THE LEONTOVICH BOUNDARY VALUE PROBLEM FOR THE TIME-HARMONIC MAXWELL EQUATIONS

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ABSTRACT. The limiting behavior of the unique solution to Maxwell's equations in an exterior domain with a Leontovich boundary condition as the impedance tends to zero is investigated. This is accomplished by reducing the impedance boundary value problem for the Maxwell equations to a system of three integral equations. It is shown that the Leontovich boundary condition leads to a singular perturbation problem for the Maxwell equations. A specific numerical treatment is required to achieve a sufficient accuracy.

KEY WORDS. Leontovich boundary condition, Maxwell's equations, singular perturbation problem, integral equations.

AMS SUBJECT CLASSIFICATIONS. 35Q60, 35B40, 35C15 .

1 INTRODUCTION

A lossy body composed of a material whose permittivity and permeability may vary as a function of position is immersed in free space and illuminated by an electromagnetic field. At high conductivity, the lossy dielectric body can be replaced by a Leontovich boundary condition applied at its boundary. This approximate boundary condition provides an approximate relationship between the electric and magnetic field on the boundary of the high conducting body.

Our main purpose in this paper is to show how this widely used approximate boundary condition yields to a singular perturbation problem for the Maxwell equations. A specific numerical treatment of the Leontovich boundary value problem for the time-harmonic Maxwell equations is then required to achieve a sufficient accuracy.

One of the first mathematical treatments of the Leontovich boundary value problem is given by Colton and Kress in [8]. Reducing this problem to that of solving a system of singular integral equations and then regularizing this system, these authors proved the existence of a unique solution for all frequencies in Hölder spaces. Here, our analysis also provides a new existence result in Sobolev spaces.

We shall briefly outline the remainder of this paper. The notations are introduced in section 2. In section 3 the main results proved in this paper are formulated. In section 4, the Leontovich boundary value problem for the time-harmonic Maxwell equations is reduced to a system of three integral equations. Section 5 is devoted to prove the convergence result and to construct the complete asymptotic expansions. In section 6, a specific numerical treatment of the Leontovich boundary condition is briefly discussed. In an appendix, the Leontovich boundary condition is derived by making use of the well-known two scale asymptotic expansion method.

2 NOTATIONS

Let Ω^i be a bounded domain in $\mathbf{R}^3$ and let Ω be the complement of $\overline{\Omega^i}$ in $\mathbf{R}^3$. The boundary Γ of Ω is assumed to be of class $\mathcal{C}^\infty$. By $\mathbf{n}$ we denote the unit normal to Γ directed into Ω. Let $\mathbf{R}(\mathbf{x})$ (resp. $c(\mathbf{x})$) be the tensor (resp. the mean value) of curvature at the point $\mathbf{x}$ on Γ. Throughout this paper, for any smooth vector field $\mathbf{u}$, we denote by $\mathbf{u}_\Gamma$ its tangential component on Γ. By $\mathbf{grad}_\Gamma$ (resp. div_Γ) we denote the tangential gradient of a function (resp. the surface divergence of a field). We also denote by $\mathbf{curl}_\Gamma$ (resp. curl_Γ) the vector rotational of a function (resp. the scalar rotational of a vector) and by Δ_Γ the Laplace-Beltrami operator on Γ. Let $\mathcal{N}(\Gamma) = \Big\{\alpha, \Delta_\Gamma\, \alpha = 0\Big\}$ be the space of harmonic fields on Γ. It is well known that $\mathcal{N}(\Gamma)$ is a finite-dimensional vectorial space and its dimension is the first Betti number. We shall also use the following notations for the usual functional spaces : $L^2(\Omega)$ is the space of complex square integrable functions defined in Ω, $L^\infty(\Gamma)$ the space of functions bounded on Γ.

$H(\mathrm{curl}, \Omega) = \Big\{\mathbf{u} \in (L^2(\Omega))^3, \mathbf{curl}\,\mathbf{u} \in (L^2(\Omega))^3\Big\}$, $\overline{H}^{\mathrm{loc}}(\mathrm{curl}, \Omega)$ is the Fréchet space of fields which are locally in $H(\mathrm{curl}, \Omega)$, $H^s(\Gamma)$ the Sobolev space of order s on Γ, $H_0^s(\Gamma) = \Big\{\varphi \in H^s(\Gamma), \langle\varphi, 1\rangle = 0\Big\}$,

$TL^2(\Gamma) = \Big\{\mathbf{u} \in (L^2(\Gamma))^3, \mathbf{u}\,.\,\mathbf{n} = 0\Big\}$,

$TH^s(\Gamma) = \Big\{\mathbf{u} \in (H^s(\Gamma))^3, \mathbf{u}\,.\,\mathbf{n} = 0\Big\}$,

$$
\begin{aligned}
TL^\infty(\Gamma) &= \Big\{ \mathbf{u} \in (L^\infty(\Gamma))^3, \mathbf{u} \,.\, \mathbf{n} = 0 \Big\}, \\
TH^s(\mathrm{curl}, \Gamma) &= \Big\{ \mathbf{u} \in (H^s(\Gamma))^3, \mathbf{u} \,.\, \mathbf{n} = 0, \mathrm{curl}_\Gamma \, \mathbf{u} \in H^s(\Gamma) \Big\}, \\
TH^s(\mathrm{div}, \Gamma) &= \Big\{ \mathbf{u} \in (H^s(\Gamma))^3, \mathbf{u} \,.\, \mathbf{n} = 0, \mathrm{div}_\Gamma \, \mathbf{u} \in H^s(\Gamma) \Big\}, \\
\mathcal{X}(\Gamma) &= \Big\{ \mathbf{u} \in (L^2(\Gamma))^3, \mathbf{u} \,.\, \mathbf{n} = 0, \mathrm{div}_\Gamma \, \mathbf{u} \in H^{-1/2}(\Gamma) \Big\}, \\
\mathcal{Y}(\Gamma) &= \Big\{ \mathbf{u} \in (H^{-1/2}(\Gamma))^3, \mathbf{u} \,.\, \mathbf{n} = 0, \mathrm{curl}_\Gamma \, \mathbf{u} \in H^{-1}(\Gamma) \Big\}, \\
\mathcal{Z}(\Gamma) &= \Big\{ \mathbf{u} \in (H^{-1/2}(\Gamma))^3, \mathbf{u} \,.\, \mathbf{n} = 0, \mathrm{curl}_\Gamma \, \mathbf{u} \in H^{1/2}(\Gamma) \Big\}.
\end{aligned}
$$

3 MAIN RESULTS

Let z_0 be a strictly positive constant. Let z be a given function on Γ with the property $\Im m(z) \geq 0$, $\Re e(z) \geq z_0$ and let k and δ be two strictly positive constants. The Leontovich boundary value problem for the time-harmonic Maxwell equations can be formulated as follows: Find E^δ and H^δ satisfying :

$$
\left\{
\begin{aligned}
\mathbf{curl}\,\mathbf{E}^\delta &= ik\,\mathbf{H}^\delta \quad \text{in } \Omega, \\
\mathbf{curl}\;\mathbf{H}^\delta &= -ik\;\mathbf{E}^\delta \quad \text{in } \Omega,
\end{aligned}
\right. \tag{1}
$$

the classical Silver-Müller radiation condition and the impedance boundary condition:

$$
\mathbf{n} \wedge (\mathbf{n} \wedge \mathbf{E}^\delta) + i\delta\, z\; \mathbf{H}^\delta \wedge \mathbf{n} \;=\; \mathbf{g}^\delta \quad \text{on } \Gamma, \tag{2}
$$

where $\mathbf{g}^\delta$ is a given tangential field on the boundary Γ.

Let $\mathbf{g}^0$ be a given tangential field on Γ. Let $(\mathbf{E}^0, \mathbf{H}^0)$ be the unique solution of the time-harmonic Maxwell equations with the Dirichlet boundary conditions:

$$
\left\{
\begin{aligned}
\mathbf{curl}\,\mathbf{E}^0 &= ik\,\mathbf{H}^0 \quad \text{in } \Omega, \\
\mathbf{curl}\;\mathbf{H}^0 &= -ik\,\mathbf{E}^0 \quad \text{in } \Omega, \\
\mathbf{n} \wedge (\mathbf{n} \wedge \mathbf{E}^0) &= \mathbf{g}^0 \quad \text{on } \Gamma,
\end{aligned}
\right. \tag{3}
$$

satisfying the classical Silver-Müller radiation condition. We shall prove the following convergence result.

