Scattering of waves by thin periodic layers at high frequencies using the on-surface radiation condition method

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Abstract

This paper deals with the scattering problem of acoustic and electromagnetic waves by thin periodic layers at high frequencies. The on-surface radiation condition method is employed to simplify the numerical computations of reflected and transmitted waves.

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1 Introduction

The electromagnetic theory of periodic structures has been studied extensively since Rayleigh's time. The advent of computers has greatly accelerated activity in this area and has given rise to several approaches and numerical methods for obtaining rigorous solutions to the problem, including differential methods, integral methods, analytical continuation, variational methods, and many others. Most of the investigations on the electromagnetic theory of periodic structures has been devoted to planar infinite ones, which allows one to model the problems in one period. Unfortunately, in the industrial applications, periodic structures are not planar nor infinite. In principle curved or finite structures can be studied by using any numerical method for the diffraction of the Helmholtz equation. The complete calculus of the scattered wave by the curved or finite structure at high frequencies leads to a great number of unknowns and consequently to very heavy (or not feasible) computations. It is well known that the direct numerical solution of the Helmholtz equation for large values of frequency ω is difficult. The fundamental difficulty is the fact that the necessary resolution is proportional to ω^{-1}. To resolve each wavelength one approximately needs 10 points. Generally speaking, the treatments of curved or finite structures are expect to be more difficult than the treatment of the plane infinite ones, with which the pseudo-periodicity of the scattering problem allows us to perform all the calculations over a single period of the periodic layer. Our motivations in this paper is to simplify the computation in the modeling of diffractive optics in curved and finite structures and to take advantage of the finite structures computationally. This is accomplished by combining an asymptotic analysis and on-surface radiation condition method. Kriegsmann et al [21] have introduced this new method for solving scattering problems for two-dimensional convex cylinders by applying a radiation boundary condition on the surface of the scatter which yields the scattered field in a simple fashion. Jones [18] has generalized their surface radiation condition to three dimensions and has given a general method of obtaining surface radiation conditions. The on-surface radiation condition theory has been extended by Arendt et al [8] to the analysis of electromagnetic scattering and penetration by homogeneous convex dielectric objects.

In this paper we study the scattering problem of acoustic and electromagnetic plane waves by thin periodic layers at high frequencies by using the on-surface radiation condition method. In the first part of this paper, we consider a thin dielectric layer covering a circular cylindrical conducting object (see figure 1). We assume that the coefficients of the dielectric material are periodic in the angular variable with period δ/R where R is the radius of the circular cylinder and δ is a small parameter of order $0(\omega^{-1})$. The thickness of the dielectric layer is also assumed to be of order $0(\omega^{-1})$. The scattering problem by a thin dielectric layer with conducting inclusions covering a circular cylindrical conducting object was first studied in [2]. In [2], the solution to the diffraction problem by the curved grating at high frequencies is expanded as an infinite series in ω^{-1} outside some transition regions corresponding to the tangent Bragg modes and to the usual transition regions separating the region of direct illumination from the region of shadow where it is expanded as an infinite series in $\omega^{-1/3}$. The most important term of the expansion of the scattered wave by the curved grating is the unique solution of the diffraction problem by an infinite plane grating. The remaining terms represent corrections to geometrical optics from finiteness of the frequency ω. These results have been extended in [1] to a thin curved layer covring a convex conducting body. In [5] a numerical method based in these results has been given and optimal error estimates for this method are established. In the present work we derive a simple numerical method compared to that set out in [5] for the calculation of high frequency asymptotic expansions of the wave scattered by a planar finite thin periodic layer and we simplify the method given in [5] for the scattering problem by thin curved layers. The accuracy and efficiency of our method stem from the fact that the scattering problem by the thin periodic layer at frequency ω is replaced by a set of scattering problems in a fixed (planar) cell at the frequency $\nu = \omega\delta$ which is much smaller than the frequency ω.

We shall briefly outline the remainder of this paper. In section 2 the scattering of acoustic waves by a thin periodic circular layer (see figure 1) will be formulated. A straightforward method of obtaining the higher-order on-surface radiation conditions is given. There an asymptotic expansion of the transmitted field in the dielectric layer will be carried by making use of the on-surface radiation conditions. Section 3 is devoted to the scattering of acoustic waves by a thin planar periodic layer. The higher-order on-surface radiation conditions, obtained from asymptotic analysis of the Steklov-Poincaré operator associated to the Helmholtz operator in a half plane, are used to derive asymptotic expansions of the transmitted field in the dielectric layer. In section 4 this approach will be applied for the scattering problem for Maxwell's equations by a finite thin planar periodic layer of dielectric material.

2 Scattering by thin curved periodic layers

The scattering of waves by thin curved layers has been of considerable recent interest in the applied mathematics literature due to its importance in the industrial area. For technological reasons, antennas or scatterers are often coated by a thin dielectric layer. When the thickness of the dielectric layer is small compared to the wave length, the dielectric layer can be replaced by a hierarchy of approximative impedance conditions (see [14], [12], [6]). These approximate boundary conditions can be used, along with the radiation condition, to compute the reflected wave without requiring detailed modeling of the wave inside the dielectric coating. Approximative impedance conditions for thin curved periodic layers were derived in [2], [4], [7]. These approximate boundary conditions fail completely when the wavelength becomes as small as the thickness of the dielectric layer. The scattering problem by thin curved layers at high frequencies is much more complicated. This problem was investigated in [2], [1] and [5]. Because the reflected and transmitted waves oscillate with a small spatial period, it is impractical to find them by direct numerical methods. Here, our objective is to give an asymptotic method based on the on-surface radiation approach to compute the the reflected and transmitted waves. No attempt is made in this paper to give rigorous error estimates for this method. However, many comments concerning the range of its validity are offered.

