A nonlinear oblique derivative boundary value problem for the heat equation

Part 1: Basic existence results for the evolution problem and steady solutions

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ABSTRACT . – We study the heat equation $B_t - \Delta B = 0$ in the half-plane with the nonlinear oblique derivative condition $B_X = KBB_Z$ on the boundary, where (B_X, B_Z) are respectively the normal and the tangential derivatives of B.
This work is divided into two parts. In this first part, we introduce self-similar solutions which verify an elliptic equation with the same nonlinear boundary condition. The main part of this first paper concerns this self-similar problem. It is well-posed and its solution is smooth. About the evolution problem itself, we prove here the global existence of a weak solution for the Cauchy problem associated to it.

Key words: Nonlinear oblique derivative condition, heat equation, self-similar solution.

RÉSUMÉ . – Nous étudions l'équation de la chaleur $B_t - \Delta B = 0$ dans le demi-plan, avec la condition aux limites $B_X = KBB_Z$ sur la frontière. Il s'agit une condition oblique non linéaire, B_X et B_Z étant respectivement les dérivées normale et tangentielle de B sur cet axe.
Ce travail est divisé en deux parties. Dans cette première partie, nous introduisons des solutions autosemblables associées à ce problème.

Ces solutions autosemblables vérifient un système elliptique, avec la même condition aux limites non linéaire sur l'axe, et l'étude de ce problème constitue la majeure partie de ce premier article. Nous prouvons qu'il est bien posé et que sa solution est régulière. Puis, concernant le problème d'évolution proprement dit, nous montrons l'existence globale d'une solution faible pour le problème de Cauchy associé.

1 Introduction and main results

The main purpose of this paper is the study of the self-similar solutions and the long-time behaviour of the following parabolic boundary value problem:

$$(1.1)\qquad \begin{cases} B_t - \Delta B = 0 & (\mathbb{R}^2_+) \\ B_X - KBB_Z = 0 & (X=0) \\ \\ B(t,-\infty,X) = 1, \quad B(t,+\infty,X) = 0 \end{cases}$$

where $B(t,Z,X) \in \mathbb{R}$,
$\mathbb{R}^2_+ = \{(Z,X) \in \mathbb{R} \times \mathbb{R}_+\}$,
$K > 0$.

This system occurs in plasma physics and describes the penetration of a magnetic field in a plasma along a perfectly conducting electrode.

Let a plasma be injected in $\mathbb{R}^2_+$ through an anode located on the axis $\{X = 0\}$. The cathode will be ignored in this paper; indeed, we assume that if it is far enough it does not perturb the phenomenon we are interested in here, which occurs near the anode. This plasma is a fluid composed of electrons and ions. In this problem, we consider the magnetic field dynamics during the propagation of an electromagnetic wave in this medium. This propagation is modified by the presence of the light electron component in the medium. This phenomenon occurs in devices such as plasma switches ([4],[6]).

The domain is supposed here to be infinite ($\mathbb{R}^2_+$) since our aim is to study self-similar solutions. After the instant $t = 0$ we apply a constant magnetic field at $z = -\infty$ (the entry of the subset).

The derivation of the model is made in the frame of the Electron Magnetohydrodynamics ([6], [7]): this is a fluid model derived from Magnetohydrodynamics where we neglect the motion of ions and the inertia of electrons. The characteristics of the plasma are constant during the time and, for the sake of simplicity ([5]), we assume that its density n is uniform. Therefore, the propagation of the field will only be diffusive. Moreover, we assume that the magnetic field has only a component orthogonal to $\mathbb{R}^2_+$, denoted $\mathcal{B}$.

The electric field $\mathcal{E}$ can be expressed in function of $\mathcal{B}$ and, finally, we can reduce all the equations of our model to the following one, governing the unknown $\mathcal{B}$.

The conservation of momentum for the electrons coupled with Maxwell equations gives a scalar equation on $\mathcal{B}(t, Z, X)$:

$$\mathcal{B}_t - \frac{1}{\mu_0 \sigma} \Delta \mathcal{B} = 0 \qquad \forall (t, Z, X) \in \mathbb{R}^*_+ \times \mathbb{R}^2_+ \tag{1.2}$$

where μ_0 is the permeability of free space
σ is the conductivity of the plasma (isotropic and uniform here)

At $Z \to -\infty$ we have the source of magnetic field $\mathcal{B} = \mathcal{B}_0$ ($\forall t \geq 0$). Equation (1.2) describes the diffusive penetration of $\mathcal{B}$ in the plasma if we let the dimensions tend to the infinity. The key point here is the influence of the electrode, through which the plasma is injected. Since it is a perfect conductor, on $X = 0$ the electric field $\mathcal{E}$ is orthogonal to the surface so it follows from the relation between $\mathcal{E}$ and $\mathcal{B}$:

$$\mathcal{B}_X - \frac{\sigma}{en} \mathcal{B} \mathcal{B}_Z = 0 \qquad \text{for } X = 0 \text{ and } \forall (t, Z) \in \mathbb{R}^*_+ \times \mathbb{R} \tag{1.3}$$

Next we rescale the equations (1.2)-(1.3) to obtain the parabolic system (1.1) of $B = \mathcal{B}/\mathcal{B}_0$, with the non dimensional constant $K = \dfrac{\sigma \mathcal{B}_0}{en}$. Formal studies of this system were made by the authors of [5] in the case where the diffusion along z is suppressed.

In the study of quasilinear parabolic systems, much has been realized in the last decade ([1], [2], [10]). Unfortunately, this problem (1.1) does not fit into these papers' assumptions on the nonlinearity of the boundary term.

To avoid inhomogeneous boundary condition at $Z \to +\infty$, let us introduce the function $\beta(t, Z)$ solution of

$$\begin{cases} \dfrac{\partial \beta}{\partial t} - \dfrac{\partial^2 \beta}{\partial Z^2} = 0 \\ \beta(t, -\infty) = 1, \quad \beta(t, +\infty) = 0 \\ \beta(t = 0) = \dfrac{1}{2\sqrt{\pi}} \displaystyle\int_Z^{+\infty} e^{-\frac{\sigma^2}{4}} d\sigma \end{cases}$$

This choice of the initial condition for β is not very important for the study of this evolution problem but was made here to be compatible further with the self-similar problem and with the function ψ. We can express

$$\beta(t, Z) = \frac{1}{2\sqrt{\pi}} \int_{Z/\sqrt{t+1}}^{+\infty} e^{-\frac{\sigma^2}{4}} d\sigma$$

If B is the solution of (1.1) then $b = B - \beta$ is homogeneous at the infinity and is solution of

$$\text{(1.4)} \qquad \begin{cases} b_t - \Delta b = 0 & (\mathbb{R}^2_+) \\ b_X - K(b+\beta)(b+\beta)_Z = 0 & (X = 0) \\ b(t, Z, X) \to 0 \ \text{ if } |Z| \text{ or } X \to +\infty \end{cases}$$

Let us introduce now a steady problem linked to (1.1). We will observe indeed that the solution of this evolution problem approaches as $t \to +\infty$ a self-similar solution U. In the reduced variables $z = \frac{Z}{\sqrt{t+1}}$ and $x = \frac{X}{\sqrt{t+1}}$, $U(z, x)$ is exactly the solution of the non homogeneous elliptic boundary value problem:

$$\text{(1.5)} \qquad \begin{cases} -\Delta U - \dfrac{1}{2}(zU_z + xU_x) = 0 & (\mathbb{R}^2_+) \\ U_x - KUU_z = 0 & (x = 0) \\ U(-\infty, x) = 1, \quad U(+\infty, x) = 0 \end{cases}$$

As we did above, we introduce the function $\psi(z)$ from the real variable, solution of

$$\begin{cases} -\psi_{zz} - \frac{1}{2} z \psi_z = 0 \\ \psi(-\infty) = 1; \psi(+\infty) = 0 \end{cases}$$

We have the analytical expression $\psi(z) = \dfrac{1}{2\sqrt{\pi}} \displaystyle\int_z^{+\infty} e^{-\frac{\sigma^2}{4}} d\sigma$ and $\psi(z) = \beta(t, Z)$.
Moreover $u = U - \psi$ verifies

$$(1.6) \qquad \begin{cases} -\Delta u - \dfrac{1}{2}(z u_z + x u_x) = 0 & (\mathbb{R}^2_+) \\ u_x - K(u+\psi)(u+\psi)_z = 0 & (x = 0) \\ \\ u \to 0 \ \text{ if } |z| \text{ or } x \to +\infty \end{cases}$$

This work is divided into two papers.

In the present part, we study the self-similar problem (1.5), (1.6) and the global existence of a solution for the Cauchy problem associated to the evolution problem (1.1), (1.4).

In the second part, we will give further results about the evolution problem and study the stability of the self-similar solutions.

The main results of this first part are contained in the following theorems and proposition:

THEOREM 1.1 (existence of the self-similar solution) *There exists a unique weak solution $u \in H^{3/2}(\mathbb{R}^2_+)$ of (1.6). Moreover we have the following properties:*

- $\exists C > 0$ *such that* $0 \le u(z,x) \le C \exp(-\frac{z^2+x^2}{8})$
- $u + \psi = U$ *is decreasing with respect to* z *and* x

THEOREM 1.2 (smoothness of the self-similar solution) *Let u the solution of (1.6) with $K < \sqrt{3}$. Then $u \in C^\infty(\overline{\mathbb{R}^2_+})$.*

PROPOSITION 1.1 (global solution of the Cauchy problem) *Let $b_0 \in L^2(\mathbb{R}^2_+)$ such that $0 \le b_0(Z,X) + \beta(0,Z) \le 1$.*
Let $T > 0$. There exists a solution b of (1.4) in the sense of distributions with the initial condition b_0 such that $b \in L^2(0,T; H^1(\mathbb{R}^2_+)) \cap \mathcal{C}^0([0,T]; L^2(\mathbb{R}^2_+))$. Moreover, we have $0 \le b + \beta \le 1$.

In the following two sections, we study the solution of (1.6). First we deal with its existence in $H^{3/2}(\mathbb{R}^2_+)$ and show Theorem 1.1: we define a regularized oblique derivative problem, whose solution converges to the solution of (1.6). Then the C^∞ regularity of u and Theorem 1.2 can be proved. In the forth part we come back to the parabolic problem (1.1). We define weak solutions and prove Proposition 1.1, then numerical simulations indicate that we cannot expect much more regularity for this problem.

2 Existence in $H^{3/2}$ and uniqueness of the selfsimilar solution

We first regularize the system (1.6). We prove that this new problem has a solution, then with a priori estimates we can pass to the limit. Finally we prove the uniqueness.

2.1 Regularized problem

If $v \in H^1(\mathbb{R}^2_+)$, we denote by γv its trace on the axis $x = 0$ (sometimes v when there is no ambiguity).
Then introduce a regularizing operator T^ε from $H^{1/2}(\mathbb{R})$ to $C^{1,\alpha}(\mathbb{R})$, where $0 < \alpha < 1$:
Let $v \in H^{1/2}(\mathbb{R})$ and $V(t,z)$ the solution of

$$\begin{cases} V(0,z) = v(z) \\ V_t - V_{zz} - \frac{1}{2} z V_z = 0 \end{cases}$$

then $T^\varepsilon v := V(\varepsilon, z)$. Thus $T^\varepsilon(\gamma u)$ (also denoted $T^\varepsilon u$) is defined as soon as $u \in H^1(\mathbb{R}^2_+)$ and we have the estimate :
$\|T^\varepsilon(\gamma u)\|_{C^{1,\alpha}(\mathbb{R})} \leq C_\varepsilon$ where C_ε depends only on ε and $\|u\|_{H^1(\mathbb{R}^2_+)}$.
Remark that, of course, $T^\varepsilon(v)$ is much more regular but we will not use this regularity. Remark also that T^ε is linear and that $T^\varepsilon \psi = \psi$. This allows us to write $T^\varepsilon(u + \psi) = T^\varepsilon u + \psi$; it is therefore equivalent to regularize the homogeneous solution (system (1.6)) or the non-homogeneous one (system (1.5)).

Define now a regularized problem associated to (1.6):

$$(2.1)\qquad \begin{cases} -\Delta u^\varepsilon - \dfrac{1}{2}(z u_z^\varepsilon + x u_x^\varepsilon) = 0 & (\mathbb{R}^2_+) \\ u_x^\varepsilon - K(T^\varepsilon u^\varepsilon + \psi)(u^\varepsilon + \psi)_z = 0 & (x = 0) \\ u^\varepsilon \to 0 \text{ at the infinity} \end{cases}$$

We will prove the existence of a smooth ($C^{2,\alpha}$) solution of this problem by the super/subsolution method. For a given ε, we have a uniform $C^{1,\alpha}$ estimate of the oblique vector $(-K(T^\varepsilon u^\varepsilon + \psi), 1)$ in the boundary condition; this will give us a solution of (2.1). For that, we need preliminary lemmas.

Let us recall that $\overline{u}$ (resp. $\underline{u}$) is a super (resp.sub) solution of (1.6) if and only if there holds, in the H^1 sense:

$$(2.2)\qquad \begin{cases} -\Delta \overline{u} - \dfrac{1}{2}(z\overline{u}_z + x\overline{u}_x) \geq 0 & (\text{resp. } \leq 0) & (\mathbb{R}^2_+) \\ \overline{u}_x - K(\overline{u} + \psi)(\overline{u} + \psi)_z \leq 0 & (\text{resp. } \geq 0) & (x = 0) \end{cases}$$

Remark that 0 is an obvious subsolution (ψ is decreasing).

