SCATTERING OF MAXWELL'S EQUATIONS WITH A LEONTOVICH BOUNDARY CONDITION IN AN INHOMOGENEOUS MEDIUM: A SINGULAR PERTURBATION PROBLEM

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ABSTRACT. The limiting behavior of the unique solution to the scattering problem for the Maxwell equations with a Leontovich boundary condition in an inhomogeneous medium as the impedance goes to zero is investigated. Making use of a Hodge decomposition lemma, it is shown that the Leontovich boundary condition leads to a singular perturbation problem for only the scalar unknown in the Hodge decomposition of the magnetic field. The use of higher order finite elements is then necessary to achieve a sufficient accuracy of this quantity.

KEY WORDS. Leontovich boundary condition, Maxwell's equations, singular perturbation problem.

AMS SUBJECT CLASSIFICATIONS. 35Q60, 35B40, 35C15 .

1 INTRODUCTION

A lossy body composed of a material whose permittivity and permeability may vary as a function of position is immersed in bounded dielectric object and illuminated by an electromagnetic field. At high conductivity, the lossy dielectric body can be replaced by a Leontovich boundary condition applied at its boundary. This approximate boundary condition provides an approximate relationship between the electric and magnetic field on the boundary of the high conducting body. We refer to our paper [2] for a formal derivation of the Leontovich boundary condition by the well-known two scale asymptotic expansion method.

Our main purpose in this paper is to show how this widely used approximate boundary condition yields to a singular perturbation problem for the Maxwell equations. A specific numerical treatement is then required to achieve a sufficient accuracy. In analyzing this singular perturbation problem, we also provide a new variational proof of the existence and uniqueness of solutions to the impedance boundary value problem for the time harmonic Maxwell equations in an inhomogeneous medium. These existence and uniqueness results generalize those of Colton and Kress [5] and Kirsch and Monk [6].

We shall briefly outline the remainder of this paper. The notations are introduced in section 2. In section 3 existence and uniqueness results for the Leontovich boundary value problem for the time-harmonic Maxwell equations are proved by making use of a Hodge decomposition of the magnetic field. Section 4 is devoted to prove the convergence result and to construct the complete asymptotic expansions. The main result in this section is that the Leontovich approximation leads to a singular perturbation problem for only the scalar unknown in the Hodge decomposition of the magnetic field. The use of higher order finite elements is then necessary to achieve a sufficient accuracy of this quantity. The computation of the vector unknown does not require the use of such higher order finite elements. A similar program was carried out by the authors in [2] for the Leontovich boundary value problem for the Maxwell equations in a homogeneous medium. For this problem, we have employed representations of electric and magnetic fields to reformulate the scattering problem as an integral equation over the scattering surface and represent the fields in terms of the tangential component of the magnetic field. By making use of a Hodge decomposition of the tangential component of the magnetic field which is the unknown, we have proved that the Leontovich approximation leads to a singular perturbation problem for only of the surface divergence of the tangential component of the magnetic field . Its numerical computation requires then the use of higher order finite elements to achieve a sufficient accuracy. If the medium is homogeneous we can use the present variational approach as well as our integral one developped in [2]. It turns out that in this case the Laplace-Beltrami operator of the scalar unknown (the quantity which appears in the present approach) is the singular part of the surface divergence of the tangential component of the magnetic field (the quantity which appears in the integral approach). The present work may be considered as a generalization of the authors results to the case of an inhomogeneous medium.

2 NOTATIONS

Let Ω^i be a bounded domain in $\mathbf{R}^3$ and let Ω^e be the complement of $\overline{\Omega^i}$ in $\mathbf{R}^3$. The boundary Γ of Ω^e is assumed to be of class $\mathcal{C}^3$. This assumption is essential to treat our singular perturbation problem. Extension of the present analysis to the case of nonregular boundaries is a nontrivial task. By $\mathbf{n}$ we denote the unit normal to Γ directed into Ω^e. We consider the scattering problem in the domain Ω^e characterized by its electric permittivity ε, and its magnetic permeability μ. We assume that the functions ε and μ are in $\mathcal{C}^2(\overline{\Omega^e})$. We also assume that $\Re e\, \varepsilon > 0$ and $\Re e\, \mu > 0$ and there exists a constant R such that outside the ball B_R, $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$, where ε_0, μ_0 are two strictly positive constants. Let $\Omega_R = \Big\{ |x| < R \Big\} \cap \Omega^e$. Throughout this paper, for any smooth vector field $\mathbf{u}$, we denote by $\mathbf{u}_\Gamma$ its tangential component on Γ. By $\mathbf{grad}_\Gamma$ (resp. div_Γ) we denote the tangential gradient of a function (resp. the surface divergence of a field). We also denote by $\mathbf{curl}_\Gamma$ (resp. curl_Γ) the vector rotational of a function (resp. the scalar rotational of a vector), and by Δ_Γ the Laplace-Beltrami operator on Γ. We shall also use the following notations for the usual functional spaces: $L^2(\Omega_R)$ is the space of complex square integrable functions defined in Ω_R, $H^s(\Omega_R)$ the usual Sobolev space of order s of functions defined in Ω_R, $H^s(\Gamma)$ the usual Sobolev space of order s of functions defined on Γ, $L^\infty(\Gamma)$ the space of bounded functions on Γ,

$H(\mathrm{curl}, \Omega_R) = \Big\{ \mathbf{u} \in (L^2(\Omega_R))^3, \mathbf{curl}\, \mathbf{u} \in (L^2(\Omega_R))^3 \Big\}$,

$TH^s(\Gamma) = \Big\{ \mathbf{u} \in (H^s(\Gamma))^3, \mathbf{u} \,.\, \mathbf{n} = 0 \Big\}, TL^\infty(\Gamma) = \Big\{ \mathbf{u} \in (L^\infty(\Gamma))^3, \mathbf{u} \,.\, \mathbf{n} = 0 \Big\}$,

$TH^s(\mathrm{curl}, \Gamma) = \Big\{ \mathbf{u} \in TH^s(\Gamma), \mathrm{curl}_\Gamma\, \mathbf{u} \in H^s(\Gamma) \Big\}$,

$\mathbb{H}(\Omega_R) = \Big\{ \mathbf{u} \in H(\mathrm{curl}, \Omega_R), \mathbf{u}_\Gamma \in TL^2(\Gamma) \Big\}$,

$\mathbb{H}_{\rm div}(\Omega_R) = \Big\{\mathbf{u} \in (H^1(\Omega_R))^3, \int_{\Omega_R} \mu\, \mathbf{u}\,.\,\mathbf{grad}\, q = 0, \forall\, q \in H^1(\Omega_R)\Big\}$
and $\mathbb{P}(\Omega_R) = \Big\{p \in H^1(\Omega_R), p|_\Gamma \in H^1(\Gamma)\Big\}$.