In order to construct the on-surface radiation conditions, we consider the scattering problem by a circular cylinder S_R of radius R in a homogeneous medium. The outgoing solution of the Helmholtz equation

$$\begin{aligned} &\Delta\, u(r,\theta) + k^2\, u(r,\theta) = 0 \quad |r| > R, \\ &u(R,\theta) = f(\theta) = \sum_{n\in\mathbb{Z}} f_n\, e^{in\,\theta}, \end{aligned}$$

where (r,θ) are polar coordinates with the centre of cylinder as pole and there exists a real s such that $f \in H^s = \Big\{ \sum_{n\in\mathbb{Z}} (1+n^2)^s\, |f_n|^2 < \infty \Big\}$, is given by

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} \frac{h_n^{(1)}(kr)}{h_n^{(1)}(kR)}\, f_n\, e^{in\,\theta}.$$

Taking the derivative of u with respect to the variable r gives

$$\partial_r\, u(r,\theta)|_{r=R} = \sum_{n\in\mathbb{Z}} k\, \frac{h_n^{(1)'}(kR)}{h_n^{(1)}(kR)}\, f_n\, e^{in\,\theta} \;=\; \mathcal{S}^{(k)}(f). \tag{1}$$

On the other hand, it is well-known that (see for instance Olver [27])

$$k\,\frac{h_n^{(1)'}(kR)}{h_n^{(1)}(kR)} = k\,\frac{i\mathcal{R}_n(kR) - \mathcal{S}_n(kR)}{\mathcal{P}_n(kR) + i\mathcal{Q}_n(kR)},$$

where $\mathcal{R}_n(z), \mathcal{S}_n(z), \mathcal{P}_n(z), \mathcal{Q}_n(z)$ have the following asymptotic expansions for $|z|$ large

$$\mathcal{R}_n(z) = 1 - \frac{(4n^2-1)(4n^2+15)}{128z^2} + 0(\frac{1}{z^4}),$$

$$\mathcal{S}_n(z) = \frac{4n^2+3}{8z} + 0(\frac{1}{z^3}),$$

$$\mathcal{P}_n(z) = 1 - \frac{(4n^2-1)(4n^2-9)}{128z^2} + 0(\frac{1}{z^4}),$$

$$\mathcal{Q}_n(z) = \frac{4n^2-1}{8z} + 0(\frac{1}{z^3}).$$

These asymptotic expansions are valid for $|z|$ large and n below and not too close to $|z|$. Putting $z = kR$ we see that

$$k\,\frac{h_n^{(1)'}(kR)}{h_n^{(1)}(kR)} = ik - \frac{1}{2R} - \frac{i}{8kR^2}(4n^2-1) + 0(\frac{1}{k^2R^3}).$$

Now, provided that kR is large enough and the major part of the function f lies in $(e^{in\theta})_n$ which are below and not too close to kR, we can approximate the identity (1) by setting

$$\partial_r\, u(r,\theta)|_{r=R} = \sum_{n\in\mathbf{Z}} \Big(ik - \frac{1}{2R} - \frac{i}{8kR^2}(4n^2-1)\Big)\, f_n\, e^{in\,\theta}. \tag{2}$$

Recalling that

$$\partial_\theta^2 \Big(\sum_{n\in\mathbf{Z}} f_n\, e^{in\theta}\Big) = -\sum_{n\in\mathbf{Z}} n^2\, f_n\, e^{in\theta},$$

we rewrite the approximation (2) in the more suitable form

$$\partial_r\, u(r,\theta)|_{r=R} \;=\; iku(r,\theta)|_{r=R} - \frac{1}{2R}u(r,\theta)|_{r=R}. \tag{3}$$

Using the terms of order z^{-2} in the asymptotics of $\mathcal{R}_n$ and $\mathcal{P}_n$ we obtain the second approximation:

$$\begin{aligned}\partial_r\, u(r,\theta)|_{r=R} \;&=\; iku(r,\theta)|_{r=R} - \frac{1}{2R}u(r,\theta)|_{r=R} + \frac{i}{2k}\,\frac{1}{R^2}\,\partial_\theta^2\, u(r,\theta)|_{r=R} \\ &\quad + \frac{i}{8k}\,\frac{1}{R^2}\, u(r,\theta)|_{r=R}.\end{aligned} \tag{4}$$

Note that the method used by Kriegsmann et al [21] was different from that set out here, as were those of Jones [17] and [18]. Our approach does not depend upon a far field representation nor upon an expansion in a neighbourhood of a phase front. Instead, an asymptotic analysis with the respect to the frequency k of the Steklov-Poincaré operator $\mathcal{S}^{(k)} : H^s \mapsto H^{s-1}$ is made. We believe that this approach can be extended to derive on-surface radiation conditions for a convex object.

We are now in position to discuss the scattering problem by a thin circular periodic layer of dielectric material at high frequencies. Let $\delta, \nu, d, h, \varepsilon_0, \mu_1, \mu_0$ be strictly positive reals. Let an incident plane wave

$$u^{in}(r, \theta; \delta) = e^{-i\delta^{-1}\nu\sqrt{\varepsilon_0\mu_0} r \sin\theta}$$

impigne on a thin dielectric layer Ω^i_δ defined by $\Omega^i_\delta = \Big\{ R < r < R + \delta\, h \Big\}$. We assume that in Ω^i_δ, the electric permittivity ε is periodic in the angular variable θ with period $\dfrac{\delta}{R}$. Let u^t and u^s be the transmitted and reflected waves solutions of the Helmholtz equations

$$\begin{aligned} \Big(\Delta + \delta^{-2}\,\nu^2\,\mu_0\,\varepsilon_0\Big)\, u^s(r,\theta;\delta) &= 0 \quad \text{in } \Omega^e_\delta = \Big\{ r > R + \delta\, h \Big\}, \\ \Big(\Delta + \delta^{-2}\,\nu^2\,\mu_1\,\varepsilon(r,\theta)\Big)\, u^t(r,\theta;\delta) &= 0 \quad \text{in } \Omega^i_\delta = \Big\{ R < r < R + \delta\, h \Big\}, \\ u^t(R,\theta;\delta) &= 0 \quad \text{on } S_R = \Big\{ r = R \Big\}, \end{aligned} \tag{5}$$

with the transmission conditions on $r = R + \delta h$

$$\begin{aligned} u^t &= u^s + u^{in}, \\ \frac{\mu_0}{\mu_1}\,\partial_r\, u^t &= \partial_r\, u^s + \partial_r\, u^{in}. \end{aligned} \tag{6}$$

We are interested in the asymptotic behavior of the transmitted and reflected waves as the parameter δ goes to zero. We look for the transmitted wave in the form phase-amplitude

$$u^t(r,\theta;\delta) = e^{-i\delta^{-1}\nu\sqrt{\varepsilon_0\mu_0} R \sin\theta} \sum_{j=0}^{+\infty} a^t_j(\theta, R\,\theta/\delta, (r-R)/\delta)\,\delta^j, \tag{7}$$

where the functions a^t_j are periodic in the variable $x = R\theta/\delta$ of period 1. Since, applied to a function $\varphi(\theta, R\,\theta/\delta, (r-R)/\delta)$, the operator $\dfrac{1}{r^2}\,\partial^2_\theta$ (resp.

