On the Cauchy problem for the full von Kármán System

Daniel Tataru*
School of Mathematics, Institute for Advanced Study
Princeton, NJ 08450, U.S.A.
and Northwestern University

Marius Tucsnak
Ecole Polytechnique, Centre de Mathématiques Appliquées,
91128 Palaiseau Cedex, France
and Université de Versailles

Abstract. *We consider here the Cauchy problem for the full system of dynamic Von Kármán equations, which is a model for the vibrations of a nonlinear elastic plate. We prove global existence and uniqueness of finite energy solutions in the case of an infinite plate. We show then that our methods and results still hold for a rectangular plate which is simply supported or clamped at the boundary. Moreover we obtain continuous dependence on the initial data.*

Key Words Nonlinear plate vibrations, Global existence, Semilinear evolution equation, Strichartz estimate
AMS Classification. Primary 35Q72, Secondary 73D35

1 Introduction

In this paper we study the existence and uniqueness of finite energy solutions of a semi-linear system of hyperbolic equations, modeling the free vibrations of an elastic plate. For this model, introduced in [6], the existence and uniqueness of strong solutions was proved in [11].

In order to state the problem consider the functions $u : \mathbf{R^2} \to \mathbf{R}^2$, $w : \mathbf{R^2} \to \mathbf{R}$ representing the in-plane, respectively the transverse displacement of

*Research supported by an Alfred P. Sloan fellowship, by the NSF grant DMS9400978 and by the NSF grant DMS9304580 to the IAS

the plate. Define the linear plane strain tensor as $\epsilon(u) = \frac{1}{2}(\nabla u + \nabla^t u)$ and denote by S the set of all second order symmetric tensors on $\mathbf{R}^2$. Define $\mathcal{C} \in \mathcal{L}(S)$ by

$$\mathcal{C}[\epsilon] = \frac{E}{d(1-\mu^2)}[\mu(tr\epsilon)I_{\mathcal{S}} + (1-\mu)\epsilon], \forall \epsilon \in S,$$

where I_S is the identity of S, d is the density, E is the Young's modulus and μ is the Poisson's ratio of the material. We shall suppose that the tensor $\mathcal{C}$ is strictly positive, i.e. it satisfies the condition:

$$\mathcal{C}[\epsilon] \cdot \epsilon \geq \lambda_0 |\epsilon|^2, \forall \epsilon \in S, \text{ where } \lambda_0 > 0. \tag{1.1}$$

This condition is verified if E and μ satisfy the usual assumptions in plate theory (cf.[6, p.13]: $E > 0$ and $0 < \mu < 1/2$.

With the above notations the full dynamic von Kármán system can be written in the following form (see [6, p.18]):

$$u'' - div\{\mathcal{C}[\epsilon(u) + f(\nabla w)]\} = 0 \quad \text{ in , } \mathbf{R}^2 \times (0,T) \tag{1.2}$$

$$w'' - \gamma\Delta w'' + D\Delta^2 w - div\{\mathcal{C}[\epsilon(u) + f(\nabla w)]\nabla w\} = 0 \quad \text{ in } \mathbf{R}^2 \times (0,T), \tag{1.3}$$

$$u(0) = u^0, u'(0) = u^1, w(0) = w^0, w'(0) = w^1 \quad \text{ in } \mathbf{R^2}, \tag{1.4}$$

where $f : \mathbf{R^2} \to S$ is defined by $f(x) = \frac{1}{2}x \otimes x$ and $D > 0$ represents the flexural rigidity of the plate. By g', g'' we denote the time derivatives of a function g.

Observe that, after formally taking the L^2 scalar product of (1.2) by u' and multiplying (1.3) by w', simple calculations show that the energy of the plate is conserved, i.e.

$$E(u(t), w(t)) = E(u(0), w(0)), \ \forall t \geq 0, \tag{1.5}$$

where

$$E(u(t), w(t)) = \frac{1}{2}\{||u'(t)||^2 + ||w'(t)||^2 + \gamma||\nabla w'(t)||^2 +$$

$$+||\Delta w(t)||^2 + (\mathcal{C}[\epsilon(u(t)) + f(\nabla w(t))], \epsilon(u(t)) + f(\nabla w(t)))\}, \tag{1.6}$$

and $||\cdot||$ and $(\cdot,\cdot)$ represent the norm, respectively the inner product, in $[L^2(\mathbf{R^2})]^k$, $k \geq 1$. Consequently, we say that a solution $\{u, w\}$ of (1.2)-(1.4) is a finite energy solution if

$$u \in C(0,T; [H_0^1(\mathbf{R^2})]^2) \cap C^1(0,T; [L^2(\mathbf{R^2})]^2), \tag{1.7}$$

$$w \in C(0,T; H_0^2(\mathbf{R^2})) \cap C^1(0,T; H_0^1(\mathbf{R^2})), \tag{1.8}$$

The main result of this paper is

Theorem 1.1 *Suppose that* $(u^0, u^1) \in [H_0^1(\mathbf{R^2})]^2 \times [L^2(\mathbf{R^2})]^2$ *and* $(w^0, w^1) \in H_0^2(\mathbf{R^2}) \times H_0^1(\mathbf{R^2})$. *Then the problem (1.2)-(1.4) admits a unique finite energy solution. (i.e. satisfying (1.7)-(1.8)).*

In the last section we prove, by using the same methods, that we have existence and uniqueness of finite energy solutions in the case of a rectangular plate (which is simply supported or clamped at the boundary).