THEOREM 3.1 *Assume that the field $\mathbf{g}^\delta$ converges to a given tangential field $\mathbf{g}^0$ on Γ in $TH^{-1/2}(\mathrm{curl}, \Gamma)$. Then $(\mathbf{E}^\delta, \mathbf{H}^\delta)$ converges strongly to $(\mathbf{E}^0, \mathbf{H}^0)$ in $\overline{H}^{\mathrm{loc}}(\mathrm{curl}, \Omega)$. Furthermore, if $g^0 \in \mathcal{Z}(\Gamma)$ then the rate of the convergence is of order $\delta + ||\mathbf{g}^\delta - \mathbf{g}^0||_{TH^{-1/2}(\mathrm{curl},\Gamma)}$.*

We also construct the complete asymptotic expansions of the fields $(\mathbf{E}^\delta, \mathbf{H}^\delta)$. These expansions depend on the smoothness of the given tangential field $\mathbf{g}^\delta$.

4 INTEGRAL EQUATION METHOD

Our basic aim in this section is to reduce the boundary value problem $(1)-(2)$ to a Fredholm system of integral equations on Γ. The main advantage of this integral approach lies in the fact that it reduces the vector problem $(1)-(2)$ defined over the unbounded domain Ω to essentially a system of two scalar equations defined on Γ. This system is used to establish the existence of a solution to $(1)-(2)$. We first consider the question of uniqueness of a solution to the Leontovich boundary value problem. From [8] we can establish by Rellich's Lemma the following uniqueness result:

LEMMA 4.1 *The Maxwell system* (1) *with the impedance boundary conditions* (2) *admits at most one solution* $(\mathbf{E}^\delta, \mathbf{H}^\delta)$ *satisfying the Silver-Müller radiation condition.*

Let $\Phi(\mathbf{x},\mathbf{y}) = e^{ik|\mathbf{x}-\mathbf{y}|}/4\pi|\mathbf{x}-\mathbf{y}|, \mathbf{x} \neq \mathbf{y}$ denote the fundamental outgoing solution to the Helmholtz equation and let $\Phi_0(\mathbf{x},\mathbf{y}) = 1/4\pi|\mathbf{x}-\mathbf{y}|$. It is well-known that the electric field E^δ solution of the Maxwell equations (1) can be represented in Ω in terms of the tangential fields $\mathbf{m}^\delta = \mathbf{E}^\delta \wedge \mathbf{n}$ and $\mathbf{j}^\delta = \mathbf{H}^\delta \wedge \mathbf{n}$ on Γ:

$$\begin{aligned} \mathbf{E}^\delta(\mathbf{x}) &= -\mathbf{curl} \int_\Gamma \mathbf{m}^\delta(\mathbf{y})\Phi(\mathbf{x},\mathbf{y})d\gamma(\mathbf{y}) \\ &\quad - \frac{i}{k}\,\mathbf{grad} \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\,\mathrm{div}_\Gamma \mathbf{j}^\delta(\mathbf{y})d\gamma(\mathbf{y}) \\ &\quad - ik \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\,\mathbf{j}^\delta(\mathbf{y})\,d\gamma(\mathbf{y}). \end{aligned} \tag{4}$$

Taking the limit of (4) as $\mathbf{x}$ tends to Γ and using the impedance boundary condition (2) yields to the integral equation:

$$\begin{aligned} &\frac{i}{2}\delta\, z(\mathbf{x})\,\mathbf{j}^\delta(\mathbf{x}) - i\delta \Big(\int_\Gamma z(\mathbf{y})\,\mathbf{grad}_x\,\Phi(\mathbf{x},\mathbf{y}) \wedge (\mathbf{j}^\delta(\mathbf{y}) \wedge \mathbf{n}(\mathbf{y}))\,d\gamma(\mathbf{y})\Big)|_\Gamma \\ &+ i\,k \Big(\int_\Gamma \Phi(\mathbf{x},\mathbf{y})\,\mathbf{j}^\delta(\mathbf{y})\,d\gamma(\mathbf{y})\Big)|_\Gamma \\ &\quad + \frac{i}{k}\mathbf{grad}_\Gamma \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\,\mathrm{div}_\Gamma\,\mathbf{j}^\delta(\mathbf{y})\,d\gamma(\mathbf{y}) = \frac{1}{2}\mathbf{g}^\delta(\mathbf{x}) \\ &\quad - \Big(\int_\Gamma \mathbf{grad}_x\,\Phi(\mathbf{x},\mathbf{y}) \wedge (\mathbf{g}^\delta(\mathbf{y}) \wedge \mathbf{n}(\mathbf{y}))\,d\gamma(\mathbf{y})\Big)|_\Gamma, \end{aligned} \tag{5}$$

where the tangential field $\mathbf{j}^\delta$ is the unknown. Similarly, the integral equation for the Dirichlet boundary value problem (3) is

$$\begin{aligned} i\,k\Big(\int_\Gamma \Phi(\mathbf{x},\mathbf{y})\,\mathbf{j}^0(\mathbf{y})\;d\gamma(\mathbf{y})\Big)|_\Gamma + \frac{i}{k}\mathbf{grad}_\Gamma \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\,\mathrm{div}_\Gamma\,\mathbf{j}^0(\mathbf{y})\;d\gamma(\mathbf{y}) \\ = \frac{1}{2}\mathbf{g}^0(\mathbf{x}) - \Big(\int_\Gamma \mathbf{grad}_x\,\Phi(\mathbf{x},\mathbf{y})\;\wedge\;(\mathbf{g}^0(\mathbf{y})\;\wedge\;\mathbf{n}(\mathbf{y}))\;d\gamma(\mathbf{y})\Big)|_\Gamma. \end{aligned} \tag{6}$$

The existence and uniqueness results for the above integral equation were given by Bendali in [6] for both the continuous and discrete case.
Now, multiplying (5) by a test field $\mathbf{u}^t$ in $\mathcal{X}(\Gamma)$ and integrating by parts gives :

$$\delta\,a^{(1)}(\mathbf{j}^\delta,\mathbf{u}^t) + a^{(2)}(\mathbf{j}^\delta,\mathbf{u}^t) \;=\; \langle \mathbf{f}^\delta,\mathbf{u}^t\rangle, \tag{7}$$

where

$$\begin{aligned}
a^{(1)}(\mathbf{u},\mathbf{u}^t) \;&=\; \frac{i}{2}\int_\Gamma z(\mathbf{x})\,\mathbf{u}(\mathbf{x})\,.\,\mathbf{u}^t(\mathbf{x})\;d\gamma(\mathbf{x}) \\
&+ \int_\Gamma\int_\Gamma z(\mathbf{y})\,\Phi(\mathbf{x},\mathbf{y})\,\mathbf{u}(\mathbf{y})\;\wedge\;\mathbf{n}(\mathbf{y})\,.\,(\mathbf{n}(\mathbf{x})-\mathbf{n}(\mathbf{y}))\,\mathrm{curl}_\Gamma\,\mathbf{u}^t(\mathbf{x})\;d\gamma(\mathbf{y})\;d\gamma(\mathbf{x}) \\
&+ \int_\Gamma\int_\Gamma z(\mathbf{y})\,(\mathbf{R}-2c)\Big(\Phi(\mathbf{x},\mathbf{y})(\mathbf{u}(\mathbf{y})\;\wedge\;\mathbf{n}(\mathbf{y}))\Big)\,.\,\mathbf{u}^t(\mathbf{x})\;d\gamma(\mathbf{y})\;d\gamma(\mathbf{x}) \\
&- \int_\Gamma\int_\Gamma z(\mathbf{y})\,\partial_{n_x}\Phi(\mathbf{x},\mathbf{y})(\mathbf{u}(\mathbf{y})\;\wedge\;\mathbf{n}(\mathbf{y}))\,.\;\mathbf{u}^t(\mathbf{x})\;d\gamma(\mathbf{y})\;d\gamma(\mathbf{x}), \\
a^{(2)}(\mathbf{u},\mathbf{u}^t) \;&=\; ik\int_\Gamma\int_\Gamma \Phi(\mathbf{x},\mathbf{y})\,\mathbf{u}(\mathbf{y})\,.\,\mathbf{u}^t(\mathbf{x})\;d\gamma(\mathbf{y})\;d\gamma(\mathbf{x}) \\
&- \frac{i}{k}\int_\Gamma\int_\Gamma \Phi(\mathbf{x},\mathbf{y})\,\mathrm{div}_\Gamma\,\mathbf{u}(\mathbf{y})\,\mathrm{div}_\Gamma\,\mathbf{u}^t(\mathbf{x})\;d\gamma(\mathbf{y})\;d\gamma(\mathbf{x}), \\
\langle \mathbf{f}^\delta,\mathbf{u}^t\rangle \;&=\; \frac{1}{2}\int_\Gamma \mathbf{g}^\delta(\mathbf{x})\,.\,\mathbf{u}^t(\mathbf{x})\;d\gamma(\mathbf{x}) \\
&- \int_\Gamma\int_\Gamma \mathbf{grad}_x\,\Phi(\mathbf{x},\mathbf{y})\;\wedge\;(\mathbf{g}^\delta(\mathbf{y})\;\wedge\;\mathbf{n}(\mathbf{y}))\,.\,\mathbf{u}^t(\mathbf{x})\;d\gamma(\mathbf{y})\;d\gamma(\mathbf{x}).
\end{aligned}$$