∂_r^2) becomes

$$\delta^{-2}\frac{1}{(1+\delta y/R)^2}(\partial_x - i\nu\sqrt{\varepsilon_0\mu_0}\cos\theta)^2$$
$$+\delta^{-1}\frac{R}{(R+\delta y)^2}\Big(\partial^2_{\theta x} - i\nu\sqrt{\varepsilon_0\mu_0}\cos\theta\,\partial_\theta + i\nu\sqrt{\varepsilon_0\mu_0}\sin\theta\Big) + \frac{1}{(R+\delta y)^2}\,\partial_\theta^2,$$

(resp. $\delta^{-2}\,\partial_y^2$), we obtain

$$\delta^2\,\Big(\Delta + \omega^2\mu_1\,\varepsilon\Big)\,u(r,\theta;\delta) \;=\; e^{-i\delta^{-1}\nu\sqrt{\varepsilon_0\mu_0}R\sin\theta}\,P^\delta\,\Big(\sum_{j=0}^{+\infty} a_j^t(\theta, R\,\theta/\delta, (r-R)/\delta)\Big),$$

where

$$P^\delta = P_0 + \delta\,P_1 + \delta^2\,P_2 + \ldots, \tag{8}$$
$$P_0 = (\partial_x - i\nu\sqrt{\varepsilon_0\mu_0}\cos\theta)^2 + \partial_y^2 + \nu^2\,\mu_1\,\varepsilon(x,y), \tag{9}$$

and

$$P_1 = -\frac{2y}{R}(\partial_x - i\nu\sqrt{\varepsilon_0\mu_0}\cos\theta)^2 + \frac{1}{R}\Big(\partial^2_{\theta x} - i\nu\sqrt{\varepsilon_0\mu_0}\cos\theta\,\partial_\theta + i\nu\sqrt{\varepsilon_0\mu_0}\sin\theta\Big).$$

Note that the angular variable θ is a parameter in P_0. Now, It follows from (8) and the Ansatz (7) of the transmitted wave u^t that the functions a_j^t satisfy

$$\begin{aligned}
&P_0\,a_0^t = 0,\\
&P_0\,a_1^t = -P_1\,a_0^t,\\
&P_0\,a_{j+2}^t = -P_1\,a_{j+1}^t - \ldots - P_{j+2}\,a_0^t, \quad j \geq 0.
\end{aligned}$$

Making use of the first on-surface radiation condition (3) we can deduce from (6) that

$$\frac{\mu_0}{\mu_1}\,\partial_r\,u^t \;=\; \partial_r\,u^{in} + i\delta^{-1}\nu\sqrt{\varepsilon_0\mu_0}(u^t - u^{in}) \quad \text{on } y = h. \tag{10}$$

The above identity leads to the boundary conditions satisfied by a_j^t on $y = h$. Inserting the Ansatz (7) of the transmitted wave u^t into (10) and identifying the coefficient of the powers δ^{-1}, δ^0 gives

$$\frac{\mu_0}{\mu_1}\,\partial_y\,a_0^t \;=\; i\nu\,\sqrt{\varepsilon_0\mu_0}\;a_0^t - i\nu\sqrt{\varepsilon_0\mu_0}\,e^{-i\nu\sqrt{\varepsilon_0\mu_0}h}(1+\sin\theta) \quad \text{on } y = h, \tag{11}$$

and

$$\frac{\mu_0}{\mu_1}\,\partial_y\, a_1^t \;=\; i\nu\,\sqrt{\varepsilon_0\mu_0}\;a_1^t - \frac{1}{2R}\,(a_0^t - e^{-i\nu\sqrt{\varepsilon_0\mu_0}h}) \quad \text{on } y=h. \tag{12}$$

The use of the second on-surface radiation condition (4) leads to the following approximate boundary conditions:

$$\begin{aligned}\frac{\mu_0}{\mu_1}\,\partial_y\, a_0^t \;&=\; i\nu\,\sqrt{\varepsilon_0\mu_0}\;a_0^t - \frac{i\nu}{2}\,\sqrt{\varepsilon_0\mu_0}\cos^2\theta\, a_0^t \\ &\quad -i\nu\sqrt{\varepsilon_0\mu_0}\,e^{-i\nu\sqrt{\varepsilon_0\mu_0}h}(1+\sin\theta - \frac{1}{2}\cos^2\theta) \quad \text{on } y=h,\end{aligned} \tag{13}$$

and

$$\begin{aligned}\frac{\mu_0}{\mu_1}\,\partial_y\, a_1^t \;&=\; i\nu\,\sqrt{\varepsilon_0\mu_0}\;a_1^t - \frac{i\nu}{2}\,\sqrt{\varepsilon_0\mu_0}\cos^2\theta\, a_1^t \\ &\quad +\cos\theta\,\partial_x\, a_0^t - \frac{1}{2R}\,(a_0^t - e^{-i\nu\sqrt{\varepsilon_0\mu_0}h}) \quad \text{on } y=h.\end{aligned} \tag{14}$$

The boundary value problems satisfied by a_0^t and a_1^t are now complete. By using the on-surface radiation condition (3) (resp. (4)), the functions a_0^t and a_1^t satisfy the boundary value problems:

$$\begin{aligned} &P_0\, a_0^t \;=\; 0 \quad \text{in }]0,1[\times]0,h[, \\ &a_0^t \;=\; 0 \quad \text{on } y=0, \\ &a_0^t \text{ satisfies (11)(resp.(13))} \quad \text{on } y=h, \\ &a_0^t \text{ is periodic in } x \text{ with period 1,}\end{aligned} \tag{15}$$

and

$$\begin{aligned} &P_0\, a_1^t \;=\; -P_1\, a_0^t \quad \text{in }]0,1[\times]0,h[, \\ &a_1^t \;=\; 0 \quad \text{on } y=0, \\ &a_1^t \text{ satisfies (12)(resp.(14))} \quad \text{on } y=h, \\ &a_1^t \text{ is periodic in } x \text{ with period 1.}\end{aligned} \tag{16}$$

It is easy to see that these mixed boundary value problems are well-posed at least for ε smooth enough. They can be numerical solved by using a finite element method similar to that of Bao [9]. We shall now make some comments on this method. The boundary value problem for the Helmholtz equation (15) is posed in the fixed planar cell $]0,1[\times]0,h[$ and at the fixed frequency

