ is positive and $r_j(u)$ is a family of eigenvectors for $B(u)$, therefore the matrix $\nabla^2 U(u_B)\cdot\big(r_j(u_B), r_j(u_B)\big)$ has positive diagonal elements. The desired result follows immediately. □

We now introduce a second set of admissible boundary values.

Definition 3.3. *Given any $u_B \in \mathcal{U}$, the set of admissible boundary values $\mathcal{E}^{\text{layer}}_{\text{viscosity}}(u_B)$, based on the boundary layer equation associated with the vanishing viscosity method is the set of all $v_\infty \in \mathcal{U}$ such that the problem*

$$\begin{aligned} &B(v)\partial_y v = f(v) - f(v_\infty),\\ &v(0) = u_B,\\ &\lim_{y\to\infty} v(y) = v_\infty. \end{aligned} \tag{3.6}$$

admits a (smooth) solution $v(y) \in \mathcal{U}$ for $y \geq 0$. □

To study the local structure of $\mathcal{E}^{\text{layer}}_{\text{viscosity}}(u_B)$, we apply the following theorem concerning the existence of invariant manifolds. Cf. Hartman [23] for a proof.

Theorem 3.4. *Consider the differential equation*

$$\frac{d\xi}{dy} = E\xi + H(\xi, \xi_0), \quad \xi(y) \in \boldsymbol{R}^N,\, y \in \boldsymbol{R}, \tag{3.7}$$

where $H : \boldsymbol{R}^N \times \boldsymbol{R}^N \to \boldsymbol{R}^N$ is of class C^1 and for each ξ_0

$$H(0, \xi_0) = \frac{dH}{d\xi}(0, \xi_0) = 0, \tag{3.8}$$

and E is a constant square matrix with d eigenvalues having negative real part, e eigenvalues having positive real part, and $N - d - e$ eigenvalues having zero real part. For every (small enough) $\xi_0 \in \boldsymbol{R}^N$, let $\xi_y = \xi(y; \xi_0)$ be the solution of (3.7) with the initial condition $\xi(0; \xi_0) = \xi_0$. Denote by T_y the mapping $\xi_0 \to \xi(y; \xi_0)$.

There exists a one-to-one mapping of class C^1, $S : \xi \to S(\xi) = (w^I, w^{II}, w^{III})$, having non-vanishing Jacobian and defined on a neighborhood of $\xi = 0 \in \boldsymbol{R}^N$ onto a neighborhood of $(w^I, w^{II}, w^{III}) = (0, 0, 0) \in \boldsymbol{R}^d \times \boldsymbol{R}^{N-d-e} \times \boldsymbol{R}^e$, such that the mapping ST_yS^{-1} takes the simple form

$$\begin{aligned} ST_yS^{-1} : \quad w_y^I &= e^{P^I y} w_0^I + W^I(y; w_0^I, w_0^{II}, w_0^{III}), \\ w_y^{II} &= e^{P^{II} y} w_0^{II} + W^{II}(y; w_0^I, w_0^{II}, w_0^{III}), \\ w_y^{III} &= e^{P^{III} y} w_0^{III} + W^{III}(y; w_0^I, w_0^{II}, w_0^{III}), \end{aligned} \tag{3.9}$$

where P^I, P^{II}, and P^{III} are constant real-valued matrices with all eigenvalues having moduli less than one so that the matrix exponentials e^{P^I}, $e^{P^{II}}$, and $e^{P^{III}}$ are well-defined, the absolute value of any eigenvalue of e^{P^I} is less than 1, and that for $e^{P^{III}}$ is greater than 1, and that for $e^{P^{II}}$ is exactly 1. Moreover the mapping W^I, W^{II}, and W^{III} are of class C^1 and their first order partial derivatives with respect to $(w_0^I, w_0^{II}, w_0^{III})$ vanish at $(0, 0, 0)$. Moreover one has

$$W^I = 0 \quad \text{and} \quad W^{II} = 0 \qquad \text{if} \quad w_0^I = 0 \quad \text{and} \quad w_0^{II} = 0, \tag{3.10}$$

and

$$W^{II} = 0 \quad \text{and} \quad W^{III} = 0 \qquad \text{if} \quad w_0^{II} = 0 \quad \text{and} \quad w_0^{III} = 0. \tag{3.11}$$

□

The condition (3.10) means that the e-dimensional plane $\{w_0^I = 0, w_0^{II} = 0\}$ is a locally invariant manifold. If $S(\xi_0)$ belongs to this plane, then $|\xi(y; \xi_0)| \to \infty$ as $y \to \infty$. The manifold $\{\xi \,/\, w_0^I = 0, w_0^{II} = 0\}$ is called the *unstable manifold* of initial data for the equation (3.7).

The condition (3.11) means that the d-dimensional plane $\{w_0^{II} = 0, w_0^{III} = 0\}$ is a locally invariant manifold.If $S(\xi_0)$ belongs to this plane, then $\xi(y; \xi_0) \to 0$ as $y \to \infty$. The manifold $\{\xi \,/\, w_0^{II} = 0, w_0^{III} = 0\}$ is called the *stable manifold.*

Using Theorem 3.4 we prove the following result.

Theorem 3.5. *Let $u_B \in \mathcal{U}$ be given and assume that, for all u in a small neighborhood of u_B,*

$$\begin{aligned} &\text{the basis } r_j(u) \text{ is a family of eigenvectors for } B(u), \\ &\text{the eigenvalues of } B(u), \text{ say } b_j(u), \text{ are positive,} \end{aligned} \tag{3.12}$$

and

$$\lambda_p(u) < 0 \le \lambda_{p+1}(u) \tag{3.13}$$

holds for some p. Then the set $\mathcal{E}^{\text{layer}}_{\text{viscosity}}(u_B)$ contains the point u_B and, locally nearby u_B, contains a manifold with dimension p at least. When $0 < \lambda_{p+1}(u_B)$, $\mathcal{E}^{\text{layer}}_{\text{viscosity}}(u_B)$ is a manifold with dimension exactly p and its tangent space at the point u_B is spanned by the eigenvectors $r_j(u_B)$, $j = 1, 2, \cdots, p$. □

A similar result has been proved by Gisclon in [19] by another method.

Proof of Theorem 3.5. The system in (3.6) can be written in the form

$$\begin{aligned}&\frac{d\tilde v}{dy} = B(v_\infty)^{-1}\nabla f(v_\infty)\tilde v + G(\tilde v, v_\infty),\\ &\tilde v(0) = u_B - v_\infty,\\ &\tilde v(\infty) = 0,\end{aligned} \tag{3.14}$$

where $\tilde v(y) = v(y) - v_\infty$ and the mapping $G(\tilde v, v_\infty)$ satisfies $G(0, v_\infty) = 0$, $\frac{\partial G}{\partial \tilde v}(0, v_\infty) = 0$. In view of the assumption (3.12), the two matrices $\nabla f(v_\infty)$ and $B(v_\infty)^{-1}\nabla f(v_\infty)$ have the same eigenvectors, and so exactly the same number of positive, zero, and negative eigenvalues. Let

$$\hat\lambda_j(v_\infty) = b_j(v_\infty)^{-1}\,\lambda_j(v_\infty)$$

be the eigenvalues of $B(v_\infty)^{-1}\nabla f(v_\infty)$. Applying Theorem 3.4 with

$$\xi(y;\xi_0) = \tilde v(y; u_B - v_\infty),$$

we see that there exists a one-to-one C^1 mapping S, defined on a neighborhood of $0 \in \boldsymbol{R}^N$, onto a neighborhood of $(w^I, w^{II}, w^{III}) = (0,0,0) \in \boldsymbol{R}^p \times \boldsymbol{R}^{N-p-1} \times \boldsymbol{R}^1$, such that the manifold

$$\mathcal{E} \;\equiv\; \big\{\tilde v \,/\, w^{II}(\tilde v) = 0, \quad w^{III}(\tilde v) = 0\big\},$$

which is of dimension p, is stable. For any point $u_B - v_\infty$ taken in this manifold as an initial data for the differential equation in (3.14), the solution $\tilde v(y)$ converges to 0 as $y \to \infty$, which is the third condition required in (3.14).

If v_∞ belongs to this manifold, then (3.14) has a solution and hence v_∞ solves the boundary layer problem. Furthermore the local structure of the ser nearby u_B can be described as follows.

Suppose that $0 < \lambda_{p+1}(u_B)$. The following estimate follows from (3.14):

$$\tilde v(y) = \sum_{j=1}^N e^{\hat\lambda_i y}\ell_j(v_\infty)\cdot(u_B - v_\infty)r_j(v_\infty) + 0(\tilde v(y))^2. \tag{3.15}$$

For the right handside of (3.15) to go to zero, we must have

$$g_j(v_\infty) \equiv \ell_j(v_\infty)\cdot(u_B - v_\infty) = 0, \qquad j = p+1, \cdots N. \tag{3.16}$$

Keeping u_B fixed, consider the map $g : \mathcal{U} \to \boldsymbol{R}^{N-p}$ with components g_j given by (3.16). We have

$$\frac{dg}{dv_\infty}(u_B) = -(\ell_{p+1}(u_B), \cdots, \ell_N(u_B)), \tag{3.17}$$

whose rank is $N-p$. By the implicit function theorem, (3.16) defines a manifold passing through u_B and of dimension p. By construction its tangent space at u_B coincides with the one for the stable manifold $\mathcal{E}$. Therefore, in view of (3.17), the tangent space at u_B for $\mathcal{E}$ is spanned by the $r_j(u_B)$, $j = 1, 2, \cdots, p$. □

A general inclusion can be proven regarding the sets introduced in the previous sections. It has been first pointed out by Serre [43] (cf. also [19]) that:

Proposition 3.6. *The two family of sets introduced in Definitions 3.1 and 3.3 satisfy the inclusion*

$$\mathcal{E}^{\text{layer}}_{\text{viscosity}}(u_B) \subset \mathcal{E}^{\text{entropy}}_{\text{viscosity}}(u_B) \tag{3.18}$$

for all $u_B \in \mathcal{U}$. □

Proof of Proposition 3.6. Let v_∞ be a point in $\mathcal{E}^{\text{layer}}_{\text{viscosity}}(u_B)$ and denote by $y \to v(y)$ the associated boundary layer function which satisfies $v(0) = u_B$ and $v(\infty) = v_\infty$. Consider the following function of the variable $y > 0$:

$$\Omega(y) \equiv F(v_\infty) - F(v(y)) + \nabla U(v(y))\big(f(v_\infty) - f(v(y))\big). \tag{3.19}$$

It is easy to see that

$$\begin{aligned}\frac{d\Omega}{dy}(y) &= \nabla^2 U(v(y))\Big(f(v_\infty) - f(v(y)), f(v_\infty) - f(v(y))\Big)\\ &\geq 0\end{aligned}$$

So the function Ω is non-decreasing, and since $\lim_{y\to\infty} \Omega(y) = 0$, we deduce that $\Omega(y) \leq 0$ for all y, in particular for $y = 0$, that is

$$F(v_\infty) - F(u_B) + \nabla U(u_B)\big(f(v_\infty) - f(u_B)\big) \leq 0.$$

Thus v_∞ belongs to $\mathcal{E}^{\text{entropy}}_{\text{viscosity}}(u_B)$. □

3.2 Finite Difference Schemes. We now turn to formulations of the boundary condition that are based on finite difference approximations. We use the notation in Section 2.2. We consider a scheme characterized by its mesh parameters τ and h with $\lambda = \tau/h$ small enough, and by its numerical flux $g(.,.)$ and its family of numerical entropy fluxes $G(.,.)$. It is tacitly assumed that the values u remain in a small neighborhood of a given state and attention is restricted to those entropies U such that the discrete entropy inequalities (2.38) are satisfied. In fact attention is mostly restricted to the Lax-Friedrichs type schemes and the Godunov scheme.

Definition 3.7. *Given $u_B \in \mathcal{U}$, the set of admissible boundary values based on the entropy inequalities associated with difference scheme is*

$$\mathcal{E}^{\text{entropy}}_{\text{scheme}}(u_B) = \big\{u_0 \in \mathcal{U};\ \textit{There exists } v_1 \textit{ s.t. for all convex } (U,F),\ G(u_B, v_1)) \geq F(u_0)\big\}. \tag{3.20}$$

□

As for $\mathcal{E}^{\text{entropy}}_{\text{scheme}}(u_B)$, this set may be too large to garantee that the boundary value problem is well posed. We also use the obvious notation $\mathcal{E}^{\text{entropy}}_{\text{Lax}}(u_B)$, $\mathcal{E}^{\text{entropy}}_{\text{splitting}}(u_B)$, and $\mathcal{E}^{\text{entropy}}_{\text{Godunov}}(u_B)$.