Next, making use of the Hodge decomposition of the Sobolev space $\mathcal{X}(\Gamma)$ (see for instance [1] and [4]), we decompose the tangential fields $\mathbf{j}^\delta$ and $\mathbf{u}^t$ in the following form:

$$\begin{aligned}
\mathbf{j}^\delta \;&=\; \mathbf{curl}_\Gamma\,\varphi^\delta + \mathbf{grad}_\Gamma\,\psi^\delta + \alpha^\delta, \\
\mathbf{u}^t \;&=\; \mathbf{curl}_\Gamma\,\varphi^t + \mathbf{grad}_\Gamma\,\psi^t + \alpha^t,
\end{aligned}$$

where the scalar functions $(\varphi^\delta, \psi^\delta)$ and (φ^t, ψ^t) are in $H_0^1(\Gamma) \times H_0^{3/2}(\Gamma)$ and the harmonic fields α^δ and α^t are in $\mathcal{N}(\Gamma)$. This yields to a system of three variational integral equations on Γ given by

$$\begin{aligned}\delta\, a_1^{(1)}(\varphi^\delta, \varphi^t) + a_1^{(2)}(\varphi^\delta, \varphi^t) + \delta b_1^{(1)}(\psi^\delta, \varphi^t) + b_1^{(2)}(\psi^\delta, \varphi^t) \\ +\delta c_1^{(1)}(\alpha^\delta, \varphi^t) + c_1^{(2)}(\alpha^\delta, \varphi^t) \ = \ \langle \mathbf{f}^\delta, \mathbf{curl}_\Gamma\, \varphi^t \rangle,\end{aligned} \tag{8}$$

$$\begin{aligned}\delta\, a_2^{(1)}(\psi^\delta, \psi^t) + a_2^{(2)}(\psi^\delta, \psi^t) + \delta b_2^{(1)}(\varphi^\delta, \psi^t) + b_2^{(2)}(\varphi^\delta, \psi^t) \\ +\delta c_2^{(1)}(\alpha^\delta, \psi^t) + c_2^{(2)}(\alpha^\delta, \psi^t) \ = \ \langle \mathbf{f}^\delta, \mathbf{grad}_\Gamma\, \psi^t \rangle,\end{aligned} \tag{9}$$

$$\begin{aligned}\delta\, a_3^{(1)}(\alpha^\delta, \alpha^t) + a_3^{(2)}(\alpha^\delta, \alpha^t) + \delta b_3^{(1)}(\varphi^\delta, \alpha^t) + b_3^{(2)}(\varphi^\delta, \alpha^t) \\ +\delta c_3^{(1)}(\psi^\delta, \alpha^t) + c_3^{(2)}(\psi^\delta, \alpha^t) \ = \ \langle \mathbf{f}^\delta, \alpha^t \rangle,\end{aligned} \tag{10}$$

where

$$\begin{aligned}
a_1^{(j)}(\varphi, \varphi^t) &= a^{(j)}(\mathbf{curl}_\Gamma\, \varphi, \mathbf{curl}_\Gamma\, \varphi^t), \\
a_2^{(j)}(\psi, \psi^t) &= a^{(j)}(\mathbf{grad}_\Gamma\, \psi, \mathbf{grad}_\Gamma\, \psi^t), \\
a_3^{(j)}(\alpha, \alpha^t) &= a^{(j)}(\alpha, \alpha^t), \\
b_1^{(j)}(\psi, \varphi^t) &= a^{(j)}(\mathbf{grad}_\Gamma\, \psi, \mathbf{curl}_\Gamma\, \varphi^t), \\
b_2^{(j)}(\varphi, \psi^t) &= a^{(j)}(\mathbf{curl}_\Gamma\, \varphi, \mathbf{grad}_\Gamma\, \psi^t), \\
b_3^{(j)}(\varphi, \alpha^t) &= a^{(j)}(\mathbf{curl}_\Gamma\, \varphi, \mathbf{curl}_\Gamma\, \alpha^t), \\
c_1^{(j)}(\alpha, \varphi^t) &= a^{(j)}(\alpha, \mathbf{curl}_\Gamma\, \varphi^t), \\
c_2^{(j)}(\alpha, \psi^t) &= a^{(j)}(\alpha, \mathbf{grad}_\Gamma\, \psi^t), \\
c_3^{(j)}(\psi, \alpha^t) &= a^{(j)}(\mathbf{grad}_\Gamma\, \psi, \alpha^t).
\end{aligned}$$

We summarize the properties of the bilinear forms $a_i^{(j)}$ and $b_i^{(j)}$ in the following lemma.

LEMMA 4.2 *Let z be in $L^\infty(\Gamma)$ with the property $\Im m(z) \geq 0$, $\Re e(z) \geq z_0$ where z_0 is a strictly positive constant. Then we have:*
$a_2^{(1)}$ is a compact perturbation of a coercive form over $H_0^1(\Gamma) \times H_0^1(\Gamma)$; $a_1^{(1)}$ is a compact perturbation of a coercive form over $H_0^1(\Gamma) \times H_0^1(\Gamma)$;
$a_2^{(2)}$ is a compact perturbation of a coercive form over $H_0^{3/2}(\Gamma) \times H_0^{3/2}(\Gamma)$;

$a_1^{(2)}$ *is a compact perturbation of a coercive form over* $H_0^{1/2}(\Gamma) \times H_0^{1/2}(\Gamma)$*;* $b_2^{(1)}$ *is continuous on* $H_0^{1/2}(\Gamma) \times H_0^{3/2}(\Gamma)$*;* $b_1^{(1)}$ *is continuous on* $H_0^{3/2}(\Gamma) \times H_0^{1/2}(\Gamma)$*;*
$b_2^{(2)}$ *is continuous on* $H_0^{s}(\Gamma) \times H_0^{1-s}(\Gamma)$*;* $b_1^{(2)}$ *is continuous on* $H_0^{1-s}(\Gamma) \times H_0^{s}(\Gamma)$ *where* $0 < s < 1$.

The Leontovich boundary value problem for the time-harmonic Maxwell equations $(1)-(2)$ is then reduced to the system of three variational integral equations $(8)-(9)-(10)$:

LEMMA 4.3 *The system* $(8)-(9)-(10)$ *is equivalent to the boundary value problem* $(1)-(2)$.

We shall prove the following result.

LEMMA 4.4 *Assume that* $\mathbf{g}^\delta$ *is in the Sobolev space* $\mathcal{Y}(\Gamma)$*. Then the system* $(8)-(9)-(10)$ *admits a unique solution in* $H_0^1(\Gamma) \times H_0^{3/2}(\Gamma) \times \mathcal{N}(\Gamma)$.

PROOF. Lemma 4.2 shows that the Fredholm theory holds for the system of variational equations $(8)-(9)-(10)$. The claim is easily obtained by combining Lemmas 4.1 and 4.3. □
From Lemma 4.4 we can conclude that the appropriate Sobolev space in which to consider the integral equation (5) is $\left\{\mathbf{u} \in TL^2(\Gamma), \mathrm{div}_\Gamma\, \mathbf{u} \in H^{-1/2}(\Gamma)\right\}$. If we assume that $\mathbf{grad}_\Gamma\, z \in TL^\infty(\Gamma)$ it follows from Lemmas 4.3 and 4.4 that $\mathbf{m}^\delta$ is in $\left\{\mathbf{u} \in TH^{-1/2}(\Gamma), \mathrm{div}_\Gamma\, \mathbf{u} \in H^{-1}(\Gamma)\right\}$.

Analogously to the treatment of the Leontovich boundary condition we have the following lemma.