$\nu\sqrt{\mu_1\varepsilon} \ll \nu\sqrt{\mu_0\varepsilon_0}\,\delta^{-1}$. This is one of the advantage of this method. The other advantages of the present method are its generality and simplicity. It is easier to be implemented and it may be used to handle more complicated periodic structures. On the other hand, it is clear that if we compare (11) and (13) we see that they disgree only in terms which involve $\cos\theta$ where θ is the angle of incidence. It follows that at normal incidence, these approximations obtained by using different order of the on-surface boundary conditions are exactly the same. The same remark holds for $(12), (14)$. We claim that at normal incidence the extra terms due to the higher order on-surface boundary conditions do not play any significant role. But, if the angle of incidence is not too close to the normal these terms can not be neglected. In this case, we must use higher order on-surface boundary conditions. There is a hierarchy in these approximations: the greater the order of the on-surface radiation condition, the better the approximation of the reflected wave. Indeed, an on-surface radiation condition of higher order is necessary to achieve a sufficient accuracy for an angle of incidence not too close to the normal.
Finally, it follows from the above asymptotic analysis for the transmitted wave u^t that the computation of the reflected wave u^s is reduced to just a line integral about the circle S_R, where the integrand is known explicitly.

3 Scattering of acoustic waves by thin finite planar layers

In this section we are interested in the scattering problem by a thin finite planar layer of dielectric material at high frequencies. We shall begin with the manner in which we construct the on-surface radiation conditions for a half space. Let (e_x, e_y) be an orthonormal basis of $\mathbf{R}^2$ related to a system of coordinates (x, y). We consider the outgoing solution of the Helmholtz equation

$$\Delta\, u + k^2\, u = 0,$$

in the half space $y \geq 0$ with the boundary condition $u = f$ on $y = 0$. We denote the dual variable to x by ξ. u can be represented by

$$u(x, y) = \int_{\mathrm{R}} e^{i\sqrt{k^2-\xi^2}\,y}\, e^{i\xi\,x}\, \hat{f}(\xi)\, d\xi.$$

Here ^ denotes Fourier transform in the variable x and

$$\begin{aligned}\sqrt{k^2-\xi^2} &= (k^2-\xi^2)^{1/2}, \quad k \geq |\xi|^2, \\ &= i(\xi^2-k^2)^{1/2}, \quad k \leq |\xi|^2.\end{aligned}$$

It follows that

$$\partial_y\, u(x,y)|_{y=0} = i \int_{\mathrm{R}} \sqrt{k^2-\xi^2}\, e^{i\xi\, x}\, \hat{f}(\xi)\; d\xi.$$

Note that if $f(x) = -e^{ik\cos\theta\, x}$ then $\partial_y\, u(x,y)|_{y=0} = ik\sin\theta\, f(x)$. We take the symbole of this pseudo-differential operator given by $i\sqrt{k^2-\xi^2}$, using the approximation

$$\sqrt{k^2-\xi^2} = k\left(1 - \frac{1}{2}\frac{\xi^2}{k^2}\right) + 0(\frac{\xi^4}{k^4})$$

and recalling that $i\,\xi$ correspond to ∂_x, we obtain
•1^{st} approximation:

$$\partial_y\, u(x,y)|_{y=0} = ik\, u(x,y)|_{y=0}, \tag{17}$$

•2^{nd} approximation:

$$\partial_y\, u(x,y)|_{y=0} = ik\, u(x,y)|_{y=0} + \frac{i}{2k}\,\partial_x^2\, u(x,y)|_{y=0}. \tag{18}$$

Using the next Taylor approximation of the square root

$$\sqrt{k^2-\xi^2} = k\left(1 - \frac{1}{2}\frac{\xi^2}{k^2} + \frac{1}{8}\frac{\xi^4}{k^4}\right) + 0(\frac{\xi^6}{k^6})$$

yields to the boundary condition,
•3^{rd} approximation:

$$\partial_y\, u(x,y)|_{y=0} = ik\, u(x,y)|_{y=0} + \frac{i}{2k}\,\partial_x^2\, u(x,y)|_{y=0} + \frac{i}{8k^3}\,\partial_x^4\, u(x,y)|_{y=0}. \tag{19}$$

More generally, the j^{th} Taylor of the square root

$$\sqrt{k^2-\xi^2} = k\left(1 - \frac{1}{2}\frac{\xi^2}{k^2} + \ldots + \alpha_j\frac{\xi^{2j}}{k^{2j}}\right) + 0(\frac{\xi^{2j+2}}{k^{2j+2}}),$$

yields to the boundary condition,
•j^{th} approximation:

$$\partial_y\, u(x,y)|_{y=0} = ik\, u(x,y)|_{y=0} + \ldots + (-1)^j\,\frac{\alpha_j}{k^{2j-1}}\,\partial_x^{2j}\, u(x,y)|_{y=0}. \tag{20}$$

Now, we consider the scattering problem for the acoustic waves by a thin planar periodic layer. We begin with a description of the geometrical setting of the problem. We assume that $\delta, \nu, d, h, \varepsilon_0, \mu_1, \mu_0$ are strictly positive reals. Let $\Omega^+ = \mathbf{R}\times]0,+\infty[$ be the upper half plan, $\Omega_\delta^- =]0,d[\times]-\delta h, 0[$ and $\mathcal{O} =]0,1[\times]-h,0[$ the unit cell. We set

$$\begin{aligned}
&\Gamma^+ =]d,+\infty[\times\{0\}, \quad \Gamma^- =]-\infty,0[\times\{0\},\\
&\Gamma^0 =]0,d[\times\{0\}, \quad \Gamma^\delta =]0,d[\times\{-\delta h\},\\
&\partial\mathcal{O}^+ =]0,1[\times\{0\}, \quad \partial\mathcal{O}^- =]0,1[\times\{-h\}.
\end{aligned}$$

We also define

$$\Sigma_\delta^+ = \{d\}\times]-\delta h,0[, \quad \Sigma_\delta^- = \{0\}\times]-\delta h,0[.$$

Finally, we set

$$\Gamma^1 = \Gamma^+ \cup \Gamma^-, \quad \Gamma_\delta^2 = \Sigma_\delta^+ \cup \Gamma^\delta \cup \Sigma_\delta^-.$$