We also show that the finite energy solutions of (1.2)-(1.4) depend continuously on the initial data. More precisely we have

Theorem 1.2 *Suppose that* (u,w) *and* $(\widetilde{u},\widetilde{w})$ *are finite energy solutions of (1.2)-(1.4) of initial energy at most* R*, defined for* $t\in[0,T]$*. Then there exists a constant* $C_R>0$ *such that*

$$\begin{aligned}
&\|u(t)-\widetilde{u}(t)\|_{[H^1(\mathbf{R^2})]^2}+\|u'(t)-\widetilde{u}'(t)\|_{[L^2(\mathbf{R^2})]^2}+\\
&+\|w(t)-\widetilde{w}(t)\|_{H^2(\mathbf{R^2})}+\|w'(t)-\widetilde{w}'(t)\|_{H^1(\mathbf{R^2})}\le\\
&\le(1+C_Rt^{1/p'})(\|u^0-\widetilde{u}^0\|_{[H^1(\mathbf{R^2})]^2}+\|u^1-\widetilde{u}^1\|_{[L^2(\mathbf{R^2})]^2}+\\
&+\|w^0-\widetilde{w}^0\|_{H^2(\mathbf{R^2})}+\|w^1-\widetilde{w}^1\|_{H^1(\mathbf{R^2})}),\ \forall t\in[0,T].
\end{aligned}\tag{1.9}$$

where $p'<\infty$*.*

A crucial intermediate step in the proof of these results is to show that any finite energy solution of (1.2)-(1.4) has a additional regularity property allowing us to prove uniqueness in a narrower class of functions

Theorem 1.3 *Let* $\{u,w\}$ *be solution of (1.2)-(1.4) satisfying (1.7), (1.8), i.e. a finite energy solution of (1.2)-(1.4). Then we have*

$$\nabla w\in L^p(0,T;L^\infty(\mathbf{R^2})),\ \forall p<\infty. \tag{1.10}$$

We shall also use the existence and uniqueness of strong solutions of (1.2)-(1.4), given by Theorem 1.4 bellow. This result is completely similar to those given in [11] for the initial and boundary value problem corresponding to a bounded plate, therefore the proof will be omitted.

Theorem 1.4 *Suppose that* $u^0\in[H^2(\mathbf{R}^2)]^2$, $u^1\in[H^1(\mathbf{R}^2)]^2$, $w^0\in H^3(\mathbf{R^2})$, $w^1\in H^2(\mathbf{R^2})$*. Then the initial value problem (1.2)-(1.4) admits a unique solution* (u,w) *having the regularity*

$$u\in C([0,\infty),[H^2(\mathbf{R}^2)]^2)\cap C^1([0,\infty),[H^1(\mathbf{R}^2)]^2),$$

$$w\in C([0,\infty),H^3(\mathbf{R}^2))\cap C^1([0,\infty),H^2(\mathbf{R}^2)).$$

For the corresponding one dimensional rod model the existence and uniqueness of global finite energy solutions is proved in [7], where damping at the boundary is also considered. For the case of a bounded plate the existence and uniqueness of strong solutions and the existence of finite energy solutions of (1.2)-(1.4) were proved in [11]. As far as we know, the uniqueness of finite energy solutions for an arbitrary bounded plate is an open problem.

The difficulty in the two dimensional case is that nonlinear terms in (1.2), (1.3) are not Lipschitz functions in the energy space. The same kind of difficulty arises for the classical von Kármán equations obtained by neglecting the term u'' in (1.2) and $\gamma\Delta w''$ in (1.3). By using the fact that without the term u'' equation (1.2) becomes elliptic and a compensated compactness technique, well-posedness in the energy space for the corresponding problem was proved in [3]. The uniqueness of finite energy solutions in the same case was obtained by a different technique in [1]. The methods used in the papers quoted above don't seem adaptable to our case as equations (1.2), (1.3) are both hyperbolic and the cancelations used for the classical von Kármán equation do not hold true in this context. This is why we shall use a different method, based on a Strichartz type estimate for the linearization of (1.3). Our main result asserts that the system (1.2)-(1.4) is well posed in the energy space. This limitation to the whole space problem or to the rectangular plate comes from the fact that Strichartz type estimates (even for the linear wave equation) are known only in the entire space (see [4] and references therein), in a rectangle (cf. section 4)or outside a convex obstacle (see [14]). Other papers studying related problems are ,[2] and [7].

2 The linearized problem

If we linearize equations (1.2)-(1.3) we obtain the following two uncoupled problems:

$$u'' - div\{\mathcal{C}[\epsilon(u)]\} = g, \text{ in } \mathbf{R}^2 \times (0,T), \tag{2.11}$$

$$u(0) = u^0, u'(0) = u^1, \text{ in } \mathbf{R^2}, \tag{2.12}$$

for u and

$$w'' - \gamma\Delta w'' + D\Delta^2 w = h, \text{ in } \mathbf{R}^2 \times (0,T), \tag{2.13}$$

$$w(0) = w^0, w'(0) = w^1, \text{ in } \mathbf{R^2}, \tag{2.14}$$

for w. Let us notice that (2.11) is the linear two dimensional elasticity equation and that (2.13) is the linear Kirchoff plate equation. Standard well-posedness results for these equations are summarized in

Proposition 2.1 *Suppose that $s \in \mathbf{R}$ and*

$$g \in L^1(0,T;[H^s(\mathbf{R^2})]^2) \qquad h \in L^1(0,T;H^{s-1}(\mathbf{R^2})).$$