LEMMA 4.5 *The Dirichlet boundary value problem for the time-harmonic Maxwell equations* (3) *is equivalent to*

$$a_1^{(2)}(\varphi^0,\varphi^t) + b_1^{(2)}(\psi^0,\varphi^t) + c_1^{(2)}(\alpha^0,\varphi^t) \;=\; \langle \mathbf{f}^0, \mathbf{curl}_\Gamma\, \varphi^t\rangle, \qquad (8)'$$

$$a_2^{(2)}(\psi^0,\psi^t) + b_2^{(2)}(\varphi^0,\psi^t) + c_2^{(2)}(\alpha^0,\psi^t) \;=\; \langle \mathbf{f}^0, \mathbf{grad}_\Gamma\, \psi^t\rangle, \qquad (9)'$$

$$a_3^{(2)}(\alpha^0,\alpha^t) + b_3^{(2)}(\varphi^0,\alpha^t) + c_3^{(2)}(\psi^0,\alpha^t) \;=\; \langle \mathbf{f}^0, \alpha^t\rangle, \qquad (10)'$$

where

$$\begin{aligned} \langle \mathbf{f}^0, \mathbf{u}^t\rangle \;&=\; \frac{1}{2}\int_\Gamma \mathbf{g}^0(\mathbf{x})\,.\,\mathbf{u}^t(\mathbf{x})\; d\gamma(\mathbf{x}) \\ &-\int_\Gamma\int_\Gamma \mathbf{grad}_\mathbf{x}\, \Phi(\mathbf{x},\mathbf{y}) \,\wedge\, (\mathbf{g}^0 \,\wedge\, \mathbf{n}(\mathbf{y}))\,.\,\mathbf{u}^t(\mathbf{x})\; d\gamma(\mathbf{y})\; d\gamma(\mathbf{x}). \end{aligned}$$

Existence and uniqueness for the system $(8)' - (9)' - (10)'$ can be established by Lemma 4.2:

LEMMA 4.6 *Assume that* $\mathbf{g}^0 \in TH^{-1/2}(\text{curl}, \Gamma)$. *Then the system of variational equations* $(8)' - (9)' - (10)'$ *admits a unique solution in* $H_0^{1/2}(\Gamma) \times H_0^{3/2}(\Gamma) \times \mathcal{N}(\Gamma)$.

Here the tangential fields $\mathbf{j}^0 = \mathbf{H}^0 \wedge \mathbf{n}$ and $\mathbf{m}^0 = \mathbf{E}^0 \wedge \mathbf{n}$ are in $TH^{-1/2}(\text{div}, \Gamma)$.

5 ASYMPTOTICS OF THE SOLUTION

Our purpose in this section is to give an asymptotic solution for the system of variational equations $(8) - (9) - (10)$ and construct complete asymptotic expansions for its solution of the form $\varphi^\delta = \sum\limits_{j=0} \delta^j \varphi^j$, $\psi^\delta = \sum\limits_{j=0} \delta^j \psi^j$, $\alpha^\delta = \sum\limits_{j=0} \delta^j \alpha^j$, where the upper limit in the sum is ∞ if $\text{curl}_\Gamma\, \mathbf{g}^\delta \in \mathcal{C}^\infty(\Gamma)$ and otherwise depends on the smoothness of $\mathbf{g}^\delta$. This section is organized as follows. In a first step, the convergence result is proved. Then, if the data $\mathbf{g}^0$ is more regular, we estimate the rate of the convergence. The third step is devoted to the construction of the complete asymptotic expansions.

5.1 CONVERGENCE RESULT

This subsection is devoted to prove the convergence of the unique solution of the system (1) with the boundary condition (2) to that of the system (3). To accomplish this we need several intermediate results. First, we obtain by integrating by parts the following lemma.

LEMMA 5.1 *Let* $0 < s < 1$. *Assume that* $z \in L^\infty(\Gamma)$ *and* $\mathbf{grad}_\Gamma\, z \in TL^\infty(\Gamma)$. *Then the bilinear form* $b_2^{(1)}$ *(resp.*$b_1^{(1)}$*) is continuous on* $H_0^s(\Gamma) \times H_0^{1-s}(\Gamma)$ *(resp.* $H_0^{1-s}(\Gamma) \times H_0^s(\Gamma)$*).*

Next, we establish the following a-priori estimates. We have:

LEMMA 5.2 *There exist two strictly positive constants* δ_0 *and* C *such that for* $0 < \delta < \delta_0$ *the following estimate holds.*

$$||\psi^\delta||_{H_0^{3/2}(\Gamma)} + \sqrt{\delta}||\varphi^\delta||_{H_0^1(\Gamma)} + ||\varphi^\delta||_{H_0^{1/2}(\Gamma)} + ||\alpha^\delta||_{\mathcal{N}(\Gamma)} \leq C.$$

PROOF. Let us prove by contradiction that the quantity

$$\gamma^\delta = ||\psi^\delta||_{H_0^{3/2}(\Gamma)} + \sqrt{\delta}||\varphi^\delta||_{H_0^1(\Gamma)} + ||\varphi^\delta||_{H_0^{1/2}(\Gamma)} + ||\alpha^\delta||_{\mathcal{N}(\Gamma)}$$

is uniformly bounded with respect to the parameter δ. Suppose that we have a subsequence, still denoted by γ^δ such that it converges to $+\infty$ as δ tends to zero. Let us define u^δ , v^δ and w^δ by

$$u^\delta = \frac{\psi^\delta}{\gamma^\delta}, \; v^\delta = \frac{\varphi^\delta}{\gamma^\delta}, \; w^\delta = \frac{\alpha^\delta}{\gamma^\delta}.$$

It follows that $(u^\delta, v^\delta, w^\delta)$ converges to (u, v, w), in $H_0^{3/2}(\Gamma) \times H_0^{1/2}(\Gamma) \times \mathcal{N}(\Gamma)$ weakly. Since $\mathcal{N}(\Gamma)$ is a finite dimensional space and w^δ converges to w in $\mathcal{N}(\Gamma)$ weakly, w^δ converges to w in $\mathcal{N}(\Gamma)$ strongly.
To obtain the claim we proceed in many steps. First, we prove that $\sqrt{\delta}\,\mathbf{curl}_\Gamma\, v^\delta$ is bounded in $TL^2(\Gamma)$. Then, since $\Re e(z) \geq z_0 > 0$, we prove by using the coerciveness properties of $a_1^{(1)}$ and $a_1^{(2)}$ that $(u^\delta, v^\delta, w^\delta)$ converges in a strong sense to $(u = 0, v = 0, w = 0)$.
We begin by rewriting the bilinear form $a_1^{(1)}$ on $H_0^1(\Gamma) \times H_0^1(\Gamma)$ as follows

$$a_1^{(1)}(\varphi, \varphi^t) = \frac{i}{2}\int_\Gamma z(x)\,\mathbf{curl}_\Gamma\,\varphi(x)\,.\,\mathbf{curl}_\Gamma\,\varphi^t(x)\;d\gamma(x) + \mathcal{R}(\varphi, \varphi^t),$$

where $\mathcal{R}$ is a continuous bilinear form on $H_0^{1/2}(\Gamma) \times H_0^{1/2}(\Gamma)$. We also rewrite $a_1^{(2)}$ in the form

$$a_1^{(2)}(\varphi, \varphi^t) \;=\; ik\int_\Gamma\int_\Gamma \Phi_0(\mathbf{x}, \mathbf{y})\,\mathbf{curl}_\Gamma\,\varphi(y)\,.\,\mathbf{curl}_\Gamma\,\varphi^{\mathbf{t}}(x)\;d\gamma(y)d\gamma(x) + \mathcal{K}(\varphi\,,\,\varphi^t)$$

where $\mathcal{K}$ is a compact form on $H_0^{1/2}(\Gamma) \times H_0^{1/2}(\Gamma)$. It is easily seen that equation (8) becomes:

$$\begin{aligned} &\delta\,\frac{i}{2}\int_\Gamma z(x)\,\mathbf{curl}_\Gamma\,v^\delta(x)\,.\,\mathbf{curl}_\Gamma\,\overline{\varphi}^t(x)\;d\gamma(x) \\ &+ik\int_\Gamma\int_\Gamma \Phi_0(\mathbf{x}, \mathbf{y})\,\mathbf{curl}_\Gamma\,\mathbf{v}^\delta(y)\,.\,\mathbf{curl}_\Gamma\,\overline{\varphi}^{\mathbf{t}}(x)\;d\gamma(y)\;d\gamma(x) \\ &= -\delta b_1^{(1)}(u^\delta, \overline{\varphi}^t) - b_1^{(2)}(u^\delta, \overline{\varphi}^t) + \delta c_1^{(1)}(\mathbf{w}^\delta, \overline{\varphi}^t) - c_1^{(2)}(\mathbf{w}^\delta, \overline{\varphi}^t) \\ &-\delta\mathcal{R}(v^\delta, \overline{\varphi}^t) - \mathcal{K}(v^\delta\,,\,\overline{\varphi}^t) + \langle\frac{\mathbf{f}^\delta}{\gamma^\delta}, \mathbf{curl}_\Gamma\,\overline{\varphi}^t\rangle \end{aligned} \qquad (11)$$