We assume that Ω_δ^- is filled with a non-homogeneous dielectric material. We also assume that the electric permittivity of this dielectric material is periodic in the variable x with period δ. The periodic dielectric layer Ω_δ^- is illuminated by an incident plane wave u^{in} given by

$$u^{in}(x,y;\delta) = e^{-i\nu\sqrt{\varepsilon_0\mu_0}(\cos\theta\, x + \sin\theta\, y)/\delta},$$

where θ is the angle of incidence. The wave number k and the angular frequency ω of the incident plane wave are

$$\begin{aligned}
k &= \nu\sqrt{\varepsilon_0\mu_0}\,\delta^{-1},\\
\omega &= \nu\,\delta^{-1}.
\end{aligned}$$

In the dielectric layer Ω_δ^-, a transmitted wave u^t is produced; and in the half plan Ω^+ a reflected wave u^r is produced. The transmitted wave u^t and the reflected wave u^r satisfy

$$\begin{aligned}
\Delta\, u^t(x,y;\delta) + \delta^{-2}\,\nu^2\,\varepsilon\mu_1\, u^t(x,y;\delta) &= 0 \quad \text{in } \Omega_\delta^-,\\
u^t(x,y;\delta) &= 0 \quad \text{on } \Gamma_\delta^2,
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
\Delta\, u^r(x,y;\delta) + \delta^{-2}\,\nu^2\,\varepsilon_0\mu_0\, u^r(x,y;\delta) &= 0 \quad \text{in } \Omega^+,\\
u^r(x,y;\delta) &= 0 \quad \text{on } \Gamma^1.
\end{aligned} \tag{22}$$

We also require the waves u^t and u^r to satisfy the transmission conditions

$$
\begin{aligned}
u^t &= u^r + u^{in} \quad \text{on } \Gamma^0, \\
\frac{1}{\mu_1}\,\partial_y\, u^t &= \frac{1}{\mu_0}\,\partial_y\,(u^r + u^{in}) \quad \text{on } \Gamma^0.
\end{aligned}
\tag{23}
$$

We recall the dielectric coefficient $\varepsilon = \varepsilon(x/\delta, y/\delta)$ is periodic in the variable x/δ with period 1. We also assume that the values of ν and h are fixed. Because the solutions of $(21) - (22) - (23)$ oscillate with a small spatial period, it is impractical to find them by direct numerical methods. We shall now show how to construct asymptotic solutions to the Helmholtz equations $(21) - (22)$ with the transmission conditions (23) for small values of δ. Following the above asymptotic analysis, we shall look for the transmitted wave $u^t = u^t(x, y; \delta)$ in the form of the asymptotic expansion

$$
u^t(x, y; \delta) = e^{-i\nu\sqrt{\varepsilon_0\mu_0}\cos\theta\, x'} \Big(a_0^t(x, x', y') + \delta\, a_1^t(x, x', y') + \dots\Big), \tag{24}
$$

where the new variables x' and y' are defined by $x' = x/\delta, y' = y/\delta$, and the functions a_j^t are periodic in the variable x' with periode 1. Applied to $e^{-i\nu\sqrt{\varepsilon_0\mu_0}\cos\theta\, x/\delta}\, \varphi(x, x/\delta, y/\delta)$ the operator ∂_x (resp. ∂_y) becomes

$$
\partial_x + \delta^{-1}(\partial_{x'} - i\nu\sqrt{\varepsilon_0\mu_0}\cos\theta)
$$

(resp. $\delta^{-1}\,\partial_{y'}$). We rewrite the Helmholtz equation (21) in the form

$$
P^\delta\, u^t = 0,
$$

where

$$
\begin{aligned}
P^\delta &= \delta^{-2}\,(\partial_{x'} - i\nu\sqrt{\varepsilon_0\mu_0}\cos\theta)^2 + \delta^{-2}\,\partial_{y'}^2 + \delta^{-2}\,\nu^2\,\varepsilon\mu_1 \\
&\quad + \delta^{-1}\partial_x\,(\partial_{x'} - i\nu\sqrt{\varepsilon_0\mu_0}\cos\theta) + \partial_x^2.
\end{aligned}
$$

We identify the coefficients of the powers $\delta^{-2}, \delta^{-1}, \delta^0$. This gives

$$
\begin{aligned}
P_0\, a_0^t &= 0 \quad \text{in } \mathcal{O}, \\
a_0^t &= 0 \quad \text{on } \partial\mathcal{O}^-,
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
P_0\, a_1^t &= -P_1\, a_0^t \quad \text{in } \mathcal{O}, \\
a_1^t &= 0 \quad \text{on } \partial\mathcal{O}^-,
\end{aligned}
\tag{26}
$$

and more generally,

$$
\begin{aligned}
P_0\, a^t_{j+2} &= -P_1\, a^t_{j+1} - P_0\, a^t_j \quad \text{in } \mathcal{O}, \\
a^t_{j+2} &= 0 \quad \text{on } \partial\mathcal{O}^-,
\end{aligned}
\tag{27}
$$

where

$$
P_0 = \Big(\partial_{x'} - i\nu\sqrt{\varepsilon_0\mu_0}\cos\theta\Big)^2 + \partial^2_{y'} + \nu^2\,\mu_1\,\varepsilon(x',y'), \tag{28}
$$

$$
P_1 = 2\,\partial_x\,\partial_{x'} - 2i\nu\sqrt{\varepsilon_0\mu_0}\cos\theta\,\partial_x
$$

and

$$
P_2 = \partial^2_x.
$$

To complete the boundary-value problems satisfied by the functions a^t_j we need to specify the boundary conditions on $y' = 0$. The 1^{st} on-surface boundary (17) condition implies that

$$
\frac{\mu_0}{\mu_1}\,\partial_{y'}\, a^t_0 = i\nu\,\sqrt{\varepsilon_0\mu_0}\;a^t_0 + i\nu\,\sqrt{\varepsilon_0\mu_0}\,(1+\sin\theta) \quad \text{on } \partial\mathcal{O}^+, \tag{29}
$$

and more generally,

$$
\frac{\mu_0}{\mu_1}\,\partial_{y'}\, a^t_{j+1} = i\nu\,\sqrt{\varepsilon_0\mu_0}\;a^t_{j+1} \quad \text{on } \partial\mathcal{O}^+, \quad j \geq 0. \tag{30}
$$