We introduce the bilinear form

$$((\mathbf{u}, \mathbf{v})) = \int_{\Omega_R} \mathbf{curl}\,\mathbf{u}\,.\,\mathbf{curl}\,\mathbf{v} + \int_{\Omega_R} \mu\, \mathbf{u}\,.\,\mathbf{v}.$$

Let z_0 be a strictly positive constant, $z \in L^\infty(\Gamma)$ a bounded function on Γ such that $\Re e(z) \geq z_0 > 0, \Im m(z) \geq 0$. Now we can state precisely the Leontovich boundary value problem for the time-harmonic Maxwell equations we shall analyze in this paper, which is to find the vector fields $\mathbf{E}^\delta$ and $\mathbf{H}^\delta$ satisfying the Maxwell equations:

$$\mathbf{curl}\,\mathbf{E}^\delta \;=\; i\omega\, \mu\, \mathbf{H}^\delta, \quad \mathbf{curl}\;\mathbf{H}^\delta \;=\; -i\omega\, \varepsilon\; \mathbf{E}^\delta \quad \text{in } \Omega^e, \tag{1}$$

the Silver-Müller radiation condition and the boundary condition:

$$-\mathbf{E}^\delta_\Gamma + i\delta\, z\; \mathbf{H}^\delta \wedge \mathbf{n} \;=\; \mathbf{g}^\delta \quad \text{on } \Gamma, \tag{2}$$

where ω is the frequency of the incident fields, $\mathbf{g}^\delta$ is a given tangential field on Γ and δ is a strictly positive constant. The parameter δ is of order of $1/\sqrt{\sigma}$ where σ is the conductivity of Ω^i and the function z is its impedance.

3 EXISTENCE AND UNIQUENESS RESULTS

The purpose of this section is to provide a variational proof of the existence and uniqueness of a solution to the Leontovich boundary value problem for the time-harmonic Maxwell equations in inhomogeneous media. Our proof which is to use exact nonlocal boundary conditions on the sphere S_R is closely related to that of Kirsch and Monk [6].

3.1 THE EXTERIOR ELECTROMAGNETIC OPERATOR

In this subsection we shall give a brief summary of the properties of the exterior electromagnetic operator mapping the tangential field $\hat{\mathbf{x}} \wedge \mathbf{H}^\delta$ on S_R onto $\hat{\mathbf{x}} \wedge \mathbf{E}^\delta$ on S_R.

Let $(\mathcal{Y}^l_m)_{(l\geq 0, -l\leq m\leq l)}$ be an orthonormal sequence of spherical harmonics of order l on the unit sphere S, normalized such that

$$\int_S \mathcal{Y}^l_m\, \mathcal{Y}^{l'}_{m'} = \delta_{ll'}\, \delta_{mm'}.$$

An orthonormal basis functions for the tangential fields on S are then

$$\mathbf{U}^l_m = \frac{1}{\sqrt{l(l+1)}}\mathbf{grad}_S \mathcal{Y}^l_m \text{ and } \mathbf{V}^l_m = \hat{\mathbf{x}} \wedge \mathbf{U}^l_m, m = -l, \ldots, l,\ l \geq 1.$$

Spaces of either scalar or vector functions on S_R can be characterized by the summability of weighted sums of their expansion coefficients into the basis $(\mathcal{Y}^l_m)_{(l\geq 0, -l\leq m\leq l)}$ or $((\mathbf{U}^l_m), (\mathbf{V}^l_m))$. Let $\mathbf{u}$ be in $TH^{-1/2}(\mathrm{div}, S_R)$. Let us define the exterior electromagnetic operator

$$\mathcal{G}^e : TH^{-1/2}(\mathrm{div}, S_R) \mapsto TH^{-1/2}(\mathrm{div}, S_R)$$

by

$$\mathcal{G}^e(\mathbf{u}) = \sqrt{\frac{\mu_0}{\varepsilon_0}} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \Big(- \frac{i\omega R}{z_l(\omega, R)} b^l_m\, \mathbf{U}^l_m + \frac{z_l(\omega, R)}{i\omega R} a^l_m\, \mathbf{V}^l_m \Big)$$

where

$$\mathbf{u} = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \Big(a^l_m\, \mathbf{U}^l_m \; + \; b^l_m\, \mathbf{V}^l_m \Big),$$

$$z_l(\omega, R) = 1 + \frac{\omega\, R\, (h_1^{(1)})'(\omega R)}{h_l^{(1)}(\omega R)},$$

and $h_l^{(1)}$ is the Hankel function of order l. It is now classical to see that ([7] and [6])

$$\mathcal{G}^e(\hat{\mathbf{x}} \wedge \mathbf{H}) = \hat{\mathbf{x}} \wedge \mathbf{E} \quad \text{on } S_R,$$

where $(\mathbf{E}, \mathbf{H})$ is any solution of the Maxwell equations (1) satisfying the Silver-Müller radiation condition. Now we summarize the properties of the exterior electromagnetic operator $\mathcal{G}^e$ in the following Lemma. We refer once again to [7] and [6] for the proofs.

LEMMA 3.1 *Let* $\mathbf{v} \in TH^{-1/2}(\mathrm{curl}, S_R), \mathbf{u} \in \mathbb{H}_{\mathrm{div}}(\Omega_R)$ *and* $p \in \mathbb{P}(\Omega_R)$. *We have:*

$$\Re e\Big(\mathcal{G}^e(\hat{\mathbf{x}} \wedge \mathbf{v}), \mathbf{v}\Big) > 0 \quad \text{if } \mathbf{v} \neq 0 \text{ on } S_R, \tag{3}$$

$$\Big(\mathcal{G}^e(\hat{\mathbf{x}} \wedge \mathbf{u}),\ (\hat{\mathbf{x}} \wedge \mathbf{grad}\, p) \wedge \hat{\mathbf{x}}\Big) = 0, \tag{4}$$

$$\Big(\mathcal{G}^e(\hat{\mathbf{x}} \wedge \mathbf{grad}\, p),\ (\hat{\mathbf{x}} \wedge \mathbf{u}) \wedge \hat{\mathbf{x}}\Big) = 0, \tag{5}$$

$$\Im m\Big(\mathcal{G}^e(\hat{\mathbf{x}}\wedge\mathbf{u}),(\hat{\mathbf{x}}\wedge\mathbf{u})\ \wedge\ \hat{\mathbf{x}}\Big)\ \geq\ c\,||\mathbf{curl}_{S_R}\mathbf{u}||^2_{H^{-1/2}(S_R)} \\ -c'\,||\mathbf{curl}_{S_R}\mathbf{u}||^2_{H^{-1}(S_R)}, \tag{6}$$

and

$$-\Im m\Big(\mathcal{G}^e(\hat{\mathbf{x}}\wedge\mathbf{grad}\,p),(\hat{\mathbf{x}}\wedge\mathbf{grad}\,p)\ \wedge\ \hat{\mathbf{x}}\Big)\ \geq\ c\,||p||^2_{H^{1/2}(S_R)} \\ -c'\,||p||^2_{H^{-1/2}(S_R)}, \tag{7}$$

where c, c' *are two strictly positive constants and* $(,)$ *is the bracket of the duality between* $TH^{-1/2}(\mathrm{div}, S_R)$ *and* $TH^{-1/2}(\mathrm{curl}, S_R)$.