LEMMA 2.1 *There exists a supersolution $\Lambda(z,x)$ of (1.6) such that*

$$\forall \sigma > 1 \ \exists C_\sigma > 0 : \quad 0 \leq \Lambda(z,x) \leq \min\left(1 - \psi, C_\sigma \exp(-\frac{z^2 + x^2}{4\sigma})\right)$$

Proof . – We seek a function Λ in the form $\Lambda(z,x) = 2\psi(x)h(z)$ with

$$h(z) = \begin{cases} 1 - \psi(z) & \text{for } z \leq 0 \\ 1 - \psi(z) - az & \text{for } 0 \leq z \leq z_1 \\ \psi(z - z_0) & \text{for } z_1 \leq z \end{cases}$$

where z_1, z_0 and a are given.
These parameters z_1, z_0 and a will be chosen so that Λ is continuous:

$$1 - \psi(z_1) - az_1 = \psi(z_1 - z_0) \qquad [C1]$$

A straightforward calculation shows that the first inequality of (2.2) is verified as soon as the constants z_1, z_0 and a are positive and as

$$h'(z_1^+) - h'(z_1^-) \leq 0 \qquad [C2]$$

This last condition comes from the $-\Lambda''$ term and from the fact that Λ is only piecewise C^1.
The second inequality of (2.2) (the condition on the boundary $\{x = 0\}$) writes differently depending on the position of z, and we can find sufficient conditions:

- $z \leq 0$: this condition is automatically checked since $-\frac{1}{\sqrt{\pi}}(1-\psi) \leq 0$.
- $0 \leq z \leq z_1$: $\quad a \leq \frac{1}{2K\sqrt{\pi}}$ $\qquad [C3]$
  (with $[C1]$ and the convexity of ψ on $\mathbb{R}_+$)
- $z_1 \leq z$: $\begin{cases} z_1 \geq z_0 + 2\sqrt{\mathrm{Log}(2K)} & \text{if } K \geq 1/2 \\ z_1 = z_0 = 0 & \text{otherwise} \end{cases}$ $\qquad [C4]$

For $K \geq 1/2$ we can see that the following parameters are convenient to ensure $[C1], [C2], [C3], [C4]$:

$$\begin{cases} z_1 = 4K\sqrt{\pi} \\ z_0 = 4K\sqrt{\pi} - 2\sqrt{\mathrm{Log}(2K)} \\ a \text{ adjusted to ensure the continuity of } h \text{ in } z_1, \text{ i.e. } [C1] \end{cases}$$

For $K \leq 1/2$, $h(z)$ is more simple:

$$h(z) = \begin{cases} 1 - \psi(z) & \text{for } z \leq 0 \\ \psi(z) & \text{for } z \geq 0 \end{cases}$$

The majoration of Λ stated in the lemma follows directly from the properties of $\psi(z) = \dfrac{1}{2\sqrt{\pi}} \displaystyle\int_z^{+\infty} e^{-\frac{\tau^2}{4}} d\tau$. □

Write now a more general oblique derivative boundary problem, with a $C^{1,\alpha}(\mathbb{R})$ positive function $\eta(z)$:

$$(2.3) \qquad \begin{cases} -\Delta u - \dfrac{1}{2}(z u_z + x u_x) = 0 & (\mathbb{R}^2_+) \\ u_x - K\eta(u+\psi)_z = 0 & (x = 0) \\ u \to 0 \text{ at the infinity} \end{cases}$$

This linear problem has a unique $C^{2,\alpha}$ solution u and we can estimate $\|u\|_{C^{2,\alpha}}$ with $\|\eta\|_{C^{1,\alpha}}$.

Let $\rho(z,x) = \exp\left(\dfrac{z^2+x^2}{8}\right)$.

We define the usual weighted Sobolev space

$H^1_\rho(\mathbb{R}^2_+) = \left\{ v \in H^1(\mathbb{R}^2_+)/\rho v \in H^1(\mathbb{R}^2_+) \right\}$ and $\|v\|_{H^1_\rho(\mathrm{R}^2_+)} = \|\rho v\|_{H^1(\mathrm{R}^2_+)}$.

LEMMA 2.2 *Let $u \in C^{2,\alpha}(\overline{\mathbb{R}^2_+})$ solution of (2.3). Suppose that $\eta \leq \Lambda + \psi$. Then u verifies the following properties:*
(i) $0 \leq u \leq \Lambda$
(ii) if η is decreasing then there exists a constant C_0, independent of η such that $\|u\|_{H^1_\rho(\mathrm{R}^2_+)} \leq C_0$
(iii) if η is decreasing then $u + \psi$ is decreasing with respect to z

Proof . –
(i): let $w = \Lambda - u$. Since Λ is a supersolution of (1.6), on $\{x = 0\}$ we have, for $x = 0$, $\Lambda_x \leq K(\Lambda + \psi)(\Lambda + \psi)_z$. Moreover $(\Lambda + \psi)_z \leq 0$. Thus, with the hypothesis on η, it yields $\Lambda_x \leq K\eta(\Lambda + \psi)_z$.
Therefore w verifies the inequalities:

$$\begin{cases} -\Delta w - \dfrac{1}{2}(zw_z + xw_x) \geq 0 & (\mathbb{R}^2_+) \\ w_x - K\eta(w+\psi)_z \leq 0 & (x = 0) \end{cases}$$

The maximum principle coupled with Hopf's lemma implies that $w \geq 0$, since $w = 0$ at the infinity.
This implies $u \leq \Lambda$. The other inequality is also a consequence of the maximum principle, directly applied on (2.3).

(ii): Let $\tilde{u} = \rho u$. Apply *(i)* and the estimate of Λ obtained in Lemma 2.1 with $\sigma = 3/2$. There exists a constant $C > 0$ such that

$$0 \leq (1 + |z|)\ \tilde{u} \leq C \exp\left(-\frac{z^2+x^2}{8}\right) \tag{2.4}$$

Moreover a direct calculation shows that $\tilde{u}$ verifies

$$\begin{cases} -\Delta\tilde{u} + \dfrac{1}{2}\left(1 + \dfrac{1}{8}(z^2+x^2)\right)\tilde{u} = 0 & (\mathbb{R}^2_+) \\ \tilde{u}_x - K\eta\tilde{u}_z = -\dfrac{K}{4}z\eta\tilde{u} - \dfrac{K}{2\sqrt{\pi}}\eta\rho^{-1} & (x = 0) \end{cases} \tag{2.5}$$

Multiply by $\tilde{u}$, integrate and apply Green's formula. It comes

$$\begin{aligned}\int_{\mathbf{R}^2_+} |\nabla \tilde{u}|^2 \quad &+ \frac{1}{2}\int_{\mathbf{R}^2_+} \left(1+\frac{1}{8}(z^2+x^2)\right)\tilde{u}^2 \; dzdx \\ &= -\int_{\mathbf{R}} \tilde{u}_x \tilde{u}\end{aligned}$$

In this paper, $\int_{\mathbf{R}}$ are always integrals calculated on the axis $(z,x) \in \mathbb{R}\times\{0\}$. To estimate this integral in the above equality, use (2.4), (2.5) and the fact that η is decreasing and bounded by $\Lambda + \psi \leq 1$. It follows

$$\begin{aligned}-\int_{\mathbf{R}} \tilde{u}_x\tilde{u} \quad &= -K\int_{\mathbf{R}} \left(\eta\tilde{u}_z\tilde{u} - \frac{1}{4}z\eta\tilde{u}^2 - \frac{1}{2\sqrt{\pi}}\eta\rho^{-1}\tilde{u}\right) \\ &= K\int_{\mathbf{R}} \eta_z\left(\frac{\tilde{u}}{2}\right)^2 + K\int_{\mathbf{R}} \eta\left(\frac{z}{4}\tilde{u} + \frac{1}{2\sqrt{\pi}}\rho^{-1}\right)\tilde{u} \\ &\leq C_1\int_{\mathbf{R}} \tilde{u} \\ &\leq C_2\end{aligned}$$

Hence $\|\tilde{u}\|_{H^1(\mathbf{R}^2_+)} \leq C_0 \implies \|u\|_{H^1_\rho(\mathbf{R}^2_+)} \leq C_0$.
This constant C_0 is independent of $\|\eta\|_{C^{1,\alpha}}$ thus of ε.

(iii): It seems to be well-known that the solution u of (2.3) is in $H^2(\mathbb{R}^2_+)$: we are in the case of a regular oblique derivative boundary problem (η is smooth) and the unbounded coefficients of the elliptic equation are taken in account easily after integrations by parts as in:

$$-\int_{\mathbf{R}^2_+} (zu_z + xu_x)u = \int_{\mathbf{R}^2_+} u^2$$

Let $v = (u+\psi)_z$. With *(ii)*, this function is in $C^{1,\alpha}(\overline{\mathbb{R}^2_+}) \cap H^1(\mathbb{R}^2_+) \cap L^2_\rho(\mathbb{R}^2_+)$ and it verifies:

$$\text{(2.6)} \qquad \begin{cases} -\Delta v - \dfrac{1}{2}(zv_z + xv_x) - \dfrac{1}{2}v = 0 & (\mathbb{R}^2_+) \\ v_x - K\eta v_z = K\eta_z v & (x=0) \end{cases}$$

Even if v was in $C^{2,\alpha}$ (indeed, the regularity of v is not a limitation here since in fact η will be the regularization by T^ε of a certain function, so v will be as smooth as we want), we could not apply the maximum principle to this function: the right-hand side of the boundary term has the wrong sign.

We may however proceed with a truncation argument.

Let $S_\delta(v) = \int_0^v S'_\delta(t)dt$ a regularization of v^+ with

$$S'_\delta(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ t/\delta & \text{for } 0 \leq t \leq \delta \\ 1 & \text{for } t \geq \delta \end{cases}$$

Since $S'_\delta(v) \in H^1(\mathbb{R}^2_+)$, we can multiply (2.6) by $S'_\delta(v)$ and integrate by parts. It comes

$$\begin{aligned} \int_{\mathbf{R}^2_+} [\frac{1}{2} S_\delta(v) \quad &+ |\nabla v|^2 (S''_\delta(v)) - \frac{1}{2}(vS'_\delta(v) - S_\delta(v)) \ ] \ dzdx \\ &= -K \int_{\mathbf{R}} \eta_z (vS'_\delta(v) - S_\delta(v)) \ dz \end{aligned}$$

Remark that $|(vS'_\delta(v) - S_\delta(v)| \leq \frac{1}{2} \min(v^+, \delta)$, that $\forall t \ S''_\delta(t) \leq 0$ and that $v \in L^1(\mathbb{R}^2_+)$ (which is a consequence of $v \in L^2_\rho(\mathbb{R}^2_+)$). Then, since we assume here that $\eta_z \leq 0$,

$$\begin{aligned} \frac{1}{2} \int_{\mathbf{R}^2_+} S_\delta v \quad &\leq \frac{1}{4} \int_{\mathbf{R}^2_+} \min(v^+, \delta) - K \frac{\delta}{2} \int_{\mathbf{R}} \eta_z \\ &\leq \frac{1}{4} \int_{\mathbf{R}^2_+} \min(v^+, \delta) + K \frac{\delta}{2} \end{aligned}$$

Let $\delta \to 0$.
We deduce from dominated convergence that $\int_{\mathbf{R}^2_+} v^+ = 0 \quad \Rightarrow \quad v \leq 0$. $\square$

We can prove now the following existence theorem:

PROPOSITION 2.1 *The regularized problem (2.1) admits a solution u^ε in $C^{2,\alpha}(\overline{\mathbb{R}^2_+})$ that verifies:*

$$\|u^\varepsilon\|_{H^1_\rho(\mathbf{R}^2_+)} \leq C_0 \quad \textit{where } C_0 \textit{ is independent of } \varepsilon \tag{2.7}$$

$$0 \leq u^\varepsilon(z,x) \leq \Lambda(z,x) \tag{2.8}$$

$$u^\varepsilon + \psi \textit{ is decreasing with respect to } z \tag{2.9}$$

$$u^\varepsilon \textit{ is decreasing with respect to } x \tag{2.10}$$

Proof . – Apply the super/subsolution method.
Let the $C^{2,\alpha}$ sequence u^k defined by: $u^0 \equiv 0$ and $\forall k \geq 0$, u^{k+1} is the solution of

$$\begin{cases} -\Delta u^{k+1} - \dfrac{1}{2}(zu_z^{k+1} + xu_x^{k+1}) = 0 & (\mathbb{R}^2_+) \\ u_x^{k+1} - K\eta^k(u^{k+1} + \psi)_z = 0 & (x = 0) \end{cases}$$

where $\eta^k = T^\varepsilon u^k + \psi$.
The definition of T^ε as the solution of a well-chosen parabolic problem and the property $-\Lambda_{zz} - \frac{1}{2}z\Lambda_z \geq 0$ on $\{x = 0\}$ imply that

$$\forall \varepsilon > 0 \quad T^\varepsilon \Lambda \leq \Lambda$$

and that

$$\left(\begin{array}{l} 0 \leq u^k \leq \Lambda \\ u^k + \psi \text{ is decreasing along } z \end{array} \right) \Longrightarrow \left(\begin{array}{l} 0 \leq \eta^k \leq \Lambda + \psi \\ \eta^k \text{ is decreasing along } z \end{array} \right)$$