The second on-surface boundary condition (18) yields to the approximate boundary conditions on $\partial\mathcal{O}^+$

$$
\begin{aligned}
\frac{\mu_0}{\mu_1}\,\partial_{y'}\, a^t_0 &= i\nu\,\sqrt{\varepsilon_0\mu_0}\;a^t_0 - \frac{i\nu}{2}\sqrt{\varepsilon_0\mu_0}\cos^2\theta\, a^t_0 - i\nu\,\sqrt{\varepsilon_0\mu_0}\,(1+\sin\theta) \\
&\quad + \frac{i\nu}{2}\sqrt{\varepsilon_0\mu_0}\cos^2\theta,
\end{aligned}
\tag{31}
$$

$$
\frac{\mu_0}{\mu_1}\,\partial_{y'}\, a^t_1 = i\nu\,\sqrt{\varepsilon_0\mu_0}\;a^t_1 - \frac{i\nu}{2}\sqrt{\varepsilon_0\mu_0}\cos^2\theta\, a^t_1 + \cos\theta\,\partial_{x'}\, a^t_0, \tag{32}
$$

and

$$
\begin{aligned}
\frac{\mu_0}{\mu_1}\,\partial_{y'}\, a^t_2 &= i\nu\,\sqrt{\varepsilon_0\mu_0}\;a^t_2 - \frac{i\nu}{2}\sqrt{\varepsilon_0\mu_0}\cos^2\theta\, a^t_2 + \cos\theta\,\partial_{x'}\, a^t_1 \\
&\quad + \frac{i}{2\nu\sqrt{\varepsilon_0\mu_0}}\partial^2_{x'}\, a^t_0.
\end{aligned}
\tag{33}
$$

We can also derive other boundary conditions by taking the higher-order on-surface radiation conditions (20). For instance, using the 3^{rd} on-surface boundary condition (19) gives the approximate boundary conditions on $\partial\mathcal{O}^+$

$$\begin{aligned}\frac{\mu_0}{\mu_1}\,\partial_{y'}\,a_0^t &= i\nu\,\sqrt{\varepsilon_0\mu_0}\;a_0^t - \frac{i\nu}{2}\,\sqrt{\varepsilon_0\mu_0}\cos^2\theta\,a_0^t + \frac{i\nu}{8}\sqrt{\varepsilon_0\mu_0}\cos^4\theta\,a_0^t\\ &\quad+\frac{i\nu}{2}\,\sqrt{\varepsilon_0\mu_0}\cos^2\theta + i\nu\,\sqrt{\varepsilon_0\mu_0}\,(1+\sin\theta) - \frac{i\nu}{8}\,\sqrt{\varepsilon_0\mu_0}\cos^4\theta,\end{aligned} \tag{34}$$

$$\begin{aligned}\frac{\mu_0}{\mu_1}\,\partial_{y'}\,a_1^t &= i\nu\,\sqrt{\varepsilon_0\mu_0}\;a_1^t - \frac{i\nu}{2}\,\sqrt{\varepsilon_0\mu_0}\cos^2\theta\,a_1^t + \frac{i\nu}{8}\sqrt{\varepsilon_0\mu_0}\cos^4\theta\,a_1^t\\ &\quad+\frac{1}{8}\cos^3\theta\,\partial_{x'}\,a_0^t,\end{aligned} \tag{35}$$

and

$$\begin{aligned}\frac{\mu_0}{\mu_1}\,\partial_{y'}\,a_2^t &= i\nu\,\sqrt{\varepsilon_0\mu_0}\;a_2^t - \frac{i\nu}{2}\,\sqrt{\varepsilon_0\mu_0}\cos^2\theta\,a_2^t + \frac{i\nu}{8}\sqrt{\varepsilon_0\mu_0}\cos^4\theta\,a_2^t\\ &\quad+\frac{1}{8}\cos^3\theta\,\partial_{x'}\,a_1^t - \frac{\cos^2\theta}{2\nu\sqrt{\varepsilon_0\mu_0}}\,\partial_{x'}\,a_0^t.\end{aligned} \tag{36}$$

We first note these approximate boundary conditions follow from $(11)-(14)$ by taking $R=+\infty$. Then, if we compare $(29), (31)$ and (34) we see that they disgree only in terms which involve $\cos\theta$ where θ is the angle of incidence. It follows that if the incident wave is normal to the thin dielectric layer, these approximations obtained by using different order of the on-surface boundary conditions are exactly the same. The same remark holds for $(30)(j=0), (32)$ and (35). We claim that at normal incidence the extra terms due to the higher order on-surface boundary conditions do not play any significant role. But, if the angle of incidence is not too close to the normal these terms can not be neglected. In this case, we must use higher order on-surface boundary conditions (20). For a θ angle of incidence the on-surface radiation condition (20) to be used have to satisfy the criterion $j\cos^{2j}\theta$ of order of the desired precision. We illustrate this in the following example. For a $\pi/4$ angle of incidence, the difference between the third on surface boundary condition (19) and (20) (for $j\geq 4$) is only of 5% of the amplitude of the transmitted wave. This indicates to be careful in choosing the appropriate on-surface radiation condition which must to be function of the angle of incidence in order to garantee a good approximation of the reflected and transmitted waves.

4 Scattering of electromagnetic waves by thin finite planar layers

The technique developed in the preceding sections is capable of yielding efficient numerical methods for other scattering problems. This section will briefly consider the scattering problem for the full Maxwell equations by thin planar periodic layer of dielectric material. As in the previous sections, we begin by deriving a set of on-surface boundary conditions. Let (e_x, e_y, e_z) be an orthonormal basis of $\mathbf{R}^3$ related to a system of coordinates (x, y, z). Let $(\mathbf{E}, \mathbf{H})$ be the outgoing solution of the Maxwell system in the upper half space

$$\begin{aligned}
\mathbf{curl}\,\mathbf{E} &= i\omega\,\mu_0\,\mathbf{H} \quad \text{in } y > 0,\\
\mathbf{curl}\;\mathbf{H} &= -i\omega\,\varepsilon_0\,\mathbf{E} \quad \text{in } y > 0,\\
-\mathbf{e}_y \wedge (\mathbf{e}_y \wedge \mathbf{E}) &= \mathbf{g} = g_x\,\mathbf{e}_x + g_z\,\mathbf{e}_z \quad \text{on } y = 0.
\end{aligned}$$