In our proof of the convergence result we shall need the following compactness result which immediately follows from Lemma 4.3 in [6]. Let the operator $\mathcal{G}^0 : TH^{-1/2}(\mathrm{div}, S_R) \mapsto TH^{-1/2}(\mathrm{div}, S_R)$ be defined by

$$\mathcal{G}^0(\mathbf{u}) = \sqrt{\frac{\mu_0}{\varepsilon_0}}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\Big(\frac{R}{z_l(i,R)}b^l_m\,\mathbf{U}^l_m - \frac{z_l(i,R)}{R}a^l_m\,\mathbf{V}^l_m\Big).$$

We have the following lemma.

LEMMA 3.2 *The operators* $\mathcal{G}^e + i\omega\,\mathcal{G}^0$ *and* $i\omega\,\mathcal{G}^e + \mathcal{G}^0$ *are compact operators on* $\mathbb{H}_{\mathrm{div}}(\Omega_R)$ *in the sense that if* $\mathbf{u}^\delta$ *is a sequence of vector fields such that* $\mathbf{u}^\delta \longrightarrow \mathbf{u}$ *in* $\mathbb{H}_{\mathrm{div}}(\Omega_R)$ *weakly when* δ *goes to zero then*

$$\Big((\mathcal{G}^e + i\omega\mathcal{G}^0)\mathbf{u}^\delta,\mathbf{u}^\delta\Big) \longrightarrow \Big((\mathcal{G}^e + i\omega\mathcal{G}^0)\mathbf{u},\mathbf{u}\Big)$$

and

$$\Big((\mathcal{G}^0 + i\omega\mathcal{G}^e)\mathbf{u}^\delta,\mathbf{u}^\delta\Big) \longrightarrow \Big((\mathcal{G}^0 + i\omega\mathcal{G}^e)\mathbf{u},\mathbf{u}\Big).$$

3.2 UNIQUENESS RESULT

We shall give a uniqueness result for the impedance boundary value problem for the Maxwell equations an inhomogeneous medium. We have the following result.

LEMMA 3.3 *The Maxwell equations* (1) *with the boundary condition* (2) *have at most one solution satisfying the Silver-Müller radiation condition.*

PROOF. Let $(\mathbf{E}^\delta, \mathbf{H}^\delta)$ be a solution of the Maxwell equations (1) satisfying the Silver-Müller radiation condition and the boundary conditions

$$\mathbf{n}\ \wedge\ (\mathbf{n}\ \wedge\ E^\delta) + i\delta\,z\ \mathbf{H}^\delta\ \wedge\ \mathbf{n} = 0 \quad \text{on } \Gamma. \tag{8}$$

Integrating by parts the equation (1) over Ω_R and making use of the boundary condition (8) gives

$$\int_{\Omega_R} \frac{1}{\mu}|\mathbf{curl}\,\mathbf{E}^\delta|^2 - \omega^2 \int_{\Omega_R} \varepsilon\,|\mathbf{E}^\delta|^2 + \delta\,\omega \int_\Gamma \overline{z}\,|\mathbf{H}^\delta \wedge \mathbf{n}|^2$$
$$-i\omega\left(\mathbf{H}^\delta \wedge \hat{\mathbf{x}}, \hat{\mathbf{x}} \wedge (\overline{\mathbf{E}^\delta} \wedge \hat{\mathbf{x}})\right) = 0.$$

By taking the imaginary part in the above expression, we obtain

$$\int_{\Omega_R} \Im m(\frac{1}{\mu})|\,\mathbf{curl}\,\mathbf{E}^\delta|^2 - \Re e\left(\mathcal{G}^e(\hat{\mathbf{x}} \wedge \mathbf{H}^\delta), \hat{\mathbf{x}} \wedge (\overline{\mathbf{H}}^\delta \wedge \hat{\mathbf{x}})\right)$$
$$+\delta\,\omega \int_\Gamma \Im m(\overline{z})\,|\mathbf{H}^\delta \wedge \mathbf{n}|^2 = 0.$$

Since $\Im m(z) \geq 0$, it follows from (3) that the tangential components of the electric and magnetic fields on S_R are equal to zero. Under the assumption on the regularity of the coefficients ε and μ we obtain the claim by the classical unique continuation principle.

□

3.3 EXISTENCE RESULT

In order to establish an existence result for the impedance boundary value problem we need a Hodge decomposition of the Sobolev space $\mathbb{H}(\Omega_R)$. We have the following Lemma.

LEMMA 3.4 *The Hodge decomposition*

$$\mathbb{H}(\Omega_R) = \mathbb{H}_{\text{div}}(\Omega_R) \oplus \mathbf{grad}\,\mathbb{P}(\Omega_R).$$

is orthogonal with respect to the product $((,))$.

PROOF. The spaces $\mathbb{H}_{\text{div}}(\Omega_R)$ and $\mathbf{grad}\,\mathbb{P}(\Omega_R)$ are obviously closed spaces in $\mathbb{H}(\Omega_R)$ and are orthogonal with respect to the product $((,))$. Therefore, there exists a bounded operator $P : \mathbb{H}(\Omega_R) \to \mathbf{grad}\,\mathbb{P}(\Omega_R)$ such that for $\mathbf{v} \in \mathbb{H}(\Omega_R)$ the identity $((P\mathbf{v} - \mathbf{v}, \mathbf{grad}\,q)) = 0$ holds for any $q \in \mathbb{P}(\Omega_R)$. If we write $\mathbf{v} = P\mathbf{v} + \mathbf{u}$, we have only to show that $\mathbf{u} \in \mathbb{H}_{\text{div}}(\Omega_R)$. From the identity

$$\int_{\Omega_R} \mu\,\mathbf{u}\,.\,\mathbf{grad}\,q = 0$$

which holds for any $q \in \mathbb{P}(\Omega_R)$, we obtain that $\mathbf{u} \in H(\mathrm{rot}, \Omega_R) \cap H(\mathrm{div}, \Omega_R)$ and $\mathbf{u}\,.\,\mathbf{n} = 0$ on $\Gamma \cup S_R$ and so, $\mathbf{u} \in (H^1(\Omega_R))^3$. This ends the proof. □

Next we proceed to reduce the impedance boundary value problem (1) – (2) to a system of two coupled variational equations. By simply using the above Hodge decomposition and integrating by parts we obtain the following lemma.