Then equations (2.11)-(2.12) admit a unique solution (u,w), such that

$$\|u\|_{L^\infty(0,T;[H^{s+1}(\mathbf{R^2}]^2))} + \|u'\|_{L^\infty(0,T;[H^s(\mathbf{R^2}]^2))} \leq C(\|u^0\|_{H^{s+1}(\mathbf{R^2})} +$$

$$\|u^1\|_{H^s(\mathbf{R^2})} + \|g\|_{L^1(0,T;H^s(\mathbf{R^2}))}), \tag{2.15}$$

$$\|w\|_{L^\infty(0,T;H^{s+2}(\mathbf{R^2}))} + \|w'\|_{L^\infty(0,T;H^{s+1}(\mathbf{R^2}))} \leq C(\|w^0\|_{H^{s+2}(\mathbf{R^2})}+$$
$$\|w^1\|_{H^{s+1}(\mathbf{R^2})} + \|h\|_{L^1(0,T;H^{s-1}(\mathbf{R^2}))}). \tag{2.16}$$

The main result of this section is

Proposition 2.2 *Suppose that $h \in L^1([0,T], H^{s-2}(\mathbf{R^2}))$, $w^0 \in H^{s+1}(\mathbf{R^2})$, $w^1 \in H^s(\mathbf{R^2})$ with $\frac{3}{4} < s \leq 1$. Then the solution w of (2.13)-(2.14) has the property*

$$\nabla w \in L^p(0,T;[L^\infty(\mathbf{R^2})]^2),\ \forall p < \frac{1}{1-s}. \tag{2.17}$$

Moreover there exists a constant $C_p > 0$ such that

$$\|\nabla w\|_{L^p(0,T;[L^\infty(\mathbf{R^2})]^2)} \leq C_p(\|w^0\|_{H^{s+1}(\mathbf{R^2})}+$$

$$+\|w^1\|_{H^s(\mathbf{R^2})} + \|h\|_{L^1(0,T;H^{s-2}(\mathbf{R^2}))}),\ \forall p < \frac{1}{1-s}. \tag{2.18}$$

Proposition 2.2 follows from an estimate for the linear wave equation. More precisely let us suppose that $c > 0$ and consider the following initial value problem:

$$v''(x,t) - c\Delta v(x,t) = 0,\ \forall (x,t) \in \mathbf{R}^2 \times (0,T), \tag{2.19}$$

$$v(x,0) = v^0(x),\ v'(x,0) = v^1(x),\ \forall x \in \mathbf{R}^2. \tag{2.20}$$

By using the Pecher estimates for the linear wave equation (see Theorem 3.1 from [9] and Theorem 1 from [10]) we can prove the following result

Lemma 2.1 *Suppose that $v^0 \in H^s(\mathbf{R}^2)$, $v^1 \in H^{s-1}(\mathbf{R}^2)$, with $\frac{3}{4} < s \leq 1$. Then the solution v of (2.19)-(2.20) satisfies*

$$\|v\|_{L^p(0,T;L^\infty(\mathbf{R^2}))} \leq C(\|v^0\|_{H^s(\mathbf{R^2})} + \|v^1\|_{H^{s-1}(\mathbf{R^2})}),\ \forall p < \frac{1}{1-s}. \tag{2.21}$$

Proof *First step :* We first prove that

$$\|v\|_{L^4(0,T;L^\infty(\mathbf{R^2}))} \leq C(\|v^0\|_{H^s(\mathbf{R^2})} + \|v^1\|_{H^{s-1}(\mathbf{R^2})}),\ \forall s > 3/4. \tag{2.22}$$

Let $-A$ denote the two dimensional Laplace operator and define

$$\widetilde{v} = A^{\frac{m}{2}} v, \tag{2.23}$$

where

$$m = s - \frac{3}{4}. \tag{2.24}$$

It is clear that $\widetilde{v}$ satisfies the conditions

$$\widetilde{v}'' - \Delta\widetilde{v} = 0, \text{ in } \mathbf{R}^2 \times (0,T),, \tag{2.25}$$

$$\widetilde{v}(0) \in H^{3/4}(\mathbf{R}^2), \ \widetilde{v}'(0) \in H^{-1/4}(\mathbf{R}^2). \tag{2.26}$$

According to Theorem 3.1 from [9] $\widetilde{v}$ satisfies

$$\|\widetilde{v}\|_{L^4(0,T;L^q(\mathbf{R}^2))} \le C(\|\widetilde{v}(0)\|_{H^{3/4}(\mathbf{R}^2)} + \|\widetilde{v}'(0)\|_{H^{-1/4}(\mathbf{R}^2)}) \quad q < \infty \tag{2.27}$$

Relations (2.23) and (2.27) obviously imply

$$\|v\|_{L^4(0,T;W^{m,q}(\mathbf{R}^2))} \le C(\|v^0\|_{H^s(\mathbf{R}^2)} + \|\widetilde{v}^1\|_{H^{s-1}(\mathbf{R}^2)}), q < \infty \tag{2.28}$$

where m is defined by (2.24). As $m > 0$ and q is arbitrary we can suppose that $mq > 2$, so $W^{m,q}(\mathbf{R}^2)$ is continuously embedded in $L^\infty(\mathbf{R}^2)$. In this case (2.28) implies that (2.22) holds true.

Second step. If E, F are Banach spaces let $\mathcal{L}(E,F)$ be the space of bounded linear operators from E to F. Consider the linear operator

$$(v_0, v_1) \to T(v_0, v_1) = v,$$

where v is the solution (2.19)-(2.20). By applying (2.22) it follows that

$$T \in \mathcal{L}(H^s(\mathbf{R}^2) \times H^{s-1}(\mathbf{R}^2), L^4(0,T;L^\infty(\mathbf{R^2}))), \ \forall\ s > 3/4. \tag{2.29}$$

On the other hand by applying Proposition 2.1 with $s = 1 + \epsilon$ combined with a Sobolev embedding theorem we obtain that

$$T \in \mathcal{L}(H^s(\mathbf{R}^2) \times H^{s-1}(\mathbf{R}^2), L^\infty(0,T;L^\infty(\mathbf{R^2}))), \ \forall\ s > 1. \tag{2.30}$$

From (2.29) and (2.30) we obtain (2.21) by interpolation (see [15, p.127-129] for the general interpolation results).