We can show that the tangential component of the magnetic field on $y = 0$ is given by

$$-\mathbf{e}_y \wedge (\mathbf{e}_y \wedge \mathbf{H})|_{y=0} = -\frac{i}{k}\sqrt{\frac{\varepsilon_0}{\mu_0}} \begin{pmatrix} \partial_y\,E_z|_{y=0} - \partial_z\,E_y|_{y=0} \\ 0 \\ \partial_x\,E_y|_{y=0} - \partial_y\,E_x|_{y=0} \end{pmatrix},$$

where $k = \omega\sqrt{\varepsilon_0\mu_0}$. Since the components (E_x, E_y, E_z) of the electric field E satisfy the Helmholtz equations

$$\begin{aligned}
&\Delta\,E_x + k^2\,E_x = 0 \quad \text{in } y > 0,\\
&E_x = g_x \quad \text{on } y = 0,
\end{aligned}$$

$$\begin{aligned}
&\Delta\,E_z + k^2\,E_z = 0 \quad \text{in } y > 0,\\
&E_z = g_z \quad \text{on } y = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\Delta\,E_y + k^2\,E_y = 0 \quad \text{in } y > 0,\\
&\partial_y\,E_y = -(\partial_x\,E_x + \partial_z\,E_z) \quad \text{on } y = 0,
\end{aligned}$$

where $k^2 = \omega^2\,\varepsilon_0\mu_0$, we have

$$\partial_y\,E_x|_{y=0} = i\int_{\mathrm{R}}\int_{\mathrm{R}} \sqrt{k^2 - \xi_x^2 - \xi_z^2}\; e^{i\xi_x\,x + i\xi_z\,z}\;\hat{g}_x(\xi_x, \xi_z)\;d\xi_x\;d\xi_z,$$

$$\partial_y E_z|_{y=0} = i \int_{\mathrm{R}} \int_{\mathrm{R}} \sqrt{k^2 - \xi_x^2 - \xi_z^2}\, e^{i\xi_x x + i\xi_z z}\, \hat{g}_z(\xi_x, \xi_z)\, d\xi_x\, d\xi_z,$$

and

$$E_y|_{y=0} = -i \int_{\mathrm{R}} \int_{\mathrm{R}} \frac{1}{\sqrt{k^2 - \xi_x^2 - \xi_z^2}}\, e^{i\xi_x x + i\xi_z z} \Big(\xi_x\, \hat{g}_x + \xi_z\, \hat{g}_z\Big)(\xi_x, \xi_z)\, d\xi_x\, d\xi_z.$$

A similar approach as in the above section yields to the following approximation:

$$\partial_y E_x|_{y=0} = ik\, E_x|_{y=0} + \frac{i}{2k}\,(\partial_x^2 + \partial_z^2)\, E_x|_{y=0} + \frac{i}{8k^3}(\partial_x^2 + \partial_z^2)^2\, E_x|_{y=0},$$

$$\partial_y E_z|_{y=0} = ik\, E_z|_{y=0} + \frac{i}{2k}\,(\partial_x^2 + \partial_z^2)\, E_z|_{y=0} + \frac{i}{8k^3}(\partial_x^2 + \partial_z^2)^2\, E_z|_{y=0},$$

and

$$E_y|_{y=0} = -\frac{1}{k}(\partial_x\, E_x|_{y=0} + \partial_z\, E_z|_{y=0}) + \frac{1}{2k^3}\,(\partial_x^3\, E_x|_{y=0} + \partial_x^2 \partial_z\, E_z|_{y=0}).$$

From these approximations, we deduce that the on-surface radiation conditions are in the form:

$$-\mathbf{e}_y \wedge (\mathbf{e}_y \wedge \mathbf{H})|_{y=0} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, \mathcal{B}_j(\mathbf{e}_y \wedge \mathbf{E}|_{y=0}),$$

where the differential operators $\mathcal{B}_j$ are given for $j = 1$ to 3 by

$$\mathcal{B}_1(\mathbf{u}) = \mathbf{u},$$

$$\mathcal{B}_2(\mathbf{u}) = \mathbf{u} + \frac{1}{k^2}\,(\partial_x^2 + \partial_z^2)\,\mathbf{u} + \frac{i}{k^2}\begin{pmatrix} -\partial_z^2 & \partial_{xz}^2 \\ \partial_{xz}^2 & -\partial_x^2 \end{pmatrix}\mathbf{u},$$

and

$$\begin{aligned}\mathcal{B}_3(\mathbf{u}) \;&=\; \mathbf{u} + \frac{1}{k^2}\,(\partial_x^2 + \partial_z^2)\,\mathbf{u} + \frac{i}{k^2}\begin{pmatrix} -\partial_z^2 & \partial_{xz}^2 \\ \partial_{xz}^2 & -\partial_x^2 \end{pmatrix}\mathbf{u} \\ &\quad + \frac{1}{8k^4}(\partial_x^4 + \partial_z^4)\,\mathbf{u} + \frac{i}{2k^4}\begin{pmatrix} -\partial_x^2\,\partial_z^2 & \partial_x^3\,\partial_z \\ \partial_x^3\,\partial_z & -\partial_x^4 \end{pmatrix}\mathbf{u}.\end{aligned}$$

Now, we consider the scattering problem for Maxwell's equations by a thin planar periodic layer. Let $\delta, \nu, d_x, d_z, h, \varepsilon_0, \mu_0$ be strictly positive reals.

Let $\Omega^+ = \mathbf{R}\times]0,+\infty[\times\mathbf{R}$ be the upper half space, Ω_δ^- the dielectric layer given by

$$\Omega_\delta^- =]0,d_x[\times]-\delta h,0[\times]0,d_z[$$

and $\mathcal{O} =]0,1[\times]-h,0[\times]0,1[$ the unit cell. We set $\Gamma^0 =]0,d_x[\times\{0\}\times]0,d_z[$,

$$\Gamma^\delta =]0,d_x[\times\{-\delta h\}\times]0,d_z[, \quad \Gamma^1 = \mathbf{R}\times\{0\}\times\mathbf{R}\setminus\overline{\Gamma^0}.$$

We also introduce

$$\begin{aligned}
\Sigma_\delta^1 &= \{0\}\times]-\delta h,0[\times]0,d_z[\cup\{d_x\}\times]-\delta h,0[\times]0,d_z[,\\
\Sigma_\delta^2 &=]0,d_x[\times]-\delta h,0[\times\{0\}\cup]0,d_x[\times]-\delta h,0[\times\{d_z\}.
\end{aligned}$$

Finally, we set

$$\Sigma_\delta = \Sigma_\delta^1\cup\Sigma_\delta^2, \quad \partial\mathcal{O}^+ =]0,1[\times\{0\}\times]0,1[, \quad \partial\mathcal{O}^- =]0,1[\times\{-h\}\times]0,1[.$$