LEMMA 3.5 *Assume that* $\mathbf{g}^\delta \in TH^{-1/2}(\Gamma)$ *and* $\mathrm{curl}_\Gamma\,\mathbf{g}^\delta \in H^{-1}(\Gamma)$. *Then the following variational formulation:*

$$\left\{ \begin{array}{l} a^0(\mathbf{u}^\delta, \mathbf{u}^t) + \delta\, a^1(\mathbf{u}^\delta, \mathbf{u}^t) = -i\omega \displaystyle\int_\Gamma (\mathbf{g}^\delta \wedge \mathbf{n})\,.\,\mathbf{u}^t_\Gamma - \delta\, a^1(\mathbf{grad}\, p^\delta, \mathbf{u}^t), \\ \hat{a}^0(p^\delta, p^t) + \delta\, a^1(\mathbf{grad}\, p^\delta, \mathbf{grad}\, p^t) = -i\omega \displaystyle\int_\Gamma (\mathbf{g}^\delta \wedge \mathbf{n})\,.\,\mathbf{grad}_\Gamma\, p^t - \delta\, a^1(\mathbf{u}^\delta, \mathbf{grad}\, p^t), \end{array} \right. \tag{9}$$

where the unknowns are $\mathbf{u}^\delta \in \mathbb{H}_{\mathrm{div}}(\Omega_R)$ *and* $p^\delta \in \mathbb{P}(\Omega_R)$ *and*

$$\left\{ \begin{array}{lcl} a^0(\mathbf{u}, \mathbf{u}^t) & = & \displaystyle\int_{\Omega_R} \frac{1}{\varepsilon} \mathrm{curl}\, \mathbf{u}\,.\mathbf{curl}\, \mathbf{u}^t - \omega^2 \int_{\Omega_R} \mu\, \mathbf{u}\,.\,\mathbf{u}^t \\ & & -i\omega\Big(\mathcal{G}^e(\hat{\mathbf{x}} \wedge \mathbf{u}), (\hat{\mathbf{x}} \wedge \mathbf{u}^t) \wedge \hat{\mathbf{x}}\Big), \\ a^1(\mathbf{u}, \mathbf{u}^t) & = & -\omega \displaystyle\int_\Gamma z\, \mathbf{u}_\Gamma\,.\,\mathbf{u}^t_\Gamma \\ \hat{a}^0(p, p^t) & = & a^0(\mathbf{grad}\, p, \mathbf{grad}\, p^t) \end{array} \right.$$

is such that $\mathbf{H}^\delta = \mathbf{u}^\delta + \mathbf{grad}\, p^\delta$ *and* $\mathbf{E}^\delta = \dfrac{i}{\omega\varepsilon}\mathbf{curl}\,\mathbf{H}^\delta$ *are solution in* $\mathbb{H}(\Omega_R) \times \mathbb{H}(\Omega_R)$ *of the Maxwell's equations* (1) *with the boundary condition* (2).

Now we are in position to prove the main result of this section. We shall show that the variational system (9) has a unique solution in $\mathbb{H}_{\mathrm{div}}(\Omega_R) \times \mathbb{P}(\Omega_R)$ by applying the Fredholm theory. This generalizes the existence and uniqueness results obtained for the impedance boundary value problem for the homogeneous Maxwell equations by Colton and Kress in [5].

THEOREM 3.1 *Assume that* $\mathbf{g}^\delta \in TH^{-1/2}(\Gamma)$ *and* $\mathrm{curl}_\Gamma\,\mathbf{g}^\delta \in H^{-1}(\Gamma)$. *The variational system* (9) *has one solution in* $\mathbb{H}_{\mathrm{div}}(\Omega_R) \times \mathbb{P}(\Omega_R)$.

PROOF. First, we observe that if $\mathrm{rot}_\Gamma\,\mathbf{g}^\delta \in H^{-1}(\Gamma)$ then the linear form

$$p \longrightarrow \int_\Gamma (\mathbf{g}^\delta \wedge \mathbf{n})\,.\,\mathbf{grad}_\Gamma\, p = -\int_\Gamma \mathrm{curl}_\Gamma(\mathbf{g}^\delta)\, p$$

is continuous over $\mathbb{P}(\Omega_R)$. Second, by using (7) we obtain

$$-\Re e\Big\{\, \hat{a}^0(p,p) + \delta\, a^1(\mathbf{grad}\, p, \mathbf{grad}\, p) \Big\} \;\geq\; \omega^2 \int_{\Omega_R} \Re e(\mu) |\mathbf{grad} p|^2$$
$$+\omega\, \delta\, z_0\, ||\mathbf{grad}_\Gamma\, p||^2_{L^2(\Gamma)} + c\,||p||^2_{H^{1/2}(S_R)} - c'\,||p||^2_{H^{-1/2}(S_R)}, \tag{10}$$

from which it follows that the Fredholm alternative holds for the bilinear form $\hat{a}^0(p,p^t) + \delta\, a^1(\mathbf{grad}\, p, \mathbf{grad}\, p^t)$ defined over $\mathbb{P}(\Omega_R) \times \mathbb{P}(\Omega_R)$.
Next we establish that the Fredholm allternative also holds for the bilinear form $a^0(\mathbf{u},\mathbf{u}) + \delta\, a^1(\mathbf{u},\mathbf{u})$ defined over $\mathbb{H}_{\mathrm{div}}(\Omega_R) \times \mathbb{H}_{\mathrm{div}}(\Omega_R)$. For this we need to estimate $\displaystyle\int_{\Omega_R} |\mathbf{grad}\,(\mu\, \mathbf{u})|^2$ where $\mathbf{u} \in \mathbb{H}_{\mathrm{div}}(\Omega_R)$. From [7] we have

$$\int_{\Omega_R} |\mathbf{grad}\,(\mu\, \mathbf{u})|^2 \leq \int_{\Omega_R} |\mathbf{curl}\,(\mu\, \mathbf{u})|^2 \;\;+\;\; c\Big\{||\hat{\mathbf{x}} \wedge \mathbf{u}||_{TH^{1/2}(S_R)}\, ||\hat{\mathbf{x}} \wedge \mathbf{u}||_{TH^{-1/2}(S_R)}$$
$$+||\mathbf{n} \wedge \mathbf{u}||_{TH^{1/2}(\Gamma)}\, ||\mathbf{n} \wedge \mathbf{u}||_{TH^{-1/2}(\Gamma)}\Big\},$$

$\forall\, \mathbf{u} \in \mathbb{H}_{\mathrm{div}}(\Omega_R)$ where c is a strictly positive constant. By the arithmetic-geometric mean inequality, it follows that

$$\left\{ \begin{array}{lll} ||\hat{\mathbf{x}} \wedge \mathbf{u}||_{TH^{1/2}(S_R)}\, ||\hat{\mathbf{x}} \wedge \mathbf{u}||_{TH^{-1/2}(S_R)} & \leq & \eta\, ||\mathbf{u}||^2_{(H^1(\Omega_R))^3} + \dfrac{1}{\eta}\, ||\mathbf{u}||^2_{H(\mathrm{curl},\Omega_R)}, \\ ||\mathbf{n} \wedge \mathbf{u}||_{TH^{1/2}(\Gamma)}\, ||\mathbf{n} \wedge \mathbf{u}||_{TH^{-1/2}(\Gamma)} & \leq & \eta\, ||\mathbf{u}||^2_{(H^1(\Omega_R))^3} + \dfrac{1}{\eta}\, ||\mathbf{u}||^2_{H(\mathrm{curl},\Omega_R)}, \end{array} \right.$$