We can now prove Proposition 2.2

Proof of Proposition 2.2: Let $s \in (3/4, 1]$. By Proposition 2.1, the solution w of (2.13)-(2.14) has the property

$$w \in C(0,T;H^{s+1}(\mathbf{R}^2)) \cap C(0,T;H^s(\mathbf{R}^2)). \tag{2.31}$$

On the other hand by applying the operator $(I - \gamma\Delta)^{-1}$ to (2.13) we obtain that

$$w'' + (I - \gamma\Delta)^{-1}\Delta^2 w = \widetilde{h} \in L^1(0,T;H^s(\mathbf{R}^2)). \tag{2.32}$$

Let us now make the notation

$$\phi = (I - \gamma\Delta)^{-1}\Delta^2 w + \gamma^{-1}\Delta w. \tag{2.33}$$

A simple calculation shows that the Fourier transform (with respect to the space variable) of ϕ is given by

$$\hat{\phi}(\xi,t) = -\frac{|\xi|^2}{\gamma + |\xi|^2}\hat{w}(\xi,t), \tag{2.34}$$

where by $\hat{\phi}$ (resp. $\hat{w}$) we denoted the Fourier transform of ϕ (resp. of w). Hence, by (2.31) and (2.34) we obtain

$$\phi \in L^\infty(0,T;H^{s+1}(\mathbf{R}^2)). \tag{2.35}$$

Relations (2.32), (2.33) and (2.35) obviously imply that

$$w'' - \gamma^{-1}\Delta w = h \in L^1(0,T;H^s(\mathbf{R}^2)). \tag{2.36}$$

If we differentiate (2.36) with respect to x_i, $i = 1,2$ and we denote $\frac{\partial w}{\partial x_i}$ by v_i relation (2.36) gives

$$v_i'' - \gamma^{-1}\Delta v_i = h_i \in L^1(0,T;H^{s-1}(\mathbf{R}^2)),\ i = 1,2.$$

As we obviously have $v_i(\cdot,0) \in H^s(\mathbf{R}^2)$ and $v_i'(\cdot,0) \in H^{s-1}(\mathbf{R}^2)$ the conclusion follows by simply applying Lemma 2.1 and the variation of parameters formula.

■

3 The semilinear equation

Proof of Theorem 1.3 : As u, w satisfy (1.7), (1.8) by simply applying Sobolev's embedding theorems we obtain

$$\mathcal{C}[\epsilon(u) + \frac{1}{2}\nabla w \otimes \nabla w]\nabla w \in L^1(0,T;H^{-\epsilon}(\mathbf{R^2})),\ \forall\epsilon > 0,$$

so we have

$$div\{\mathcal{C}[\epsilon(u) + \frac{1}{2}\nabla w \otimes \nabla w]\nabla w\} \in L^1(0,T;H^{-1-\epsilon}(\mathbf{R^2})),\ \forall\epsilon > 0. \tag{3.37}$$

Apply Proposition 2.2 to w, with $s = 1 - \epsilon$. Due to (3.37) we get

$$\nabla w \in L^p(0,T;L^\infty(\mathbf{R^2})),\ \forall p \geq 1. \tag{3.38}$$

□

Proof of Theorem 1.1 : Define

$$X_p = L^p(0,T;[L^2(\mathbf{R^2})]^2) \times L^p(0,T;H^{-1}(\mathbf{R^2})),\ \forall p \geq 1. \tag{3.39}$$

Note that Hölder's inequality gives

$$||(f,g)||_{X_1} \leq T^{\frac{1}{p'}} ||(f,g)||_{X_p}, \tag{3.40}$$

where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

Denote also

$$G(u,v) = \left(\frac{1}{2} div\{\mathcal{C}[\nabla w \otimes \nabla w]\}, div\{\mathcal{C}[\epsilon(u) + \frac{1}{2} \nabla w \otimes \nabla w] \nabla w\} \right)$$

Theorem 1.3 shows that the linear map

$$(u_0, u_1, v_0, v_1, f, g) \longrightarrow (u,v),$$

where (u,w) satisfy (2.11)-(2.14)), is bounded from

$$\mathcal{A} = H^2(R^2) \times H^1(R^2) \times H^1(R^2) \times L^2(R^2) \times X_1$$

into

$$L^\infty(0,T; H^1(\mathbf{R^2}) \times (L^\infty(0,T; H^2(\mathbf{R^2})) \cap L^p(0,T; W^{1,\infty}(\mathbf{R^2})),$$

for all $p \geq 1$. This implies that the quadratic map

$$S_T : \mathcal{A} \longrightarrow X_p$$

$$S_T(u_0, u_1, v_0, v_1, f, g) = G(u,v) \tag{3.41}$$

is well-defined and bounded. Hence,

$$|S_T(\alpha) - S_T(\tilde{\alpha})|_{X_p} \leq C_p R |\alpha - \tilde{\alpha}|_{\mathcal{A}} \tag{3.42}$$

provided that

$$|\alpha|_{\mathcal{A}}, |\tilde{\alpha}|_{\mathcal{A}} \leq R$$

Fix u_0, u_1, v_0, v_1. For $(g,h) \in X_1$ we define the map

$$N_T(g,h) : X_1 \longrightarrow X_p$$

by

$$N_T(g,h) = S_T(u_0, u_1, v_0, v_1, f, g).$$

By Theorem 1.3 it is clear that (u,w) is a solution of the nonlinear problem (1.2)-(1.4) if and only if $G(u,v)$ is a fixed point of N_T.