For general systems and the diagonalizable spltings, i.e. those such that the vectors r_j form a basis of eigenvectors for the matrices $\nabla f^\pm$, we have:

Proposition 3.8. *Consider a Lax-Friedrichs type scheme or, more generally an diagonalizable, entropy flux-splitting scheme. Fix a state $u_B \in \mathcal{U}$ and suppose that (3.4) holds for some p. Then the set obtained by formally linearizing the inequalities in the definition of $\mathcal{E}^{\text{entropy}}_{\text{scheme}}(u_B)$ is an affine manifold of dimension p containing u_B and spanned by the vectors $r_j(u_B)$, $j = 1, 2, \cdots, p$.* □

Proof of Proposition 3.8. We formally plug the second order expansion

$$F^\pm(u_0) \approx F^\pm(u_B) + \nabla F^\pm(u_B)(u_0 - u_B) + \nabla^2 F^\pm(u_B)\big(u_0 - u_B, u_0 - u_B\big) \tag{3.21}$$

and obtain the second order version of the inequalities in (3.20):

$$\nabla F(u_B)(u_0 - u_B) + \nabla^2 F(u_B)\big(u_0 - u_B, u_0 - u_B\big) \leq \nabla F^-(u_B)(v_1 - u_B) + \nabla^2 F^-(u_B)\big(v_1 - u_B, v_1 - u_B\big).$$

Using the trivial entropies (i.e. choose for F the components of f), we get an (second order) expression for v_1:

$$\nabla f^-(u_B)(v_1 - u_B) + \nabla^2 f^-(u_B)\big(v_1 - u_B, v_1 - u_B\big) = \nabla f(u_B)(u_0 - u_B) + \nabla^2 f(u_B)\big(u_0 - u_B, u_0 - u_B\big),$$

which can be used to rewrite the above inequality:

$$\nabla^2 U(u_B)\nabla f(u_B)\big(u_0 - u_B, u_0 - u_B\big) \leq \nabla^2 U(u_B)\nabla f^-(u_B)\big(v_1 - u_B, v_1 - u_B\big).$$

At the first order, v_1 is given by

$$\nabla f^-(u_B)(v_1 - u_B) \;=\; \nabla f(u_B)(u_0 - u_B)$$

so we arrive at the inequality

$$-\nabla f^+(u_B)^T \nabla f^-(u_B)^{-T} \nabla^2 U(u_B) \nabla f(u_B)\big(u_0 - u_B, u_0 - u_B\big) \;\leq\; 0.$$

The desired result follows immediatly since r_j is a basis of eigenvectors for the matrices ∇f^+, ∇f^-, and ∇f, and the function U is convex. □

The second family of sets is now defined.

Definition 3.9. *Given any $u_B \in \mathcal{U}$, the set of admissible boundary values $\mathcal{E}^{\text{layer}}_{\text{scheme}}(u_B)$, based on the boundary layer equation associated with the difference scheme is the set of all $v_\infty \in \mathcal{U}$ such that the problem*

$$\begin{aligned}
&g(v(y), v(y+1)) \;=\; f(v_\infty),\\
&v(y) = u_B \quad \textit{for } y \in [0,1),\\
&\lim_{y\to\infty} v(y) = v_\infty,
\end{aligned} \tag{3.22}$$

admits a (piecewise constant) solution $v(y) \in \mathcal{U}$ for $y \geq 0$. □

To study the local structure of $\mathcal{E}^{\text{layer}}_{\text{scheme}}(u_B)$, we apply the following theorem concerning the existence of discrete invariant manifolds. (Cf. Hartman [23] for a proof.)

Theorem 3.10. *Let $T : \boldsymbol{R}^N \to \boldsymbol{R}^N$, $\xi_0 \to \xi_1$, be a mapping of the form*

$$\xi_1 = \Gamma \xi_0 + E(\xi_0), \tag{3.23}$$

where $E(\xi_0)$ is of class C^1 for small ξ_0 and satisfy $E(0) = 0$ and $\frac{DE}{D\xi_0}(0) = 0$, and the matrix Γ is constant, non-singular, and has $d \geq 0$, $N-d-e$, $e \geq 0$ eigenvalues of absolute value less than 1, equal to 1, and greater than 1, respectively.

There exists a map S of a neighborhood of $\xi_0 = 0$ onto a neighborhood of the origin in the space of $(w_0^I, w_0^{II}, w_0^{III}) \in \boldsymbol{R}^d \times \boldsymbol{R}^{N-d-e} \times \boldsymbol{R}^e$ such that S is of class C^1 with non-vanishing Jacobian and STS^{-1} takes the simple form

$$\begin{aligned}
STS^{-1}: \quad w_1^I &= A^I w_0^I + W^I(w_0^I, w_0^{II}, w_0^{III}),\\
w_1^{II} &= A^{II} w_0^{II} + W^{II}(w_0^I, w_0^{II}, w_0^{III}),\\
w_1^{III} &= A^{III} w_0^{III} + W^{III}(w_0^I, w_0^{II}, w_0^{III}),
\end{aligned} \tag{3.24}$$

where P^I, P^{II}, and P^{III} are $d \times d$, $(N-d-e) \times (N-d-e)$, and $e \times e$ square matrices with eigenvalues of absolute value less than 1, equal to 1, greater than 1, respectively, and the mapping W^I, W^{II}, and W^{III} are of class C^1 and their first order partial derivatives with respect to $(w_0^I, w_0^{II}, w_0^{III})$ vanish at $(0,0,0)$. Moreover one has

$$W^I = 0 \quad \textit{and} \quad W^{II} = 0 \qquad \textit{if} \quad w_0^I = 0 \quad \textit{and} \quad w_0^{II} = 0, \tag{3.25}$$

and

$$W^{II} = 0 \quad \textit{and} \quad W^{III} = 0 \qquad \textit{if} \quad w_0^{II} = 0 \quad \textit{and} \quad w_0^{III} = 0. \tag{3.26}$$

□

The condition (3.25) means that the plane $v_0 = 0, w_0 = 0$ of dimension d is locally invariant manifold and if $R(\xi_0)$ belongs to this manifold then $T^n \xi_0 \to 0$ as $n \to \infty$.

The condition (3.26) means that the plane $u_0 = 0, w_0 = 0$ is a locally invariant manifold and if $R(\xi_0)$ belongs to this manifold, $| T^n \xi_0 | \to \infty$ as $n \to \infty$.

Using this theorem we shall prove:

Theorem 3.11. *Consider a Lax-Friedrichs type scheme. Let $u_B \in \mathcal{U}$ be given and assume that (3.13) holds for some p. Then the set $\mathcal{E}_{\text{Lax}}^{\text{layer}}(u_B)$ contains the point u_B and, locally nearby u_B, contains a manifold with dimension p. When $0 < \lambda_{p+1}(u_B)$, $\mathcal{E}_{\text{Lax}}^{\text{layer}}(u_B)$ is a manifold with dimension exactly p and its tangent space at the point u_B is spanned by the eigenvectors $r_j(u_B)$, $j = 1, 2, \cdots, p$.* □

Proof of Theorem 3.11. We search for all v_∞ that solve the problem:

$$\begin{aligned} &H(v(y), v(y+1), v_\infty) \;=\; 0 \\ &v(0) = 0, \\ &v(\infty) = v_\infty \end{aligned} \tag{3.27}$$

with

$$H(v(y), v(y+1), v_\infty) \equiv v(y+1) - v(y) - \frac{\lambda}{2Q}\big(f(v(y)) + f(v(y+1)) - 2f(v_\infty)\big). \tag{3.28}$$

Using the notation $H = H(v, w, v_\infty)$, we compute

$$\begin{aligned} &\frac{\partial H}{\partial v}(v, w, v_\infty) = Id + \frac{\lambda}{2Q}\nabla f(v), \\ &\frac{\partial H}{\partial w}(v, w, v_\infty) = Id - \frac{\lambda}{2Q}\nabla f(w). \end{aligned} \tag{3.29}$$

For $\lambda/(2Q)$ small enough, the matrix $\partial H/\partial w$ is invertible and its inverse is uniformly bounded w.r.t the variables v, w, and v_∞. By the global implicit function theorem (see J.T. Schwartz [45]) the system (3.27) can be solved for $v(y+1)$. So there exists a smooth mapping $K(v(y), v_\infty)$ such that

$$v(y+1) \;=\; K(v(y), v_\infty) \tag{3.30}$$

and $K(v_\infty, v_\infty) = 0$. Moreover one has

$$\frac{\partial K}{\partial v}(v(y), v_\infty) = \big(Id - \frac{\lambda}{2Q}\nabla f(v(y+1))\big)^{-1}\big(Id + \frac{\lambda}{2Q}\nabla f(v(y))\big). \tag{3.31}$$

The system (3.30) can be linearized around v_∞:

$$\begin{aligned} v(y+1) = &- \big(Id - \frac{\lambda}{2Q}\frac{\partial f}{\partial u}(v_\infty)\big)^{-1}\big(Id + \frac{\lambda}{2Q}\frac{\partial f}{\partial u}(v_\infty)\big)v(y) \\ &+ K(v(y), v_\infty) + \big(Id - \frac{\lambda}{2Q}\frac{\partial f}{\partial u}(v_\infty)\big)^{-1}\big(Id + \frac{\lambda}{2Q}\frac{\partial f}{\partial u}(v_\infty)\big)v(y). \end{aligned}$$

Set $v^*(y+1) = v(y+1) - v_\infty$. The system can be written as

$$\begin{aligned} v^*(y+1) = &- \big(Id - \frac{\lambda}{2Q}\frac{\partial f}{\partial u}(v_\infty)\big)^{-1}\big(Id + \frac{\lambda}{2Q}\frac{\partial f}{\partial u}(v_\infty)\big)v^*(y) \\ &+ G(v^*(y) + v_\infty, v_\infty) + \big(Id - \frac{\lambda}{2Q}\frac{\partial f}{\partial u}(v_\infty)\big)^{-1}\big(Id + \frac{\lambda}{2Q}\frac{\partial f}{\partial u}(v_\infty)\big)v^*(y). \end{aligned}$$

In other words

$$v^*(y+1) = A(v_\infty)v^*(y) + K^*(v^*(y), v_\infty), \tag{3.32}$$

where

$$A(v_\infty) \equiv \big(Id - \frac{\lambda}{2Q}\nabla f(v_\infty)\big)^{-1}\big(Id + \frac{\lambda}{2Q}\frac{\partial f}{\partial u}(v_\infty)\big) \tag{3.33a}$$

and

$$K^* \text{ and } \frac{\partial K^*}{\partial v^*(y)} \text{ vanish at } v^*(y) = 0. \tag{3.33b}$$

We observe that

$$\text{The eigenvalues of the matrix } A(v_\infty) \text{ are } \frac{1 + \lambda\lambda_i(v_\infty)}{1 - \lambda\lambda_i(v_\infty)} \tag{3.34}$$

where we recall that $\lambda_i(v_\infty)$ are the eigenvalues of $\nabla f(v_\infty)$.

Namely (3.34) follows from the fact that the following two statements

(1) a is an eigenvalue of $A(v_\infty)$,
(2) There exists $r \neq 0$ such that $A(v_\infty)r = ar$,

are equivalent.

Using the expression (3.21) of $A(v_\infty)$ and simplifying the resulting equation, we get

$$\nabla f(v_\infty)r = \frac{(a-1)}{\lambda(1+a)}r.$$

So a is an eigenvalue of $A(v_\infty)$ if and only if $\frac{a-1}{\lambda(1+a)}$ is an eigenvalue of $\partial f \partial u(v_\infty)$ with right eigenvector r; so

$$\frac{a-1}{\lambda(a+1)} = \lambda_i(v_\infty) \tag{3.35}$$

for some i with left eigenvector $\ell_i(v_\infty)$ and right eigenvector $r_i(v_\infty)$. Solving (3.35) for a we get ith eigenvalue of $A(v_\infty)$

$$a_i = \frac{1 + \lambda\lambda_i(v_\infty)}{1 - \lambda\lambda_i(v_\infty)}. \tag{3.36}$$

let T be a matrix which diagonalize $\nabla f(v_\infty)$. Then the same matrix diagonalize $A(v_\infty)$:

$$TAT^{-1} = \text{diag}(a_1, a_2, \cdots a_n).$$

Set $w^*(y+1) = Tv^*(y+1)$, we get

$$w^*(y+1) = \begin{pmatrix} a_1 & & \\ & a_2 & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} w^*(y) + L^*(T^{-1}w^*(y), v_\infty)$$

where G^* and $\frac{\partial G^*}{\partial w^*(y)}$ are zero at $w^*(y) = 0$.