Taking $\varphi^t = v^\delta$ in (11) shows that the quantity $b_1^{(2)}(u^\delta,\overline{v}^\delta)$ converges to $b_1^{(2)}(u,v)$. Using Lemmas 4.2 and 5.1, passing to the limit when δ tends to zero, we see that the terms $\delta b_1^{(1)}(u^\delta,\overline{u}^\delta), \delta c_1^{(1)}(\mathbf{w}^\delta,\overline{u}^\delta)$ and $\delta\mathcal{R}(v^\delta,\overline{u}^\delta)$ vanish and then

$$\begin{aligned}
&\delta\,\frac{i}{2}\int_\Gamma z(x)\,\mathbf{curl}_\Gamma\, v^\delta(x)\,.\,\mathbf{curl}_\Gamma\,\overline{v}^\delta(x)\;d\gamma(x)\\
&+ik\int_\Gamma\int_\Gamma \Phi_0(\mathbf{x},\mathbf{y})\,\mathbf{curl}_\Gamma\,\mathbf{v}^\delta(y)\,.\,\mathbf{curl}_\Gamma\,\overline{\mathbf{v}}^\delta\;d\gamma(y)\;d\gamma(x)\\
&\qquad\qquad\longrightarrow\quad -b_1^{(2)}(u,v)-c_1^{(2)}(w,v)-\mathcal{K}(v,v)
\end{aligned}\tag{12}$$

Combining the fact that the imaginary part of the left-hand terms is positive ($\Re e(z(x))\ge z_0>0$) and the fact that the sum is uniformly bounded with respect to the parameter δ, gives that $\sqrt{\delta}\;\mathbf{curl}_\Gamma\, v^\delta$ converges weakly in $TL^2(\Gamma)$. Next, since $\sqrt{\delta}\;\mathbf{curl}_\Gamma\, v^\delta$ is uniformly bounded in $TL^2(\Gamma)$, it follows that the quantity

$$\begin{aligned}
&\delta\,\frac{i}{2}\int_\Gamma z(x)\,\mathbf{curl}_\Gamma\, v^\delta(x)\,.\,\mathbf{curl}_\Gamma\,\varphi^t\;d\gamma(x)\\
&=\frac{1}{2}\sqrt{\delta}\int_\Gamma\Big(\sqrt{\delta}\mathbf{curl}_\Gamma\, z(x)\,v^\delta(x)\Big)\,.\,\mathbf{curl}_\Gamma\,\varphi^t\;d\gamma(x)
\end{aligned}$$

goes to zero. On the other hand, from the fact that the relation (11) holds for all $\varphi^t\in\mathcal{C}^\infty(\Gamma)$, we see that the terms $\delta b_1^{(1)}(u^\delta,\overline{\varphi}^t), \delta c_1^{(1)}(\mathbf{w}^\delta,\overline{\varphi}^t)$ and $\delta\mathcal{R}(v^\delta,\overline{\varphi}^t)$ go to zero when δ goes to zero. Therefore, the right-hand side of the equation (11) converges to $-b_1^{(2)}(u,\varphi^t)-c_1^{(2)}(\mathbf{w},\varphi^t)-\mathcal{K}(v^\delta,\,\overline{\varphi}^t)$. Finally, it follows that

$$a_1^{(2)}(v,\varphi^t)+b_1^{(2)}(u,\varphi^t)+c_1^{(2)}(\mathbf{w},\varphi^t)\;=\;0,\tag{13}$$

and the density of $\mathcal{C}^\infty(\Gamma)$ in the Sobolev space $H_0^{1/2}(\Gamma)$ gives the claim.
From (9), we get the corresponding limiting identity by using the continuous properties of the operators proved in lemma 4.2 and lemma 5.1. In fact, we have that:

$$\begin{aligned}
&\delta\,a_2^{(1)}(u^\delta,\psi^t)+a_2^{(2)}(u^\delta,\psi^t)+\delta b_2^{(1)}(v^\delta,\psi^t)+b_2^{(2)}(v^\delta,\psi^t)\\
&\quad+\delta c_2^{(1)}(\mathbf{w}^\delta,\psi^t)+c_2^{(2)}(\mathbf{w}^\delta,\psi^t)\;=\;\langle\frac{\mathbf{f}^\delta}{\gamma^\delta},\mathbf{grad}_\Gamma\,\psi^t\rangle.
\end{aligned}\tag{14}$$

Therefore, (u,v,w) is such that

$$a_2^{(2)}(u,\psi^t)+b_2^{(2)}(v,\psi^t)+c_2^{(2)}(\mathbf{w},\psi^t)=0,\tag{15}$$

In a similar manner, we can show that

$$a_3^{(2)}(\mathbf{w},\alpha^t) + b_3^{(2)}(\alpha^t, v) + c_3^{(2)}(\alpha^t, u) = 0. \tag{16}$$

Lemma 4.6 gives $u = v = w = 0$.
To obtain the claim we have to prove the convergence in a strong sense of u^δ, v^δ and $\sqrt{\delta} v^\delta$ to 0 in $H_0^{3/2}(\Gamma), H_0^{1/2}(\Gamma)$ and $H_0^1(\Gamma)$, respectively.
Using the fact that the imaginary part of both of the left-hand terms of equation (12) is positive and the sum converges uniformly to zero when the parameter δ goes to zero, we obtain that

$$\delta \frac{1}{2} \int_\Gamma z(x)\, \mathbf{curl}_\Gamma\, v^\delta(x) \,.\, \mathbf{curl}_\Gamma\, \overline{v}^\delta(x)\; d\gamma(x) \longrightarrow 0,$$

$$\int_\Gamma \int_\Gamma \Phi_0(\mathbf{x},\mathbf{y})\, \mathbf{curl_\Gamma}\, \mathbf{v}^\delta(y) \,.\, \mathbf{curl_\Gamma}\, \overline{\mathbf{v}}^\delta(x)\; d\gamma(y)\; d\gamma(x) \longrightarrow\; 0.$$

Thus, $\sqrt{\delta}\, v^\delta$ (resp. v^δ) converges to 0 in $H_0^1(\Gamma)$ (resp. $H_0^{1/2}(\Gamma)$) strongly. The convergence in a strong sense of u^δ to zero follows from the fact that $a_2^{(2)}(u^\delta, u^\delta)$ converges to zero as δ tends to zero.
Since

$$||u^\delta||_{H_0^{3/2}(\Gamma)} + \sqrt{\delta}||v^\delta||_{H_0^1(\Gamma)} + ||v^\delta||_{H_0^{1/2}(\Gamma)} + ||w^\delta||_{\mathcal{N}(\Gamma)} \;=\; 1,$$

the above conclusion leads to a contradiction and the proof is complete. $\square$

LEMMA 5.3 *Assume that the field $\mathbf{g}^\delta$ converges to $\mathbf{g}^0$ in $TH^{-1/2}(\mathrm{curl},\Gamma)$. Let $(\varphi^0, \psi^0, \alpha^0)$ be defined as the unique solution of $(8)' - (9)' - (10)'$ in $H_0^{1/2}(\Gamma) \times H_0^{3/2}(\Gamma) \times \mathcal{N}(\Gamma)$. Then,*

$$\begin{aligned} \varphi^\delta &\longrightarrow \varphi^0 \quad H_0^{1/2}(\Gamma) \text{ strongly}, \\ \psi^\delta &\longrightarrow \psi^0 \quad H_0^{3/2}(\Gamma) \text{ strongly}, \\ \alpha^\delta &\longrightarrow \alpha^0 \quad \mathcal{N}(\Gamma) \text{ strongly}. \end{aligned}$$