So, recurrently, Lemma 2.2 shows that these properties are verified $\forall k \geq 0$.
It also gives the property $\|u^k\|_{H^1_\rho(\mathrm{R}^2_+)} \leq C_0$.
Prove now that the sequence u^k is increasing.
Let $w^{k+1} = u^{k+1} - u^k$.
We have $w^1 = u^1 \geq 0$. Since T^ε is linear, this function verifies, for $k \geq 0$,

$$\begin{cases} -\Delta w^{k+1} - \dfrac{1}{2}(zw_z^{k+1} + xw_x^{k+1}) = 0 & (\mathbb{R}^2_+) \\ w_x^{k+1} = K\eta^{k-1}w_z^{k+1} + KT^\varepsilon w^k(u^{k+1} + \psi)_z & (x = 0) \end{cases}$$

Suppose that $w^k \geq 0$.
Then $KT^\varepsilon w^k(u^{k+1} + \psi)_z \leq 0$ and w^{k+1} verifies

$$w_x^{k+1} - K\eta^{k-1}w_z^{k+1} \leq 0 \qquad (x = 0)$$

We deduce from the maximum principle and Hopf's lemma that $w^{k+1} \geq 0$.
Therefore, recurrently we prove that $u^k \leq u^{k+1}$ and obtain finally

$$0 \leq u^1 \leq u^2 \leq ... \leq u^k \leq u^{k+1} \leq ... \leq \Lambda$$

Moreover, with $\|u^k\|_{H^1(\mathrm{R}^2_+)} \leq C$ and the definition of T^ε, if yields

$$\|\eta^k\|_{C^{1,\alpha}(\mathbf{R})} \leq C_\varepsilon \text{ and } \|u^k\|_{C^{2,\alpha}(\mathrm{R}^2_+)} \leq C'_\varepsilon$$

where C'_ε depends on ε but not on k. The last estimate comes from the properties of the regular problem (2.3).
Hence we can pass to the limit as $k \to +\infty$ and get $u^\varepsilon \in C^{2,\alpha'}(\overline{\mathbb{R}^2_+})$ solution of (2.1). We also pass to the limit for the properties (2.8) and (2.9) and apply Lemma 2.2 once again, with $\eta = T^\varepsilon u^\varepsilon + \psi$ to obtain the property (2.7).

To prove (2.10), let $v^\varepsilon = u^\varepsilon_x$. This function verifies

$$\begin{cases} -\Delta v^\varepsilon - \dfrac{1}{2}(z v^\varepsilon_z + x v^\varepsilon_x) - \dfrac{1}{2} v^\varepsilon = 0 & (\mathbb{R}^2_+) \\ v^\varepsilon = K(u^\varepsilon + \psi)(u^\varepsilon + \psi)_z \leq 0 & (x = 0) \end{cases}$$

Multiply the first equation by $(v^\varepsilon)^+$ and integrate; with $(v^\varepsilon)^+_{(x=0)} = 0$ it comes $(v^\varepsilon)^+ \equiv 0$ on $\overline{\mathbb{R}^2_+}$. So u^ε is decreasing with respect to x. □

2.2 Further estimates

The crucial point that will allow us to pass to the limit is the following estimate.

PROPOSITION 2.2 *Let u^ε the solution of the regularized problem* (2.1). *There exists a constant C independent of ε such that*

$$\|u^\varepsilon\|_{H^{3/2}(\mathrm{R}^2_+)} \leq C \tag{2.11}$$

Proof . – We proceed in two steps: first we estimate $\|\gamma u^\varepsilon_z\|_{L^2]-\infty,a]}$ for $a \in \mathbb{R}$, then we extend to $\|u^\varepsilon\|_{H^{3/2}(\mathrm{R}^2_+)}$.
Step 1. $u^\varepsilon \in H^2(\mathbb{R}^2_+)$ so we can take u^ε_z as a test function in the variational formulation associated to (2.1):

$$\int_{\mathrm{R}^2_+} \nabla u^\varepsilon \nabla u^\varepsilon_z - \frac{1}{2}\int_{\mathrm{R}^2_+} (z u^\varepsilon_z + x u^\varepsilon_x) u^\varepsilon_z = -K \int_{\mathrm{R}} (T^\varepsilon u^\varepsilon + \psi)(u^\varepsilon_z + \psi_z) u^\varepsilon_z$$

Apply (2.7), (2.8), (2.9) and $\int_{\mathrm{R}^2_+} \nabla u^\varepsilon \nabla u^\varepsilon_z = 0$, it follows

$$K \int_{\mathrm{R}} (T^\varepsilon u^\varepsilon + \psi) {u^\varepsilon}^2_z \leq C$$

where C is independent of ε.
Since $\psi \leq (T^\varepsilon u^\varepsilon + \psi)$ and ψ decreases, we obtain

$$\forall a \in \mathbb{R} \qquad \|\gamma u^\varepsilon_z\|_{L^2(]-\infty,a])} \leq \frac{C}{\psi(a)} \tag{2.12}$$

Step 2. In this part we estimate u^ε_z on $[a, +\infty[$ thanks to (2.12). Let us first state two well-known lemma; the second one is a consequence of the first one.

LEMMA 2.3 *Let $g \in L^2(\mathbb{R})$ and $u \in H^{3/2}(\mathbb{R}^2_+)$ verifying*

$$\begin{cases} -\Delta u - \frac{1}{2}(zu_z + xu_x) = 0 & (\mathbb{R}^2_+) \\ u_x = g & (x = 0) \end{cases}$$

Then $\|u\|_{H^{3/2}(\mathrm{R}^2_+)} \leq C\|g\|_{L^2(\mathrm{R})}$

LEMMA 2.4 *Let $a > 0$ and $u \in H^{3/2}(\mathbb{R}^2_+)$ verifying $-\Delta u - \frac{1}{2}(zu_z + xu_x) = 0$. Then there exists a constant C such that*

$$\|u\|_{H^{3/2}(]a,+\infty[\times \mathrm{R}_+)} \leq C(\|u\|_{L^2(]a-1,+\infty[\times \mathrm{R}_+)} + \|\gamma u_x\|_{L^2(]a-1,+\infty[)}) \tag{2.13}$$

End of the proof of Proposition (2.2). – We have seen that $T^\varepsilon u^\varepsilon \leq \Lambda$, so with the decrease of Λ and ψ at $z \to +\infty$, there exists a constant C such that

$$\text{for } z \geq a - 1, \qquad K(T^\varepsilon u^\varepsilon + \psi) \leq Ce^{-a^2/8}$$

and

$$\|u^\varepsilon\|_{L^2(]a-1,+\infty[)\times \mathrm{R}_+} \leq Ce^{-a^2/8}$$

Thus if we replace γu^ε_x by $K(T^\varepsilon u^\varepsilon + \psi)(u^\varepsilon_z + \psi_z)$ in (2.13), it follows

$$\|u^\varepsilon\|_{H^{3/2}(]a,+\infty[\times \mathrm{R}_+)} \leq \left(C_1 + C_2\|\gamma u^\varepsilon_z\|_{L^2(]a-1,+\infty[)}\right) e^{-a^2/8}$$

Then use a trace theorem: $\|\gamma u_z^\varepsilon\|_{L^2(]a,+\infty[)} \leq C_3 \|u^\varepsilon\|_{H^{3/2}(]a,+\infty[\times \mathbf{R}_+)}$. This works because $\partial/\partial z$ is a tangential derivative.
We chose $a > 0$ such that $C_2 C_3 e^{-a^2/8} \leq 1/2$: it gives

$$\|u^\varepsilon\|_{H^{3/2}(]a,+\infty[\times \mathbf{R}_+)} \leq C(1 + \|\gamma u^\varepsilon\|_{L^2(]a-1,a[)})$$

With the result of *Step 1* we can conclude the proof. □

2.3 Passing to the limit

Let $\varepsilon \to 0$ and take (2.7), (2.8), (2.11): we can extract from u^ε a subsequence that converges toward u in $H^{3/2}(\mathbb{R}^2_+)$ weak, $L^2(\mathbb{R}^2_+)$ strong and $L^\infty(\mathbb{R}^2_+)$ weak *. Indeed, $H^1_\rho(\mathbb{R}^2_+)$ is compact in $L^2(\mathbb{R}^2_+)$.

Then let $\varphi \in C_c^\infty(\overline{\mathbb{R}^2_+})$. We know that $\|\gamma u^\varepsilon\|_{H^1(\mathbf{R})} \leq C$. Thus if $\Xi \subset \mathcal{O} \subset \mathbb{R}$ is the compact support of $\gamma\varphi$ and $\mathcal{O}$ is bounded then after another extraction, γu^ε converges to γu in $L^2(\mathcal{O})$ strong. Write the variational formulation of (2.1) applied to the test-function φ and let $\varepsilon \to 0$.
The crucial term is the nonlinear boundary term, that we write:

$$\int_{\mathbf{R}} (T^\varepsilon u^\varepsilon u_z^\varepsilon - u u_z)\varphi = \int_{\mathbf{R}} (T^\varepsilon u^\varepsilon - u^\varepsilon) u_z^\varepsilon \varphi + \int_{\mathbf{R}} (u^\varepsilon - u) u_z^\varepsilon \varphi + \int_{\mathbf{R}} u(u_z^\varepsilon - u_z)\varphi$$

These three integrals converge to zero:

- For the first one we use the fact that $T^\varepsilon \to \mathrm{Id}$ uniformly on every bounded subset of $H^{1/2}(\mathbb{R})$ and that $\gamma u_z^\varepsilon \varphi$ is bounded in $H^{-1/2}(\mathbb{R})$.

- For the second one, $\gamma u^\varepsilon \to \gamma u$ in $L^2(\mathcal{O})$ strong and $\gamma u_z^\varepsilon \varphi$ is bounded in $L^2(\Xi)$

- The third one converges to zero since $\gamma u_z^\varepsilon \to \gamma u_z$ in $L^2(\mathbb{R})$ weak.

We have proved then that

$$\forall \varphi \in C_c^\infty(\overline{\mathbb{R}^2_+})$$
$$\int_{\mathbf{R}^2_+} \nabla u \nabla \varphi - \frac{1}{2} \int_{\mathbf{R}^2_+} (z u_z + x u_x)\varphi = -K \int_{\mathbf{R}} (u + \psi)(u + \psi)_z \varphi$$

The second hand side makes sense since $\gamma(u+\psi)\varphi \in H^{1/2}(\mathbb{R})$ and $\gamma(u+\psi)_z \in H^{-1/2}(\mathbb{R})$ but we can also notice that

$$\frac{1}{2}\gamma[(u+\psi)^2]_z \in H^{-1/2}(\mathbb{R})$$

so we can extend by density to $\varphi \in H^1(\mathbb{R}^2_+)$, writing

$$\forall \varphi \in H^1(\mathbb{R}^2_+)$$
$$\int_{\mathbb{R}^2_+} \nabla u \nabla \varphi - \frac{1}{2}\int_{\mathbb{R}^2_+} (zu_z + xu_x)\varphi = -K\left\langle \frac{1}{2}[(u+\psi)^2]_z, \varphi \right\rangle_{H^{-1/2}(\mathrm{R}),H^{1/2}(\mathrm{R})}$$

Thus u is a weak solution of (1.6). We can also pass to the limit in (2.9), (2.10), (2.8) and take Lemma 2.1 with $\sigma = 2$ to prove the properties stated in theorem 1.1. To end the proof of this theorem, it remains to prove the uniqueness of the solution.

2.4 Uniqueness

In this section we end the proof of Theorem 1.1 and prove the uniqueness of the solution of (1.6) for two classes of functions : first for $u \in H^1(\mathbb{R}^2_+)$ such that $u+\psi$ is decreasing with respect to z; then for $u \in H^{3/2}(\mathbb{R}^2_+)$. The solution constructed above verifies both properties, but in section 4 we will only have the first one for the solution of the evolution problem (1.4).