Note that

$$a_1 < a_2 < \cdots a_p < 1 \leq a_{p+1} < \cdots < a_n. \tag{3.37}$$

and

$$a_{p+1} = 1 \Leftrightarrow \lambda_{p+1}(v_\infty) = 0.$$

Since all the hypothesis of Theorem 3.10 are satisfied, there exists a p-dimensional invariant manifold defined near 0 such that, if the data v_0^* belongs to this manifold, then $w^*(y+1) \to 0$ as $y \to \infty$. In fact in terms of the original variable $v(y+1)$, we have the expansion

$$v(y+1) - v_\infty = \sum_{j=1}^{N} a_j^y < \ell_j(u), v_b - v_\infty > r_j(v_\infty) + 0(| v(y+1) - v_\infty |)^2. \tag{3.38}$$

In order for this to go to zero, as $y \to 0$ we must have

$$< \ell_j(v_\infty), u_B - v_\infty >= 0, \qquad j = p+1, \cdots N. \tag{3.39}$$

This for fixed u_B defines a map from $R^N \to R^{N-p}$ and whose Jacobian at $u_B = v_\infty$ is the matrix whose $N-p$ rows are $\ell_j(v_\infty)$. Since $\ell_j(v_\infty)$ are linearly independent by implicit function theorem we deduce that (3.39) defines a p dimensional manifold passing through u_B and if v_∞ is in this manifold then there exist a solution to (3.29) whose local structure is given by (3.39). □

The following general inclusion can be proven:

Proposition 3.12. *The two family of sets introduced in Definitions 3.7 and 3.8 satisfy, for all $u_B \in \mathcal{U}$,*

$$\mathcal{E}^{\text{layer}}_{\text{scheme}}(u_B) \subset \mathcal{E}^{\text{entropy}}_{\text{scheme}}(u_B). \tag{3.40}$$

□

Proof of Proposition 3.12. We consider as before a difference scheme that satisfies discrete entropy inequalities. For every v_∞ in the set $\mathcal{E}^{\text{layer}}_{\text{scheme}}(u_B$, there exists a corresponding boundary layer profile $v(y)$, solution of

$$g(v(y), v(y+1)) \; = \; f(v_\infty).$$

The function $v(y)$ is actually a stationnary solution to the scheme since

$$v(y) - v(y) + \lambda\big(g(v(y), v(y+1)) - g(v(y-1), v(y))\big) = 0.$$

Therefore for every convex entropy pair (U, F), it satifies the entropy inequality

$$U(v(y)) - U(v(y)) + \lambda\big(G(v(y), v(y+1)) - G(v(y-1), v(y))\big) \; \leq \; 0,$$

which is nothing but

$$G(v(y), v(y+1)) - G(v(y-1), v(y) \; \leq \; 0$$

Since $\lim_{y \to -\infty} v(y) = v_\infty$, we get

$$G(v(y), v(y+1)) \; \geq \; F(v_\infty)$$

and so with $y = 0$, since $v(y) = u_B \quad$ for $y \in [0, 1)$,

$$G(u_B, v_1)) \geq F(u_0)$$

with $v_1 = v(1)$. That establishes that v_∞ belongs to the set $\mathcal{E}^{\text{entropy}}_{\text{scheme}}(u_B)$. □

Finally we treat the Godunov scheme. The sets $\mathcal{E}^{\text{layer}}_{\text{Godunov}}(u_B)$ and $\mathcal{E}^{\text{entropy}}_{\text{Godunov}}(u_B)$ are defined by Definitions 3.7 and 3.8. We now prove:

Theorem 3.13. *Consider the Godunov scheme and let $u_B \in \mathcal{U}$ be given. We have*

$$\mathcal{E}^{\text{layer}}_{\text{Godunov}}(u_B) \; = \; \mathcal{E}^{\text{entropy}}_{\text{Godunov}}(u_B). \tag{3.41}$$

This set can also be described as the set

$$\mathcal{E}^{\text{Riemann}}(u_B) \; = \; \big\{ R(u_B, w) \, / \, w \in \mathcal{U} \big\},$$

where $R(u_B, w)$ denotes the value at $x/t = 0+$ of the solution of the Riemann problem with data u_B and w on the left and right, respectively. Moreover when (3.4) *holds for some p, the set above contains the point u_B and, locally nearby u_B, is a manifold with dimension p and with tangent space at the point u_B spanned by the eigenvectors $r_j(u_B)$, $j = 1, 2, \cdots, p$.* □

Observe that the Godunov scheme does not produce any boundary layer, in the sense that the layer contains no interior point.

Proof of Theorem 3.13. We recall that the set $\mathcal{E}^{\text{layer}}_{\text{Godunov}}(u_B)$ is defined by the equation

$$
\begin{aligned}
& f(u_B) \;=\; f(R(v(y), v(y+1)), \\
& v(y) = u_B \quad \text{for all } \; y \in [0,1), \\
& \lim_{y \to \infty} v(y) = v_\infty,
\end{aligned} \tag{3.42}
$$

while the set $\mathcal{E}^{\text{entropy}}_{\text{Godunov}}(u_B)$ is defined by the inequalities

$$
F(R(u_B, v_1)) \geq F(u_0) \qquad \text{for all convex pair } \; (U, F) \tag{3.43}
$$

and for some $v_1 \in \mathcal{U}$. So it is not hard to see from the definition that

$$
\mathcal{E}^{\text{Riemann}}(u_B) \subset \mathcal{E}^{\text{layer}}_{\text{Godunov}}(u_B).
$$

On the other hand the inclusion

$$
\mathcal{E}^{\text{layer}}_{\text{Godunov}}(u_B) \subset \mathcal{E}^{\text{entropy}}_{\text{Godunov}}(u_B)
$$

also holds in view of Proposition 3.12.

It remains to show that

$$
\mathcal{E}^{\text{entropy}}_{\text{Godunov}}(u_B) \subset \mathcal{E}^{\text{Riemann}}(u_B).
$$

Consider a pair (u_0, v_1) that solves (3.43). Then w need show that there exists w such that

$$
R(u_B, w) = u_0. \tag{3.44}
$$

Using the trivial entropies, we get

$$
f(R(u_B, v_1)) = f(u_0)
$$

which, combined with the inequality (3.43), shows that the pair of states $(R(u_B, v_1), u_0)$ is an entropy satisfying, stationnary shock wave. On the other hand the Riemann problem with left and right initial data u_B and $R(u_B, v_1)$, respectively, contains only waves with non-positive speeds. Therefore the Riemann solution, with u_B as a left state and u_0 as a right state, only contains waves with non-positive speeds. This function takes the value u_0 in the whole half-interval $x/t > 0$ and thus $R(u_B, u_0) = u_0$, which proves (3.44) with $w = u_0$. □

4. Selected Examples I.

In this section, we investigate first the convex scalar conservation laws and establish that all the sets introduced in Section 4 are essentially the same. Some remarks are then given for the linear hyperbolic systems. Next we return to the scalar equation and treat a non-convex flux function, showing again that the sets are the same with the exception of the set based on the boundary layer equations.

4.1. Scalar Conservation laws: Convex Fluxes.

We consider a scalar conservation law with strictly convex flux, i.e. $f''(u) > 0$ and analyze the boundary layer equation. Let u_* be the unique point such that $f'(u_*) = 0$. To the state u_B, when $u_B \neq u_*$, we associate the solution $u_B^* \neq u_B$ of the equation $f(u_B^*) = f(u_B)$.

We show here that some of the sets introduced in Section 3 coincide in this case. We also recover the formulation of the boundary condition discovered by Bardos-Leroux-Nedelec [4] and Leroux [34].

Theorem 4.1. *Consider a scalar conservation laws with convex flux.*

1) For any $u_B \in \mathcal{U} \equiv \boldsymbol{R}$, the sets of admissible boundary values $\mathcal{E}^{\text{entropy}}_{\text{viscosity}}(u_B)$, $\mathcal{E}^{\text{layer}}_{\text{Godunov}}(u_B)$, and $\mathcal{E}^{\text{entropy}}_{\text{Godunov}}(u_B)$, coincide with

$$
\mathcal{E}^{\text{Riemann}}(u_B) \;=\; \begin{cases} (-\infty, u_B^*] \cup \{u_B\} & \text{if } \; u_B > u_*, \\ (-\infty, u_*] & \text{if } \; u_B \leq u_*. \end{cases} \tag{4.1}
$$

and

$$
\mathcal{E}^{\text{layer}}_{\text{viscosity}} \; (u_B) = \mathcal{E}_{\text{Rieman}}(u_B) - \{u_B^*\}
$$

2) Given $u_B \in \mathcal{U} \equiv [-M, M]$ for a fixed value of $M > 0$, we set $\|f'\|_\infty = \sup_{w \in [-8M, 8M]} |f'(w)|$ and consider a Lax-Friedrichs type scheme with coefficient λ and Q satisfying $\|f'\|_\infty \lambda / Q \leq 1$, then

$$\begin{aligned} \mathcal{E}^{\text{layer}}_{\text{Lax}}(u_B) \cap [-M, M] &= \mathcal{E}^{\text{Riemann}}(u_B) \cap [-M, M] - \{u_B^*\} \\ \mathcal{E}^{\text{entropy}}_{\text{Lax}}(u_B) \cap [-M, M] &= \mathcal{E}^{\text{Riemann}}(u_B) \cap [-M, M] \end{aligned} \tag{4.2}$$

□

Proof of Theorem 4.1.
Step 1: The set $\mathcal{E}^{\text{layer}}_{\text{viscosity}}(u_B)$.

The problem to be solved is

$$B(v)\partial_y v = f(v) - f(v_\infty) \tag{4.3}$$

with the boundary conditions u_B and v_∞ at $y = 0$ and $y = \infty$ respectively. We need show that (4.3) has a solution if and only if v_∞ belongs to the set described in (4.1).

Case 1 : $u_B > u_*$. In this case $f'(u_B) > 0$ and u_B is an "entering" data.

If $v_\infty > u_B$ (4.1) has no solution because, at $y = 0$, $\partial_y v < 0$ and hence v is decreasing at $y = 0$ and hence all later points.

If $u_B > v_\infty > u_B^*$ then $f(u_B) - f(v_\infty) > 0$, and hence v is increasing at $y = 0$ and at every point for the same reason and hence there does not exist solution.

If $v_\infty < v_0^*$, then $f(u_B) - f(v_\infty) < 0$ and v is decreasing at 0 and for all points for a similar reason. Since $u(y)$ cannot cross v_∞, because v_∞ is a critical point $v(y)$ converges to v_∞ as $y \to \infty$.

If $v_\infty = u_B^*$, then $f(u_B) = f(v_\infty)$

$$\partial_y v(0) = 0.$$

Now the equation (4.3) with initial conditions $v(0) = u_B, \partial_y v(0) = 0$ has a unique solution namely $v(y) = u_B$. Hence $v(y)$ does not go to u_B^* as $y \to \infty$. Hence no solution.

Case 2 : $u_B \leq u_*$.

If $v_\infty \leq u_*$ reasoning the same way as before we get existence of solution.

If $v_\infty > u_*$, since we want $u(y) \to v_\infty$ as $y \to \infty$, there exists y_1 such that $u_* < v(y_1) < v_\infty$ and at $y_1, f(v(y_1)) - f(v_\infty) < 0$ and hence $v(y_1)$ is decreasing at y_1 and hence $v(y)$ cannot go to v_∞.

Step 2: The set $\mathcal{E}^{\text{layer}}_{\text{Lax}}(u_B)$.

Recall that the boundary layer equation here is

$$\frac{\lambda}{2Q}(f(v(y+1)) - f(v_\infty)) + \frac{\lambda}{2Q}(f(v(y)) - f(v_\infty)) = v(y+1) - v(y) \tag{4.4}$$

We show:

$$\text{Either } v(y+1) = v_\infty = u_B \text{ for all } y \text{ or } v(y+1) > v(y) \text{ for all } y \text{ or } v(y+1) < v(y) \text{ for all } y. \tag{4.5}$$

To show (4.5), we subtract the equation (4.4) with y replaced by $y - 1$ to the original equation (4.4). Using the mean value theorem we get

$$\frac{\lambda}{2Q}f'(\xi_1)(v(y+1) - v(y)) + \frac{\lambda}{2Q}f'(\xi_2)(v(y) - v(y-1)) = (v(y+1) - v(y)) - (v(y) - v(y-1)).$$

Rearranging the terms, we arrive to

$$1 - \frac{\lambda}{2Q}f'(\xi_1)(v(y+1) - v(y)) = (1 + \frac{\lambda}{2Q}f'(\xi_2))(v(y) - v(y-1)).$$

The claim (4.5) follows since $(1-\frac{\lambda}{2Q}f'(\xi_1))$ and $(1+\frac{\lambda}{2Q}f'(\xi_2))$ are positive.

