Low-frequency electromagnetic scattering

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Abstract

The main result of this paper is to reduce the calculation of higher-order terms in the asymptotic expansions of the electric and magnetic fields at low frequencies to the solutions of certain canonical problems. We also provide a new and simple variational proof of the convergence of the electric and magnetic fields solutions of the scattering problem for the Maxwell equations as the frequency goes to zero. Besides its theoretical interest, our analysis is motivated by its application to the numerical computation of the higher-order terms.

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1 Introduction and statement of the problem

The scattering of electromagnetic waves from bounded objects whose dimensions are small compared with the length of the incident wave has been the subject of considerable study for more than a century. This problem is of interest in geophysics, astrophysics, electrical engineering, physics of the atmosphere and ocean, medicine, biology, and other fields. The study of wave scattering at low frequencies was pioneered by Rayleigh [23] who continued this work until his death. His contributions in this area provide the foundation on which almost all subsequent work is based. A low-frequency asymptotic for Maxwell's boundary value problem and transmission problem has been given for instance by Stevenson [24], Werner [31] and [29], Picard [19], Kleinman-Senior [13], Kress [14] Ramm et al. [22], Ramm-Somersalo [21]. Kleinman and Senior systematized the calculation of the dominant term in the low-frequency limit for electromagnetic problems involving impenetrable, penetrable, non-lossy and lossy obstacles. With boundary integral methods, Werner [31], [29] and Kress [14] obtained the limit of the solution to the scattering problem by a perfectly conducting object for the Maxwell equations. Our present work is closely related to the work of Picard [19] who proved the convergence for non-homogeneous Maxwell's equations using Hilbert space concepts. The approach of Picard [19] is based on the limiting absorption principle for a modified first-order Maxwell system to overcome the fact that the case $\omega = 0$ is embedded in the continuous spectrum of the modified first-order Maxwell operator. He reformulated Maxwell's equations in the following way:

$$\Big(M + N - \omega\Big)\, U^\omega \;=\; F^\omega,$$

where

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{i}{\varepsilon}\,\overrightarrow{\mathrm{curl}} & 0 \\ 0 & \dfrac{i}{\mu}\,\overrightarrow{\mathrm{curl}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & i\,\mathrm{div}\,\varepsilon & 0 & 0 \\ i\,\overrightarrow{\mathrm{grad}} & 0 & 0 & 0 \\ 0 & 0 & 0 & i\,\overrightarrow{\mathrm{grad}} \\ 0 & 0 & i\,\mathrm{div}\,\mu & 0 \end{pmatrix},$$

and

$$U^{\omega} = \begin{pmatrix} 0 \\ E^{\omega} \\ H^{\omega} \\ 0 \end{pmatrix}.$$

Here ε is the electric permittivity, μ the magnetic permeability, E^{ω} the electric field, H^{ω} the magnetic field and F^{ω} a given vector function with a bounded support. His method can not be easily extended to obtain the rate of the convergence of the electric and magnetic fields with higher-order terms since the assumption on the support of F^{ω} is crucial for his proof (Theorem 3 p.70). In none of the works cited above is a variational method for calculating the higher-order terms given and convergence with these terms analyzed. This is our basic aim in the present paper. We also provide a new and simple variational proof of the convergence of the electric and magnetic fields as the frequency goes to zero.

We first formulate the scattering problem equivalently on a ball B_R of radius R containing the inhomogeneity by making use of an adequate Steklov-Poincaré operator on the sphere $S_R = \partial B_R$ called, throughout what follows, the electromagnetic operator. Either the expression of this pseudo-diffrential operator or the variational formulation used here for Maxwell's equations is now well-known (see for instance [11] and [18]). Then, the electric and magnetic fields, as well as the electromagnetic operator, are expanded in power series with respect to the frequency. Making use of some properties of the Hankel functions, we show that in the asymptotic expansion of the electromagnetic operator with respect to the frequency all the coefficients which are pseudo-differential operators are of order less than the order of the electromagnetic operator. This important result is an essential tool which will permit us to derive the appropriate boundary conditions satisfied by the higher order terms of the electric and magnetic fields on the sphere S_R. These boundary conditions on the fictive surface S_R complete the reduction of the calculation of the higher-order terms in the asymptotic expansions of the electric and magnetic fields to two canonical problems which are uniquely solvable. A variational proof of the convergence with these higher-order terms will be given in the last part of this paper. Let us note that in the addition of the difficulty coming from the asymptotic expansion of the electromagnetic operator, there is another that is due to a lack of coerciverness in the variational formulation. This difficulty is overcome by using a compactness result due to Abboud [1] (Lemma 5.2).