$\forall\, \eta > 0$. Therefore, since $\mu \in \mathcal{C}^2(\Omega_R)$ there exists a strictly positive constant c'' such that

$$\int_{\Omega_R} |\mathbf{curl}\, \mathbf{u}|^2 \;\geq\; c'' \int_{\Omega_R} |\mathbf{grad}\, \mathbf{u}|^2, \quad \forall\, \mathbf{u} \in \mathbb{H}_{\mathrm{div}}(\Omega_R). \tag{11}$$

Now, using the inequality

$$\left\{ \begin{array}{lll} \Re e(a^0(\mathbf{u},\mathbf{u}) + \delta\, a^1(\mathbf{u},\mathbf{u})) & \geq & \displaystyle\int_{\Omega_R} \frac{1}{\varepsilon}\, |\mathrm{curl}\, \mathbf{u}|^2 - \omega^2 \int_{\Omega_R} \Re e(\mu)\, |\mathbf{u}|^2 \\ & & \displaystyle -\delta\, \omega \int_\Gamma \Re e(z)\, |\mathbf{u}_\Gamma|^2 + c\,||\mathbf{curl}_{S_R} \mathbf{u}||^2_{H^{-1/2}(S_R)} \\ & & -c'\,||\mathbf{curl}_{S_R} \mathbf{u}||^2_{H^{-1}(S_R)}, \quad \forall\, \mathbf{u} \in \mathbb{H}_{\mathrm{div}}(\Omega_R), \end{array} \right. \tag{12}$$

together with (11), we can conclude that the Fredholm alternative holds for the bilinear form $a^0(\mathbf{u},\mathbf{u}) + \delta\, a^1(\mathbf{u},\mathbf{u})$ defined over $\mathbb{H}_{\mathrm{div}}(\Omega_R) \times \mathbb{H}_{\mathrm{div}}(\Omega_R)$.
The form

$$a^1(\mathbf{grad}\, p, \mathbf{u}) = -\omega \int_\Gamma z\, \mathbf{grad}\, p \,.\, \mathbf{u}_\Gamma,$$

defined over $\mathbb{P}(\Omega_R)\times\mathbb{H}_{\mathrm{div}}(\Omega_R)$ is compact if $z \in L^\infty(\Gamma)$. Since the impedance boundary value problem for the Maxwell equations $(1)-(2)$ is uniquely solvable, Lemma 3.5 gives the claim. □

4 ASYMPTOTICS OF THE SOLUTION

Our purpose in this section is to give a complete asymptotic analysis for the system of variational equations (9) when the parameter δ goes to zero. This section is organized as follows. In a first step, the convergence result is proved. Then if the data $\mathbf{g}^0 = \lim_{\delta\to 0}\mathbf{g}^\delta$ in $TH^{-1/2}(\mathrm{curl},\Gamma)$ is more regular, we estimate the rate of the convergence. The final step is devoted to the construction of the complete asymptotic expansions of $\mathbf{u}^\delta$ and p^δ.

4.1 CONVERGENCE RESULT

We first state the main result of this subsection. We shall prove the following theorem.

THEOREM 4.1 *Assume that $\mathbf{g}^\delta$ converges in $TH^{-1/2}(\mathrm{curl},\Gamma)$ to the tangential field $\mathbf{g}^0$ on Γ and $\mathbf{grad}_\Gamma\, z \in TL^\infty(\Gamma)$. The pair $(\mathbf{u}^\delta, p^\delta)$ converges in $\mathbb{H}_{\mathrm{div}}(\Omega_R)\times H^1(\Omega_R)/\mathbb{C}$ strongly to $(\mathbf{u}^0,p^0)$ defined as the unique solution in $\mathbb{H}_{\mathrm{div}}(\Omega_R)\times H^1(\Omega_R)/\mathbb{C}$ of the variational system:*

$$\left\{\begin{array}{ll} a^0(\mathbf{u}^0,\mathbf{u}^t) = -i\omega \displaystyle\int_\Gamma (\mathbf{g}^0\ \wedge\ \mathbf{n})\,.\,\mathbf{u}^t_\Gamma, & \forall\ \mathbf{u}^t \in \mathbb{H}_{\mathrm{div}}(\Omega_R),\\ \hat{a}^0(p^0,p^t) = -i\omega \displaystyle\int_\Gamma (\mathbf{g}^0\ \wedge\ \mathbf{n})\,.\,\mathbf{grad}_\Gamma\, p^t, & \forall\ p^t \in H^1(\Omega_R)/\mathbb{C}\,. \end{array}\right. \tag{13}$$

To prove this theorem we need several intermediate results. First, we can show by similar arguments that used in section 3 the unique solvability of the limiting problem.

LEMMA 4.1 *Assume that $\mathbf{g}^0 \in TH^{-1/2}(\mathrm{curl},\Gamma)$. The variational system (13) has one solution in $\mathbb{H}_{\mathrm{div}}(\Omega_R)\times H^1(\Omega_R)/\mathbb{C}$.*

The fields $\mathbf{H}^0 = \mathbf{u}^0 + \mathbf{grad}\, p^0$ and $\mathbf{E}^0 = \dfrac{i}{\omega\varepsilon}\mathbf{curl}\,\mathbf{H}^0$ are then solutions in the space $H(\mathrm{curl},\Omega_R)\times H(\mathrm{curl},\Omega_R)$ of the Maxwell equations (1) with the boundary conditions of Dirichlet on Γ: $-\mathbf{E}^0_\Gamma = \mathbf{g}^0 \in TH^{-1/2}(\mathrm{curl},\Gamma)$. In what follows we assume that $\mathbf{grad}_\Gamma\, z \in TL^\infty(\Gamma)$.
Next, we establish the following a priori estimate. We have:

LEMMA 4.2 *There exists a strictly positive constant C independent of δ such that*

$$||\mathbf{u}^\delta||_{\mathbb{H}_{\mathrm{div}}(\Omega_R)} + ||p^\delta||_{H^1(\Omega_R)/\mathbb{C}} + \sqrt{\delta}\,||p^\delta||_{\mathbb{P}(\Omega_R)} \leq C. \tag{14}$$

PROOF. Let us prove by contradiction that the quantity

$$\gamma^\delta = ||\mathbf{u}^\delta||_{\mathbb{H}_{\mathrm{div}}(\Omega_R)} + ||p^\delta||_{H^1(\Omega_R)/\mathbb{C}} + \sqrt{\delta}\,||p^\delta||_{\mathbb{P}(\Omega_R)}$$

is uniformly bounded with respect to the parameter δ. Assume that we have a subsequence, still denoted by γ^δ such that it converges to $+\infty$ as δ goes to zero. Let us define $\mathbf{v}^\delta$ and q^δ by

$$\mathbf{v}^\delta = \frac{\mathbf{u}^\delta}{\gamma^\delta}, \; q^\delta = \frac{p^\delta}{\gamma^\delta}$$