By the implicit function theorem, given u_B, there exists a solution $v(1), v(2), \cdots, v(y+1)$ to (4.4) on the interval $[0, y+1]$. We have to find v_∞ for which $v(y+1) \to v_\infty$ and show that this set is $\mathcal{E}(u_B)^{\mathrm{Riemann}} \cap [-M, M]$.

Case 1 : $u_B > u_*$.

If $v_\infty > u_B$ there is no solution. Namely, if there is a solution we must have $u_* < u_B < v(y) < v(y+1) < v_\infty$. This implies on one hand $v(y+1) - v(y) > 0$ and on the other hand from (4.4), $v(y+1) - v(y) < 0$, since both terms on the left are < 0.

If $u_B^* < v_\infty < u_B$, there is no solution. Namely if there is a solution we must have $v(1) < u_B$ and

$$\frac{\lambda}{2Q}(f(v(1)) - f(v_\infty)) + \frac{\lambda}{2Q}(f(u_B) - f(v_\infty)) = v(1) - u_B \tag{4.6}$$

Since $u_B^* < v_\infty < u_B, f(u_B) - f(v_\infty) > 0$ and hence from (4.6), we get

$$\frac{\lambda}{2Q}(f(u_1) - f(u_B)) < u_1 - u_B.$$

By the mean value theorem $(-1+\frac{\lambda}{2Q}f'(\xi_1))(u_1 - u_B) < 0$, which implies $v(1) - u_B > 0$ contradicting $v(1) < u_B$.

If $v_\infty < u_B^*$, then there exists a solution to (4.3). Indeed from (4.4) we have

$$u_B > v(1) > v(2) > \cdots v(y+1) > v(y+2) \cdots$$

We have to show that $v(y+1) > v_\infty$ for all y. Otherwise, there exists y_0 such that $v(y_0 - 1) > v_\infty > v(y_0)$. But then $f(v(y_0)) > f(v(y_0 - 1))$ and from (4.4) we have

$$\frac{\lambda}{2Q}(f(v(y_0-1)) - f(v(y_0)) < vy_0 - v(y_0 - 1).$$

This implies $(1+\frac{\lambda}{2Q}f'(\xi))(v(y_0) - v(y_0-1)) > 0$, which is not possible since $v(y_0) < v(y_0-1)$. Hence $v(y) > v_\infty$. Since $v(y+1)$ is a monotone sequence, there exists u_∞ su that

$$v(y+1) \to u_\infty \qquad \text{as } y \to \infty.$$

Letting $y \to \infty$ in (4.4) we get

$$f(u_\infty) - f(v_\infty) = 0.$$

Since u_∞ and v_∞ are less that u_*, we deduce that $u_\infty = v_\infty$.

If $v_\infty = u_B^*$ then

$$\frac{\lambda}{2Q}(f(v_1) - f(v_\infty)) = v_1 - u_B.$$

Since $f(v_\infty) = f(u_B)$, we get

$$\frac{\lambda}{2Q}(f(v_1) - f(u_B)) = v_1 - u_B.$$

That is $(1-\frac{\lambda}{2Q}f'(\xi))(v_1 - u_B) = 0$. Since $(1-\frac{\lambda}{2Q}f'(\xi)) > 0$, we get $v_1 = u_B$ and hence $v(y) = u_B$ for all y. Thus $v(y)$ cannot converge to v_∞.

Case 2 : $u_B < u_*$.

By an arguments similar to that we have done above we can show that the set of v_∞ for which (4.4) has a solution is $(-\infty, u_*] \cap [-M, M]$.

Step 3: The set $\mathcal{E}^{\text{entropy}}_{\text{viscosity}}(u_B)$.

This is the set of all $u_0 \in \boldsymbol{R}$ such that

$$F(u_B) + \nabla U(u_B)(f(u_0) - f(u_B)) \geq F(u_0) \tag{4.7}$$

for all convex entropy pairs (U, F). It is well known that for scalar conservation laws it is enough to consider Kruzhov entropics: $U(u) = | u - k |, F(u) = \text{sgn}(u-k)(f(u) - f(k))$ for $k \in \boldsymbol{R}$. In this case (4.7) reduces to

$$\big(\text{sgn}(u_B - k) - \text{sgn}(u_0 - k)\big)(f(u_0) - f(k)) \geq 0 \tag{4.8}$$

for all $k \in \boldsymbol{R}$. This inequality holds trivially if k is not in $[\min(u_0, u_B), \max(u_0, u_B)]$. We determine the set of all u_0 such that (4.8) holds for all $k \in [\min(u_0, u_B), \max(u_0, u_B)]$. We need to consider several cases.

Case 1 : $u_B > u_*$.

If $u_0 > u_B$, then for (4.8) to hold we must have $-2(f(u_0) - f(k)) \geq 0$ for $k \in [u_B, u_0]$, which is not possible as $f(u_0) - f(k) > 0$ for $k \in (u_B, u_0)$.

If $u_B^* < u_0 < u_B$, then we must have $f(u_0) - f(k) \geq 0$ for $k \in (u_0, u_B)$. This is not possible for $k > u_0^*$.

If $u_0 \leq u_B^* < u_B$, then we must have $f(u_0) - f(k) \geq 0$ for all $k \in [u_0, u_B]$, which is true. Thus we get

$$\mathcal{E}^{\text{entropy}}_{\text{viscosity}}\ (u_B) = (-\infty, u_B^*] \text{ if } u_B > u_*.$$

Case 2 : $u_B \leq u_*$.

If $u_0 \leq u_B \leq u_*$, then for (4.8) to hold we must have $f(u_0) - f(k) \geq 0$ for all $k \in [u_0, u_B]$, which is true.

If $u_B < u_0 \leq u_*$, we must have $-(f(u_0) - f(k)) \geq 0$ for all $k \in [u_B, u_0]$, which is true.

If $u_0 > u_*$, then we must have $-(f(u_0) - f(k)) \geq 0$ for all $k \in [u_B, u_0]$. This is not true because $u_* \in [u_B, u_0]$ and $f(u_0) - f(u_*) > 0$.

Thus we get

$$\mathcal{E}^{\text{entropy}}_{\text{viscosity}}\ (u_B) = (-\infty, u_*],\ \text{if } u_B \leq u_*.$$

Step 4. $\mathcal{E}^{\text{entropy}}_{\text{Lax}}\ (u_B) \cap [-M, M]$.

This is the set of all $u_0 \in [-M, M]$ such that there exists v_1 for which

$$F(u_B) + F(v_1) + \frac{Q}{2\lambda}(U(u_B) - u(v_1)) \geq F(u_0) \tag{4.10}$$

for all convex entropy pairs. Since the Kruzkov entropies

$$U(k) = | u - k |, F(u) = \text{sgn}(u-k)(f(u) - f(k)),$$

generates the set of all convex functions, (4.10) reduces to

$$\begin{aligned} &\text{sgn}(u_B - k)[\frac{\lambda}{2Q}(f(u_B) - f(k)) + (u_B - k)] + \\ &\text{sgn}(v_1 - k)[\frac{\lambda}{2Q}(f(v_1) - f(k)) - (v_1 - u)] + \\ &\text{sgn}(u_0 - k)[\frac{\lambda}{Q}(f(u_0) - f(k)) \geq 0. \end{aligned} \tag{4.11}$$

for all $k \in \boldsymbol{R}$. If there exists v_1 then by taking large negative and positive k we get v_1 must satisfy

$$\frac{\lambda}{2Q}(f(u_B)+f(v_1))+u_B-v_1=\frac{\lambda}{Q}f(u_0). \tag{4.12}$$

From (4.12) we get

$$\mid v_1-\frac{\lambda}{2Q}f(v_1)\mid\leq 4M \tag{4.13}$$

provided we choose λ and Q such that $\frac{\lambda}{Q}\max_{\xi\in[-M,M]}\mid f'(\xi)\mid\leq 1$. From (4.13) we get $\mid 1-\frac{\lambda}{2Q}f'(\xi)\mid\mid v_1\mid\leq 4M$, for some ξ between 0 and v_1. Now if we choose λ and Q such that $\frac{\lambda}{Q}\max_{|\xi|\leq 8M}\mid f'(\xi)\mid\leq 1$, then $(1-1/2)\mid v_1\mid\leq 4M$. In otherwords if we choose λ and Q such that

$$\frac{\lambda}{Q}\max_{|\xi|\leq 8M}\mid f'(\xi)\mid\leq 1 \tag{4.14}$$

then there exists a solution v_1 of (4.12) and v_1 has the estimate

$$\mid v_1\mid\leq 8M. \tag{4.15}$$

Let $I(u_B,u_0,v_1)$ be the closed interval $[\min(u_B,u_0,v_1),\max(u_B,u_0,v_1)]$. Then for all k outside $I(u_B,u_0,v_1)$ the inequality (4.11) is trivially satisfied. Thus u_0 is in $\mathcal{E}_{\text{Lax}}^{\text{entropy}}(u_B)\cap[-M,M]$ iff v_1 satisfy (4.11) for all k in $I(u_B,u_0,v_1)$. Rewriting (4.12) and applying mean value theorem we get

$$(1-\frac{\lambda}{2Q}f'(\xi_1))(v_1-u_0)=(1+\frac{\lambda}{2Q}f'(\xi_2))(u_B-u_0)$$

for some ξ_1 in between v_1 and u_0 and ξ_2 in between u_B and u_0. This says by (4.14) and (4.15)

$$u_B>u_0\Longleftrightarrow v_1>v_0, u_B<u_0\Longleftrightarrow v_1<u_0, u_B=u_0\Longleftrightarrow v_1=u_0. \tag{4.15}$$

So far we have not used convexity of $f(u)$. Now consider $f(u)$ is convex.

Case 1 : $u_B>u_*$.

If $u_B<u_0$, then by (4.15) $v_1<u_0$. On the other hand from (4.12) $\frac{\lambda}{2Q}(f(v_1)-f(u_B))+u_B-v_1)>0$. By the mean value theorem, this implies $(1-\frac{\lambda f'}{2Q}(\xi))(u_B-v_1)>0$ for some ξ in between v_1 and u_B. This means that $u_B>v_1$. Now for (4.11) to hold for $k=u_B$, we must have

$$-\frac{\lambda}{2Q}(f(v_1)-f(u_B))+(v_1-u_B)-\frac{\lambda}{Q}(f(u_0)-f(u_B))\geq 0.$$

Since $f(u_0)-f(u_B)>0$, we must have $v_1-u_B-\frac{\lambda}{2Q}(f(v_1)-f(u_B))>0$. Applying mean value theorem we get $v_1-v_B>0$. This contradicts $u_B-v_1>0$. Thus u_0 is not admissible. If $u_B^*<u_0<u_B$, as before we get $u_0<u_B<v_1$. By taking $k=u_B$, we can show that u_0 is not admissible. If $u_0\leq u_B^*$, then we get $u_0<v_1\leq u_B$. Now let $k\in[u_0,v_1]$ in (4.11) we must have

$$\frac{\lambda}{2Q}(f(u_B)-f(k))+(u_B-k)+\frac{\lambda}{2Q}(f(v_1)-f(k))-(v_1-k)+\frac{\lambda}{Q}(f(u_0)-f(k))\geq 0.$$

Using (4.12) this is equivalent to $(f(u_0)-f(k))\geq 0$, which is true since $u_0\leq u_B^*$ and $v_1\leq u_B$ and $k\in[u_0,v_1]$. Now if $k\in(v_1u_B]$, we need to check

$$\frac{\lambda}{2Q}(f(u_B)-f(k))+(u_B-k)-\frac{\lambda}{2Q}(f(v_1)-f(k))+(v_1-k)+\frac{\lambda}{Q}(f(u_0)-f(k))\geq 0.$$

Using (4.12) this is equivalent to

$$\frac{\lambda}{2Q}(f(u_B) - f(k)) + (u_B - k) \geq 0, \text{ for all } k \in (v_1, u_B].$$

Byy mean value theorem it follows that this is true. Thus we have the admissible set

$$\mathcal{E}_{\text{Lax}}^{\text{entropy}}(u_B) \cap [-M, M] = [-M, u_B^*] \text{ if } u_B > u_*.$$

Case 2 : $u_B \leq u_*$.