Solutions in $H^1(\mathbb{R}^2_+)$ such that $(u+\psi)_z \leq 0$.
Let u_1 and u_2 two solutions of (1.6) that verify those two properties and let $w = u_1 - u_2$.
We proceed as for the proof of decrease, but with $R_\delta(.)$ a regularization of $|.|$ instead of S_δ. Define first R'_δ an odd function that writes on $\mathbb{R}^*_+$ as:

$$R'_\delta(t) = \begin{cases} t/\delta & \text{for } 0 < t \leq \delta \\ 1 & \text{for } t \geq \delta \end{cases}$$

$R'_\delta(w)$ is a $H^1(\mathbb{R}^2_+)$ function that approaches $\mathrm{sign}(w)$ and that we take as a test function in the variational formulation of the system verified by w.
Let $R_\delta(w) = \int_0^w R'_\delta(t)dt$; R_δ is even and

$$R_\delta(w) = \begin{cases} \dfrac{w^2}{2\delta} & \text{for } 0 \leq w \leq \delta \\ w - \dfrac{\delta}{2} & \text{for } w \geq \delta \end{cases}$$

After a calculation we obtain

$$\int_{\mathrm{R}^2_+} R''_\delta(w)\,|\nabla w|^2 + \int_{\mathrm{R}^2_+} R_\delta(w)$$
$$+\frac{K}{2}\left\langle (u_1+\psi+u_2+\psi)_z, (wR'_\delta(w)-R_\delta(w))\right\rangle_{H^{-1/2}(\mathrm{R}),H^{1/2}(\mathrm{R})} = 0$$

With the properties

$$\begin{aligned} &\forall t,\quad 0 \le tR'_\delta(t) - R_\delta(t) \le \frac{\delta}{2} \\ &(u_1+\psi+u_2+\psi)_z \le 0 \\ \text{and}\quad &(u_1+\psi+u_2+\psi)(-\infty) = 2;\ (u_1+\psi+u_2+\psi)(+\infty) = 0 \end{aligned}$$

We can prove that

$$-\frac{K}{2}\left\langle (u_1+\psi+u_2+\psi)_z, (wR'_\delta(w)-R_\delta(w))\right\rangle_{H^{-1/2}(\mathrm{R}),H^{1/2}(\mathrm{R})} \le \frac{K\delta}{2}$$

Moreover, $R''_\delta(w) \ge 0$ so $0 \le \displaystyle\int_{\mathrm{R}^2_+} R_\delta(w) \le C\delta$.
A simple study of the function R_δ shows that $R_\delta(w)$ is an increasing sequence of functions as $\delta \to 0$.
Since $R_\delta(w) \overset{\delta\to 0}{\longrightarrow} |w|$ a.e. and $\sup\limits_\delta \displaystyle\int_{\mathrm{R}^2_+} |R_\delta(w)| < +\infty$, Beppo-Levi theorem implies that $\|w\|_{L^1(\mathrm{R}^2_+)} = 0 \implies u_1 = u_2$.

Solutions in $H^{3/2}(\mathbb{R}^2_+)$
Now u_1 and u_2 are two solutions of (1.6) in $H^{3/2}(\mathbb{R}^2_+)$. Remark that we only suppose the $H^{3/2}$ regularity of these solutions; we do not know *a priori* that they are bounded by Λ or that $u_1+\psi$ or $u_2+\psi$ is decreasing with respect to z.
We proceed as above, but with :

$$R'_\delta(t) = \begin{cases} \left(\dfrac{t}{\delta}\right)^\delta & \text{for } 0 < t \le \delta \\ 1 & \text{for } t \ge \delta \end{cases}$$

$$R_\delta(w) = \begin{cases} \dfrac{w^{\delta+1}}{(1+\delta)\delta^\delta} & \text{for } 0 < w \le \delta \\ w - \frac{\delta^2}{1+\delta} & \text{for } w \ge \delta \end{cases}$$

We have now

$$\int_{\mathbf{R}^2_+} R''_\delta(w)\ |\nabla w|^2 + \int_{\mathbf{R}^2_+} R_\delta(w)\ + \frac{K}{2}\int_{\mathbf{R}} (u_1+\psi+u_2+\psi)_z (wR'_\delta(w)-R_\delta(w)) = 0$$

$\gamma(u_1+\psi+u_2+\psi)_z \in L^2(\mathbb{R})$ and $\gamma(wR'_\delta(w)-R_\delta(w)) \in L^2(\mathbb{R})$ so we can apply Schwartz's formula to the boundary term $(B.T.)$:

$$|B.T.| \leq \frac{K}{2}\left(\int_{\mathbf{R}} (u_1+\psi+u_2+\psi)_z^2 . \int_{\mathbf{R}} (wR'_\delta(w)-R_\delta(w))^2\right)^{1/2}$$

Remark that $\forall t\ 0 \leq tR'_\delta(t) - R_\delta(t) \leq \min\left(\frac{\delta^2}{(1+\delta)}, \frac{t^{1+\delta}}{(1+\delta)\delta^{\delta-1}}\right)$.

To control $\int_{\mathbf{R}} (wR'_\delta(w)-R_\delta(w))^2$, use now a balance between the two terms of the min function: let $(.)^2 = (.)^{2(1-\alpha)}(.)^{2\alpha}$ with $\alpha = 1/(1+\delta)$. It gives

$$\begin{aligned}\int_{\mathbf{R}} (wR'_\delta(w)-R_\delta(w))^2 &\leq \left(\frac{\delta^2}{(1+\delta)}\right)^{\frac{2\delta}{1+\delta}} \int_{\mathbf{R}} \left(\frac{w^{1+\delta}}{(1+\delta)\delta^{\delta-1}}\right)^{\frac{2}{1+\delta}} \\ &\leq \frac{\delta^2}{(1+\delta)^2}\int_{\mathbf{R}} w^2\end{aligned}$$

Hence $|B.T.| \leq C\delta \|(u_1+u_2+2\psi)_z\|_{H^{1/2}(\mathbf{R}^2_+)} . \|w\|_{H^{1/2}(\mathbf{R}^2_+)}$

Since $R''_\delta(w) \geq 0$, if we fix w then $0 \leq \int_{\mathbf{R}^2_+} R_\delta(w) \leq C\delta$.

The end of the proof is the same; we verify indeed that Beppo Levi theorem can be applied. □

3 Smoothness of the self-similar solution

In this part, we prove the C^∞ regularity of the solution u of (1.6) in the case $K < \sqrt{3}$. We proceed as follows:
First, we prove that $u \in H^2(\mathbb{R}^2_+)$, then that $\gamma u \in C^{1,\alpha}(\mathbb{R})$, finally we deduce the C^∞ regularity of u in $\mathbb{R}^2_+$ by an iterative method.

3.1 $H^2(\mathbb{R}^2_+)$ estimate

Let us first prove the lemma:

LEMMA 3.1 *Let $K < \sqrt{3}$ and u the solution of (1.6). Then there exists $C > 0$ such that $\|\gamma u_z\|_{L^3(\mathbf{R})} < C$.*

Proof . – We come back to the regularized problem (2.1): u^ε is a solution of this system. Since $u^\varepsilon \in H^2(\mathbb{R}^2_+) \cap C^{2,\alpha}(\overline{\mathbb{R}^2_+})$ the following functions are in $H^1(\mathbb{R}^2_+)$:

$$\begin{aligned}\varphi_1 &= (u^\varepsilon_z)^2\\ \varphi_2 &= (u^\varepsilon_x)^2\\ \varphi_3 &= (u^\varepsilon_z)(u^\varepsilon_x)\end{aligned}$$

Moreover, with Sobolev embeddings and $\|u^\varepsilon\|_{H^{3/2}(\mathbf{R}^2_+)} \leq C$, these functions are bounded in $L^2(\mathbb{R}^2_+)$, independently of ε.
Remark that in this lemma we deal with $\|\gamma u_z\|_{L^3(\mathrm{R})}$ so the problem doesn't change if we add $\psi_z = -\frac{1}{2\sqrt{\pi}}\exp(-\frac{z^2}{4})$ to u_z:
let $U = u + \psi$, solution of the non homogeneous problem (1.5) and $U^\varepsilon = u^\varepsilon + \psi$, solution of its regularized associated problem. We can write a variational formulation of (1.6):

$$\begin{aligned}&\forall \varphi \in H^1(\mathbb{R}^2_+)\\ &\int_{\mathbf{R}^2_+} \nabla u^\varepsilon \nabla \varphi - \frac{1}{2}\int_{\mathrm{R}^2_+} (z u^\varepsilon_z + x u^\varepsilon_x)\varphi + K\int_{\mathbf{R}} T^\varepsilon U^\varepsilon U^\varepsilon_z \varphi = 0\end{aligned} \tag{3.1}$$

Denote by (3.1.i) the equation (3.1) with $\varphi = \varphi_i$ for $i = 1, 2, 3$.
With (2.7), the functions zu^ε_z and xu^ε_x are bounded in $L^2(\mathbb{R}^2_+)$. Hence, for $i = 1, 2$ or 3, the second integral of (3.1.i): $\frac{1}{2}\int_{\mathbf{R}^2_+} (z u^\varepsilon_z + x u^\varepsilon_x)\varphi$ is bounded independently of ε.
For the first integrals, after developments and integrations by parts, it comes

$$\int_{\mathbf{R}^2_+} \nabla u^\varepsilon \nabla \varphi_2 - \int_{\mathrm{R}^2_+} \nabla u^\varepsilon \nabla \varphi_1 = -\frac{2}{3}\int_{\mathrm{R}} (u^\varepsilon_x)^3 = -\frac{2}{3}\int_{\mathbf{R}} (U^\varepsilon_x)^3$$

and

$$\int_{\mathbf{R}^2_+} \nabla u^\varepsilon \nabla \varphi_3 = -\frac{1}{2}\int_{\mathrm{R}} \left((U^\varepsilon_x)^2 u^\varepsilon_z + \frac{1}{3}(u^\varepsilon_z)^3 \right)$$

Since on $(x=0)$ we have $u_x^\varepsilon = U_x^\varepsilon = KT^\varepsilon U^\varepsilon U_z^\varepsilon$, the third integrals write:

$$K\int_{\mathrm{R}} T^\varepsilon U^\varepsilon \; U_z^\varepsilon \; (u_z^\varepsilon)^2 \quad (i=1)$$

$$K^3\int_{\mathrm{R}} (T^\varepsilon U^\varepsilon)^3 \; (U_z^\varepsilon)^3 \quad (i=2)$$

and

$$K^2\int_{\mathrm{R}} (T^\varepsilon U^\varepsilon)^2 \; (U_z^\varepsilon)^2 \; u_z^\varepsilon \quad (i=3)$$

Consider β a real coefficient and the linear combinaison (3.1.2) − (3.1.1) + 2β(3.1.3). We obtain

$$\int_{\mathrm{R}}\left(\frac{K^3}{3}(T^\varepsilon U^\varepsilon)^3(U_z^\varepsilon)^3 - K(T^\varepsilon U^\varepsilon)(u_z^\varepsilon)^2 U_z^\varepsilon + \beta K^2 (T^\varepsilon U^\varepsilon)^2 (U_z^\varepsilon)^2 u_z^\varepsilon - \frac{\beta}{3}(u_z^\varepsilon)^3\right)$$

$$= \frac{1}{2}\int_{\mathrm{R}_+^2} (zu_z^\varepsilon + xu_x^\varepsilon)(\varphi_2 - \varphi_1 + 2\beta\varphi_3)$$

Replace all the u_z^ε terms with $U_z^\varepsilon - \psi_z$ in this formula, and develop it. Remark then that γU_z^ε is bounded in $L^2(\mathbb{R})$, independently of ε. If we keep in the right-hand side only the third-order terms in U_z^ε, we finally obtain

$$\left|\int_{\mathrm{R}}\left(\frac{K^3}{3}(T^\varepsilon U^\varepsilon)^3 + \beta K^2 (T^\varepsilon U^\varepsilon)^2 - K(T^\varepsilon U^\varepsilon) - \frac{\beta}{3}\right)(U_z^\varepsilon)^3\right| \le C_{K,\beta}$$

which can be written

$$\left|\int_{\mathrm{R}} f_\beta(KT^\varepsilon U^\varepsilon)(U_z^\varepsilon)^3\right| \le C_{K,\beta}$$

with

$$f_\beta(y) = \frac{y^3}{3} + \beta y^2 - y - \frac{\beta}{3}$$

The study of the function f_β shows that

$$\text{If } K < \sqrt{3} \text{ then } \exists(\beta,\eta) \in (\mathbb{R}_+^*)^2 : \forall y \in [0,K] \;\; f_\beta(y) < -\eta$$

With this choice of (β,η) and the properties $U_z^\varepsilon \le 0$ and $0 \le KT^\varepsilon U^\varepsilon \le K$, it follows

$$\|U_z^\varepsilon\|_{L^3(\mathrm{R})}^3 \le \frac{C_{K,\beta}}{\eta}$$

This bound is independent of ε so we let $\varepsilon \to 0$, then write $u_z^\varepsilon = U_z^\varepsilon - \psi_z$ to conclude the proof of Lemma 3.1. □

To prove the H^2 regularity of the self-similar solution, we will use the Nirenberg translations method. For that, introduce some notations and a technical lemma:

$$\begin{aligned} \text{for } h \in \mathbb{R}^* \quad & \tau_h u(z,x) = u(z+h,x) \\ & D_h u = \frac{\tau_h u - u}{|h|} \end{aligned}$$

LEMMA 3.2 *D_h possesses the following properties:*
(i) D_h is linear and $\forall s,p$ $W^{s,p}(\mathbb{R}^2_+)$ is stable by D_h
(ii) D_h and ∇ commute
(iii) $\forall u,v \quad D_h(u^2) = 2uD_hu + |h|(D_hu)^2$
$D_h(zu) = zD_hu + \frac{h}{|h|}\tau_h u$
$D_h(uv) = vD_hu + (D_hv)(\tau_h u)$
(iv) $\forall (u,v) \in (L^2(\mathbb{R}^2_+))^2 \quad \int_{\mathrm{R}^2_+} u.D_{-h}v = \int_{\mathrm{R}^2_+} D_h u.v$
(v) $\forall u \in H^1(\mathbb{R}^2_+) \quad \|D_hu\|_{L^2(\mathrm{R}^2_+)} \le \|\nabla u\|_{L^2(\mathrm{R}^2_+)}$
(vi) $\forall v \in W^{1,3}(\mathbb{R}) \quad \left|\int_{\mathbf{R}} v_z (D_h v)^2\right| \le \|v_z\|^3_{L^3(\mathbf{R})}$

Proof . – $(i)(ii)$ and (iii) are immediate and come from the expression of D_h after straightforward calculations.
(iv) is also obvious after an integration by parts.
(v) is proved in [3]
(vi) can be proved as (v): first for $v \in C^1(\mathbb{R})$, after an integration:

$$D_h v = \frac{h}{|h|}\int_0^1 v'(z+th)dt$$

then

$$\begin{aligned} \left|\int_{\mathbf{R}} v_z (D_h v)^2 dz\right| &\le \int_0^1 \left(\int_{\mathrm{R}} |v_z|.(v_z(z+th))^2 dz\right) dt \\ &\le \|v_z\|_{L^3(\mathbf{R})} \left(\int_{\mathrm{R}} (v_z(z+th))^3 dz\right)^{2/3} \\ &\qquad\qquad \text{with Hölder inequality} \\ &\le \|v_z\|^3_{L^3(\mathbf{R})} \end{aligned}$$

We conclude with a density argument. □

PROPOSITION 3.1 *Let $K < \sqrt{3}$ and u the solution of (1.6). Then there exists $C > 0$ such that $\|u\|_{H^2(\mathbf{R}^2_+)} < C$.*

Proof . – Let us first do the following remark:

Suppose that $\|u_z\|_{H^1(\mathbf{R}^2_+)} < C_1$; then γu is bounded in $H^{3/2}(\mathbb{R})$. Thus, since u verifies $-\Delta u - \frac{1}{2}(zu_z + xu_x) = 0$ in $(\mathbb{R}^2_+)$, Agmon-Douglis-Nirenberg theorem gives us the estimate in $H^2(\mathbb{R}^2_+)$.