The approach adopted in this work is different from those mentionned above. To the best of our knowledge, the technique developed in this paper, although quite natural, does not seem to be applied for the scattering of electromagnetic waves at low frequencies.
Let us now introduce the scattering problem. Suppose that a bounded inhomogeneity characterized by permittivity ε and permeability μ is illuminated by a time-harmonic electromagnetic wave given by $(E^{in}_\omega, H^{in}_\omega)$ (where E^{in}_ω is the electric field and H^{in}_ω is the magnetic field). The incoming wave $(E^{in}_\omega, H^{in}_\omega)$ is assumed to be a classical solution of the Maxwell system

$$\begin{cases} \overrightarrow{\text{curl}}\, H^{in}_\omega \;=\; -i\omega\varepsilon_0\, E^{in}_\omega, \\ \overrightarrow{\text{curl}}\, E^{in}_\omega \;=\; i\omega\mu_0\, H^{in}_\omega, \end{cases}$$

in all of R^3 except possibly for a finite number of points located outside a sphere strictly containing the inhomogeneity. This incoming field will interact with the inhomogeneity to give rise to a scattered field $(E^{sc}_\omega, H^{sc}_\omega)$. Let ε, μ be two functions of the spatial variable x; ε_0 and μ_0 two positive constants; and $r = |x|$. The total field (E^ω, H^ω) satisfies the time harmonic Maxwell system

$$\begin{cases} \overrightarrow{\text{curl}}\, H^\omega \;=\; -i\omega\varepsilon\, E^\omega \quad \text{in } R^3, \\ \overrightarrow{\text{curl}}\, E^\omega \;=\; i\omega\mu\, H^\omega \quad \text{in } R^3, \\ H^\omega \;=\; H^{sc}_\omega + H^{in}_\omega \quad \text{in } R^3, \\ E^\omega \;=\; E^{sc}_\omega + E^{in}_\omega \quad \text{in } R^3, \\ \lim_{r\to+\infty}\Big(\sqrt{\mu_0}\, H^{sc}_\omega \wedge x - r\sqrt{\varepsilon_0}\, E^{sc}_\omega\Big) \;=\; 0. \end{cases} \tag{1}$$

We assume that the electric permittivity and the magnetic permeability, ε and μ are in $L^\infty(R^3)$. In the next section, we will add some restrictive conditions regarding the regularity of these coefficients to ensure the uniqueness of a solution to (1). We also assume that the inhomogeneity is bounded so that there exists a constant $R_0 > 0$ such that $\varepsilon(x) = \varepsilon_0$ and $\mu(x) = \mu_0$ if $r = |x| \geq R_0$. The support of the function $\varepsilon - \varepsilon_0$, denoted by $\text{Supp}(\varepsilon - \varepsilon_0)$, and that of $\mu - \mu_0$ are included in the ball $B_{R_0} = \Big\{|x| < R_0\Big\}$.
For some fixed $R > R_0$ we define the ball $B_R = \Big\{x \in R^3, |x| < R\Big\}$, and the sphere $S_R = \Big\{x \in R^3, |x| = R\Big\}$.
The present study is concerned with the analysis of the behavior of the fields