If $u_* < u_0 \leq u_B^*$, then as before $u_B \leq v_1 < u_0$. For u_0 to be admissible from (4.11) for all $k \in (v_1, u_0]$ we must have $f(k) - f(u_0) \geq 0$ which is not possible since $u_0 > u_*$.

By a similar argument we can show that if $u_0 > u_B^*, u_0$ is not admissible.

If $u_B < u_0 \leq u_*$, then we get $u_B < v_1 < v_0 \leq u_*$. If $k \in [u_B, v_1]$ (4.11) is equivalent to $\frac{\lambda}{2Q}(f(k) - f(u_B)) + (k - u_B) \geq 0$, which is true. If $k \in (v_1, u_0]$ (4.11) is equivalent to $f(k) - f(u_0) \geq 0$ which again is true. Thus u_0 is admissible.

If $u_0 \leq u_B$, it can be shown by the same reasoning as above u_0 is admissible. Thus we have

$$\mathcal{E}_{\text{Lax}}^{\text{entropy}}(u_B) \cap [-M, M] = (-M, u_*] \text{ if } u_B \leq u_*.$$

□

4.2 Linear Hyperbolic Systems.

It is not hard to prove that for a linear and strictly hyperbolic system, the sets defined in Section 3 are all equivalent. We only consider here the case of the discrete boundary layer based on the Lax-Friedrichs scheme.

We also focus attention in this section to establish that the restriction (3.12) on the viscosity matrix is essential to our purpose here, as was observed in another context by Majda-Pego [40] in their study of traveling wave solutions to (2.1). The following example shows a situation where the viscosity matrix is a positive diagonal matrix, and does not satisfy (3.12), while the formulation may lead to a “wrong” boundary condition.

We consider the linear system

$$\partial_t u + \begin{pmatrix} 5 & -5 \\ 3 & -3 \end{pmatrix} \partial_x u = \epsilon \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \partial_{xx} u. \tag{4.16}$$

According to our earlier analysis, the boundary layer equation is

$$\partial_{yy} v(y) = \begin{pmatrix} 1/5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & -5 \\ 3 & -3 \end{pmatrix} \partial_y v(y),$$

i.e.

$$\partial_{yy} v(y) = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \partial_y v(y).$$

Integrating this equation once and using $v(+\infty) = v_\infty$, we get

$$\partial_y v(y) = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} (v - v_\infty). \tag{4.17}$$

Now the eigenvalues of $\begin{pmatrix} 5 & -5 \\ 3 & -3 \end{pmatrix}$ are $\lambda_1 = 0$ and $\lambda_2 = 2$. On the other hand, the eigenvalues of $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}$ are $\mu_1 = -2$ and $\mu_2 = 0$. The solution of (4.17) with the initial condition $v(0) = v_B - v_\infty$ is

$$v(y) - v_\infty =< \bar{\ell}_1, v_B - v_\infty > \bar{r}_1 \bar{e}^{-2y} + < \bar{\ell}_2, v_B - v_\infty > \bar{r}_2,$$

where

$$\bar{\ell}_1 = \left(\frac{1}{\sqrt{2}} \, \frac{-1}{\sqrt{2}} \right), \quad \bar{\ell}_2 = \left(\frac{-3}{2}, \frac{1}{2} \right), \quad \bar{r}_1 = \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix}, \quad \bar{r}_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

In order for $v(y) \to v_\infty$ as $y \to \infty$, we must have $< \bar{\ell}_2, v_B - v_\infty >= 0$ or

$$< \bar{\ell}_2, v_\infty >=< \bar{\ell}_2, v_B > .$$

This requires that we prescribe $< \bar{\ell}_2, u >$ at the boundary. But the correct boundary condition for the hyperbolic system

$$\partial_t u + \begin{pmatrix} 5 & -5 \\ 3 & -3 \end{pmatrix} \partial_x u = 0$$

is to prescribe $< \ell_2, u >$ where $\ell_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$.

Let us now consider the numerical boundary layer for a general linear and strictly hyperbolic system. Set $f(u) = Au$, where A is a constant matrix. The boundary layer equation becomes

$$\frac{\lambda}{2} A \, v(y+1) + \frac{\lambda}{2} A \, v(y) - \frac{1}{2} \big(v(y+1) - v(y) \big) = \lambda \, A \, v_\infty, \tag{4.18}$$

$$v(0) = v_B, \qquad v(\infty) = v_\infty.$$

For a given u_B, we search for the set of states v_∞ for which this problem has a solution. Set $v^\infty(y) = v(y+1) - v_\infty$. The first equation in (4.18) becomes

$$(\lambda A - I) v^\infty(y) = -(\lambda A + I) v^\infty(y-1). \tag{4.19}$$

Let ℓ_j and r_j be the left- and right- eigenvectors for A associated with the eigenvalues λ_j. Set $C^j(y) =< \ell_j, v(y+1) >$. From (4.19) we get

$$(1 - \lambda \, \lambda_j) C(y)^j = (1 + \lambda \, \lambda_j) C^j(y-1)$$

or

$$C^j(y) = \left(\frac{1 + \lambda \, \lambda_j}{1 - \lambda \, \lambda_j} \right) C^j(y-1)$$

with

$$C_0^j =< \ell_j, v_B - v_\infty > .$$

Integrating this, we get

$$C^j(y) =< \ell_j, v_B - v_\infty > \left(\frac{1 + \lambda \lambda_j}{1 - \lambda \lambda_j} \right)^y$$

or

$$v(y+1) - v_\infty = v^\infty(y) = \sum_{j=1}^n \left(\frac{1 + \lambda \lambda_j}{1 - \lambda \lambda_j} \right)^y < \ell_j, u_B - v_\infty) r_j.$$

For $v(y+1) \to v_\infty$, we need $< \ell_j, v_B - v_\infty >= 0, j = p+1, \cdots n$ because $\lambda_1 < \lambda_2 < \cdots \lambda_p < 0 \leq \lambda_{p+1} < \cdots \lambda_n$. This gives correct boundary condition when the eigenvalues are not zero; i.e. to prescribe

$$< \ell_j, u > \qquad \text{for } j = p+1, \cdots, N.$$

4.3 Scalar Conservation Laws: Non-Convex Fluxes.

We return to scalar conservation laws but now with non-convex fluxes. For definiteness we treat the case of the cubic flux given by

$$f(u) = \frac{1}{2}(u^3 - 3u), \tag{4.20}$$

which has one minima and one maxima; indeed

$$f(1) = -1,\ f'(1) = 0,\ f''(1) = 3,\ f(-1) = 1,\ f'(-1) = 0,\ f''(-1) = -3.$$

For a given $u_B \in \boldsymbol{R}$ and the function f given by (4.20), we shall need the solution of the equation

$$f(u) = f(u_B), u \neq u_B. \tag{4.21}$$

If $u_B < -2$ or $u_B > 2$, there is no solution for (4.21). If $u_B \in (-2,-1) \cup (1,2)$, then (4.21) has exactly two solutions. In this case we denote by u_B^ℓ and u_B^s the largest and smallest solutions of (4.21), respectively. If $u_B = -2, -1, 1$, or 2, then (4.21) has exactly one solution; namely $1, 2, -2$, and -1, respectively.

For the formulation of the results in this subsection, it will be convenient to introduce the following set, which is either the empty set or contains a single element:

$$E(u_B) = \begin{cases} \emptyset, & \text{if } u_B \in (-\infty,-2) \cup [-1,1] \cup (2,\infty) \\ \{1\} & \text{if } u_B = -2 \\ \{u_B^s\}, & \text{if } -2 < u_B < -1 \\ \{u_B^\ell\}, & \text{if } 1 < u_B < 2. \\ \{-1\} & \text{if } u_B = 2. \end{cases} \tag{4.22}$$

When $E(u_B)$ is non-empty, we denote by u_B^{**} its element.

Theorem 4.2. *Consider the scalar conservation law with the non-convex flux* (4.20).

1) For any $u_B \in \mathcal{U} = R$, the set of admissible boundary values $\mathcal{E}^{\text{entropy}}_{\text{viscosity}}(u_B), \mathcal{E}^{\text{layer}}_{\text{Godunov}}(u_B)$, and $\mathcal{E}^{\text{entropy}}_{\text{Godunov}}(u_B)$ coincide with

$$\mathcal{E}^{\text{Riemann}}(u_B) = \begin{cases} \{u_B\}, & \text{if } u_B < -2 \\ \{-2, 1\}, & \text{if } u_B = -2 \\ [u_B^s, 1] \cup \{u_B\}, & \text{if} -2 < u_B < -1 \\ [-1, 1] & \text{if } -1 \leq u_B \leq 1 \\ [-1, u_B^\ell] \cup \{u_B\}, & \text{if } 1 < u_B < 2 \\ \{u_B\}, & \text{if } u_B > 2 \\ \{2, -1\}, & \text{if } u_B = 2 \end{cases} \tag{4.23}$$

and

$$\mathcal{E}^{\text{layer}}_{\text{viscosity}}(u_B) = \mathcal{E}^{\text{Riemann}}(u_B) - E(u_B).$$

2) Given any state $u_B \in \mathcal{U} = [-M, M]$ for a fixed value $M > 2$, we set $\| f' \|_\infty = \sup_{w \in [-8M, 8M]} | f'(w) |$ and consider a Lax-Friedrichs type scheme with coefficient λ and Q satisfying $\| f' \|_\infty \frac{\lambda}{Q} \leq 1$. Then

$$\begin{aligned} \mathcal{E}^{\text{layer}}_{\text{Lax}}(u_B) \cap [-M, M] &= \mathcal{E}^{\text{Riemann}}(u_B) \cap [-M, M] - E(u_B), \\ \mathcal{E}^{\text{entropy}}_{\text{Lax}}(u_B) \cap [-M, M] &= \mathcal{E}^{\text{Riemann}}(u_B) \cap [-M, M]. \end{aligned} \tag{4.24}$$

□

Proof of Theorem 4.2.

Step 1. $\mathcal{E}^{\text{layer}}_{\text{viscosity}}\,(u_B)$.

This is the set of all v_∞ such that the problem

$$\begin{aligned} &B(v)\,\partial_y v = f(v) - f(v_\infty), \\ &v(0) = u_B, \qquad v(\infty) = v_\infty, \end{aligned} \tag{4.25}$$

has a solution. First of all we note that, for any u_B, the state $v_\infty = u_B$ is admissible. On the other hand, any solution of (4.25), if it exists, should be strictly monotone or constant throughout the interval.

Case 1 : $u_B < -2$.

If $v_\infty < u_B$, then $\partial_y v > 0$ at $y = 0$ and $v(y)$ increasing at $y = 0$ and hence at all later points. Thus (4.25) cannot have a solution.

If $v_\infty > u_B$, then $f(v_\infty) > f(u_B)$ and hence $\partial_y v < 0$ at $y = 0$ and, thus, for all $y > 0$. Therefore (4.25) does not have solution.

Thus we get

$$\mathcal{E}^{\text{layer}}_{\text{viscosity}}\,(u_B) = \{u_B\} \qquad \text{if } u_B < -2.$$

Case 2 : $-2 \le u_B < -1$.

If $v_\infty < u_B$ or $u_B < v_\infty < u_B^s$, arguments similar to Case 1 shows that (4.25) has not solution. If $v_\infty = u_B^s$, then from the definition of u_B^s and the ODE in (4.25) we get $\partial_y v(0) = 0$. But this, together with $v(0) = u_B$, uniquely determine the solution of $B(v)\partial_y v = f(v) - f(v_\infty)$, namely $v = u_B$. Thus (4.25) has no solution since $v(y)$ cannot go to v_∞. (For $u_B = -2$, we used notation $u_B^s = 1$.) If $u_B^s < v_\infty < 1$, then $f(u_B) - f(v_\infty) = f(u_B^s) - f(v_\infty) > 0$. Hence $\partial_y v > 0$ at $y = 0$ and for all y such that $v(y) < 1$. Since v_∞ is a critical point $v(y)$ cannot cross v_∞ which is less than one and $v(y) \to v_\infty$.

If $-2 < u_B < -1$ and $v_\infty = 1$, by the same argument as above there is solution for (4.25).