PROOF. Using Lemma 5.2 we obtain that $(\sqrt{\delta}\varphi^\delta, \varphi^\delta, \psi^\delta, \alpha^\delta)$ converges to $(\theta, \varphi, \psi, \alpha)$ in $H_0^1(\Gamma) \times H_0^{1/2}(\Gamma) \times H_0^{3/2}(\Gamma) \times \mathcal{N}(\Gamma)$ weakly. We shall prove that $(\varphi, \psi, \alpha) = (\varphi^0, \psi^0, \alpha^0)$. Taking $\varphi^t \in \mathcal{C}^\infty(\Gamma)$ as a test function in (8) yields

$$\begin{aligned} &\sqrt{\delta}\, a_1^{(1)}(\sqrt{\delta}\varphi^\delta, \varphi^t) + a_1^{(2)}(\varphi^\delta, \varphi^t) + \delta b_1^{(1)}(\psi^\delta, \varphi^t) + b_1^{(2)}(\psi^\delta, \varphi^t) \\ &\quad + \delta c_1^{(1)}(\alpha^\delta, \varphi^t) + c_1^{(2)}(\alpha^\delta, \varphi^t) \;=\; \langle \mathbf{f}^\delta, \mathbf{curl}_\Gamma\, \varphi^t \rangle. \end{aligned} \tag{17}$$

Making use of the fact that $a_1^{(1)}(\sqrt{\delta}\varphi^\delta, \varphi^t)$ is uniformly bounded with respect to δ gives that $\sqrt{\delta} a_1^{(1)}(\sqrt{\delta}\varphi^\delta, \varphi^t) \longrightarrow 0$. Therefore,

$$a_1^{(2)}(\varphi, \varphi^t) + b_1^{(2)}(\psi, \varphi^t) + c_1^{(2)}(\alpha, \varphi^t) = \langle \mathbf{f}^0, \mathbf{curl}_\Gamma \, \varphi^t \rangle.$$

Next, from (9), we get

$$\begin{aligned} \delta \, a_2^{(1)}(\psi^\delta, \psi^t) + a_2^{(2)}(\psi^\delta, \psi^t) + \delta b_2^{(1)}(\varphi^\delta, \psi^t) + b_2^{(2)}(\varphi^\delta, \psi^t) & \\ + \delta c_1^{(1)}(\alpha^\delta, \psi^t) + c_1^{(2)}(\alpha^\delta, \psi^t) \; = \; \langle \mathbf{f}^\delta, \mathbf{grad}_\Gamma \, \psi^t \rangle. & \end{aligned} \tag{18}$$

The terms $\delta \, a_2^{(1)}(\psi^\delta, \psi^t)$, $\delta b_2^{(1)}(\varphi^\delta, \psi^t)$ and $\delta c_1^{(1)}(\alpha^\delta, \psi^t)$ converge to zero when δ goes to zero. This gives

$$a_2^{(2)}(\psi, \psi^t) + b_2^{(2)}(\varphi, \psi^t) + c_2^{(2)}(\alpha, \psi^t) = \langle \mathbf{f}^0, \mathbf{grad}_\Gamma \, \psi^t \rangle.$$

Similar arguments for (10) yield

$$a_3^{(2)}(\alpha, \alpha^t) + b_3^{(2)}(\varphi, \alpha^t) + c_3^{(2)}(\psi, \alpha^t) \; = \; \langle \mathbf{f}^0, \alpha^t \rangle$$

and the claim is obtained by Lemma 4.6. □

5.2 RATE OF CONVERGENCE

If $\mathbf{g}^0$ is more regular, we can determine the rate of the convergence of $(\varphi^\delta, \psi^\delta, \alpha^\delta)$ to $(\varphi^0, \psi^0, \alpha^0)$.

THEOREM 5.1 *Assume that* $\mathbf{g}^0 \in TH^{-1/2}(\Gamma)$ *is such that* $\mathrm{curl}_\Gamma \, \mathbf{g}^0 \in H^{1/2}(\Gamma)$. *Then there exists a constant* C *independent of* δ *such that*

$$||\varphi^\delta - \varphi^0||_{H_0^{1/2}(\Gamma)} + ||\psi^\delta - \psi^0||_{H_0^{3/2}(\Gamma)} + ||\alpha^\delta - \alpha^0||_{\mathcal{N}(\Gamma)} \; \leq \; C \left(\delta + ||\mathbf{g}^\delta - \mathbf{g}^0||_{TH^{-1/2}(\mathrm{curl},\Gamma)} \right)$$

PROOF. Let us prove by contradiction that the quantity

$$c^\delta = ||\psi^\delta - \psi^0||_{H_0^{3/2}(\Gamma)} + ||\varphi^\delta - \varphi^0||_{H_0^{1/2}(\Gamma)} + ||\alpha^\delta - \alpha^0||_{\mathcal{N}(\Gamma)}$$

is bounded by the quantity

$$(\delta + ||\mathbf{g}^\delta - \mathbf{g}^0||_{TH^{-1/2}(\mathrm{curl},\Gamma)}).$$

We suppose that we have a subsequence

$$\frac{c^\delta}{\delta + ||\mathbf{g}^\delta - \mathbf{g}^0||_{TH^{-1/2}(\mathrm{curl},\Gamma)}}$$

such that it converges to $+\infty$ as δ tends to zero.
The quantities $\dfrac{\varphi^\delta - \varphi^0}{c^\delta}, \dfrac{\psi^\delta - \psi^0}{c^\delta}, \dfrac{\alpha^\delta - \alpha^0}{c^\delta}$ are bounded in $H_0^{3/2}(\Gamma) \times H_0^{1/2}(\Gamma) \times \mathcal{N}(\Gamma)$. It follows that $(\dfrac{\varphi^\delta - \varphi^0}{c^\delta}, \dfrac{\psi^\delta - \psi^0}{c^\delta}, \dfrac{\alpha^\delta - \alpha^0}{c^\delta})$ converges to $(\varphi^\star, \psi^\star, \alpha^\star)$ in $H_0^{1/2}(\Gamma) \times H_0^{3/2}(\Gamma) \times \mathcal{N}(\Gamma)$ weakly. If $\mathrm{curl}_\Gamma\, \mathbf{g}^0 \in H^{1/2}(\Gamma)$ then $\varphi^0 \in H_0^{3/2}(\Gamma)$. So, $\varphi^\delta - \varphi^0 \in H_0^{1/2}(\Gamma)$ and can be taken as a test function in (8). Straightforward calculations show that:

$$\begin{aligned} &\delta a_1^{(1)}(\varphi^\delta - \varphi^0, \varphi^\delta - \varphi^0) + a_1^{(2)}(\varphi^\delta - \varphi^0, \varphi^\delta - \varphi^0) \\ &+ b_1^{(2)}(\psi^\delta - \psi^0, \varphi^\delta - \varphi^0) + c_1^{(2)}(\alpha^\delta - \alpha^0, \varphi^\delta - \varphi^0) \\ &= \langle \mathbf{f}^\delta - \mathbf{f}^0, \mathbf{curl}_\Gamma\, (\varphi^\delta - \varphi^0) \rangle - \delta\, a_1^{(1)}(\varphi^0, \varphi^\delta - \varphi^0) \\ &\quad - \delta b_1^{(1)}(\psi^\delta, \varphi^\delta - \varphi^0) - \delta c_1^{(1)}(\alpha^\delta, \varphi^\delta - \varphi^0) \end{aligned} \tag{19}$$

So, the right-hand of the above expression is bounded by

$$\Big(\delta + ||g^\delta - g^0||_{TH^{-1/2}(\mathrm{curl},\Gamma)}\Big) ||\varphi^\delta - \varphi^0||_{H_0^{1/2}(\Gamma)}$$

since $a_1^{(1)}(\varphi^0, \varphi^\delta - \varphi^0) \leq C\, ||\varphi^0||_{H_0^{3/2}(\Gamma)}\, ||\varphi^\delta - \varphi^0||_{H_0^{1/2}(\Gamma)}$. Therefore,

$$\begin{aligned} &\delta a_1^{(1)}(\frac{\varphi^\delta - \varphi^0}{c^\delta}, \frac{\varphi^\delta - \varphi^0}{c^\delta}) + a_1^{(2)}(\frac{\varphi^\delta - \varphi^0}{c^\delta}, \frac{\varphi^\delta - \varphi^0}{c^\delta}) \\ &+ b_1^{(2)}(\frac{\psi^\delta - \psi^0}{c^\delta}, \frac{\varphi^\delta - \varphi^0}{c^\delta}) + c_1^{(2)}(\frac{\alpha^\delta - \alpha^0}{c^\delta}, \frac{\varphi^\delta - \varphi^0}{c^\delta}) \\ &\leq \frac{\delta + ||\mathbf{g}^\delta - \mathbf{g}^0||_{TH^{-1/2}(\mathrm{curl},\Gamma)}}{c^\delta} \frac{||\varphi^\delta - \varphi^0||_{H_0^{1/2}(\Gamma)}}{c^\delta}. \end{aligned} \tag{20}$$