(E^ω, H^ω) solutions of (1) as the frequency ω tends to zero. Our aim in this paper is to prove the convergence of the fields (E^ω, H^ω) and to reduce the calculation of higher order-terms to solutions of certain canonical problems. A similar program was carried out by the authors for the scattering problem from a small conductor embedded in a homogeneous chiral media [4]. For this problem, we have employed representations of electric and magnetic fields to reformulate the scattering problem as an integral equation over the scattering surface and represent the fields in terms of surface fields. Using a Hodge decomposition of the tangent fields, we have characterized the dependence of the asymptotics on the topological properties of the scatterer and we have shown the effect of the chirality admittance. The extension of the present study to the scattering of electromagnetic waves in a chiral media from a small inhomogeneity does not lead to any specify difficulty apart from much more cumbersome notations and calculations.
Now, we introduce some standard notations. For any smooth vector field w, we denote by w_{S_R} its tangential component on S_R

$$w_{S_R} = -n \wedge (n \wedge w) \quad \text{on } S_R.$$

For any smooth function f or any vector field u defined on S_R, we get an extension of each one to $R^3 \setminus \overline{B_R}$ by setting :

$$\forall\, x \in R^3 \setminus \overline{B_R}, \tilde{f}(x) = f(\frac{x}{|x|}), \tilde{u}(x) = u(\frac{x}{|x|}).$$

We also introduce the following boundary differential operators :
the tangential gradient of a function f : $\overrightarrow{\mathrm{grad}}_{S_R} f = \overrightarrow{\mathrm{grad}}\, \tilde{f}|_{S_R}$,
the surface divergence of a field u : $\mathrm{div}_{S_R} u = \mathrm{div}\, \tilde{u}|_{S_R}$,
the vector rotational of a function : $\overrightarrow{\mathrm{curl}}_{S_R} f = \overrightarrow{\mathrm{curl}}\,(fn)|_{S_R}$,
the scalar rotational of a vector : $curl_{S_R} u = n \,.\, \overrightarrow{\mathrm{curl}}\, \tilde{u}|_{S_R}$,
and the Laplace-Beltrami operator on S_R defined on scalar functions by

$$\Delta_{S_R} f = \mathrm{div}_{S_R} \overrightarrow{\mathrm{grad}}_{S_R} f = -curl_{S_R} \overrightarrow{\mathrm{curl}}_{S_R} f.$$

To state our boundary-value problem in a suitable mathematical form, we shall use the following notations for the usual functional spaces :
$\mathcal{D}'(B_R)$ is the space of distributions in B_R.
$L^2(B_R)$ is the space of complex square integrable functions defined in B_R.
$L^\infty(R^3)$ is the space of bounded functions defined in R^3.
$H^1(B_R) = \Big\{ \varphi \in L^2(B_R), \overrightarrow{\mathrm{grad}}\, \varphi \in (L^2(B_R))^3 \Big\}$.

$H_0^1(B_R) = \Big\{\varphi \in H^1(B_R), \varphi = 0 \,\text{on}\, S_R\Big\}.$
$H(\text{curl}, B_R) = \Big\{u \in (L^2(B_R))^3, \overrightarrow{\text{curl}}\, u \in (L^2(B_R))^3\Big\}.$
$H_0(\text{curl}, B_R) = \Big\{ u \in H(\text{curl}, B_R), u \wedge n = 0 \text{ on } B_R \Big\}.$
$TL^2(S_R) = \Big\{c \in (L^2(S_R))^3, c\,.\,n = 0\Big\}.$
$TH^s(S_R) = \Big\{c \in (H^s(S_R))^3, c\,.\,n = 0\Big\}.$
$TH^{-1/2}(\text{curl}, S_R) = \Big\{c \in (H^{-1/2}(S_R))^3, c\,.\,n = 0, curl_{\partial\Omega}\, c \in H^{-1/2}(S_R)\Big\}.$
$TH^{-1/2}(\text{div}, S_R) = \Big\{c \in (H^{-1/2}(S_R))^3, c\,.\,n = 0, \text{div}_{S_R}\, c \in H^{-1/2}(S_R)\Big\}.$
$TH^{-1/2}(\text{curl}, S_R)$ and $TH^{-1/2}(\text{div}, S_R)$ are Hilbert spaces. Finally, we recall without proof the following well-known duality result (see e.g. [18] for a proof) : $TH^{-1/2}(\text{curl}, S_R) = \Big(TH^{-1/2}(\text{div}, S_R)\Big)'.$