Therefore the main difficulty here is to estimate u_z in $H^1(\mathbb{R}^2_+)$. As in Lemma 3.1, $U_z = u_z + \psi_z$, where U is the solution of the non homogeneous problem (1.5). We have

$$\begin{aligned} &\forall \varphi \in H^1(\mathbb{R}^2_+) \\ &\int_{\mathbf{R}^2_+} \nabla u \nabla \varphi - \frac{1}{2}\int_{\mathbf{R}^2_+} (zu_z + xu_x)\varphi + \frac{K}{2}\int_{\mathrm{R}} (U^2)_z \varphi = 0 \end{aligned} \tag{3.2}$$

The idea of the proof is to take $\varphi = u_{zz}$ in (3.2), but we only know that $u_{zz} \in H^{-1/2}(\mathbb{R}^2_+)$: it is not an acceptable test function. To avoid this difficulty, we use the Nirenberg translations method:
With Lemma 3.2 (i)

$$\varphi = D_{-h}D_h u \in H^1(\mathbb{R}^2_+)$$

Then, with the properties $(i), (ii), (iii)$ from this lemma, we calculate and estimate the different terms of (3.2):

$$\int_{\mathbf{R}^2_+} \nabla u \nabla \varphi = \int_{\mathrm{R}^2_+} (D_h(\nabla u))^2$$

$$\left| -\frac{1}{2}\int_{\mathbf{R}^2_+} xu_x\varphi \right| = \frac{1}{4}\int_{\mathrm{R}^2_+} (D_h u)^2 \leq \frac{1}{4}\|\nabla u\|^2_{L^2(\mathrm{R}^2_+)} \text{ with } (v)$$

$$\begin{aligned} \left| -\frac{1}{2}\int_{\mathbf{R}^2_+} zu_z\varphi \right| &= \left| \frac{1}{4}\int_{\mathrm{R}^2_+} (D_h u)^2 + \frac{h}{2|h|}\int_{\mathrm{R}^2_+} u_z D_{-h}u \right| \\ &\leq \frac{1}{4}\|\nabla u\|^2_{L^2(\mathbf{R}^2_+)} + \frac{1}{2}\|u_z\|_{L^2(\mathbf{R}^2_+)}\ \|D_{-h}u\|_{L^2(\mathbf{R}^2_+)} \\ &\qquad \text{with } (v) \text{ and Cauchy-Schwarz formula} \\ &\leq \frac{3}{4}\|\nabla u\|^2_{L^2(\mathbf{R}^2_+)} \end{aligned}$$

$$\begin{aligned}
\frac{K}{2}\int_{\mathrm{R}}(U^2)_z\varphi \quad &= \frac{K}{2}\int_{\mathrm{R}}(U^2)_z D_{-h}D_hU - \frac{K}{2}\int_{\mathrm{R}}(U^2)_z D_{-h}D_h\psi \\
&= -\frac{K}{2}\int_{\mathbf{R}} 2U(D_hU)(D_hU)_z - \frac{K}{2}|h|\int_{\mathbf{R}}(D_hU)^2\cancel{(D_hU)_z} \\
&\quad +\frac{K}{2}\int_{\mathrm{R}}(U^2)D_{-h}D_h\psi_z \\
&= \frac{K}{2}\int_{\mathbf{R}} U_z(D_hU)^2 + \frac{K}{2}\int_{\mathbf{R}}(U^2)D_{-h}D_h\psi_z \\
\left|\frac{K}{2}\int_{\mathrm{R}}(U^2)_z\varphi\right| \quad &\le \frac{K}{2}\|\gamma U_z\|^3_{L^3(\mathbf{R})} + C\|\gamma U\|^2_{L^2(\mathrm{R})} \\
&\qquad\qquad \text{with Lemmas 3.1, } 3.2(vi)
\end{aligned}$$

Finally

$$\int_{\mathbf{R}^2_+}(D_h(\nabla u))^2 \le \|\nabla u\|^2_{L^2(\mathbf{R}^2_+)} + \frac{K}{2}\|\gamma U_z\|^3_{L^3(\mathbf{R})} + C\|\gamma U\|^2_{L^2(\mathbf{R})} \le C'$$

As in [3], we can deduce from this inequality that $\|\nabla u_z\|_{L^2(\mathbf{R}^2_+)} \le C$.
We conclude as explained at the beginning of the proof. □

REMARK 3.1 *Case* $K \ge \sqrt{3}$

If $K \ge \sqrt{3}$, with the same technique, we can prove that the solution U of (1.5) is in H^2 over $\mathbb{R}^2_+$ except near the points of the boundary where $U = \frac{\sqrt{3}}{K}$, thus in $H^2(\mathbb{R}^2_+ \setminus \Upsilon_\omega)$, where $\omega > 0$ is as small as we want and

$$\Upsilon_\omega = \left\{ z \in \mathbb{R}^2_+ \ / \left| U(z,0) - \frac{\sqrt{3}}{K} \right| \le \omega \right\} \times [0,\omega]$$

Let us only sketch the proof: we proceed with the same two steps, $L^3(\mathbb{R})$ then $H^2(\mathbb{R}^2_+)$. For the first one (Lemma 3.1), we take the truncation function $\chi_1(z) \in C^\infty(\mathbb{R})$:

$$\begin{cases}
\chi_1(z) = 1 & \text{if } U(z,0) \ge \frac{\sqrt{3}}{K} + \omega' \\
\chi_1(z) = 0 & \text{if } -\omega \le U(z,0) - \frac{\sqrt{3}}{K} \le \omega \\
\chi_1(z) = -1 & \text{if } U(z,0) \le \frac{\sqrt{3}}{K} - \omega'
\end{cases}$$

where $\omega' > \omega$; we take the test functions $\chi_1\varphi_i$ for $i = 1,2,3$ instead of φ_i.

For the second step, we multiply U by another truncation function χ_2 that vanishes on Υ_ω and takes the value 1 on $\mathbb{R}^2_+ \setminus \Upsilon_{\omega'}$. The system verified by $\chi_2 U$ has the same high-order terms as the one verified by U; the presence of χ_2 creates only low-order terms (one or zero for the equation in $\mathbb{R}^2_+$ and zero for the boundary condition).

3.2 Smoothness of the solution

Two lemmas will lead us to the conclusion.

LEMMA 3.3 *Let $K < \sqrt{3}$ and u the solution of (1.6). If $\gamma u_z \in L^\infty(\mathbb{R})$ then $u \in C^{1,\alpha}(\overline{\mathbb{R}^2_+})$.*

Proof . – We shall prove that $u \in H^3(\mathbb{R}^2_+)$ which implies the result with Sobolev embeddings.
With Proposition 3.1 we already have $u \in H^2(\mathbb{R}^2_+)$.
Let $v = -u_z$ and $V = -U_z = v - \psi_z$; the assumption of the lemma is $L = \|\gamma v\|_{L^\infty(\mathrm{R})} < +\infty$.
v is solution of the system :

$$\text{(3.3)} \qquad \begin{cases} -\Delta v - \dfrac{1}{2}(zv_z + xv_x) - \dfrac{1}{2}v = 0 & (\mathbb{R}^2_+) \\ v_x = KUV_z - KV^2 & (x = 0) \end{cases}$$

Thus

$$\forall \varphi \in H^1(\mathbb{R}^2_+)$$
$$\int_{\mathrm{R}^2_+} \nabla v \nabla \varphi \; - \frac{1}{2}\int_{\mathrm{R}^2_+} (zv_z + xv_x)\varphi \; - \frac{1}{2}\int_{\mathrm{R}^2_+} v\varphi \; + K\int_{\mathrm{R}} (UV_z - V^2)\varphi = 0$$

Remark that all the terms are well defined.
As in last section, use the Nirenberg translations and take $\varphi = D_{-h}D_h v \in H^1(\mathbb{R}^2_+)$. A calculation similar as before, with the technical lemma 3.2, gives:

$$\int_{\mathrm{R}^2_+} \nabla v \nabla \varphi = \int_{\mathrm{R}^2_+} (D_h(\nabla v))^2$$

and

$$\left| -\frac{1}{2}\int_{\mathrm{R}^2_+} (zv_z + xv_x + v)\varphi \right| \leq \frac{1}{2}\|\nabla v\|^2_{L^2(\mathrm{R}^2_+)}$$

The new term is the boundary term:

$$K\int_{\mathbf{R}}(UV_z - V^2)\varphi = \cdots = -K\underbrace{\int_{\mathbf{R}} vD_h uD_h v_z}_{I_1} - \frac{K}{2}\underbrace{\int_{\mathbf{R}}\tau_h v(D_h v)^2}_{I_2} + I_3$$

where I_3 is a lower order term in u, v, also involving ψ and its derivative; this term is easily estimated.
We can estimate the two integrals I_1 and I_2 with the assumption on L:

$$\begin{aligned}
|I_2| &\leq L\int_{\mathbf{R}}(D_h v)^2 \\
&\leq CL\|D_h v\|^2_{H^{1/2}(\mathrm{R}^2_+)} \qquad \text{with a trace theorem} \\
&\leq \frac{CL}{2A}\|D_h v\|^2_{H^1(\mathbf{R}^2_+)} + \frac{C'LA}{2}\|D_h v\|^2_{L^2(\mathrm{R}^2_+)} \quad \forall A > 0 \\
&\leq \frac{L}{2}(\frac{C}{A} + C'A)\|\nabla v\|^2_{L^2(\mathrm{R}^2_+)} + \frac{C'L}{2A}\|\nabla D_h v\|^2_{L^2(\mathbf{R}^2_+)} \\
&\qquad \text{with Lemma 3.2 } (v)
\end{aligned}$$

$$\begin{aligned}
|I_1| &= \langle vD_h u, D_h v_z\rangle_{H^{1/2}(\mathbf{R}),H^{-1/2}(\mathbf{R})} \\
&\leq C\|vD_h u\|_{H^1(\mathbf{R}^2_+)}\|D_h v_z\|_{L^2(\mathrm{R}^2_+)}
\end{aligned}$$

v is a $H^1(\mathbb{R}^2_+)$ function that verifies

$$-\Delta v - \frac{1}{2}(zv_z + xv_x) - \frac{1}{2}v = 0$$

so since $\|\gamma v\|_{L^\infty}(\mathbb{R}) \leq L$, it yields $\|v\|_{L^\infty}(\mathbb{R}^2_+) \leq L$ (indeed, proceed as in the proof of Lemma 2.2, with a truncation argument).
Moreover, since $v = -u_z$, we have $\|D_h u\|_{L^\infty(\mathbf{R}^2_+)} \leq L$.

Therefore $\|vD_h u\|_{H^1(\mathrm{R}^2_+)} \leq C_L\|v\|_{H^1(\mathbf{R}^2_+)}$ and, since this last norm is controlled,

$$|I_1| \leq C_1\|\nabla D_h v\|_{L^2(\mathrm{R}^2_+)}$$

Finally

$$\begin{aligned}
&\forall A > 0 \\
&\|\nabla D_h v\|^2_{L^2(\mathrm{R}^2_+)} \leq \quad C_1\|\nabla D_h v\|_{L^2(\mathbf{R}^2_+)} + C_2(A)\|v\|^2_{H^1(\mathrm{R}^2_+)} \\
&\qquad\qquad + \frac{C_3}{A}\|\nabla D_h v\|^2_{L^2(\mathbf{R}^2_+)} + C_4
\end{aligned}$$

We chose $A = 2C_3$, thus $\|\nabla D_h v\|^2_{L^2(\mathrm{R}^2_+)} - 2C_1\|\nabla D_h v\|_{L^2(\mathrm{R}^2_+)} - C_5 \leq 0$. Remark that all these constants are independent of h.