An incident plane wave

$$\mathbf{E}^{in} = \mathbf{E}_0^{in}\, e^{i\nu\sqrt{\varepsilon_0\mu_0}(p_x x+p_y y+p_z z)/\delta},$$

impinges on the dielectric layer Ω_δ^-. Then, the reflected fields $(\mathbf{E}^r,\mathbf{H}^r)$ and the transmited fields $(\mathbf{E}^t,\mathbf{H}^t)$ satisfy

$$\begin{aligned}
\mathbf{curl}\,\mathbf{E}^r &= i\omega\,\mu_0\,\mathbf{H}^r \quad \text{in } \Omega^+,\\
\mathbf{curl}\,\mathbf{H}^r &= -i\omega\,\varepsilon_0\,\mathbf{E}^r \quad \Omega^+,
\end{aligned}$$

$$\begin{aligned}
\mathbf{curl}\,\mathbf{E}^t &= i\omega\,\mu\,\mathbf{H}^t \quad \text{in } \Omega_\delta^-,\\
\mathbf{curl}\,\mathbf{H}^t &= -i\omega\,\varepsilon\,\mathbf{E}^t \quad \text{in } \Omega_\delta^-,
\end{aligned}$$

with the transmission conditions

$$\begin{aligned}
(\mathbf{E}^r+\mathbf{E}^{in})\wedge e_y &= \mathbf{E}^t\wedge\mathbf{e}_y \quad \text{on } \Gamma^0,\\
(\mathbf{H}^r+\mathbf{H}^{in})\wedge\mathbf{e}_y &= \mathbf{H}^t\wedge\mathbf{e}_y \quad \text{on } \Gamma^0.
\end{aligned}$$

Expanding $(\mathbf{E}^t,\mathbf{H}^t)$ in the following form

$$\begin{aligned}
\mathbf{E}^t(x,y,z) &= e^{i\nu\sqrt{\varepsilon_0\mu_0}(p_x x'+p_z z')}\Big(\mathbf{E}_0^t(x',y',z')+\delta\,\mathbf{E}_1^t(x',y',z')+\dots\Big),\\
\mathbf{H}^t(x,y,z) &= e^{i\nu\sqrt{\varepsilon_0\mu_0}(p_x x'+p_z z')}\Big(\mathbf{H}_0^t(x',y',z')+\delta\,\mathbf{H}_1^t(x',y',z')+\dots\Big),
\end{aligned} \tag{37}$$

we get

$$\begin{aligned}\mathbf{curl}'\,\mathbf{E}_j^t &= i\nu\,\mu(x',y',z')\,\mathbf{H}_j^t \quad \text{in } \mathcal{O}^-,\\ \mathbf{curl}'\,\mathbf{H}_j^t &= -i\nu\,\varepsilon(x',y',z')\,\mathbf{E}_j^t \quad \text{in } \mathcal{O}^-,\end{aligned}$$

where the operator $\mathbf{curl}'$ is defined by

$$\mathbf{curl}' = \left|\begin{array}{l}\partial_{x'} + i\nu\sqrt{\varepsilon_0\mu_0}\,p_x\\ \partial_{y'}\\ \partial_{z'} + i\nu\sqrt{\varepsilon_0\mu_0}\,p_z\end{array}\right. \wedge$$

The approximate boundary condition

$$-\mathbf{e}_y \wedge (\mathbf{e}_y \wedge \mathbf{H}^r) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\,\mathcal{B}_j(\mathbf{e}_y \wedge \mathbf{E}^r) \quad \text{on } \Gamma^0,$$

gives

$$-\mathbf{e}_y \wedge (\mathbf{e}_y \wedge \mathbf{H}^t) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\,\mathcal{B}_j(\mathbf{e}_y \wedge \mathbf{E}^t) - \mathbf{e}_y \wedge (\mathbf{e}_y \wedge \mathbf{H}^{in}) - \sqrt{\frac{\varepsilon_0}{\mu_0}}\,\mathcal{B}_j(\mathbf{e}_y \wedge \mathbf{E}^{in}) \quad \text{on } \Gamma^0.$$

Therefore, by inserting (37) into the above approximation and identifying the coefficients of the powers δ^{-1}, δ^0, we obtain the following approximate boundary conditions on Γ^0.

•1^{st} approximation:

$$-\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\mathbf{e}_y \wedge (\mathbf{e}_y \wedge \mathbf{H}_0^t) = \mathbf{e}_y \wedge \mathbf{E}_0^t - \mathbf{e}_y \wedge (\mathbf{e}_y \wedge \mathbf{H}_0^{in}) - \mathbf{e}_y \wedge \mathbf{E}_0^{in}.$$

•2^{nd} approximation:

$$\begin{aligned}-\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\mathbf{e}_y \wedge (\mathbf{e}_y \wedge \mathbf{H}_0^t) &= \mathbf{e}_y \wedge \mathbf{E}_0^t - \mathbf{e}_y \wedge (\mathbf{e}_y \wedge \mathbf{H}_0^{in}) - \mathbf{e}_y \wedge \mathbf{E}_0^{in}\\ &\quad -\nu^2\,\varepsilon_0\mu_0(p_x^2+p_z^2)\,\mathbf{e}_y \wedge (\mathbf{E}_0^t - \mathbf{E}_0^{in})\\ &\quad +i\varepsilon_0\mu_0\begin{pmatrix} p_z^2 & -p_z\,p_x\\ -p_z\,p_x & p_x^2\end{pmatrix}\mathbf{e}_y \wedge (\mathbf{E}_0^t - \mathbf{E}_0^{in}).\end{aligned}$$

$$\begin{aligned}-\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\mathbf{e}_y \wedge (\mathbf{e}_y \wedge \mathbf{H}_1^t) &= \mathbf{e}_y \wedge \mathbf{E}_1^t - \nu^2\,\varepsilon_0\mu_0(p_x^2+p_z^2)\,\mathbf{e}_y \wedge \mathbf{E}_1^t\\ &\quad +i\varepsilon_0\mu_0\begin{pmatrix} p_z^2 & -p_z\,p_x\\ -p_z\,p_x & p_x^2\end{pmatrix}\mathbf{e}_y \wedge \mathbf{E}_1^t\\ &\quad +\frac{\sqrt{\varepsilon_0\mu_0}}{\nu}\begin{pmatrix} p_z\,\partial_{z'} & -(p_z\,\partial_{x'} + p_x\partial_{z'})\\ -(p_z\,\partial_{x'} + p_x\partial_{z'}) & p_x\,\partial_{x'}\end{pmatrix}\mathbf{e}_y \wedge \mathbf{E}_0^t\end{aligned}$$

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