Fix $R > 0$, and let B_R be the ball of radius R in X_1. We claim that N_T is a strict contraction in B_R, for T small enough. Indeed, by (3.42) we get

$$||N_T(g_1,h_1) - N_T(g_2,h_2)||_{X_p} \leq CR ||(g_1,h_1) - (g_2,h_2)||_{X_1}, \tag{3.43}$$

Now by (3.40)

$$\|N_T(g_1,h_1)-N_T(g_2,h_2)\|_{X_1}\le C_pRT^{1/p'}\|(g_1,h_1)-(g_2,h_2)\|_{X_1},$$

$$\|N_T(0,0)\|_{X_1}\le C_pT^{1/p'}$$

This shows that if T is sufficiently small then N_T is a contraction from B_R into itself, which implies the small time existence result.

Second step. We prove small time uniqueness. Let (f,g), $(\widetilde{f},\widetilde{g})\in X_p$ be two fixed points of N_{T_0} for some $T_0>0$. Let $R>0$. Then (3.40) shows that (f,g), $(\widetilde{f},\widetilde{g})\in B_R$ if T is sufficiently small. Hence, the fixed point argument above shows that $(f,g)=(\widetilde{f},\widetilde{g})$.

Third step. Global existence. It obviously suffices to show that the energy of the local solution (obtained in the first two steps) doesn't explode in finite time. Let us consider the sequences

$$(u^0)_n\subset[H^2(\mathbf{R^2})]^2,\ (u^1)_n\subset[H^1(\mathbf{R^2})]^2,$$

$$(w^0)_n\subset H^3(\mathbf{R^2}),\ (w^1)_n\subset H^2(\mathbf{R^2}),$$

such that

$$u_n^0\to u^0,\ \text{in } [H^1(\mathbf{R^2})]^2,\ u_n^1\to u^1,\ \text{in } [L^2(\mathbf{R^2})]^2,$$
$$w_n^0\to w^0,\ \text{in } H^2(\mathbf{R^2}),\ w_n^1\to w^1,\ \text{in } H^1(\mathbf{R^2}).$$

By Theorem 1.4 problem (1.2)-(1.4) with initial data $(u_n^0,u_n^1,w_n^0,w_n^1)$ admits a strong solution

$$u_n\in C([0,T],[H^2(\mathbf{R}^2)]^2)\cap C^1([0,T],H^1(\mathbf{R}^2)]^2),$$

$$w_n\in C([0,T],H^3(\Omega))\cap C^1([0,T],H^2(\Omega)).$$

Moreover it is obvious that this solution satisfies the energy estimate of

$$E(u_n(t),w_n(t))=E(u_n(0),w_n(0)),\ \forall t\in[0,T], \tag{3.44}$$

where the energy $E(u,w)$ is defined by (1.6). From (3.44), by passing to a subsequence, we obtain the existence of

$$u\in L^\infty(0,T;[H^1(\mathbf{R^2})]^2)\cap W^{1,\infty}(0,T;[L^2(\mathbf{R^2})]^2), \tag{3.45}$$

$$w\in L^\infty(0,T;H^2(\mathbf{R^2}))\cap W^{1,\infty}(0,T;H^1(\mathbf{R^2})), \tag{3.46}$$

such that:

$$u_n\to u,\ \text{in } L^\infty(0,T;[H^1(\mathbf{R^2})]^2),\text{weak }*, \tag{3.47}$$

$$u_n'\to u',\ \text{in } L^\infty(0,T;[L^2(\mathbf{R^2})]^2),\text{weak }*, \tag{3.48}$$

$$w_n\to w,\ \text{in } L^\infty(0,T;H^2(\mathbf{R^2}),\text{weak }*, \tag{3.49}$$

$$w_n' \to w', \text{ in } L^\infty(0,T;H^1(\mathbf{R^2})), \text{weak } *. \tag{3.50}$$

Let us now consider an arbitrary bounded open set $\Omega \subset \mathbf{R^2}$. By using a compactness result proved in [13], from (3.49)-(3.50) we obtain that:

$$\nabla w_n \to \nabla w, \text{dans } L^\infty([0,T], H^{1-\epsilon}(\Omega)), \text{ strongly}, \forall \epsilon > 0. \tag{3.51}$$

As, in two space dimensions, we have the embedding $H^{1-\epsilon}(\Omega) \subset L^{2/\epsilon}(\Omega)$ the relation (3.51) implies that, for ϵ small, we have:

$$\nabla w_n \to \nabla w, \text{dans } L^\infty([0,T], L^6(\Omega)), \text{ strongly }. \tag{3.52}$$

On the other hand from (3.47) we obtain that

$$\epsilon_{ij}(u_n) \to \epsilon_{ij}(u), \text{dans } L^\infty([0,T], L^2(\Omega)]), \text{weak } *. \tag{3.53}$$

As the open bounded set Ω is arbitrary, from (3.47)-(3.53) we easily get that the nonlinear terms in (1.2) and (1.3) pass to the limit, so u, w satisfy (1.2)-(1.4). Moreover, by using a result from [8] the regularity (3.45), (3.46) can be lifted in time to (1.7), (1.8). As at the second step we proved small time uniqueness it follows that the solution (u,w) obtained by passing to the limit in (u_n, w_n) coincides with those obtained in the first two steps. By combining now the fact that the functional $(u,w) \to E(u,w)$ is weakly lower semicontinuous and (3.44) we obtain that

$$E(u(t), w(t)) \leq E(u(0), w(0)), \ \forall t \in [0,T),$$

so finite energy solutions are global.