2 The exterior electromagnetic operator for Maxwell's equations

In this section we shall give an explicit construction of the electromagnetic operator T^ω defined for a tangential vector field $g^\omega \in TH^{-1/2}(\text{curl}, S_R)$ by

$$T^\omega(g^\omega) = \mathcal{H}^\omega \wedge n, \tag{2}$$

where

$$\begin{cases} \overrightarrow{\text{curl}}\,\mathcal{H}^\omega = -i\omega\varepsilon_0\,\mathcal{E}^\omega \quad \text{in } R^3 \setminus \overline{B_R}, \\ \overrightarrow{\text{curl}}\,\mathcal{E}^\omega = i\omega\mu_0\,\mathcal{H}^\omega \quad \text{in } R^3 \setminus \overline{B_R}, \\ \mathcal{E}^\omega_{S_R} = g^\omega \quad \text{on } S_R, \\ \lim_{r\to+\infty}\Big(\sqrt{\mu_0}\,\mathcal{H}^\omega \wedge x - r\sqrt{\varepsilon_0}\,\mathcal{E}^\omega\Big) = 0. \end{cases} \tag{3}$$

In order to prove the convergence of solutions to the Maxwell equations (1) as $\omega \to 0$, we also need the operator D^ω defined by

$$D^\omega : g^\omega \in TH^{-1/2}(\text{curl}, S_R) \mapsto \mathcal{E}^\omega\,.\,n. \tag{4}$$

Since $\mathcal{E}^\omega$ satisfies

$$\mathcal{E}^\omega = \frac{i}{\omega\varepsilon_0}\overrightarrow{\text{curl}}\mathcal{H}^\omega,$$

it follows that

$$D^\omega(g^\omega) = \frac{i}{\omega\varepsilon_0}\text{div}_{S_R} T^\omega(g^\omega),$$

and then the explicit expression of the operator D^ω will be deduced from that of the operator T^ω. Our basic aim in this section is to analyze the dependence of both the operators T^ω and D^ω on the frequency ω. We start by representing the boundary data in terms of suitable vector basis functions on S_R. Following [7], let $(Y_l^m)_{-l\leq m\leq l}$ be an orthonormal sequence of spherical harmonics of order l on the unit sphere Σ, normalized such that

$$\int_\Sigma Y_l^m \, . \, \overline{Y}_{l'}^{m'} \; = \; \delta_{l,l'} \, \delta_{m,m'} .$$

The basis functions for tangential fields on S_R are then

$$G_l^m \; = \; \frac{1}{\sqrt{l(l+1)}} \nabla_{S_R} Y_l^m \text{ and } R_l^m \; = \; n \wedge G_l^m, \quad \text{for } -l \leq m \leq l, \; l \geq 1.$$

The tangential vector fields G_l^m and R_l^m are an orthonormal basis on the unit sphere (in the L^2 inner product). It follows that any tangential vector field $g \in TL^2(S_R)$ can be written in the form

$$g \; = \; \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \alpha_l^m \, G_l^m + \beta_l^m \, R_l^m. \tag{5}$$

Spaces of either scalar or vector functions on S_R can be characterized by the summability of weighted sums of their expansion coefficients. The norm on the space $TH^s(S_R)$ can be characterized using the series coefficients by (see [11])

$$||g||_{TH^s(S_R)} \; = \; \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \Big(1 + l(l+1)\Big)^s \, \Big(|\alpha_l^m|^2 \, + |\beta_l^m|^2\Big),$$

the norm on the space $H^s(S_R)$ by

$$||\varphi||_{H^s(S_R)} \; = \; \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \Big(1 + l(l+1)\Big)^s \, |\varphi_l^m|^2,$$

where

$$\varphi = \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \varphi_l^m \, Y_l^m,$$

the norm on the space $TH^{-1/2}(\text{curl}, S_R)$ by

$$||g||_{TH^{-1/2}(\text{curl},S_R)} \; = \; \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \sqrt{1 + l(l+1)} \, |\beta_l^m|^2 \; + \; \frac{1}{\sqrt{1 + l(l+1)}} \, |\alpha_l^m|^2,$$

and the norm on the space $TH^{-1/2}(\mathrm{div}, S_R)$ by

$$||g||_{TH^{-1/2}(\mathrm{div},S_R)} = \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \sqrt{1+l(l+1)}\, |\alpha_l^m|^2 + \frac{1}{\sqrt{1+l(l+1)}} |\beta_l^m|^2.$$