If $u_B = -2$ and $v_\infty > 1$, or $-2 < u_B < -1$ and $v_\infty > u_B^\ell$, there is no solution for (4.25) for $\partial_y v(0) > 0$. If $-2 < u_B < -1$ and $1 < v_\infty < u_B^\ell$, then $\partial_y v(0) > 0$ and $v(y)$ increases at zero and for all $u_B < v(y) < u_\infty^\ell$. Also $v(y)$ has to take all values between u_B and v_∞ if $v(y) \to v_\infty$. But since $\partial_y v$ is decreasing if $v(y)$ lies in $(v_\infty^\ell, v_\infty), v(y)$ cannot tend to v_∞ as $y \to \infty$. Thus we get

$$\mathcal{E}^{\text{layer}}_{\text{viscosity}}\,(u_B) = \begin{cases} \{-2\} & \text{if } u_B = -2 \\ (u_B^s, 1] \cup \{u_B\} & \text{if } u_B \in (-2, -1). \end{cases}$$

Case 3 : $-1 \le u_B \le 1$.

Repeating the same argument above it can be easily seen that

$$\mathcal{E}^{\text{layer}}_{\text{viscosity}} = [-1, 1].$$

Case 4 : $1 < u_B \le 2$.

By the same proof as the one in Case 2, we have

$$\mathcal{E}^{\text{layer}}_{\text{viscosity}}\,(u_B) = \begin{cases} \{2\} & \text{if } u_B = 2 \\ [-1, u_B^\ell) & \text{if } 1 < u_b < 2. \end{cases}$$

Case 5 : $u_B > 2$.

By the same proof as the one in Case 1, we have

$$\mathcal{E}^{\text{layer}}_{\text{viscosity}}(u_B) = \{u_B\}.$$

Step 2. $\mathcal{E}^{\text{layer}}_{\text{Lax}}\ (u_B)$, where $u_B \in \mathcal{U} = [-M, M]$ for some $M > 0$.

In this case the boundary layer equation is

$$\frac{\lambda}{2Q}(f(v(y+1)) - f(v_\infty)) + \frac{\lambda}{2Q}(f(v(y)) - f(v_\infty)) = v(y+1) - v(y). \tag{4.26}$$

$$v(0) = u_B, \qquad \lim_{y\to\infty} v(y) = v_\infty. \tag{4.27}$$

As in the earlier case, once $v(0) = u_B$ is given, the equation (4.25) has a unique solution $v(0), v(1), v(2), \cdots$, which is either strictly monotone or constant throughout. We determine the set of v_∞ in $[-M, M]$ for which $v(y) \to v_\infty$ as $y \to \infty$. This is the set $\mathcal{E}^{\text{layer}}_{\text{lax}}\ (u_B) \cap [-M, M]$ by definition. Note that u_B is always in this set. In the following we take M large enough so that all points under consideration are in $[-M, M]$.

Case 1 : $u_B < -2$.

If $v_\infty < u_B$, then $\frac{\lambda}{2Q}(f(v_1) - f(u_B)) < v_1 - u_B$, since $f(u_B) - f(v_\infty) > 0$. This implies $(1 - \frac{\lambda}{\alpha 0} f'(\xi))(v_1 - u_B) > 0$ for some ξ in the interval (min (v_B, v_1) max (u_B, v_1). Hence $v_1 > u_B$. Thus the sequence $v(y)$ cannot decrease to v_∞ as $y \to \infty$.

If $v_\infty > u_B$, by a similar reasoning, the problem (4.26)-(4.27) does not have a solution. We get

$$\mathcal{E}^{\text{layer}}_{\text{lax}}\ (u_B) \cap [-M, M] = \{u_B\}.$$

Case 2 : $u_B = -2$.

As in Case 1, it can be easily seen that (4.26)-(4.27) does not have a solution if $v_\infty \neq -2$ or 1. When $v_\infty = 1$, then $v_1 - u_B = \frac{\lambda}{2Q} f(v_1) - f(u_B)$, where we used $f(u_B) = f(-2) = f(1)$. This implies $(1 - \frac{\lambda}{\alpha\varphi} f'(\xi))(v_1 - u_B) = 0$ because of our choice of λ and Q and $v_1 = u_B$. Thus $v(y) = u_B$ for all y. Thus $v(y)$ does not go to 1 as $y \to \infty$.

Case 3 : $-2 < u_B < -1$.

If $v_\infty < u^s_B$ or $v_\infty > u^\ell_B$, following the same reasoning as Case 1 gives that (4.26)-(4.27) has no solution.

If $v_\infty = u^s_B$, we can argue as in the second part of Case 2 to show that a solution does not exist.

If $u^s_B < v_\infty \le 1$, then $v_1 - u_B > \frac{\lambda}{2Q}(f(v_1) - f(u_B))$ since $f(u_B) > f(v_\infty)$. This implies $(1 - \frac{\lambda}{2Q}) f'(\xi)(v_1 - u_B) > 0$ and thus $v_1 > u_B$ and $v(y)$ is a strictly increasing function. We show that $v(y) < v_\infty$. From (4.26) we have

$$\frac{\lambda}{2Q}(f(v(y+1)) - f(v_\infty)) + \frac{\lambda}{2Q}(f(v(y)) - f(v_\infty)) = (v(y+1) - v_\infty) - (v(y) - v_\infty).$$

Applying the mean value theorem and rearranging the terms, we get

$$(1 - \frac{\lambda}{2Q} f'(\xi_1))(v(y+1) - v_\infty) = (1 + \frac{\lambda}{2Q} f'(\xi_2))(v(y) - v_\infty).$$

Since $1 - \frac{\lambda}{2Q} f'(\xi_1) > 0, 1 + \frac{\lambda}{2Q} f'(\xi_2) > 0$, we find that $v(y) - v_\infty$ is positive or negative. In our case $v_1 - v_\infty$, is negative and hence $v(y) < v_\infty$. Hence $v(y) \to \overline{v} \le v_\infty$. From (4.26) we get

$$f(\overline{v}) = f(v_\infty).$$

This equation has only one solution $\overline{v} = v_\infty$ in the interval $(u_B, v_\infty]$. So (4.26)-(4.27) has a solution.

If $1 < v_\infty < u_B^\ell$, there is no solution. For if there is a solution then $u_B < v(1) < v(2) < \cdots v(y) \to v_\infty$ and except for a finite number of integers y, the state $v(y)$ lies in $(1, v_\infty)$. But then $f(v(y)) - f(v_\infty) < 0$ for all y except for a finite number of integer values of y. Using this fact in (4.26), we get $v(y+1) < v(y)$ except for a finite number of integers y which is not possible.

As in the second part of Case 2, it can be seen that, if $v_\infty = u_B^\ell$, there is no solution. Finally we have

$$\mathcal{E}_{\text{Lax}}^{\text{layer}}\ (u_B) = (u_B^s, 1] \cup \{u_B\} \qquad \text{if } u_B \in (-2, -1).$$

Case 4 : $u_B = -1$.

If $v_\infty < u_B = -1$, then $f(v_\infty) < f(u_B)$. Using this in (4.26), we get $v_1 - u_B > \frac{\lambda}{2Q}(f(v_1) - f(u_B))$. This implies $(1 - \frac{\lambda}{2Q} f'(\xi))(v_1 - u_B) > 0$ and hence $v_1 > u_B$. Hence there is no solution.

If $-1 < v_\infty \leq 1$, then $f(u_B) - f(v_\infty) > 0$ and, as before, $v_1 > u_B$. It is easily shown as in Part 3 of Case 3 that there exists a solution to (4.26)-(4.27). If $1 < v_\infty < 2$ there is no solution. Proof of this fact is same as in Case 1.

If $2 < v_\infty$, then using the same reasoning as for Case 1, we see that there is no solution.

If $v_\infty = 2$, there is no solution because we can easily show that $(1 - \frac{\lambda}{2Q} f'(\xi))(v_1 - v_B) = 0$ and hence $v_1 = u_B$. Combining the two, we get

$$\mathcal{E}_{\text{Lax}}^{\text{layer}}\ (u_B) = [-1, 1], \text{ if } u_B = 1.$$

Case 5 : $-1 < u_B \leq 1$.

By the same arguments as above we get

$$\mathcal{E}_{\text{Lax}}^{\text{layer}}\ (u_B) = [-1, 1], \text{ if } u_B \in (-1, 1].$$

Proofs the following cases are repetition of earlier cases and are omitted.

Case 6 : $2 > u_B > 1$. Then $\mathcal{E}_{\text{Lax}}^{\text{layer}}\ (u_B) = (u_B^\ell, 1] \cup \{u_B\}$.

Case 7 : $u_B = 2$. Then $\mathcal{E}_{\text{Lax}}^{\text{layer}}\ (u_B) = \{2\}$.

Case 8 : $u_B > 2$. Then $\mathcal{E}_{\text{Lax}}^{\text{layer}}\ (u_B) = \{u_B\}$.

Step 3. The set $\mathcal{E}_{\text{viscosity}}^{\text{entropy}}(u_B)$.

As observed in the convex case, this is the set of all $u_0 \in \boldsymbol{R}$ such that

$$\big(\operatorname{sgn}(u_B - k) - \operatorname{sgn}(u_0 - k)\big)\big(f(u_0) - f(k)\big) \geq 0 \tag{4.28}$$

holds for all $k \in [\min(u_B, u_0), \max(u_B, u_0)]$.

Case 1 : $u_B < -2$.

If $u_0 < u_B$, then, for u_0 to be admissible, we must have from (4.28) $f(u_0) - f(k) \geq 0$ for all $k \in [u_0, u_B]$, which is not possible.

If $u_B < u_0 \leq -1$, for u_0 to be admissible we should have $f(k) - f(u_0) \geq 0$ for all $k \in [u_B, u_0]$, which again is not possible.

If $u_0 > -1$, then plugging $k = -2$ in (4.28) gives $f(-2) - f(u_0) \geq 0$, which is not possible. Thus

$$\mathcal{E}_{\text{viscosity}}^{\text{entropy}}\ (u_B) = \{u_B\} \qquad \text{if } u_B < -2.$$

Case 2 : $u_B = -2$.

If $u_0 \neq 1$ or -2, then by the same argument as above u_0 is not admissible. If $u_0 = 1$ then (4.28) to hold for all $k \in [-2, 1]$ we must have $f(k) - f(u_0) \geq 0$, which is true. Thus

$$\mathcal{E}^{\text{entropy}}_{\text{viscosity}}\ (-2) = \big\{ -2, 1 \big\}.$$

Case 3 : $-2 < u_B < -1$.

If $u_0 < u_B$, then from (4.28) we get u_0 is admissible if $f(u_0) - f(k) \geq 0$ for all $k \in [u_0, u_B]$, which not true. If $u_B < u_0 < u_B^s$ for admissibility we should have for $k \in [u_B, u_0]$, $f(k) - f(u_0) \geq 0$ which is not possible.

If $u_B^s \leq u_0 \leq 1$ it follows as above that (4.28) is satisfied and if $u_0 > 1$ (4.28) is not satisfied for $k = 1$. Thus

$$\mathcal{E}^{\text{entropy}}_{\text{viscosity}}\ (u_B) = [u_B^s, 1] \cup \big\{u_B\big\}.$$

Case 4 : $u_B = -1$.

If $u_0 < -1$, (4.28) is violated for $k \in (u_0, -1)$ and if $1 < u_0 < \infty$, (4.28) is violated for example for $k = 1$. If $-1 < u_0 \leq 1$ then (4.28) is satisfied for all $k \in [u_B, u_0]$. Thus

$$\mathcal{E}^{\text{entropy}}_{\text{viscosity}} = [-1, 1].$$

In a similar way we can show that

Case : $-1 < u_B \leq 1$. Then $\mathcal{E}^{\text{entropy}}_{\text{viscosity}}\ (u_B) = [-1, 1]$

Case : $-1 < u_B < 2$. Then $\mathcal{E}^{\text{entropy}}_{\text{viscosity}}\ (u_B) = \big\{u_B\big\} U [-1, u_B^\ell]$

Case : $u_B = 2$. Then $\mathcal{E}^{\text{entropy}}_{\text{viscosity}}\ (2) = \big\{ -2, -1 \big\}$

Case : $u_B > 2$. Then $\mathcal{E}^{\text{entropy}}_{\text{viscosity}}\ (u_B) = \big\{u_B\big\}$.

Step 4. The set $\mathcal{E}^{\text{entropy}}_{\text{Lax}}\ (u_B) \cap [-M, M]$.