Remark 3.1 *By the method used in the last step of the proof above one can easily obtain the existence (but no uniqueness) of finite energy solutions for an arbitrary bounded plate (see Theorem 5.2 from [11] for the details)*

Proof of Theorem 1.2 : Since the energy is non increasing on trajectories, it suffices to prove the estimate for sufficiently small T. By applying (3.42) combined with Hölder's inequality we obtain

$$|G(u,v) - G(\widetilde{u}, \widetilde{w})|_{X_p} \leq C_R((||u^0 - \widetilde{u}^0||_{[H^1(\mathbf{R^2})]^2} + ||u^1 - \widetilde{u}^1||_{[L^2(\mathbf{R^2})]^2} +$$

$$+||w^0 - \widetilde{w}^0||_{H^2(\mathbf{R^2})} + ||w^1 - \widetilde{w}^1||_{H^1(\mathbf{R^2})}) + T^{1/p'}|G(u,v) - G(\widetilde{u}, \widetilde{w})|_{X_p} \tag{3.54}$$

Hence, if T is sufficiently small then we get

$$|G(u,v) - G(\widetilde{u}, \widetilde{w})|_{X_p} \leq C_R((||u^0 - \widetilde{u}^0||_{[H^1(\mathbf{R^2})]^2} + ||u^1 - \widetilde{u}^1||_{[L^2(\mathbf{R^2})]^2} +$$

$$+||w^0 - \widetilde{w}^0||_{H^2(\mathbf{R^2})} + ||w^1 - \widetilde{w}^1||_{H^1(\mathbf{R^2})}) \tag{3.55}$$

and further

$$|G(u,v) - G(\widetilde{u}, \widetilde{w})|_{X_1} \leq C_R T^{1/p'}((||u^0 - \widetilde{u}^0||_{[H^1(\mathbf{R^2})]^2} + ||u^1 - \widetilde{u}^1||_{[L^2(\mathbf{R^2})]^2} +$$

$$+\|w^0-\widetilde{w}^0\|_{H^2(\mathbf{R^2})}+\|w^1-\widetilde{w}^1\|_{H^1(\mathbf{R^2})}) \tag{3.56}$$

On the other hand,

$$\|u(t)-\widetilde{u}(t)\|_{[H^1(\mathbf{R^2})]^2}+\|u'(t)-\widetilde{u}'(t)\|_{[L^2(\mathbf{R^2})]^2}+$$

$$+\|w(t)-\widetilde{w}(t)\|_{H^2(\mathbf{R^2})}+\|w'(t)-\widetilde{w}'(t)\|_{H^1(\mathbf{R^2})}\leq$$

$$\leq C(\|u^0-\widetilde{u}^0\|_{[H^1(\mathbf{R^2})]^2}+\|u^1-\widetilde{u}^1\|_{[L^2(\mathbf{R^2})]^2}+\|w^0-\widetilde{w}^0\|_{H^2(\mathbf{R^2})}$$

$$+\|w^1-\widetilde{w}^1\|_{H^1(\mathbf{R^2})}+|G(u,v)-G(\widetilde{u},\widetilde{w})|_{X_1^t}),\quad \forall t\in[0,T], \tag{3.57}$$

where X_1^t is the space obtained by replacing T by t in (3.39) Combining the last two estimates yields the desired conclusion.

4 An initial and boundary value problem for a rectangular plate

In this section we shall consider the case of a rectangular plate, occupying the domain $\Omega=(0,a)\times(0,b)$ with $a,b>0$, which is simply supported at the boundary. More precisely we shall consider the following initial and boundary value problem:

$$u''-div\{\mathcal{C}[\epsilon(u)+f(\nabla w)]\}=0 \quad \text{in } \Omega\times(0,T) \tag{4.58}$$

$$w''-\gamma\Delta w''+D\Delta^2 w-div\{\mathcal{C}[\epsilon(u)+f(\nabla w)]\nabla w\}=0 \quad \text{in } \Omega\times(0,T), \tag{4.59}$$

$$u=0,\ w=\Delta w=0,\ \text{on } \partial\Omega\times(0,T), \tag{4.60}$$

$$u(0)=u^0, u'(0)=u^1, w(0)=w^0, w'(0)=w^1 \quad \text{in } \Omega. \tag{4.61}$$

The main result of this section is the following

Theorem 4.1 *Suppose that $(u^0,u^1)\in[H_0^1(\Omega)]^2\times[L^2(\Omega)]^2$ and $(w^0,w^1)\in[H_0^2(\Omega)\cap H_0^1(\Omega)]\times H_0^1(\Omega)$. Then the problem (4.58)-(4.61) admits a unique solution satisfying*

$$u\in C(0,T;[H_0^1(\Omega)]^2)\cap C^1(0,T;[L^2(\Omega)]^2), \tag{4.62}$$

$$w\in C(0,T;H^2(\Omega))\cap C^1(0,T;H_0^1(\Omega)). \tag{4.63}$$

The proof of Theorem 4.1 is completely similar to the proof of Theorem 1.1 provided we obtain the analogue of Proposition 2.2 in the case of the rectangular plate. More precisely consider the following linear initial and boundary value problem:

$$w''-\gamma\Delta w''+D\Delta^2 w=h,\ \text{in } \Omega\times(0,T), \tag{4.64}$$

$$w=\Delta w=0,\ \text{on } \partial\Omega\times(0,T), \tag{4.65}$$

$$w(0)=w^0, w'(0)=w^1,\ \text{in } \Omega, \tag{4.66}$$

Proposition 4.1 *Suppose that $h \in L^1([0,T], H^{s-2}(\Omega))$, $w^0 \in H^{s+1}(\Omega)$, $w^1 \in H^s(\Omega)$ with $\frac{3}{4} < s < 1$. Then the solution w of (4.64)-(4.66) has the property*

$$\nabla w \in L^p(0,T;[L^\infty(\Omega)]^2), \ \forall p < \frac{1}{1-s}. \tag{4.67}$$