It follows that $(\mathbf{v}^\delta, q^\delta)$ converges to a certain $(\mathbf{v}, q), \mathbb{H}_{\mathrm{div}}(\Omega_R) \times H^1(\Omega_R)/\mathbb{C}$ weakly. We shall prove that $q = 0$ and $\mathbf{v} = 0$. We begin with the assertion $q = 0$. For this, We rewrite the second equation in (9) as

$$\left\{ \begin{array}{l} -\omega^2 \displaystyle\int_{\Omega_R} \mu \,\mathbf{grad}\, q^\delta . \mathbf{grad}\, q^t - \omega\,\delta \int_\Gamma z\, \mathbf{grad}_\Gamma q^\delta . \mathbf{grad}\, q^t \\ +(\mathcal{G}^0(\hat{\mathbf{x}} \wedge \mathbf{grad}\, q^\delta), (\hat{\mathbf{x}} \wedge \mathbf{grad}\, q^t) \wedge \hat{\mathbf{x}}) = -i\omega \displaystyle\int_\Gamma \frac{1}{\gamma^\delta}(\mathbf{g}^\delta \wedge \mathbf{n}) \,.\, \mathbf{grad}_\Gamma\, q^t \\ +((i\omega\,\mathcal{G}^e + \mathcal{G}^0)(\hat{\mathbf{x}} \wedge \mathbf{grad}\, q^\delta), (\hat{\mathbf{x}} \wedge \mathbf{grad}\, q^t) \wedge \hat{\mathbf{x}}) - \delta \displaystyle\int_\Gamma z\, \mathbf{grad}_\Gamma\, q^t \,.\, \mathbf{v}^\delta. \end{array} \right. \tag{15}$$

Taking $q^t = q^\delta$ and passing to the limit when δ tends to zero, we see that the terms

$$\int_\Gamma \frac{1}{\gamma^\delta}(\mathbf{g}^\delta \wedge \mathbf{n}) \,.\, \mathbf{grad}_\Gamma\, q^\delta = -\int_\Gamma \frac{1}{\gamma^\delta} \mathrm{curl}_\Gamma\, \mathbf{g}^\delta \; q^\delta$$

and

$$\delta \int_\Gamma z\, \mathbf{grad}_\Gamma\, q^\delta \,.\, \mathbf{v}^\delta = -\delta \Big\{ \int_\Gamma z\, q^\delta\, \mathrm{div}_\Gamma\, \mathbf{v}^\delta + \int_\Gamma \mathbf{grad}_\Gamma\, z \,.\, \mathbf{v}^\delta\, q^\delta \Big\}$$

goe to zero and the term

$$((i\omega\,\mathcal{G}^e + \mathcal{G}^0)(\hat{\mathbf{x}} \wedge \mathbf{grad}\, q^\delta), (\hat{\mathbf{x}} \wedge \mathbf{grad}\, q^\delta) \wedge \hat{\mathbf{x}})$$

converges to $((i\omega\,\mathcal{G}^e + \mathcal{G}^0)(\hat{\mathbf{x}} \wedge \mathbf{grad}\, q), (\hat{\mathbf{x}} \wedge \mathbf{grad}\, q) \wedge \hat{\mathbf{x}})$ from the compactness properties of the operator $(i\omega\,\mathcal{G}^e + \mathcal{G}^0)$ given in Lemma 3.2. Therefore,

$$\begin{array}{l} -\omega^2 \displaystyle\int_{\Omega_R} \mu\, |\mathbf{grad}\, q^\delta|^2 - \omega\,\delta \int_\Gamma z\, |\mathbf{grad}_\Gamma q^\delta|^2 + (\mathcal{G}^0(\hat{\mathbf{x}} \wedge \mathbf{grad}\, q^\delta), (\hat{\mathbf{x}} \wedge \mathbf{grad}\, q^\delta) \wedge \hat{\mathbf{x}}) \\ \longrightarrow \quad ((i\omega\,\mathcal{G}^e + \mathcal{G}^0)(\hat{\mathbf{x}} \wedge \mathbf{grad}\, q), (\hat{\mathbf{x}} \wedge \mathbf{grad}\, q) \wedge \hat{\mathbf{x}}) \end{array} \tag{16}$$

Combining the fact that the real part of all the left-hand terms is negative if $\Re e(z(x)) \geq z_0 > 0$ together with the fact that the sum is uniformly bounded with respect to the parameter δ gives that $\sqrt{\delta}\,\mathbf{grad}_\Gamma\, q^\delta$ converges weakly in $TL^2(\Gamma)$. Next, since $\sqrt{\delta}\,\mathbf{grad}_\Gamma\, q^\delta$ is uniformly bounded in $TL^2(\Gamma)$, it follows that the quantity

$$\omega\,\delta \int_\Gamma z\,\mathbf{grad}_\Gamma q^\delta \,.\, \mathbf{grad}_\Gamma q^t = \omega\sqrt{\delta}\int_\Gamma z\,\sqrt{\delta}\,\mathbf{grad}_\Gamma q^\delta \,.\, \mathbf{grad}_\Gamma q^t$$

goes to zero for all $q^t \in \mathcal{C}^\infty(\overline{\Omega_R})$. Finally, we obtain

$$-\omega^2 \int_{\Omega_R} \mu\,\mathbf{grad}\, q \,.\, \mathbf{grad}\, q^t - i\omega\,(\mathcal{G}^e(\hat{\mathbf{x}} \wedge \mathbf{grad}\, q), (\hat{\mathbf{x}} \wedge \mathbf{grad}\, q^t)\;\wedge\;\hat{\mathbf{x}}) = 0, \quad (17)$$

for all $q^t \in \mathcal{C}^\infty(\overline{\Omega_R})$. From the density of $\mathcal{C}^\infty(\overline{\Omega_R})$ in the Sobolev space $H^1(\Omega_R)$, the coerciveness property of the bilinear form $\hat{a}_0$ over $H^1(\Omega_R)/\mathbb{C} \times \mathrm{H}^1(\Omega_\mathrm{R})/\mathbb{C}$ gives $q = 0$.

Now, we prove that $\mathbf{v} = 0$. From the first equation of the variational system (9), we obtain the limiting problem simply by using continuous properties of the operator $\mathcal{G}^e$ and the bilinear form a^1 over $\mathbb{H}_{\mathrm{div}}(\Omega_R) \times \mathbb{H}_{\mathrm{div}}(\Omega_R)$ and the fact that $a^1(\mathbf{v}^\delta, \mathbf{v}^t) \longrightarrow a^1(\mathbf{v}, \mathbf{v}^t)$. We also use that from the fact $\sqrt{\delta}\,\mathbf{grad}_\Gamma\, q^\delta$ is bounded in $TL^2(\Gamma)$, we can conclude that $\delta\, a^1(\mathbf{grad}\, q^\delta, \mathbf{v}^t)$ goes to zero δ goes to zero. Thus, $a^0(\mathbf{v}, \mathbf{v}^t) = 0$ and Lemma (4.1) gives that $\mathbf{v} = 0$.

Finally, we show that the convergence of $\mathbf{v}^\delta, q^\delta$ and $\sqrt{\delta}\, q^\delta$ to 0 in $\mathbb{H}_{\mathrm{div}}(\Omega_R), H^1(\Omega_R)/\mathbb{C}$ and $\mathbb{P}(\Omega_R)$ respectively is in a strong sense.