It follows that the last term converges to zero when δ goes to zero.
A similar argument as in the proof of Lemma 5.2 gives that $(\varphi^\star, \psi^\star, \alpha^\star)$ is $(0,0,0)$ and that $(\dfrac{\varphi^\delta - \varphi^0}{c^\delta}, \dfrac{\psi^\delta - \psi^0}{c^\delta}, \dfrac{\alpha^\delta - \alpha^0}{c^\delta})$ converges to $(\varphi^\star, \psi^\star, \alpha^\star)$ in $H^{1/2}(\Gamma) \times H^{3/2}(\Gamma) \times \mathcal{N}(\Gamma)$ strongly. This is a contradiction to the fact that

$$\frac{||\varphi^\delta - \varphi^0||_{H_0^{1/2}(\Gamma)}}{c^\delta} + \frac{||\psi^\delta - \psi^0||_{H_0^{3/2}(\Gamma)}}{c^\delta} + \frac{||\alpha^\delta - \alpha^0||_{\mathcal{N}(\Gamma)}}{c^\delta} = 1.$$

□

5.3 THE COMPLETE CONSTRUCTION OF THE ASYMPTOTIC EXPANSIONS

In this section, we assume that $\mathbf{g}^\delta$ in $\mathcal{Y}(\Gamma)$ has the following asymptotic form:

$$\mathbf{g}^\delta = \mathbf{g}^0 + \delta\,\mathbf{g}^1 + \ldots + \delta^N\,\mathbf{g}^N + \ldots$$

where $\mathbf{g}^j \in \mathcal{Y}(\Gamma)$ for $j \geq 0$. We shall show how to construct the complete asymptotic expansions of $\varphi^\delta, \psi^\delta$ and α^δ:

$$\left\{\begin{array}{l}\varphi^\delta = \varphi^0 + \delta\,\varphi^1 + \ldots + \delta^N\,\varphi^N + \varphi^\delta_{N+1},\\ \psi^\delta = \psi^0 + \delta\,\psi^1 + \ldots + \delta^N\,\psi^N + \psi^\delta_{N+1},\\ \alpha^\delta = \alpha^0 + \delta\,\alpha^1 + \ldots + \delta^N\,\alpha^N + \alpha^\delta_{N+1}.\end{array}\right. \tag{21}$$

The standard construction of the functions $\varphi^j, \psi^j, \alpha^j, j \geq 0$, uses the formal series (21) which are inserted into the system (8) – (9) – (10). Identifying different terms following increasing powers of δ gives the sequence of the variational systems

$$\begin{aligned}
a_1^{(2)}(\varphi^0,\varphi^t) + b_1^{(2)}(\psi^0,\varphi^t) + c_1^{(2)}(\alpha^0,\varphi^t) &= \langle \mathbf{f}^0, \mathbf{curl}_\Gamma\,\varphi^t\rangle,\\
a_2^{(2)}(\psi^0,\psi^t) + b_2^{(2)}(\varphi^0,\psi^t) + c_2^{(2)}(\alpha^0,\psi^t) &= \langle \mathbf{f}^0, \mathbf{grad}_\Gamma\,\psi^t\rangle,\\
a_3^{(2)}(\alpha^0,\alpha^t) + b_3^{(2)}(\varphi^0,\alpha^t) + c_3^{(2)}(\psi^0,\alpha^t) &= \langle \mathbf{f}^0, \alpha^t\rangle,
\end{aligned} \tag{22}$$

and more generally,

$$\begin{aligned}
a_1^{(2)}(\varphi^{j+1},\varphi^t) + b_1^{(2)}(\psi^{j+1},\varphi^t) + c_1^{(2)}(\alpha^{j+1},\varphi^t) &= \langle \mathbf{f}^{j+1}, \mathbf{curl}_\Gamma\,\varphi^t\rangle\\
-a_1^{(1)}(\varphi^j,\varphi^t) - b_1^{(1)}(\psi^j,\varphi^t) - c_1^{(1)}(\alpha^j,\varphi^t),&\\
a_2^{(2)}(\psi^{j+1},\psi^t) + b_2^{(2)}(\varphi^{j+1},\psi^t) + c_2^{(2)}(\alpha^{j+1},\psi^t) &= \langle \mathbf{f}^{j+1}, \mathbf{grad}_\Gamma\,\psi^t\rangle\\
-a_2^{(1)}(\psi^j,\psi^t) - b_2^{(1)}(\varphi^j,\psi^t) - c_2^{(1)}(\alpha^j,\psi^t),&\\
a_3^{(2)}(\alpha^{j+1},\alpha^t) + b_3^{(2)}(\varphi^{j+1},\alpha^t) + c_3^{(2)}(\psi^{j+1},\alpha^t) &= \langle \mathbf{f}^{j+1}, \alpha^t\rangle\\
-a_3^{(1)}(\alpha^j,\alpha^t) - b_3^{(1)}(\varphi^j,\alpha^t) - c_3^{(1)}(\psi^j,\alpha^t).&
\end{aligned} \tag{23}$$

THEOREM 5.2 *Let* $N \in \mathbb{N}$. *Let* $\mathcal{Y}^j(\Gamma) = \Big\{ \mathbf{u} \in TH^{-1/2}(\Gamma), \mathrm{curl}_\Gamma \, \mathbf{u} \in H^{1/2+N-j}(\Gamma) \Big\}$. *For* $\mathbf{g}^\delta \in \mathcal{Y}(\Gamma)$ *such that* $\mathbf{g}^j \in \mathcal{Y}^j(\Gamma)$ *for* $j = 0$ *to* N, *let* $(\varphi^\delta, \psi^\delta, \alpha^\delta)$ *defined by* $(8)-(9)-(10)$. *Let* $\varphi^j, \psi^j, \alpha^j$ *be constructed according to the sequence of variational systems* (23). *Then* $(\varphi^\delta, \psi^\delta, \alpha^\delta)$ *admit the asymptotic expansions* (21), *where the remainders* $\varphi^\delta_{N+1}, \psi^\delta_{N+1}$ *and* α^δ_{N+1} *satisfy the estimate*

$$||\varphi^\delta_{N+1}||_{H_0^{1/2}(\Gamma)} + ||\psi^\delta_{N+1}||_{H_0^{3/2}(\Gamma)} + ||\alpha^\delta_{N+1}||_{\mathcal{N}(\Gamma)} \leq C\,\delta^{N+1} \sum_{j=0}^{N} ||\mathbf{g}^j||_{\mathcal{Y}^j(\Gamma)}. \tag{24}$$

Here the constant C *does not depend on* δ *and* $\mathbf{g}^j$.

PROOF. It suffices to apply the previous estimates of $\varphi^\delta - \varphi^0, \psi^\delta - \psi^0$ and $\alpha^\delta - \alpha^0$ established in Theorem 5.1 to the problem satisfied by $(\varphi^\delta_{N+1}, \psi^\delta_{N+1}, \alpha^\delta_{N+1})$. □

6 CONCLUDING REMARKS

To conclude this paper, we shall make the following important remarks. We have proved that the Leontovich boundary value problem for the time-harmonic Maxwell equations $(1)-(2)$ leads to a singular perturbation problem $(8)-(9)-(10)$. If we are interested in the numerical computations of the tangential field $\mathbf{j}^\delta$ the use of higher order finite elements or the refinement of the mesh is necessary to achieve a sufficient accuracy of the quantity $\mathrm{curl}_\Gamma \, \mathbf{j}^\delta$. The computation of the quantity $\mathrm{div}_\Gamma \, \mathbf{j}^\delta$ does not require the use of such higher order finite elements nor the refinement of the mesh. Using the key variational formulations $(8)-(9)-(10)$, we are able to prove the following error estimates:

$$\begin{cases} ||\varphi^\delta - \varphi^\delta_h||_{H_0^1(\Gamma)} = 0(\dfrac{h}{\delta}), \\ ||\varphi^\delta - \varphi^\delta_h||_{H_0^{1/2}(\Gamma)} = 0(h), \\ ||\psi^\delta - \psi^\delta_h||_{H_0^{3/2}(\Gamma)} = 0(h), \end{cases}$$

and for $\mathbf{j}^\delta$

$$\begin{cases} ||\mathbf{j}^\delta - \mathbf{j}_h^\delta||_{TL^2(\Gamma)} = 0(\frac{h}{\delta}), \\ ||\mathbf{j}^\delta - \mathbf{j}_h^\delta||_{TH^{-1/2}(\mathrm{div},\Gamma)} = 0(h), \end{cases}$$

where h is a small parameter associated with the mesh and $\varphi_h^\delta, \psi_h^\delta$ and $\mathbf{j}_h^\delta$ are the approximate functions obtained from the variational problem $(8)-(9)-(10)$ by the Galerkin method. The rate $0(\frac{h}{\delta})$ is connected to the fact that $(8)-(9)-(10)$ is a singular perturbation problem.
To approximate the scattering problem from a lossy body at high conductivity it is more appropriate to expand the electric and magnetic fields with respect to the square root of the inverse of the conductivity and compute the terms of these asymptotic expansions term by term instead of the use of the Leontovich boundary condition.
A similar conclusion has been made by Bendali and Lemrabet [7] for the effective boundary conditions which approximate the effect of thin coatings on the scattering of a time-harmonic wave for the Helmholtz equation. More surprisingly, this fact has been also observed for a long time in the construction of higher-order absorbing boundary conditions (see the paper of Engquist and Majda [10]).