It comes $\|\nabla D_h v\|^2_{L^2(\mathrm{R}^2_+)} \leq C$.
This implies $v_z \in H^1(\mathbb{R}^2_+)$, then γu_z is in $H^{3/2}(\mathbb{R}) \Longrightarrow \gamma u \in H^{5/2}(\mathbb{R})$.
Therefore $\gamma u \in C^{1,\alpha}(\mathbb{R})$ □

REMARK 3.2 Although we did not let appear above the dependence of C on $L = \|\gamma u_z\|_{L^\infty(\mathrm{R})}$ and on $\|u\|_{H^2(\mathrm{R}^2_+)}$, it is easy to obtain the following result:
Let Ξ a compact subset of $[0, \sqrt{3}[$, if $K \in \Xi$ and u^K is the solution of (1.6), then $\|u^K\|_{H^2(\mathrm{R}^2_+)}$ is uniformly bounded on Ξ.
Moreover, there exists $C(L, \Xi)$ such that if $K \in \Xi$ and $|\gamma u_z\|_{L^\infty(\mathrm{R})} \leq L$ then $\|u^K\|_{C^{1,\alpha}(\overline{\mathrm{R}^2_+})} \leq C(L, \Xi)$.

LEMMA 3.4 *Let $K < \sqrt{3}$.*
Then the solution u of (1.6) verifies $\|\gamma u_z\|_{L^\infty(\mathrm{R})} < \infty$.

Proof . – For $K < \sqrt{3}$ denote by u^K the unique solution of (1.6) corresponding to K and $M(K) = \|\gamma u^K_z\|_{L^\infty(\mathrm{R})} \in [0, +\infty]$.

We will prove the lemma by the contrary: assume that $\exists K_1 \in [0, \sqrt{3}[$ such that $M(K_1) = +\infty$.

Define $K_0 = \inf\{K \in [0, \sqrt{3}[\ / \ M(K) = +\infty\}$: with the assumption on K_1 we have $K_0 \leq K_1 < \sqrt{3}$. Moreover, the implicit functions theorem applied to our system in a neighbourhood of $(u = 0, K = 0)$ implies that for K small enough the solution u^K is very smooth, so it is bounded (indeed, this function vanishes at the infinity since $u^K_z \in H^1(\mathbb{R}^2_+)$). Therefore $K_0 > 0$.

Let $K < K_0$. For a given K $M(K) < \infty$ so Lemma 3.3 implies $u^K \in C^{1,\alpha}(\overline{\mathbb{R}^2_+})$. We will prove now that $(M(K))_{K<K_0}$ is not bounded.

Indeed, if it is bounded by a constant C_0, then with the Remark (3.2) and if we take $\Xi = [0, K_0]$ and $L = C_0$,

$$\|u^K\|_{C^{1,\alpha}(\overline{\mathrm{R}^2_+})} \text{ is uniformly bounded for } K < K_0$$

and

$$\|u^K\|_{H^2(\mathbf{R}^2_+)} \text{ is uniformly bounded for } K \leq K_0$$

It follows that $u^K \xrightarrow{K \to K_0} u^{K_0}$ in $C^{1,\alpha'}(\overline{\mathbb{R}^2_+})$ so u^{K_0} is $C^{1,\alpha'}$, where $\alpha' < \alpha$. Hence $u^{K_0} \in C^{2,\alpha}(\overline{\mathbb{R}^2_+})$ since it verifies now a regular elliptic boundary value problem. We can apply again the implicit functions theorem to (u^{K_0}, K_0) to find that there exists $\eta > 0$ such that on $]K_0 - \eta, K_0 + \eta[$ the system (1.6) has a regular $(C^{2,\alpha})$ solution.

This leads to a contradiction with the definition of K_0, so $(M(K))_{K<K_0}$ *is not bounded.*

The remainder of the proof relies on a blowup technique.
Define the sequence

$$K_n \xrightarrow{n \to +\infty} K_0 \text{ with } 0 < K_n < K_0$$

such that

$$M(K_n) \xrightarrow{n \to +\infty} +\infty$$

Then denote $v_n = -u_z^{K_n}$

$\lambda_n = (M(K_n))^{-1} \xrightarrow{n \to +\infty} 0$

and z_n such that $v_n(z_n, 0) = \max_{\mathbf{R}}(\gamma v_n) = M(K_n)$

Remark that $v_n \to 0$ at the infinity so $|z_n| < +\infty$.
Now rescale the function v_n:

$$w_n(z, x) = \lambda_n v_n(z_n + \lambda_n z, \lambda_n x)$$

We will prove the following assertions:

(i) $w_n(0, 0) = 1$
(ii) $w_n \longrightarrow 0$ in $H^1(\mathbb{R}^2_+)$ weak
$(iii) \exists C : \|\gamma w_n\|_{H^{3/2}(\mathbf{R})} \leq C$

These three properties lead to a contradiction. Indeed with the Sobolev embeddings *(iii)* implies that a subsequence of w_n converges to a limit w in $C^{0,\alpha}$. Then with *(i)* and *(ii)* we have both $w \equiv 0$ and $w(0, 0) = 1$.
This contradiction ends the proof :

$$\forall K < \sqrt{3} \;\; M(K) < +\infty$$

It remains to prove the three assertions.
(i) is immediate
(ii) comes from $\|v_n\|_{H^1(\mathrm{R}^2_+)} \leq \|u_n\|_{H^2(\mathrm{R}^2_+)} \leq C$ and

$$\begin{cases} \|w_n\|_{L^2(\mathrm{R}^2_+)} = \|v_n\|_{L^2(\mathrm{R}^2_+)} \leq C \\ \|\nabla w_n\|_{(L^2(\mathrm{R}^2_+))^2} = -\lambda_n \|\nabla v_n\|_{(L^2(\mathrm{R}^2_+))^2} \leq C\lambda_n \end{cases}$$

The key point that will imply *(iii)* is that $\max_{\mathrm{R}}(\gamma w_n) = 1$ with this rescaling. We already know that $\|w_n\|_{H^1(\mathrm{R}^2_+)}$ is bounded but *(iii)* needs a new estimate. It fact it is nearly the same as for Lemma 3.3: from a L^∞ estimate of γw_n we obtain a $H^{3/2}(\mathbb{R})$ estimate of the same function, with a constant independent of n.

The proof is more simple here since, from Lemma 3.3, we know more regularity on w_n, at least that it is in $H^2(\mathbb{R}^2_+)$.
Let $\mu_n = \frac{\partial}{\partial z} w_n$ (we change here the notation for the derivative because of the n, which is only an index). This function is in $H^1(\mathbb{R}^2_+)$, its $L^2(\mathbb{R}^2_+)$ norm is bounded by a constant independent of n and it verifies

$$(3.4)\begin{cases} -\Delta\mu_n - \dfrac{\lambda_n^2}{2}(z\dfrac{\partial}{\partial z}\mu_n + x\dfrac{\partial}{\partial x}\mu_n) - \lambda_n^2\mu_n - \dfrac{\lambda_n z_n}{2}\dfrac{\partial}{\partial z}\mu_n = 0 & (\mathbb{R}^2_+) \\ \dfrac{\partial}{\partial x}\mu_n = K_n \check{U}_n \dfrac{\partial}{\partial z}\boldsymbol{\mu}_n - 3K_n W_n \boldsymbol{\mu}_n & (x=0) \end{cases}$$

In the boundary term, $\check{U}_{n(z,x)} = (u_n + \psi)_{(z_n + \lambda_n z, \lambda_n x)}$
$W_{n(z,x)} = (w_{n(z,x)} + \lambda_n \psi_{(z_n+\lambda_n z, \lambda_n x)})$
and $\boldsymbol{\mu}_{n(z,x)} = (\mu_{n(z,x)} + \lambda_n^2 \psi_{z(z_n+\lambda_n z, \lambda_n x)})$
Multiply by μ_n, integrate by parts and estimate the different terms like for the proof of lemma 3.3. It comes

$$\|\nabla\mu_n\|^2_{(L^2(\mathrm{R}^2_+))^2} \leq C$$

Thus

$$\begin{aligned} \|\mu_n\|_{H^1(\mathrm{R}^2_+)} \leq C \quad &\Longrightarrow \|\gamma\mu_n\|_{H^{1/2}(\mathrm{R})} \leq C' \\ &\Longrightarrow \|\gamma w_n\|_{H^{3/2}(\mathrm{R})} \leq C'' \end{aligned}$$

□

Proof of Theorem 1.2. – Let $K < \sqrt{3}$. With Lemma 3.4, $\gamma u_z \in L^\infty(\mathbb{R})$.
Hence with Lemma 3.3, $u \in C^{1,\alpha}(\overline{\mathbb{R}^2_+})$.
Thus we have already seen that it implies $u \in C^{2,\alpha}(\overline{\mathbb{R}^2_+})$.

Remark that because of the elliptic regularity, u is already C^∞ inside $\mathbb{R}^2_+$.
We have to prove the smoothness on the boundary.
Write the system verified by $V = U_z$:

$$\text{(3.5)} \qquad \begin{cases} -\Delta V - \dfrac{1}{2}(zV_z + xV_x) - \dfrac{1}{2}V = 0 & (\mathbb{R}^2_+) \\ V_x = KUV_z - KV^2 & (x = 0) \end{cases}$$

Since we know that U and V are in $C^{1,\alpha}(\overline{\mathbb{R}^2_+})$, this is a regular oblique derivative problem so $U_z \in C^{2,\alpha}(\overline{\mathbb{R}^2_+})$.
Hence the boundary condition of the system (1.5) verified by U, which is $U_x = KUU_z$, gives the $C^{2,\alpha}$ regularity of U_x on the boundary.

Finally in this step we proved that $U \in C^{3,\alpha}(\overline{\mathbb{R}^2_+})$.
Recurrently, by successive derivations of (1.5), we obtain the C^∞ regularity of U on $\overline{\mathbb{R}^2_+}$.

□

4 The evolution problem

We now return to the evolution problem (1.1) or to its homogeneous formulation (1.4). First we deal with the Cauchy problem and Proposition 1.1. Then numerical simulations seem to show that the solution of (1.1) is not as smooth as we want.

4.1 The Cauchy problem

To emphasize the difference between the two systems of variables (the initial one and the self-similar one), we introduce a new notation for the

initial variables (they were (t, Z, X) in the introduction for the sake of simplicity). This notation is not very useful in this section since we only deal with the initial problem (1.1), but will become essential in the second part of the paper.

Let (τ, η, ξ) the new variables for the initial evolution problem. This problem writes now, with $\mathbb{R}^2_+ = \{(\eta, \xi) \in \mathbb{R} \times \mathbb{R}_+\}$:

$$(4.1) \qquad \begin{cases} b_\tau - \Delta b = 0 & (\mathbb{R}^2_+) \\ b_\xi - K(b+\beta)(b+\beta)_\eta = 0 & (\xi = 0) \\ b(\tau, \eta, \xi) \to 0 \ \text{ if } |\eta| \text{ or } \xi \to +\infty \end{cases}$$

We recall that

$$\beta(\tau, \eta) = \frac{1}{2\sqrt{\pi}} \int_{\eta/\sqrt{\tau+1}}^{+\infty} e^{-\frac{\sigma^2}{4}} d\sigma$$

and that

$$\beta(0, \eta) = \psi(\eta)$$

Let b_0 an initial condition in $L^2(\mathbb{R}^2_+)$ such that $0 \leq b_0 + \psi \leq 1$. We will prove the existence of a global solution of the Cauchy problem associated to (4.1).

First we write a variational formulation for our problem, then we find a solution by the Faedo-Galerkin method as in [9] for the evolution Navier-Stokes problem.

The maximum principle applied to classical solutions of (4.1) shows that these solutions will be bounded. Hence we will seek some weak solutions in L^∞.