Moreover there exists a constant $C_p > 0$ such that

$$||\nabla w||_{L^p(0,T;[L^\infty(\Omega)]^2)} \leq C_p(||w^0||_{H^{s+1}(\Omega)}+$$

$$+||w^1||_{H^s(\Omega)} + ||h||_{L^1(0,T;H^{s-2}(\Omega))}), \ \forall p < \frac{1}{1-s}. \tag{4.68}$$

In order to prove Proposition 4.1 let us first notice that the Strichartz type estimate from Lemma 2.1 implies the following estimate for the solutions of the wave equation with homogenous Neumann boundary conditions.

Lemma 4.1 *Suppose that $c > 0$, $\Omega \subset \mathbf{R}^2$ is a rectangle and that $v^0 \in H^s(\Omega)$, $v^1 \in H^{s-1}(\Omega)$, with $\frac{3}{4} < s < 1$. Then the solution v of*

$$v''(x,t) - c\Delta v(x,t) = 0, \ \forall (x,t) \in \Omega \times (0,T), \tag{4.69}$$

$$\frac{\partial v}{\partial \nu} = 0, \ \text{on } \partial\Omega \times (0,T), \tag{4.70}$$

$$v(x,0) = v^0(x), \ v'(x,0) = v^1(x), \ \text{in } \Omega, \tag{4.71}$$

satisfies

$$||v||_{L^p(0,T;L^\infty(\Omega))} \leq C(||v^0||_{H^s(\Omega)} + ||v^1||_{H^{s-1}(\Omega)}), \ \forall p < \frac{1}{1-s}. \tag{4.72}$$

Proof : Let $\Omega = (0,a) \times (0,b)$ and suppose that $(v^0, v^1) \in H^2(\Omega) \times H^1(\Omega)$ with

$$\frac{\partial v^0}{\partial \nu} = 0, \ \text{on } \partial\Omega. \tag{4.73}$$

The solution v of (4.69)-(4.71) will satisfy

$$v \in C(0,T;H^2(\Omega)) \cap C^1(0,T;H^1(\Omega)). \tag{4.74}$$

We shall now extend v by periodicity outside Ω. More precisely if $(x,y) \in \mathbf{R}^2$ take $m, n \in \mathbf{Z}$ and $(\widetilde{x}, \widetilde{y}) \in \Omega$ such that

$$x = ma + \widetilde{x}, \ y = nb + \widetilde{y},$$

and consider $\widetilde{v} : \mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}$ defined by

$$\widetilde{v}(x, y, t) = v(\widetilde{x}, \widetilde{y}, t).$$

It is clear that $\widetilde{v} \in C(0, T; H^2_{\text{loc}}(\Omega))$ satisfies the conditions:

$$\widetilde{v}''(x, t) - c\Delta\widetilde{v}(x, t) = 0, \ \forall (x, t) \in \mathbf{R^2} \times (\mathbf{0}, \mathbf{T}), \tag{4.75}$$

$$\frac{\partial \widetilde{v}}{\partial \nu} = 0, \ \text{on } \partial\Omega \times (0, T), \tag{4.76}$$

$$\widetilde{v}(x, 0) = \widetilde{v}^0(x), \ \widetilde{v}'(x, 0) = \widetilde{v}^1(x), \ \text{in } \Omega, \tag{4.77}$$

where $\widetilde{v}^0$(resp. $\widetilde{v}^1$) is the extension by periodicity of v^0 (resp. v^1).

Consider now $(\bar{v}^0, \bar{v}^1) \in H^2(\mathbf{R}^2) \times H^1(\mathbf{R}^2)$ such that

$$\bar{v}^i(x) = \widetilde{v}^i(x), \ |x| \leq R, \ i = 1, 2,$$

$$\|\bar{v}^0\|_{H^s(\mathbf{R^2})} + \|\bar{v}^1\|_{H^{s-1}(\mathbf{R^2})} \leq C_R \left(\|v^0\|_{H^s(\Omega)} + \|\bar{v}^1\|_{H^{s-1}(\Omega)} \right), \ \frac{3}{4} < s \leq 1.$$

We define $\bar{v} \in C(0, T; H^2(\mathbf{R}^2)$ as the unique solution of the Cauchy problem

$$\bar{v}''(x, t) - c\Delta\bar{v}(x, t) = 0, \ \forall (x, t) \in \mathbf{R^2} \times (\mathbf{0}, \mathbf{T}), \tag{4.78}$$

$$\bar{v}(x, 0) = \bar{v}^0(x), \ \bar{v}'(x, 0) = \bar{v}^1(x), \ \text{in } \Omega. \tag{4.79}$$

By using now the finite sppeed of propagation for the solutions of the wave equation we obtain (provided that R is large enough) that

$$v(x, t) = \widetilde{v}(x, t) = \bar{v}(x, t), \ \forall x \in \Omega, \ \forall t \in (0, T). \tag{4.80}$$