ACKNOWLEDGEMENT. The partial support for H.A. from the Royal Institute of Technology (Sweden) is gratefully acknowledged.

7 APPENDIX. DERIVATION OF THE LEONTOVICH BOUNDARY CONDITION

In this appendix, we apply the well-known two scale asymptotic expansion method to derive impedance boundary conditions at the curved surface of a highly conducting body. One of the first derivations of these approximate boundary conditions was by Rytov [17] who used a WKB method. We also refer to the papers [12], [14] and [15] co-authored by MacCamy on the skin effect in metals.
Let Ω_1 be a bounded domain in $\mathbf{R}^3$ and let $\omega, \delta, \mu_1, \mu_2, \varepsilon_1$, and ε_2 be strictly positive reals. By $\mathbf{n}$ we denote the unit normal to Γ the boundary of Ω_1 directed into $\mathbf{R}^3 \setminus \overline{\Omega_1}$. Let (s,n) be a local parametrization of Γ. $\mathbf{s}$ (resp. n)

is the tangential (resp. the normal) variable. Maxwell's equations are

$$\left\{\begin{array}{l}\mathbf{curl}\,\mathbf{E}_1^\delta = i\omega\mu_1\,\mathbf{H}_1^\delta \quad \text{in } \Omega_1,\\ \mathbf{curl}\,\mathbf{H}_1^\delta = -i\omega\Big(\varepsilon_1 + \dfrac{i}{\delta^2}\Big)\,\mathbf{E}^\delta \quad \text{in } \Omega_1,\\ \mathbf{curl}\,\mathbf{E}_2^\delta = i\omega\mu_2\,\mathbf{H}_2^\delta \quad \text{in } \mathbf{R}^3\setminus\overline{\Omega_1},\\ \mathbf{curl}\,\mathbf{H}_2^\delta = -i\omega\varepsilon_2\,\mathbf{E}^\delta \quad \text{in } \mathbf{R}^3\setminus\overline{\Omega_1},\\ \mathbf{E}_1^\delta\,\wedge\,\mathbf{n} = \mathbf{E}_2^\delta\,\wedge\,\mathbf{n} \quad \text{on } \Gamma,\\ \mathbf{H}_1^\delta\,\wedge\,\mathbf{n} = \mathbf{H}_2^\delta\,\wedge\,\mathbf{n} \quad \text{on } \Gamma,\end{array}\right. \tag{25}$$

where the index 1 (resp. 2) denotes the interior (resp. exterior) fields. On the assumption that δ is small (Ω_1 is highly conducting), we seek a boundary condition that can be applied at Γ to simulate the effect of the material. We assume that inside the highly conducting body Ω_1, the electric and magnetic fields have two scale asymptotic expansions of the form:

$$\left\{\begin{array}{l}\mathbf{E}_1^\delta(\mathbf{s},n) = \mathbf{E}_1^0(\mathbf{s},n,\tau) + \delta\,\mathbf{E}_1^1(\mathbf{s},n,\tau) + \delta^2\,\mathbf{E}_1^2(\mathbf{s},n,\tau) + \ldots,\\ \mathbf{H}_1^\delta(\mathbf{s},n) = \mathbf{H}_1^0(\mathbf{s},n,\tau) + \delta\,\mathbf{H}_1^1(\mathbf{s},n,\tau) + \delta^2\,\mathbf{H}_1^2(\mathbf{s},n,\tau) + \ldots\end{array}\right. \tag{26}$$

where the new variable $\tau = \dfrac{n}{\delta}$. Using the following well-known formula

$$\mathbf{curl}\,\mathbf{u} = \mathrm{curl}_{\Gamma_n}(u_{\Gamma_n})\,\mathbf{n} + \partial_n(\mathbf{n}\,\wedge\,\mathbf{u}) - \mathbf{curl}_{\Gamma_n}(\mathbf{u}\,.\,\mathbf{n}) + (\mathbf{R} - 2c)(\mathbf{s},n)\,\mathbf{n}\,\wedge\,\mathbf{u}, \tag{27}$$

where $\Gamma_n = \Gamma + n\,\mathbf{n}$, $\mathbf{R}(\mathbf{s},n)$ (resp. $c(\mathbf{s},n)$) is the tensor (resp. the mean value) of curvature at the point $\mathbf{x} = (\mathbf{s},n)$, we obtain that

$$\mathbf{curl}\,\mathbf{E}_1^j = \frac{1}{\delta}\partial_\tau(\mathbf{n}\,\wedge\,\mathbf{E}_1^j) + \mathbf{curl}_{(\mathbf{s},n)}\,\mathbf{E}_1^j,$$

Inserting the asymptotic expansions (26) into the Maxwell equations (25) and identifying the coefficients of the powers δ^{-1}, δ^0 gives

$$\left\{\begin{array}{l}\mathbf{E}_1^0 = 0,\\ \partial_\tau(\mathbf{H}_1^0\,\wedge\,\mathbf{n}) = \omega\,\mathbf{E}_1^1,\\ \partial_\tau(\mathbf{E}_1^1\,\wedge\,\mathbf{n}) = i\omega\mu_1\,\mathbf{H}_1^0.\end{array}\right. \tag{28}$$

This yields

$$\mathbf{E}_1^0(\mathbf{s},n,\tau)\,\wedge\,\mathbf{n}(\mathbf{s}) = 0,$$

$$\mathbf{E}_1^1(\mathbf{s},n,\tau)\,\wedge\,\mathbf{n}(\mathbf{s}) = \mathbf{E}_1^1(\mathbf{s},n,\tau=0)\,\wedge\,\mathbf{n}(\mathbf{s})\;e^{\omega\sqrt{\mu_1}(i-1)\tau/\sqrt{2}}.$$

From (28), it follows that

$$\frac{1}{\sqrt{2\mu_1}}(1+i)\,\mathbf{E}_1^1(\mathbf{s},n=0,\tau=0)\wedge\mathbf{n}(\mathbf{s})+\mathbf{n}\wedge(\mathbf{n}\wedge\mathbf{H}_1^0(\mathbf{s},n=0,\tau=0))=0.$$

Finally, we obtain:
• the 1^{st} approximation or the perfect conductor approximate boundary condition:

$$\begin{cases} \mathbf{curl}\,\mathbf{E}_2^\delta = i\omega\mu_2\,\mathbf{H}_2^\delta \quad \text{in } \mathbf{R}^3\setminus\overline{\Omega_1}, \\ \mathbf{curl}\,\mathbf{H}_2^\delta = -i\omega\varepsilon_2\,\mathbf{E}^\delta \quad \text{in } \mathbf{R}^3\setminus\overline{\Omega_1}, \\ \mathbf{E}_2^\delta\wedge\mathbf{n} = 0(\delta) \quad \text{on } \Gamma. \end{cases} \tag{29}$$

• the 2^{nd} approximation or the Leontovich approximate boundary condition:

$$\begin{cases} \mathbf{curl}\,\mathbf{E}_2^\delta = i\omega\mu_2\,\mathbf{H}_2^\delta \quad \text{in } \mathbf{R}^3\setminus\overline{\Omega_1}, \\ \mathbf{curl}\,\mathbf{H}_2^\delta = -i\omega\varepsilon_2\,\mathbf{E}_2^\delta \quad \text{in } \mathbf{R}^3\setminus\overline{\Omega_1}, \\ \mathbf{E}_2^\delta\wedge\mathbf{n}+\sqrt{\mu_1}(1-i)\,\delta\,\mathbf{n}\wedge(\mathbf{n}\wedge\mathbf{H}_2^\delta) = 0(\delta^2) \quad \text{on } \Gamma. \end{cases} \tag{30}$$

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