To prove the existence of a solution, we will write a variational formulation for our system (4.1), then we will regularize some terms in this formulation. For that purpose, let us transform the boundary integral into an integral over $\mathbb{R}^2_+$; we will see that this will allow us further to pass to the limit in the approximate system with less regularity than $H^{3/2}(\mathbb{R}^2_+)$. Indeed, write, for $b \in H^1(\mathbb{R}^2_+) \cap L^\infty(\mathbb{R}^2_+)$ and $v \in H^1(\mathbb{R}^2_+)$

$$\int_{\mathbf{R}} (b+\beta)(b+\beta)_\eta v = \int_{\mathbf{R}^2_+} (b+\beta)[(b+\beta)_\xi v_\eta - (b+\beta)_\eta v_\xi]$$

Hence, formally, if we multiply (4.1) by v and integrate, it comes

$$(4.2)\quad \int_{\mathrm{R}^2_+} b'v + \int_{\mathrm{R}^2_+} \nabla b\ \nabla v + K \int_{\mathrm{R}^2_+} (b+\beta)[(b+\beta)_\xi v_\eta - (b+\beta)_\eta v_\xi] = 0$$

where we denote by $'$ the derivative in time.
Let us now define the bilinear and trilinear forms:

$$\begin{aligned}
\forall (u,v) \in\ & \left(H^1(\mathbb{R}^2_+)\right)^2 \\
& f(u,v) = \int_{\mathrm{R}^2_+} \nabla u \nabla v + K \int_{\mathrm{R}^2_+} \beta(u_\xi v_\eta - u_\eta v_\xi) - K \int_{\mathrm{R}^2_+} \beta_\eta u v_\xi \\
\forall (u,v,w)\ & \in L^\infty(\mathbb{R}^2_+) \times \left(H^1(\mathbb{R}^2_+)\right)^2 \\
& g(u,v,w) = K \int_{\mathrm{R}^2_+} u(v_\xi w_\eta - v_\eta w_\xi)
\end{aligned}$$

Remark that

$$(4.3)\qquad f(u,u) = \int_{\mathrm{R}^2_+} |\nabla u|^2 - K \int_{\mathrm{R}^2_+} \beta_\eta u u_\xi$$

$$(4.4)\qquad g(u,v,v) = 0$$

We can prove that a variational formulation of (4.1) is:

$$(4.5)\quad \begin{cases} b \in L^2(0,T;H^1(\mathbb{R}^2_+)) \cap C^0(0,T;L^2(\mathbb{R}^2_+)) \\ 0 \le b+\beta \le 1 \\ \displaystyle\int_{\mathrm{R}^2_+} b'v + f(b,v) + g(b,b,v) = K \int_{\mathrm{R}^2_+} \beta\beta_\eta v_\xi \quad \forall v \in H^1(\mathbb{R}^2_+) \\ b(0) = b_0 \end{cases}$$

To solve (4.5), we will first solve a modified problem. Let Θ^ε the regularizing operator, with ρ^ε an approximation of the Dirac mass:

$$\Theta^\varepsilon u = \rho^\varepsilon * u$$

Then we modify (4.2) as follows:

$$\int_{\mathrm{R}^2_+} (b^\varepsilon)'v + \int_{\mathrm{R}^2_+} \nabla b^\varepsilon\ \nabla v + K \int_{\mathrm{R}^2_+} \Theta^\varepsilon(b^\varepsilon+\beta)[(b^\varepsilon+\beta)_\xi v_\eta - (b^\varepsilon+\beta)_\eta v_\xi] = 0$$

Hence, if

$$f^\varepsilon(u,v) = \int_{\mathrm{R}^2_+} \nabla u \nabla v + K \int_{\mathrm{R}^2_+} \Theta^\varepsilon(\beta)(u_\xi v_\eta - u_\eta v_\xi) - K \int_{\mathrm{R}^2_+} \Theta^\varepsilon(u)\beta_\eta v_\xi$$

and

$$g^\varepsilon(u,v,w) = K\int_{\mathbf{R}^2_+} \Theta^\varepsilon(u)(v_\xi w_\eta - v_\eta w_\xi)$$

the modified variational problem (4.5) becomes:

$$(4.6)\begin{cases} b^\varepsilon \in L^2(0,T;H^1(\mathbb{R}^2_+)) \cap C^0(0,T;L^2(\mathbb{R}^2_+)) \\ \displaystyle\int_{\mathbf{R}^2_+} (b^\varepsilon)' v + f^\varepsilon(b^\varepsilon, v) + g^\varepsilon(b^\varepsilon, b^\varepsilon, v) = K\int_{\mathbf{R}^2_+} \Theta^\varepsilon(\beta)\beta_\eta v_\xi \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall v \in H^1(\mathbb{R}^2_+) \\ b^\varepsilon(0) = b_0 \end{cases}$$

Note that we do not need any more to impose a L^∞ condition on $b^\varepsilon + \beta$.

Solving the modified problem (4.6)

As in [9], introduce a discretization of $H^1(\mathbb{R}^2_+)$ with $V_p = \mathrm{Vect}(w_1, \cdots, w_p)$. Then $b^{\varepsilon,p}(\tau) \in V_p$ verifies the ODE system:

$$(4.7)\begin{cases} \displaystyle\int_{\mathbf{R}^2_+} (b^{\varepsilon,p})' w_j + f^\varepsilon(b^{\varepsilon,p}, w_j) + g^\varepsilon(b^{\varepsilon,p}, b^{\varepsilon,p}, w_j) = K\int_{\mathbf{R}^2_+} \Theta^\varepsilon(\beta)\beta_\eta (w_j)_\xi \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \le j \le p \\ b^{\varepsilon,p}(\tau = 0) = b_0^p \end{cases}$$

where $b_0^p \in V_p$ and $b_0^p \to b_0$ in $L^2(\mathbb{R}^2_+)$ as $p \to +\infty$.

The solution of the discretized problem exists on a maximal interval that we will identify with $[0,T]$ after some estimates.
We have, indeed, the property $(*)$:

$b^{\varepsilon,p}$ remains in a bounded subset of $L^2(0,T;H^1(\mathbb{R}^2_+)) \cap L^\infty(0,T;L^2(\mathbb{R}^2_+))$

To prove that, and if $b^{\varepsilon,p}(\tau) = \displaystyle\sum_{1\le j\le p} \lambda_j(\tau) w_j$, let us multiply (4.6) by λ_j and sum on j. It comes, after an integration over τ between 0 and τ :

$$\begin{aligned} \frac{1}{2}\int_{\mathbf{R}^2_+} (b^{\varepsilon,p})^2 \quad &+ \int_0^\tau \int_{\mathbf{R}^2_+} |\nabla b^{\varepsilon,p}|^2 \\ &= \frac{1}{2}\int_{\mathbf{R}^2_+} (b_0^p)^2 + K\int_0^\tau \int_{\mathbf{R}^2_+} (\beta_\eta \Theta^\varepsilon(b^{\varepsilon,p})(b^{\varepsilon,p})_\xi + \Theta^\varepsilon(\beta)\beta_\eta (b^{\varepsilon,p})_\xi) \\ &\le \frac{1}{2}\int_{\mathbf{R}^2_+} (b_0^p)^2 + C\left\{\frac{A}{2} + \frac{A}{2}\int_0^\tau \int_{\mathbf{R}^2_+} (b^{\varepsilon,p})^2 + \frac{1}{A}\int_0^\tau \int_{\mathbf{R}^2_+} |\nabla b^{\varepsilon,p}|^2\right\} \end{aligned}$$

where we have used $\int_{\mathrm{R}^2_+} (\Theta^\varepsilon(u))^2 \le C \int_{\mathrm{R}^2_+} u^2$

This estimate is true for any $A > 0$. With a good choice of A, it comes

$$\int_{\mathbf{R}^2_+} (b^{\varepsilon,p})^2 \le C_1 + C_2 \int_0^\tau \int_{\mathrm{R}^2_+} (b^{\varepsilon,p})^2$$

and

$$\int_0^\tau \int_{\mathrm{R}^2_+} |\nabla b^{\varepsilon,p}|^2 \le C_3 + C_4 \int_0^\tau \int_{\mathbf{R}^2_+} (b^{\varepsilon,p})^2$$

The result $(*)$ follows after application of Gronwall's lemma on the first inequality.

To obtain compactness, we need another estimate, that we will obtain following the proof of [9]. Indeed, we prove that a fractional derivative in time of $b^{\varepsilon,p}$ is bounded.
Let $\tilde{b}^{\varepsilon,p}$ (resp. $\tilde{\beta}$) the function defined on $\mathbb{R} \times \mathbb{R}^2_+$ which coincides with $b^{\varepsilon,p}$ (resp. β) on $[0,T]$ and vanishes outside this interval. We have, $\forall j$

$$\begin{aligned}\int_{\mathbf{R}^2_+} (\tilde{b}^{\varepsilon,p})' w_j = \; & -\int_{\mathrm{R}^2_+} \nabla \tilde{b}^{\varepsilon,p} \, \nabla w_j \\ & -K \int_{\mathrm{R}^2_+} \Theta^\varepsilon(\tilde{b}^{\varepsilon,p} + \tilde{\beta})[(\tilde{b}^{\varepsilon,p} + \tilde{\beta})_\xi w_{j\eta} - (\tilde{b}^{\varepsilon,p} + \tilde{\beta})_\eta w_{j\xi}] \\ & -\delta_T \int_{\mathbf{R}^2_+} b^{\varepsilon,p}(T) w_j + \delta_0 \int_{\mathrm{R}^2_+} b^{\varepsilon,p}(0) w_j\end{aligned}$$

Let us denote by $\mathcal{F}$ the Fourier transform in time and by τ' the frequency variable associated to the time variable τ. It comes

$$\begin{aligned}2i\pi\tau' \int_{\mathbf{R}^2_+} \mathcal{F}(\tilde{b}^{\varepsilon,p}) w_j = \; & -\int_{\mathbf{R}^2_+} \mathcal{F}(\nabla \tilde{b}^{\varepsilon,p}) \, \nabla w_j \\ & -K \int_{\mathbf{R}^2_+} \mathcal{F}\left[\Theta^\varepsilon(\tilde{b}^{\varepsilon,p} + \tilde{\beta})(\tilde{b}^{\varepsilon,p} + \tilde{\beta})_\xi\right] w_{j\eta} \\ & +K \int_{\mathbf{R}^2_+} \mathcal{F}\left[\Theta^\varepsilon(\tilde{b}^{\varepsilon,p} + \tilde{\beta})(\tilde{b}^{\varepsilon,p} + \tilde{\beta})_\eta\right] w_{j\xi} \\ & -e^{-2i\pi T\tau'} \int_{\mathbf{R}^2_+} b^{\varepsilon,p}(T) w_j + \int_{\mathbf{R}^2_+} b_0^p w_j\end{aligned}$$

Since $\mathcal{F}(\tilde{b}^{\varepsilon,p}) \in V_p$ we can replace w_j by $\mathcal{F}(\tilde{b}^{\varepsilon,p})$ in this formula. Moreover, from $(*)$ and Parseval's formula we get, for a given σ:

$$\int_{-\infty}^{+\infty} \frac{1}{1+|\tau'|^\sigma} \|\mathcal{F}(\nabla \tilde{b}^{\varepsilon,p})\|^2_{(L^2(\mathbf{R}^2_+))^2} d\tau' \leq \|b^{\varepsilon,p}\|^2_{L^2(0,T;H^1(\mathbf{R}^2_+))} \leq C_1$$

$$\left| \int_{-\infty}^{+\infty} \frac{1}{1+|\tau'|^\sigma} \int_{\mathbf{R}^2_+} \mathcal{F}\left[\Theta^\varepsilon(\tilde{b}^{\varepsilon,p}+\tilde{\beta})(\tilde{b}^{\varepsilon,p}+\tilde{\beta})_\eta\right] \mathcal{F}(\tilde{b}^{\varepsilon,p})_\xi \, d\eta \, d\xi \, d\tau' \right| \leq C_2$$

$$\left| \int_{-\infty}^{+\infty} \frac{1}{1+|\tau'|^\sigma} \int_{\mathbf{R}^2_+} \mathcal{F}\left[\Theta^\varepsilon(\tilde{b}^{\varepsilon,p}+\tilde{\beta})(\tilde{b}^{\varepsilon,p}+\tilde{\beta})_\xi\right] \mathcal{F}(\tilde{b}^{\varepsilon,p})_\eta \, d\eta \, d\xi \, d\tau' \right| \leq C_2$$

where C_2 depends only on $\|b^{\varepsilon,p}\|_{L^2(0,T;H^1(\mathbf{R}^2_+))}$
and on $\|\Theta^\varepsilon(b^{\varepsilon,p}+\beta)\|_{L^\infty((0,T)\times\mathbf{R}^2_+)}$

$$\|b^{\varepsilon,p}(T)\|_{L^2(\mathbf{R}^2_+)} \leq \|b^{\varepsilon,p}\|_{L^\infty(0,T;L^2(\mathbf{R}^2_+))} \leq C_3$$

Thus, as in [9] if we take $\sigma > 1/2$, then we obtain

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{|\tau'|}{1+|\tau'|^\sigma} \|\mathcal{F}\tilde{b}^{\varepsilon,p}\|^2_{L^2(\mathbf{R}^2_+)} d\tau' &\leq C + C' \int_{-\infty}^{+\infty} \frac{1}{1+|\tau'|^\sigma} \|\mathcal{F}\tilde{b}^{\varepsilon,p}\|_{L^2(\mathbf{R}^2_+)} d\tau' \\ &\leq C + C'' \|b^{\varepsilon,p}\|_{L^2(0,T;L^2(\mathbf{R}^2_+))} \leq C'''\end{aligned}$$

This gives that $D_\tau^{1/4-\epsilon} b^{\varepsilon,p}$ is bounded in $L^2(0,T;L^2(\mathbb{R}^2_+))$ and we will apply a compacity lemma on bounded subsets included in $\overline{\mathbb{R}^2_+}$.

Finally, after extraction of a subsequence we have, as $\mu \to \infty$,

$$b^{\varepsilon,\mu} \to b^\varepsilon \qquad \text{in } L^2(0,T;H^1(\mathbb{R}^2_+)) \text{ weak and } L^\infty(0,T;L^2(\mathbb{R}^2_+)) \text{ weak } *$$

Moreover, for every $\mathcal{O}$ bounded subset in $\overline{\mathbb{R}^2_+}$, after another extraction of a subsequence,

$$\begin{array}{ll} b^{\varepsilon,\mu} \to b^\varepsilon & \text{in } L^2(0,T;L^2(\mathcal{O})) \text{ strong and a.e. in } \mathcal{O} \\ \text{so } \Theta^\varepsilon(b^{\varepsilon,\mu}) \to \Theta^\varepsilon(b^\varepsilon) & \text{in } L^2(0,T;L^\infty(\mathcal{O})) \text{ strong } (\varepsilon \text{ is fixed}) \end{array}$$

This implies that, if we have chosen w_j such that Supp w_j is compact in $\mathbb{R}^2_+$, then

$$g^\varepsilon(b^{\varepsilon,\mu}, b^{\varepsilon,\mu}, w_j) \to g^\varepsilon(b^\varepsilon, b^\varepsilon, w_j) \qquad \text{in } L^\infty(0,T) \text{ weak } *$$

We can pass to the limit and obtain, in $\mathcal{D}'(0,T)$:

$$\int_{\mathbf{R}^2_+} b^{\varepsilon\prime} w_j + f^\varepsilon(b^\varepsilon, w_j) + g^\varepsilon(b^\varepsilon, b^\varepsilon, w_j) = K \int_{\mathbf{R}^2_+} \Theta^\varepsilon(\beta)\beta_\eta(w_j)_\xi \qquad \forall j$$

Hence if we let $V_p \to H^1(\mathbb{R}^2_+)$ then b^ε is solution of (4.6) in the sense of distributions and, if V' is the dual space of $H^1(\mathbb{R}^2_+)$, we have $(b^\varepsilon)' \in L^2(0,T;V')$ directly from (4.6).