Making use of the fact that the real part of each term in the left-hand side in (16) is negative and the sum converges uniformly to zero when the parameter δ goes to zero, we obtain

$$\int_{\Omega_R} \Re e(\mu)\,|\mathbf{grad}\, q^\delta|^2 \longrightarrow 0,$$

$$\delta \int_\Gamma \Re e(z)\,|\mathbf{grad}_\Gamma q^\delta|^2 \longrightarrow \; 0.$$

Thus, $\sqrt{\delta}\, q^\delta$ (resp. q^δ) converges to 0 in $\mathbb{P}(\Omega_R)$ (resp. $H^1(\Omega_R)/\mathbb{C}$) strongly. The strong convergence of $\mathbf{v}^\delta$ to zero follows from the fact that $a^0(\mathbf{v}^\delta, \mathbf{v}^\delta)$ converges to zero as δ tends to zero.

Since

$$||\mathbf{v}^\delta||_{\mathbb{H}_{\mathrm{div}}(\Omega_R)} + \sqrt{\delta}\,||q^\delta||_{H^1(\Omega_R)/\mathbb{C}} + ||q^\delta||_{\mathbb{P}(\Omega_R)} \;=\; 1,$$

this leads to a contradiction and the proof is complete. □

LEMMA 4.3 *Assume that the field* $\mathbf{g}^\delta$ *converges to* $\mathbf{g}^0$ *in* $TH^{-1/2}(\mathrm{curl},\Gamma)$. *Let* $(\mathbf{u}^0,p^0)$ *be defined as the unique solution of* (13) *in* $\mathbb{H}_{\mathrm{div}}(\Omega_R)\times H^1(\Omega_R)/\mathbb{C}$. *Then,*

$$\mathbf{u}^\delta \longrightarrow \mathbf{u}^0 \quad \mathbb{H}_{\mathrm{div}}(\Omega_R) \text{ strongly},$$
$$p^\delta \longrightarrow p^0 \quad H^1(\Omega_R)/\mathbb{C} \text{ strongly},$$

PROOF. Using Lemma (4.2) we obtain that $(\sqrt{\delta}p^\delta, p^\delta, \mathbf{u}^\delta)$ converges to $(q', q, \mathbf{v})$ in $\mathbb{P}(\Omega_R)\times H^1(\Omega_R)/\mathbb{C}\times \mathbb{H}_{\mathrm{div}}(\Omega_R)$ weakly. We shall prove that $(q,\mathbf{v}) = (p^0,\mathbf{u}^0)$. Taking $q^t \in \mathcal{C}^\infty(\overline{\Omega_R})$ as a test function in the second equation of (9) yields

$$\begin{aligned} &\hat{a}^0(p^\delta, q^t) + \sqrt{\delta}\, a^1(\sqrt{\delta}\,\mathbf{grad}\, p^\delta, \mathbf{grad}\, q^t) = \\ &\quad -i\omega \int_\Gamma (\mathbf{g}^\delta \wedge \mathbf{n}) \,.\, \mathbf{grad}_\Gamma\, q^t - \delta\, a^1(\mathbf{u}^\delta, \mathbf{grad}\, q^t). \end{aligned} \tag{18}$$

Making use of the fact that $a^1(\sqrt{\delta}\,\mathbf{grad}\, p^\delta, \mathbf{grad}\, q^t)$ is uniformly bounded with respect to δ gives that

$$\sqrt{\delta}\, a^1(\sqrt{\delta}\,\mathbf{grad}\, p^\delta, \mathbf{grad}\, q^t) \longrightarrow 0.$$

Therefore,

$$\hat{a}^0(q, p^t) = -i\omega \int_\Gamma (\mathbf{g}^0 \wedge \mathbf{n}) \,.\, \mathbf{grad}_\Gamma\, p^t,$$

On the other hand, we easily obtain that

$$a^0(\mathbf{v}, \mathbf{u}^t) = -i\omega \int_\Gamma (\mathbf{g}^0 \wedge \mathbf{n}) \,.\, \mathbf{u}^t_\Gamma \tag{19}$$

Lemma 4.1 on the unique solvability of the Maxwell equations (with Dirichlet boundary conditions) together with the density of $\mathcal{C}^\infty(\overline{\Omega_R})$ in $H^1(\Omega_R)$ gives the claim. □

4.2 RATE OF CONVERGENCE

If $\mathbf{g}^0$ is more regular, we can obtain the rate of the convergence of $(\mathbf{u}^\delta, p^\delta)$ to $(\mathbf{u}^0, p^0)$.

THEOREM 4.2 *Assume that* $\mathbf{g}^0 \in TH^{-1/2}(\Gamma)$ *is such that* $\mathbf{curl}_\Gamma\, \mathbf{g}^0 \in H^{1/2}(\Gamma)$. *Then there exists a constant* C *independent of* δ *such that*

$$||\mathbf{u}^\delta - \mathbf{u}^0||_{\mathbb{H}_{\mathrm{div}}(\Omega_R)} + ||p^\delta - p^0||_{H^1(\Omega_R)/\mathbb{C}} \leq C\left(\delta + ||\mathbf{g}^\delta - \mathbf{g}^0||_{TH^{-1/2}(\mathrm{curl},\Gamma)}\right).$$

PROOF. The key point in this proof is to use the extra regularity of p^0 when $\mathrm{curl}_\Gamma\, \mathbf{g}^0 \in H^{1/2}(\Gamma)$ and the boundary Γ is of class $\mathcal{C}^3$. Standard regularity results for the Laplace equation ensure that $p^0 \in H^2(\Omega_R)$ providing $\mathrm{curl}_\Gamma\, \mathbf{g}^0 \in H^{1/2}(\Gamma)$. The function p^0 is now a valid test function to the variational equation (9). Making use of this extra regularity result, let us prove by contradiction that the quantity

$$c^\delta = ||\mathbf{u}^\delta - \mathbf{u}^0||_{\mathbb{H}_{\mathrm{div}}(\Omega_R)} + ||p^\delta - p^0||_{H^1(\Omega_R)/\mathbb{C}}$$

is bounded by the quantity

$$(\delta + ||\mathbf{g}^\delta - \mathbf{g}^0||_{TH^{-1/2}(\mathrm{curl},\Gamma)}).$$

If we suppose that we have a subsequence

$$\frac{c^\delta}{\delta + ||\mathbf{g}^\delta - \mathbf{g}^0||_{TH^{-1/2}(\mathrm{curl},\Gamma)}}$$

such that it converges to $+\infty$ as δ tends to zero, the pair $(\dfrac{\mathbf{u}^\delta - \mathbf{u}^0}{c^\delta}, \dfrac{p^\delta - p^0}{c^\delta})$ converges to a certain $(\mathbf{u}^\star, p^\star)$ in $\mathbb{H}_{\mathrm{div}}(\Omega_R) \times H^1(\Omega_R)/\mathbb{C}$ weakly. Since $\mathrm{curl}_\Gamma\, \mathbf{g}^0 \in H^{1/2}(\Gamma)$, $p^\delta - p^0$ can be used as a test function in (9). This shows that:

$$\begin{aligned}\hat{a}^0(p^\delta - p^0, p^\delta - p^0) + \delta\, a^1(\mathbf{grad}\,(p^\delta - p^0), \mathbf{grad}\,(p^\delta - p^0)) = \\ -i\omega \int_\Gamma ((\mathbf{g}^\delta - \mathbf{g}^0) \wedge \mathbf{n}) \,.\, \mathbf{grad}_\Gamma\,(p^\delta - p^0), \\ -\delta\, a^1(\mathbf{grad}\, p^0, \mathbf{grad}\,(p^\delta - p^0))\end{aligned} \tag{20}$$