Let $u_B \in [-M, M]$. Here we take $M > 2$ to include all the interesting cases. The set $\mathcal{E}^{\text{entropy}}_{\text{Lax}}\ (u_B) \cap [-M, M]$ is the set of all $u_0 \in [-M, M]$ for which there exists v_1 such that

$$\operatorname{sgn}(u_B - k)\Big[\frac{\lambda}{2Q}(f(u_B) - f(k)) + (u_B - k)\Big] + \operatorname{sgn}(v_1 - k)\Big[\frac{\lambda}{2Q}(f(v_1) - f(k)) - (v_{-}k)\Big] - \operatorname{sgn}(u_0 - k)\Big[\frac{\lambda}{Q} f(u_0) - f(k))\Big] \geq 0. \tag{4.29}$$

We have seen in Step 4 of Theorem 4.1 (for any smooth flux f) that if such a v_1 exists for a given $u_0 \in [-M, M]$ then it must satisfy

$$\frac{\lambda}{2Q}(f(u_B) + f(v_1)) + u_B - v_1 = \frac{\lambda}{Q} f(u_0). \tag{4.30}$$

There, we also have seen that if λ and Q are chosen such that $\frac{\lambda}{Q}\max_{|\xi| \leq 8M} \mid f'(\xi) \mid < 1$, then (4.30) has a unique solution v_1 satisfying $\mid v_1 \mid \leq 8M$ and

$$u_B > u_0 \Longleftrightarrow v_1 > u_0, \qquad u_B < u_0 \Longleftrightarrow v_1 < u_0, \qquad u_B = u_0 \Longleftrightarrow v_1 = u_0. \tag{4.31}$$

Further if k is outside the interval $I(u_B, u_0, v_1)$ limited by the states u_B, u_0, v_1, then (4.29) is automatically satisfied. Thus the admissible values u_0 in $[-M, M]$ are those for which u_B, u_0 and the solution v_1 of (4.30) satisfy (4.29) for all k in $I(u_B, u_0, v_1) = [min(u_B, u_0, v_1), \max(u_B, u_0, v_1)]$.

Case 1 : $u_B < -2$.

If $u_0 < u_B$, then, from (4.30) and (4.31), we get $u_0 < u_B < v_1$. Take $k = u_B$ in (4.29) and use (4.30); we get, $(u_B - v_1) + \frac{\lambda}{2Q}(f(v_1) - f(u_B)) \geq 0$, which by the mean value theorem and our choice of λ and Q implies $u_B - v_1 \geq 0$, contradicting $u_B < v_1$.

If $u_0 > u_B$, then as before v_1 must satisfy $v_1 < u_B < u_0$. So, for $k = u_B$, (4.29) is not satisfied and

$$\mathcal{E}_{\text{Lax}}^{\text{entropy}}(u_B) = \{u_B\} \text{ if } u_B < -2.$$

Case 2 : $u_B = -2$.

By the same argument as in Case 1, it can be seen that no point in the set $[-M, M]\{-2, 1\}$ is in $\mathcal{E}_{\text{Lax}}^{\text{entropy}}(-2)$. If $u_0 = 1$, then we get from (4.30) that $v_1 = u_B$. Now for $u_0 = 1$ to be admissible from (4.29) we must have $f(k) - f(-2) \geq 0$ for all $k \in [-2, 1]$, which is true.

Thus $\mathcal{E}_{\text{Lax}}^{\text{entropy}}(-2) = \{-2, 1\}$.

Case 3 : $-2 < u_B < -1$.

If $u_0 < u_B$, then $u_0 < u_B < v_1$ and for $k = u_B$, (4.29) is not satisfied.

If $u_B < u_0 < u_B^S$, then $v_1 < u_B < u_0$ and for $k = u_B$, (4.29) is not satisfied.

If $u_B < u_0 < u_B^s$, then $v_1 < u_B < u_0$ and for $k = u_B$, (4.29) is not satisfied.

If $u_B^s \leq u_0 \leq 1$, then from (4.30) and (4.31) we have $u_B < v_1 < u_0$ and it can be easily seen that (4.29) is satisfied for all $k \in [u_B, u_0]$.

If $u_0 > 1$, it can be easily shown that u_0 is not admissible. Thus we have

$$\mathcal{E}_{\text{Lax}}^{\text{entropy}}(u_B) = [u_B^s, 1]U\{u_B\}, \text{ if } -2 < u_B < -1.$$

Case 4 : $-1 \leq u_B \leq 1$.

If $u_0 < -1$, then by (4.31) $u_0 < v_1$. If $f(u_0) - f(u_B) < 0$, then from (4.30) $u_B < v_1$ and thus $u_0 < u_B < v_1$. It can be seen easily that for $k = u_B$ (4.29) is not satisfied.

If $f(u_0) = f(u_B)$ then $v_1 = u_B$ and for $k = -1$, (4.29) is not satisfied.

If $f(u_0) - f(u_B) > 0$ then as before $u_0 < v_1 < u_B$. Now take $k \in (u_0, \min(v_1, -1)$, for u_0 to be admissible from (4.29) and (4.30) we must have have $f(u_0) \geq f(k)$. This is not true for k in $(u_0, \min\,(v_1 - 1))$. Thus u_0 is not admissible if $u_0 < -1$.

If $-1 \leq u_0 < u_B$, we have $u_0 < v_1 < u_B$. If $k \in [u_0, v_1]$ (4.29) is equivalent to $f(u_0) - f(k)) \geq 0$, which is true. Similarly (4.29) holds for $k \in (v_1, u_B]$. Thus u_0 is admissible.

If $u_0 > 1$, it can be easily checked that u_0 is not admissible. Thus we have

$$\mathcal{E}_{\text{Lax}}^{\text{entropy}}(u_B) = [-1, 1] \text{ if } u_B \in [-1, 1].$$

In the following cases the proofs are repetition of earlier cases and are omitted.

Case : $-1 < u_B < 2$. Then $\mathcal{E}_{\text{Lax}}^{textentropy}(u_B) = \{u_B\}U[-1, u_B^\ell]$.

Case : $u_B = 2$. Then $\mathcal{E}_{\text{Lax}}^{textentropy}(u_B) = \{2, -1\}$.

Case : $u_B > 2$. Then $\mathcal{E}_{\text{Lax}}^{textentropy}(u_B) = \{u_B\}$. □

5. Selected Examples II.

5.1 Nonlinear Elastodynamics. The system considered now arises in the modeling of elastic materials [10]:

$$\begin{aligned} \partial_t v - \partial_x u &= 0, \\ \partial_t u - \partial_x \sigma(v) &= 0. \end{aligned} \tag{5.1}$$

It describes the evolution of a nonlinear material with deformation gradient v and velocity u. The stress function σ is assumed to be smooth enough and satisfy the following conditions:

$$\sigma'(v) > 0, \qquad v\,\sigma''(v) > 0. \tag{5.2}$$

Let us discuss the vanishing viscosity approximation for the viscosity matrix $B(u) = I$. The boundary layer problem to be studied here is

$$\begin{aligned} -\partial_y u \;&=\; \partial_y^2 v, \\ -\partial_y \sigma(v) \;&=\; \partial_y^2 u, \\ v(0) = v_B, \qquad v(\infty) &= v_\infty, \\ u(0) = u_B, \qquad u(\infty) &= u_\infty \end{aligned} \tag{5.3}$$

We need determine the set of (v_∞, u_∞) for which (5.3) has a solution. Integrating once the ODE'S and using the boundary condition at infinity, we get

$$\partial_y v = u_\infty - u, \qquad u_y = \sigma(v_\infty) - \sigma(v). \tag{5.4}$$

Cross multiplying the equations and integrating, we get

$$\frac{(u - u_\infty)^2}{2} = \int_{v_\infty}^{v} (\sigma(s) - \sigma(v_\infty))\, ds,$$

so

$$(u - u_\infty)^2 = \pm \left(\int_{v_\infty}^{v} 2\,(\sigma(s) - \sigma(v_\infty))\, ds \right)^{1/2}. \tag{5.5}$$

Note that $\int_{v_\infty}^{v} (\sigma(s) - \sigma(v_\infty)) ds \geq 0$ because of the condition $\sigma'(v) > 0$. From (5.5) it follows that

$$v(y) = v_\infty \quad \Leftrightarrow \quad u(y) = u_\infty.$$

Since we are interested in a solution connecting (v_B, u_B) at $y = 0$ to (v_∞, u_∞) at $y = \infty$, we get from (5.4) that either

$$\begin{aligned} &v_B < v(y) < v_\infty \quad \text{and} \quad u_B > u(y) > u_\infty \\ &\text{or} \\ &v_B > v(y) > v_\infty \quad \text{and} \quad u_B < u(y) < u_\infty. \end{aligned} \tag{5.6}$$

This determines the sign in (5.5):

$$u = \begin{cases} u_\infty - \left(\displaystyle\int_{v_\infty}^{v} 2\,(\sigma(s) - \sigma(\bar{v}))\, ds \right)^{1/2} & \text{if}\, v > v_\infty \\ u_\infty + \left(\displaystyle\int_{v_\infty}^{v} 2\,(\sigma(s) - \sigma(\bar{v}))\, ds \right)^{1/2} & \text{if}\, v < v_\infty. \end{cases}$$

Since we need (v_B, u_B) to be on this curve, we obtain that the set of (v_∞, u_∞) so that (5.3) has a solution lies on the curve

$$u_\infty = \begin{cases} u_B + \left(\int_{v_\infty}^{u_B} 2\,(\sigma(s) - \sigma(\bar{v}))\,ds \right)^{1/2} & \text{if } v_\infty < v_B \\ u_B - \left(\int_{v_\infty}^{u_B} 2\,(\sigma(s) - \sigma(\bar{v}))\,ds \right)^{1/2} & \text{if } v_\infty > v_B, \end{cases} \tag{5.7}$$

where (v_B, u_B) is fixed.

Let us now turn to the Lax Friedrichs scheme. For the system (5.1), the discrete boundary layer equation is

$$H(v(y), v(y+1), u_\infty, v_\infty) \equiv \begin{pmatrix} \lambda(v(y+1) + v(y)) + v(y) - v(y-1) - 2\,\lambda\, u_\infty) \\ \lambda(\sigma(v(y)) + \sigma(v(y-1)) + v(y+1) - v(y) - 2\,\lambda\,\sigma(v_\infty)) \end{pmatrix} = 0, \tag{5.8}$$

$$(v, u)(0) = (u_B, u_0), \qquad (v, u)(\infty) = (v_\infty, u_\infty).$$

Here the eigenvalues of the system (5.1) are

$$\lambda_2(v_\infty, u_\infty) = -\lambda_1(v_\infty, u_\infty) = \sigma'(v_\infty)^{1/2}$$

and, with the notations of Section 3,

$$a_1(v_\infty, u_\infty) = \frac{1 - \lambda\sigma'(v_\infty)^{1/2}}{1 + \lambda\sigma'(v_\infty)^{1/2}}, \qquad a_2(v_\infty, u_\infty) = \frac{1 + \lambda\sigma'(v)^{1/2}}{1 - \lambda\sigma'(v_\infty)^{1/2}}.$$

Thus $0 < a_1(v_\infty, u_\infty) < 1$, $a_2(v_\infty, u_\infty) > 1$. By the analysis of Section 3, it follows that the set of (v_∞, u_∞) near (v_B, u_B) for which (5.8) has a solution lie on a curve passing through (v_B, u_B).