Solving the initial problem (4.5)

It remains to pass to the limit as $\varepsilon \to 0$. For that, we need a L^∞ estimate of b^ε.

Since $\beta_\tau - \beta_{\eta\eta} = 0$, if we take $v = (b^\varepsilon + \beta)^-$, then $v = (b^\varepsilon + \beta - 1)^+$ in (4.6), it comes

$$\begin{array}{lll} \forall \tau \in [0,T] & 0 \le b^\varepsilon + \beta \le 1 & \text{a.e. in } \mathbb{R}^2_+ \\ \text{then} & |\Theta^\varepsilon(b^\varepsilon)| \le C & \text{in } \overline{\mathbb{R}^2_+} \text{ for } C \text{ large enough} \end{array}$$

Moreover, we observe that the $L^2(0,T;H^1(\mathbb{R}^2_+)) \cap L^\infty(0,T;L^2(\mathbb{R}^2_+))$ estimate of b^ε is independent of ε.
Then, since with (4.6) $\|(b^\varepsilon)'\|_{L^2(0,T;V')}$ depends only on $\|b^\varepsilon\|_{L^2(0,T;H^1(\mathrm{R}^2_+))}$ and on $\|\Theta^\varepsilon(b^\varepsilon)\|_{L^\infty(0,T;L^\infty(\mathrm{R}^2_+))}$, this quantity does not depend on ε.

Thus we can extract a subsequence such that, as $\varepsilon \to 0$,

$$b^\varepsilon \to b \qquad \text{in } L^2(0,T;H^1(\mathbb{R}^2_+)) \text{ weak and } L^\infty(0,T;L^2(\mathbb{R}^2_+)) \text{ weak } *$$

$$b^\varepsilon \to b \qquad \text{in } L^\infty(0,T;L^\infty(\mathbb{R}^2_+)) \text{ weak } *$$

$$(b^\varepsilon)' \to b' \qquad \text{in } L^2(0,T;V') \text{ weak}$$

and for $\mathcal{O}$ bounded subset of $\overline{\mathbb{R}^2_+}$

$$\begin{array}{ll} b^\varepsilon \to b & \text{in } L^2(0,T;L^2(\mathcal{O})) \text{ strong and a.e. in } \mathcal{O} \\ \Theta^\varepsilon(b^\varepsilon) \to b & \text{in } L^2(0,T;L^2(\mathcal{O})) \text{ strong} \end{array}$$

Let $\varphi \in C^\infty_c(\overline{\mathbb{R}^2_+})$. The nonlinear term writes

$$\int_{\mathbf{R}^2_+} \Theta^\varepsilon(b^\varepsilon)(b^\varepsilon_\xi \varphi_\eta - b^\varepsilon_\eta \varphi_\xi)$$

We can pass to the limit and, in the sense of distributions:

$$\int_{\mathbf{R}^2_+} b'\varphi + f(b,\varphi) + g(b,b,\varphi) = K \int_{\mathbf{R}^2_+} \beta\beta_\eta\varphi_\xi$$

By density we extend to $\varphi \in H^1(\mathbb{R}^2_+)$. The continuity property in time of (4.5), thanks to $b' \in L^2(0,T;V')$, is a classical result.

b is solution of (4.5): the existence of a solution is proved. Remark that the boundary condition makes sense if we write it:

$$b_\xi = K \left[\frac{(b+\beta)^2}{2} \right]_\eta$$

4.2 Numerical simulations

This part deals with numerical simulations we have computed to get some information about the regularity of the solution of the evolution problem. Can we expect more regularity than $H^1 \cap L^\infty$? Is $\partial_\eta B$ bounded ?

Indeed, in the proof of the regularity of the self-similar solution (1.5) the crucial step was the $H^{3/2}(\mathbb{R}^2_+)$ estimate, obtained in Proposition 2.2.
For the parabolic problem, we cannot obtain directly the same estimate because of the $\partial_\tau b$ term.

We present here some simulations of the parabolic problem 1.1, i.e., the heat equation

$$\partial_\tau B - \Delta B = 0 \quad \text{in the rectangle } (\eta,\xi) \in [0,L]\text{x}[0,l]$$

with the following boundary conditions:

$$\begin{cases} \partial_\xi B = K\ B\ \partial_\eta B & \text{on } [0,L]\text{x}\{0\} \\ \partial_\xi B = 0 & \text{on } [0,L]\text{x}\{l\} \\ B = 1 & \text{on } \{0\}\text{x}[0,l] \\ B = 0 & \text{on } \{L\}\text{x}[0,l] \end{cases}$$

Scheme
Let us describe briefly the scheme we chose here. We use an explicit finite difference scheme.

A time iteration is made in two steps: first a prediction on $[\tau_k, \tau_{k+1/2}]$, then a correction on $[\tau_k, \tau_{k+1}]$ using the previously calculated values of B at $\tau_{k+1/2}$. For the space discretization, we take a special care to the η variable. The scheme for the heat equation is the most simple we can take with a five points Laplacian, the difficulty being the boundary condition on $[0, L]$x$\{0\}$. It can be seen as a Burgers equation, thus discretized with a nonlinear hyperbolic method [8]: we take a second order (in η) TVD slope-limiter scheme.

Numerical results
In the following, take the notations:

B is the solution of the continuous problem
B_k is the iterated solution, at the instant τ_k
D_η will be the discrete derivation with respect to η

Of course, B_k is only defined on the mesh. The long-time behaviour of the solution is well observed, with different initial data, but we will highlight a phenomenon that occurs in a finite time.

We consider here the behaviour of the function $B(\tau, \eta, 0)$, on the boundary $\eta \in [0, L]$, $\xi = 0$, and we observe its derivative in η. The simulations seem to show that this derivative *explodes* at a given point of the boundary in a finite time τ^*; then, for $\tau > \tau^*$, this function is smooth.

For the two first figures, the mesh is given.
The facts described above are difficult to observe on the function itself: Figure 1 shows the initial datum (solid line) and the solution B_k in function of η at different iterations: $\tau_k \in \{10, 20, 40, 60, 100\}$ with a renormalized time unity. We can observe the propagation towards the increasing η.

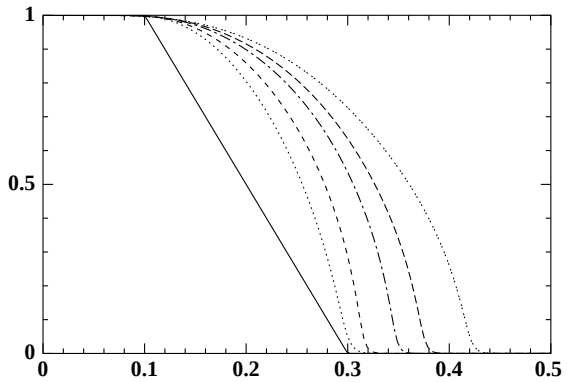

Figure 1 : $B_k(\tau_k, \eta, 0)$ *for* $\tau_k \in \{0, 10, 20, 40, 60, 100\}$

Consider now the derivative $D_\eta B_k(\tau_k, \eta, 0)$ and let $\eta^*(\tau_k)$ the point on the boundary where this function is extremum.
On figure 2 we have represented this derivative in function of η for the same iterations $\tau_k \in \{10, 20, 40, 60, 100\}$.

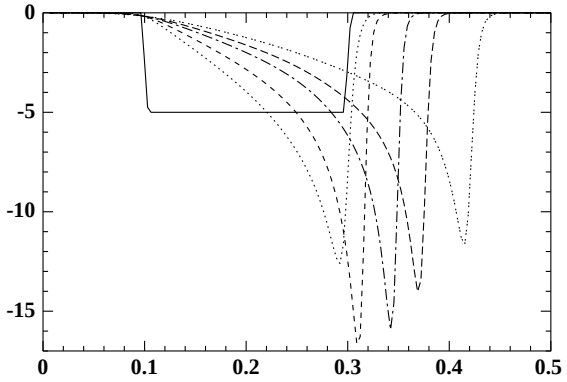

Figure 2 : $D_\eta B_k(\tau_k, \eta, 0)$ *for* $\tau_k \in \{0, 10, 20, 40, 60, 100\}$

For a given τ, as η goes from 0 to L, $D_\eta B_k$ first decreases and reaches a negative minimum in η^*, then increases to 0. Moreover, if we consider this minimum $D_\eta B_k(\tau, \eta^*(\tau_k), 0)$, it is of course negative, and reaches itself a

minimum for a given τ_k^*.

Let $M = |D_\eta B_k(\tau_k^*, \eta^*(\tau_k), 0)|$. We let now appear the influence of the mesh. We characterize the mesh here with $\Delta\eta$, the step for the discretization of $[0, L]$. The other steps $(\Delta\tau, \Delta\xi)$ follow with different CFL conditions. We denote here $M(\Delta\eta)$ the extremum of $D_\eta B_k$.

Now we let $\Delta\eta = L/N_\eta \to 0$.
We represent on Figure 3 the function $D_\eta B_n(\tau_n^*, \eta, 0)$ in function of η for different values of N_η, $(50, 75, 100, 200, 300, 400)$: the function becomes sharper as N_η is increased (the diffusive role of the mesh appears here). The Figure 4 shows $M(\Delta\eta) = |D_\eta B_k|(\tau_k^*, \eta^*, 0)$ versus the mesh width $\Delta\eta$.

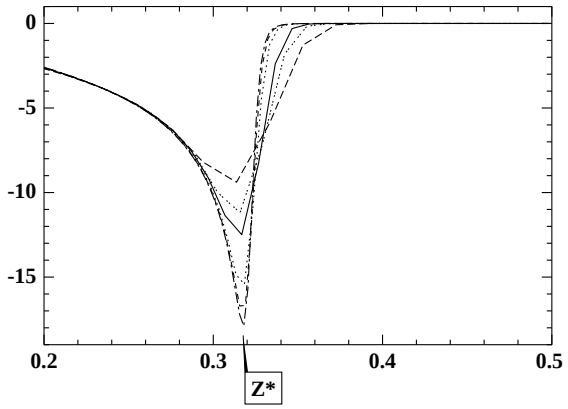

Figure 3 : $D_\eta B_k, (\tau_k^*, \eta, 0)$ *for* $N_\eta \in \{50, 75, 100, 200, 300, 400\}$

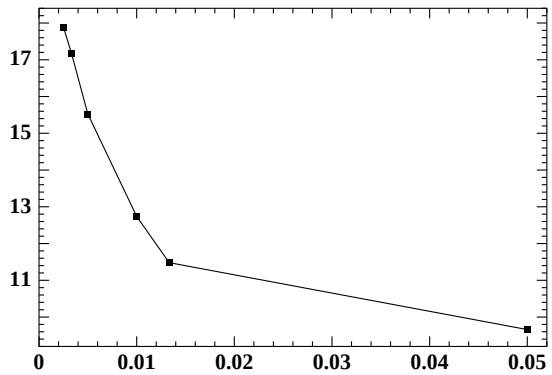

Figure 4 : *Maximum of the derivative of B versus the mesh width*

Figures 3 and 4 seem to show that $M(\Delta\eta) \to +\infty$ as $\Delta\eta \to 0$: the solution of the continuous problem seems to have an infinite derivative at $(\tau^*, \eta^*, 0)$.

We will conjecture that, for some initial data, there exists a pair (τ^*, η^*) such that $b_\eta(\tau^*, \eta^*, 0) = -\infty$ and for $\tau > \tau^*$ the solution $b(\tau)$ is regular.

5 Conclusion of the first part

In this first part of the paper we mainly studied the solution of the elliptic problem (1.5). We proved that this problem is well posed, and that for $K < \sqrt{3}$ its solution is $C^\infty(\overline{\mathbb{R}^2_+})$. If this condition is not fulfilled then we already know that the solution is in $H^{3/2}(\mathbb{R}^2_+)$. To go further we have to isolate the points on the boundary where the function takes the value $\frac{\sqrt{3}}{K}$ to prove the H^2 regularity, but after that the previous method seem to be not very adapted for the proof of the C^∞ regularity in this case.

About the initial evolution problem (1.1), we only proved in section 4 the existence of a global weak solution, i.e. in $H^1(\mathbb{R}^2_+) \cap L^\infty(\mathbb{R}^2_+)$. Numerical simulations indicate that we cannot expect much more regularity for this function.

Some problem linked to the maximum principle and the parabolic problem (1.1) such as the uniqueness of its solution will be treated in the second part of this paper. Our major task in this part will be the study of the long-time behaviour of this solution and the stability of the self-similar solution.

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