By applying now Lemma 2.1 to $\bar{v}$ relation (4.80) implies estimate (4.72) for $(v^0, v^1) \in H^2(\Omega) \times H^1(\Omega)$. By a simple density argument the conclusion still holds true for $(v^0, v^1) \in H^s(\Omega) \times H^{s-1}(\Omega), \ \frac{3}{4} < s \leq 1$. □

We can now prove Proposition 4.1

Proof of Proposition 4.1: For the sake of simplicity we shall detail the proof only in the case a square plate, more precisely we suppose that $\Omega = (0, \pi) \times (0, \pi)$. Let $s \in (3/4, 1]$. By classical energy methods the solution w of (4.64)-(4.66) has the property

$$w \in C(0, T; H^{s+1}(\Omega)) \cap C(0, T; H^s(\Omega)), \tag{4.81}$$

which implies that

$$w(x, y, t) = \sum_{m,n \geq 1} w_{mn}(t) \sin(mx) \sin(ny), \tag{4.82}$$

with

$$\max_{0\le t\le T}\sum_{m,n\ge 1}(m^2+n^2)^{s+1}w_{mn}^2(t)<\infty. \tag{4.83}$$

On the other hand by applying the operator $(I-\gamma\Delta)^{-1}$ to (4.64) we obtain that

$$w''+(I-\gamma\Delta)^{-1}\Delta^2 w=\widetilde{h}\in L^1(0,T;H^s(\Omega)). \tag{4.84}$$

Let us now make the notation

$$\phi=(I-\gamma\Delta)^{-1}\Delta^2 w+\gamma^{-1}\Delta w. \tag{4.85}$$

A simple calculation using (4.82) and (4.85) shows that

$$\phi(x,t)=-\frac{1}{\gamma}\sum_{m,n\ge 1}\frac{m^2+n^2}{1+\gamma(m^2+n^2)}w_{mn}(t)\sin{(mx)}\sin{(ny)}. \tag{4.86}$$

Hence, by combining (4.81) and (4.86) we obtain

$$\phi\in L^\infty(0,T;H^{s+1}(\Omega)). \tag{4.87}$$

Relations (4.84), (4.85) and (2.35) obviously imply that

$$w''-\gamma^{-1}\Delta w=h\in L^1(0,T;H^s(\Omega)). \tag{4.88}$$

If we differentiate (4.88) with respect to $x_i,\ i=1,2$ and we denote $\nabla w=v$ and $\frac{\partial w}{\partial x_i}$ by v_i relation (4.88) gives

$$v_i''-\gamma^{-1}\Delta v_i=h_i\in L^1(0,T;H^{s-1}(\Omega)),\ i=1,2.$$

On the other hand relations div $v=\Delta w$ and curl $v=0$ imply that

$$\frac{\partial v_i}{\partial\nu}=0,\ i=1,2.$$

As we obviously have $v_i(\cdot,0)\in H^s(\Omega)$ and $v_i'(\cdot,0)\in H^{s-1}(\Omega)$ the conclusion follows by simply applying Lemma 4.1 and the variation of parameters formula. ∎

Remark 4.1. *The results proved in the last section can be extended for a rectangular plate clamped at the boundary. However in this case the proof becomes slightly more involved as the explicit form of eigenvalues is not avaible. However one can easily prove the analogue of Proposition 4.1 by using the asymptotic behaviour of eigenvalues given by some extension of Weyl's formula (cf. e.g. [5])*

References

[1] A. Boutet de Montvel and I. Chueskov, Uniqueness theorem for weak solutions of non Kármán evolution equations, preprint

[2] M. Bradley and I.Lasiecka, Global decay rates for the solutions to a von Kármaán plate without geometric conditions, J. Math. Anal. Appl., **181**(1994), p.254-276

[3] A. Favini, I. Lasiecka, M.A. Horn and D. Tătaru, Global existence, uniqueness and regularity of solutions to a von Kármán system with nonlinear boundary dissipation, to appear in Diff. and Int. Eq.

[4] J. Ginibre, G. Velo Generalized Strichartz inequalities for the wave equation, J. Funct. Anal., **133**(1995), p.50-68

[5] A. Haraux and S. Jaffard, Pointwise and spectral control of plate vibrations, Rev. Mat. Iberoamericana, 7(1991), p.1-23

[6] J.E. Lagnese, Boundary stabilization of thin plates,SIAM publications, Boston, 1989

[7] J.E.Lagnese and G.Leugering, Uniform stabilization of a nonlinear beam by nonlinear boundary feedback, J.Diff.Eq., **91**, pp.355-388(1991)

[8] I. Lasiecka and R. Triggiani, A lifting theorem for the time regularity of solutions to absrtact equations of unbounded operators ans applications ti hyperbolic equations, Proc. Amer.Math.Soc., **104**, pp.745-755(1988)

[9] H. Lindblad, C.D. Sogge, On existence and scattering with minimal regularity for semilinear wave equation, Journal of Functional Analysis, **130**, pp.357-426 (1995).

[10] H. Pecher, Nonlinear small data scattering for the wave and Klein-Gordon equations, Math.Z.,**185**, pp.261-270(1984)

[11] J.P. Puel and M. Tucsnak, Global existence for the full von Kármán system, to appear in Appl.Math. and Opt.

[12] J.P. Puel and M. Tucsnak, Boundary stabilization for the von Kármán equations, SIAM J. on Control, **33**, pp.255-273(1995)

[13] J. Simon, Compact sets in the space $L^p(0,T;B)$, Annali di Matematica Pura ed Applicata, IV, **CXLVI**, pp 65-96(1987)

[14] H. Smith, C.D. Sogge, On the critical semilinear wave equation outside convex obstacles, preprint

[15] H. Triebel, Interpolation theory, function spaces, differential operators, North Holland, Amsterdam,1978