5.2. Eulerian Isentropic Gas Dynamics. We now consider the isentropic approximation to the compressible Euler system. The system is composed of the two conservation laws for the mass and the momentum of a gas [10]:

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0, \\ &\partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) = 0. \end{aligned} \tag{5.9}$$

The main unknowns are the specific density ρ and the velocity u. The pressure is a function of the density and, for simplicity, we shall restrict to a polytropic perfect gas:

$$p(\rho) = \rho^\gamma, \qquad \gamma \in (1, \infty). \tag{5.10}$$

We consider the boundary layer equation generated by the vanishing viscosity method with $B(u) = I$:

$$\begin{aligned} &\partial_y(\rho u) = \partial_y^2 \rho \\ &\partial_y(\rho u^2 + p(\rho)) = \partial_y^2 u \\ &\rho(0) = \rho_B, \quad u(0) = u_B, \quad \rho(\infty) = \rho_\infty, \quad u(\infty) = u_\infty. \end{aligned} \tag{5.11}$$

Integrating the ODE'S and using the boundary condition at infinity, we get

$$\begin{aligned} &\partial_y \rho = \rho u - \rho_\infty\, u_\infty \\ &\partial_y u = \rho u^2 + p(\rho) - \rho_\infty u_\infty^2 - p(\rho_\infty) \\ &\rho(0) = \rho_B, \quad u(0) = u_B, \quad \rho(\infty) = \rho_\infty, u(\infty) = u_\infty. \end{aligned} \tag{5.12}$$

The eigenvalues of the matrix obtained by linearizing the R.H.S. of (5.12) around (ρ_∞, u_∞) are

$$\lambda_1(\rho_\infty, u_\infty) = u_\infty - c(\rho_\infty), \lambda_2(\rho_\infty, u_\infty) = u_\infty + c(\rho_\infty) \tag{5.13}$$

where $c^2(\rho_\infty) = p'(\rho)$. We have to distinguish between five different cases. We define the following regions in (ρ, u)–plane:

$$\begin{aligned} \Omega_I &= \{(\rho, u) : u - c(\rho) < 0, u + c(\rho) < 0\} \\ \Omega_{II} &= \{(\rho, u) : u - c(\rho) < 0, u + c(\rho) = 0\} \\ \Omega_{III} &= \{(\rho, u) : u - c(\rho) < 0, u + c(\rho) > 0\} \\ \Omega_{IV} &= \{(\rho, u) : u - c(\rho) = 0, u + c(\rho) > 0\} \\ \Omega_V &= \{(\rho, u) : u - c(\rho) > 0, u + c(\rho) > 0\} \end{aligned} \tag{5.14}$$

Thus in Ω_I both eigenvalues are negative, whereas in Ω_{II} one has $\lambda_1 < 0, \lambda_2 = 0$. In Ω_{III}, one has $\lambda_1 < 0, \lambda_2 < 0$, wheras in Ω_{IV}, one has $\lambda_1 = 0, \lambda_2 > 0$ and in Ω_V, $\lambda_1 > 0$ and $\lambda_2 > 0$. Following the analysis that we did for the proof of Theorem 3.2, it is not hard to get the following local result.

Case 1 : $(\rho_B, u_B) \in \Omega_I$. In this case the set of (ρ_∞, u_∞) close to (ρ_B, u_B) for which (5.12) has a solution is an open neighborhood of (ρ_B, u_B).

Case 2 : $(\rho_B, u_B) \in \Omega_{II}$. In this case the set of (ρ_∞, u_∞) close to (ρ_B, u_B) for which (5.12) has a solution is a union of a two-dimensional region U in Ω_I and a curve in Ω_{III} through (ρ_B, u_B) intersecting U.

Case 3 : $(\rho_B, u_B) \in \Omega_{III}$. In this case the set of states (ρ_∞, u_∞) close to (ρ_B, u_B) for which (5.12) has a solution is a curve through (ρ_B, u_B)

Case 4 : $(\rho_B, u_B) \in \Omega_{IV}$. In this case the set of states (ρ_∞, u_∞) near (ρ_B, u_B) for which (5.12) has a solution lies in a curve in Ω_{III} through (ρ_B, u_B). This does not extend to Ω_V.

Case 5 : $(\rho_B, u_B) \in \Omega_V$. There cannot be any point (ρ_∞, u_∞) in Ω_V for which (5.12) has a solution.

Nest we consider the Lax-Friedrichs scheme. The discrete boundary layer problem to be solved is

$$\begin{aligned} &\lambda(\rho(y)v(y+1) + \rho(y-1)v(y)) - 2\lambda\rho_\infty u_\infty - (\rho(y) - \rho(y-1)) = 0 \\ &\lambda(\rho(y)v(y+1)^2 + \rho(y-1)v(y)^2) - 2\lambda\rho_\infty u_\infty^2 - (\rho(y)v(y+1) - \rho(y-1)v(y)) = 0 \end{aligned} \tag{5.15}$$

(ρ_B, u_B) given and $(\rho, u)(\infty) = (\rho_\infty, u_\infty)$.

Given (ρ_B, u_B) we determine (ρ_∞, u_∞) close to (ρ_B, u_B) for which (5.15) has a solution. Following the analysis of the proof of Theorem (3.4), we get the eigenvalues of the linearized matrix at (ρ_∞, u_∞) are

$$a_1 = a_1(\rho_\infty, u_\infty) = \frac{1 + \lambda\lambda_1(\rho_\infty, u_\infty)}{1 - \lambda\lambda_1(\rho_\infty, u_\infty)}, a_2 = a_2(\rho_\infty, u_\infty) = \frac{1 + \lambda\lambda_2(\rho_\infty, u_\infty)}{1 - \lambda\lambda_2(\rho_\infty, u_\infty)}$$

where λ_1 and λ_2 are given by (5.13). If $(\rho_\infty, u_\infty) \in \Omega_I, a_1 < 1, a_2 < 1$, if $(\rho_\infty, u_\infty) \in \Omega_{II}, a_1 < 1, a_2 = 1$, if $(\rho_\infty, u_\infty) \in \Omega_{III}, a_1 < 1, a_2 > 1$, if $(\rho_\infty, u_\infty) \in \Omega_{IV}, a_1 = 1, a_2 > 1$ and if $(\bar{\rho}, u_\infty) \in \Omega_V, a_1 > 1, a_2 > 1$. It follows from the proof of Theorem (3.4), that if $(\rho_B, u_B) \in \Omega_I$, then the set of states (ρ_∞, u_∞) near (ρ_B, u_B) for which (5.15) has a solution connecting (ρ_B, u_B) to (ρ_∞, u_∞) is a neighborhood of (ρ_B, u_B). If $(\rho_B, u_B) \in \Omega_{II}$ this set is a union of an open set U in Ω_I and a curve in Ω_{III} through (ρ_B, u_B) which interset U. If $(\rho_B, u_B) \in \Omega_{III}$ this set of (ρ_∞, u_∞) near (ρ_B, u_B) consists of a curve through (ρ_B, u_B) and if $(\rho_B, u_B) \in \Omega_{IV}$ this set consists of a curve in Ω_{III} through (ρ_B, u_B). If $(\rho_B, u_B) \in \Omega_V$ no point $(\rho_\infty, u_\infty) \in \Omega_V$ can be connected by a solution of (5.15) from (ρ_B, u_B).

5.3. Lagrangian Isentropic Gas Dynamics. Finally, we consider the system of gas dynamics in Lagrangian coordinates

$$\begin{aligned} &\partial_t v_t - \partial_x u = 0, \\ &\partial_t u + \partial_x \left(\frac{1}{v}\right) = 0, \end{aligned} \tag{5.16}$$

where u is the velocity and $v > 0$ is the specific density. The eigenvalues of the system (5.16) are

$$\lambda_1 = -\frac{1}{v} < 0, \qquad \lambda_2 = \frac{1}{v} > 0; \tag{5.17}$$

hence the boundary $x = 0$ is not characteristic.

The purpose of this section is to provide an explicit formula for the boundary layer set associated with the Lax-Friedrichs scheme. The boundary layer equation takes the form

$$\begin{aligned} &\lambda(u(y+1) + u(y)) - 2\,\lambda\,u_\infty + v(y+1) - v(y) = 0 \\ &\lambda\left(\frac{1}{v(y+1)} + \frac{1}{v(y)}\right) - 2\,\frac{\lambda}{v_\infty} - u(y+1) + u(y) = 0 \end{aligned} \tag{5.18}$$

with

$$(v(0), u(0)) = (v_B, u_B), \qquad (v, u)(\infty) = (v_\infty, u_\infty). \tag{5.19}$$

We restrict attention to $v_B > \delta > 0$ for fixed δ, and we determine the set of (v_∞, u_∞) for which (5.18) has a solution. We set

$$w(y) = \frac{v(y)}{\lambda} \tag{5.20}$$

so that (5.18) becomes

$$\begin{aligned} &\frac{1}{w(y+1)} + \frac{1}{w(y)} - u(y+1) + u(y) = \frac{2}{w_\infty} \\ &w(y+1) - w(y) + u(y+1) + u(y) = 2\,u_\infty. \end{aligned}$$

Adding the two equalities, we get

$$w(y+1) + \frac{1}{w(y+1)} + \frac{1}{w(y)} - w(y) + 2u(y) = \frac{2}{w_\infty} + 2\,u_\infty.$$

Setting

$$N(y) = -2\,u(y) + 2\,u_\infty - \frac{1}{w(y)} + \frac{2}{w_\infty} + w(y),$$

we obtain a quadratic equation for $w(y+1)$:

$$w^2(y+1) - N(y)\,w(y+1) + 1 = 0. \tag{5.21}$$

Therefore

$$w(y+1) = \frac{1}{2}\left(N(y) \pm \left(N(y)^2 - 4\right)^{1/2}\right)$$

from which we get an expression for $u(y+1)$ as well:

$$u(y+1) = \frac{\lambda}{2}\,N(y) \pm \frac{\lambda}{2}\,(N(y)^2 - 4)^{1/2}. \tag{5.22}$$

Observe that $N(\infty) = w_\infty + 1/w_\infty$, where $w_\infty = v_\infty/\lambda$ and $N(\infty)^2 - 4 = (w_\infty - 1/w_\infty)^2$. The product of the two roots of (5.21) is equal to one. Stability requires $w_infty > 1$ so we choose the larger root in (5.22). We have finally from (5.22) and (5.18).

$$\begin{aligned} v(y+1) &= \frac{\lambda}{2}N(y) + \frac{\lambda}{2}(N(y)^2-4)^{1/2} \\ u(y+1) &= 2u_\infty - u(y) + \frac{v(y)}{\lambda} - \frac{N(y)}{2} - \frac{1}{2}(N(y)^2-4)^{1/2}. \end{aligned} \tag{5.23}$$

The Jacobian of the R.H.S. of (5.23) at (v_∞, v_∞) is easily seen to be

$$A(v_\infty, u_\infty) = \begin{pmatrix} \dfrac{w_\infty^2+1}{w_\infty^2-1} & \dfrac{-2\lambda w_\infty^2}{w_\infty^2-1} \\ \dfrac{-2}{\lambda(w_\infty^2-1)} & \dfrac{w_\infty^2+1}{w_\infty^2-1} \end{pmatrix},$$

whose eigenvalues are

$$a_1 = \frac{w_\infty - 1}{w_\infty + 1}, \qquad a_2 = \frac{w_\infty + 1}{w_\infty - 1}.$$

In terms of v_∞, we have

$$a_1 = \frac{1 - \frac{\lambda}{v_\infty}}{1 + \frac{\lambda}{v_\infty}}, \qquad a_2 = \frac{1 + \frac{\lambda}{v_\infty}}{1 - \frac{\lambda}{v_\infty}}.$$

If the data for the Lax-Friedrichs scheme are chosen such that the v component is bounded away from zero, then so is the approximate solution. Hence we can restrict attnetion to $v_\infty > \delta'$ for some $\delta' > 0$. For λ small enough, we have

$$0 < a_1 < 1 \quad \text{and} \quad a_2 > 1,$$

and Theorem 3.10 applies. We deduce that the set of all states (v_∞, u_∞) near (v_B, u_B) for which (5.18)-(5.19) has a solution is a curve passing through (v_B, u_B).

6. Concluding Remarks.

Given a family of sets such as those introduced in this paper, we can formulate the boundary condition for the hyperbolic problem. When the solutions u under consideration are functions of bounded variation, the traces exist in a strong sense and one can require that

$$u(0+, t) \in \mathcal{E}(u_B(t)), \qquad t > 0, \tag{6.1}$$

holds for all, except countably many, t. This type of regularity has been recently proven by Amadori by the front tracking scheme and for the family of sets $\mathcal{E}_{\text{Godunov}}$ $(= \mathcal{E}^{\text{layer}}_{\text{Godunov}} = \mathcal{E}^{\text{entropy}}_{\text{Godunov}})$.

When considering L^∞ solutions constructed by the vanishing viscosity method, the boundary condition

$$\text{supp } \nu_{0,t} \subset \mathcal{E}^{\text{entropy}}_{\text{viscosity}} \tag{6.2}$$

has been rigorously derived in Theorem 2.1. When the method of compensated compactness applies [12], an existence theorem for the boundary-value problem (1.1)–(1.3), (6.2) follows immediatly from Theorem 2.1. Such a result is satisfactory provided the condition (6.2) yields, for simple enough initial and boundary data at least, a well-posed problem. This is the case for the scalar equations and the linear systems, but more difficult to answer for systems.

The formulation based on boundary layers may not be appropriate as it is when the boundary is characteristic. On the hand, the formulation based on entropy inequality seems to capture all of the features in the solution near the boundary, but it is more difficult to work with it analytically. Further study of the connection between the two sets for systems is in progress [26].

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