Therefore, the right-hand of the above expression is bounded by

$$\Big(\delta + ||\mathbf{g}^\delta - \mathbf{g}^0||_{TH^{-1/2}(\mathrm{curl},\Gamma)}\Big)||p^\delta - p^0||_{H^1(\Omega_R)/\mathbb{C}}$$

since $a^1(\mathbf{grad}\, p^0, \mathbf{grad}\,(p^\delta - p^0)) \le C\, ||p^0||_{H^2(\Gamma)}\, ||p^\delta - p^0||_{H^1(\Omega_R)/\mathbb{C}}$. Estimate previously given then yields

$$\begin{aligned}|\hat{a}^0(\frac{(p^\delta - p^0)}{c^\delta}, \frac{(p^\delta - p^0)}{c^\delta}) + \delta\, a^1(\frac{\mathbf{grad}\,(p^\delta - p^0)}{c^\delta}, \frac{\mathbf{grad}\,(p^\delta - p^0)}{c^\delta})| \\ \le \frac{\delta + ||\mathbf{g}^\delta - \mathbf{g}^0||_{TH^{-1/2}(\mathrm{curl},\Gamma)}}{c^\delta} \frac{||p^\delta - p^0||_{H^1(\Omega_R)/\mathbb{C}}}{c^\delta}.\end{aligned} \tag{21}$$

As a direct consequence, we obtain that $\dfrac{p^\delta - p^0}{c^\delta}$ converges to zero in $H^1(\Omega_R)/\mathbb{C}$, strongly. A similar analysis can be used for $\dfrac{\mathbf{u}^\delta - \mathbf{u}^0}{c^\delta}$ to prove that it converges to zero in $\mathbb{H}_{\mathrm{div}}(\Omega_R)$, strongly. This is a contradiction to the fact that

$$\frac{||p^\delta - p^0||_{H^1(\Omega_R)/\mathbb{C}}}{c^\delta} + \frac{||\mathbf{u}^\delta - \mathbf{u}^0||_{\mathbb{H}_{\mathrm{div}}(\Omega_R)}}{c^\delta} = 1.$$

The proof is then complete. □

4.3 COMPLETE CONSTRUCTION OF THE ASYMPTOTIC EXPANSIONS

In this subsection, we assume that $\mathbf{g}^\delta$ in $TH^{-1/2}(\mathrm{curl}, \Gamma)$ has the following asymptotic form:

$$\mathbf{g}^\delta = \mathbf{g}^0 + \delta\, \mathbf{g}^1 + \ldots + \delta^N\, \mathbf{g}^N + \ldots$$

where $\mathbf{g}^j \in TH^{-1/2}(\mathrm{curl}, \Gamma)$ for $j \geq 0$. We shall show how to construct the full asymptotic expansions of $\mathbf{u}^\delta$ and p^δ. We can easily obtain from Theorems 4.1 and 4.2 the following result.

THEOREM 4.3 *Let $N \in \mathbb{N}$. Suppose that $\mathbf{g}^\delta = \mathbf{g}^0 + \ldots + \delta^N\, \mathbf{g}^N + \ldots$, where $\mathrm{curl}_\Gamma\, \mathbf{g}^j \in H^{1/2+N-j}(\Gamma), 0 \leq j \leq N$. Then, the functions $(\mathbf{u}^\delta, p^\delta)$ have the asymptotic expansions:*

$$\begin{cases} \mathbf{u}^\delta = \mathbf{u}^0 + \delta\, \mathbf{u}^1 + \ldots + \delta^N\, \mathbf{u}^N + \delta^{N+1}\, \mathbf{u}_r^{N+1}, \\ p^\delta = p^0 + \delta\, p^1 + \ldots + \delta^N\, p^N + \delta^{N+1}\, p_r^{N+1}, \end{cases}$$

where the functions $(\mathbf{u}^j, p^j) \in \mathbb{H}_{\mathrm{div}}(\Omega_R) \times H^1(\Omega_R)/\mathbb{C}$ are such that

$$\begin{cases} a^0(\mathbf{u}^{j+1}, \mathbf{u}^t) = -i\omega \displaystyle\int_\Gamma (\mathbf{g}^{j+1} \wedge \mathbf{n}) \,.\, \mathbf{u}_\Gamma^t - a^1(\mathbf{u}^j, \mathbf{u}^t) - a^1(\mathbf{grad}\, p^j, \mathbf{u}^t), \\ \hat{a}^0(p^{j+1}, p^t) = -i\omega \displaystyle\int_\Gamma (\mathbf{g}^{j+1} \wedge \mathbf{n}) \,.\, \mathbf{grad}_\Gamma\, p^t - a^1(\mathbf{grad}\, p^j, \mathbf{grad}\, p^t) - a^1(\mathbf{u}^j, \mathbf{grad}\, p^t), \end{cases}$$

for all $\mathbf{u}^t \in \mathbb{H}_{\mathrm{div}}(\Omega_R)$ and $p^t \in H^1(\Omega_R)/\mathbb{C}$. Furthermore, the quantities $(\mathbf{u}_r^{N+1}, p_r^{N+1})$ are such that

$$||\mathbf{u}_r^{N+1}||_{\mathbb{H}_{\mathrm{div}}(\Omega_R)} + ||p_r^{N+1}||_{H^1(\Omega_R)/\mathbb{C}} \leq C\, \delta^{N+1} \sum_{j=0}^{N} ||\mathbf{g}^j||_{\mathcal{Y}^j(\Gamma)},$$

where $\mathcal{Y}^j(\Gamma) = \Big\{ \mathbf{u} \in TH^{-1/2}(\Gamma), \mathrm{curl}_\Gamma\, \mathbf{u} \in H^{1/2+N-j}(\Gamma) \Big\}$ *and the constant* C *is independant of the parameter* δ.

5 CONCLUSION

We shall make the following conclusions on the use of the Leontovich approximation in electromagnetics. We have proved that the Leontovich boundary value problem for the time harmonic Maxwell equations $(1) - (2)$ in an inhomogeneous medium leads as in [2] to a singular perturbation problem. If we are interested in the numerical computations of the magnetic field $\mathbf{H}^\delta$ the use of higher order finite elements is necessary to achieve a sufficient accuracy of the quantity p^δ. Our variational formulation (9) does not yield to a nice numerical result for the scalar unknown p^δ since we have to approximate it in the Sobolev space $\mathbb{P}(\Omega_R)$ and not in $H^1(\Omega_R)$. The computation of the quantity $\mathbf{u}^\delta$ does not require the use of such higher order finite elements.

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