Throughout this paper, we assume for simplicity in exposition that $\varepsilon_0\, \mu_0 = 1$ and $R = 1$. Now, if we expand the tangential vector field g^ω in the form

$$g^\omega = \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \alpha_l^m(\omega)\, G_l^m + \beta_l^m(\omega)\, R_l^m, \tag{6}$$

we have the following explicit representation for $T^\omega(g^\omega$ (see appendix A)

$$T^\omega(g^\omega) = \sqrt{\frac{\varepsilon_0}{\mu_0}} \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \frac{\alpha_l^m(\omega)}{\gamma_l(\omega)}\, G_l^m + \frac{\beta_l^m(\omega)\gamma_l(\omega)}{i\omega}\, R_l^m, \tag{7}$$

where

$$\gamma_l(\omega) = 1 + \omega \frac{(h_l^{(1)})'(\omega)}{h_l^{(1)}(\omega)}. \tag{8}$$

From (7), we deduce that the operator D^ω introduced in (4) is given by

$$D^\omega(g^\omega) = \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \frac{\sqrt{l(l+1)}}{\gamma_l(\omega)}\, \alpha_l^m(\omega)\, Y_l^m, \tag{9}$$

for any $g^\omega \in TH^{-1/2}(\mathrm{curl}, S_R)$ in the form (6).
Now, we summarize the mapping properties of the exterior electromagnetic operator T^ω.

Lemma 2.1 *There exists a constant C such that for any $g \in TH^{-1/2}(\mathrm{curl}, S_R)$ the following inequality holds*

$$||T^\omega(g)||_{TH^{-1/2}(\mathrm{div},S_R)} \leq C\, ||g||_{TH^{-1/2}(\mathrm{curl},S_R)}. \tag{10}$$

Furthermore,

$$\Re e\left(T^\omega(g), g\right) > 0, \tag{11}$$

for any $g \neq 0$ in $TH^{-1/2}(\mathrm{curl}, S_R)$ where $(,)$ denotes the duality between $TH^{-1/2}(\mathrm{div}, S_R)$ and $TH^{-1/2}(\mathrm{curl}, S_R)$.

Proof. (10) is proved in [11]. Thus, only the last part of the lemma needs proving. From

$$\Big(T^\omega(g), g\Big) \;=\; \sqrt{\frac{\varepsilon_0}{\mu_0}} \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} i\omega \frac{|\alpha_l^m|^2}{\gamma_l(\omega)} \;+\; \frac{|\beta_l^m|^2\, \gamma_l(\omega)}{i\omega},$$

we obtain that

$$\Re e \Big(T^\omega(g), g\Big) \;=\; \sqrt{\frac{\varepsilon_0}{\mu_0}} \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \omega \frac{|\alpha_l^m|^2}{|\gamma_l(\omega)|^2}\, \Im m\,(\gamma_l(\omega)) \;+\; \frac{|\beta_l^m|^2}{\omega}\, \Im m\,(\gamma_l(\omega)).$$

The fact that $\Im m\,(\gamma_l(\omega)) > 0$ (see for instance [18]) implies that (10) holds for any $g \neq 0$ in $TH^{-1/2}(\mathrm{curl}, S_R)$. $\square$

Our next result shows that D^ω is continuous as a map

$$D^\omega : TH^{-1/2}(\mathrm{curl}, S_R) \mapsto H^{-1/2}(S_R).$$

Lemma 2.2 *There exists a constant C such that for any $g \in TH^{-1/2}(\mathrm{curl}, S_R)$ the following inequality holds*

$$||D^\omega(g)||_{H^{-1/2}(S_R)} \;\leq\; C\, ||g||_{TH^{-1/2}(\mathrm{curl}, S_R)}. \tag{12}$$

Proof. Using the fact that there exist positive constants C_1 and C_2 such that for all l (see for instance [7])

$$C_1\, l \;\leq\; |\gamma_l(\omega)| \;\leq\; C_2\, l,$$

we obtain

$$\begin{aligned}
||D^\omega(g)||^2_{H^{-1/2}(S_R)} \;&=\; \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \frac{l(l+1)}{|\gamma_l|^2(\omega)} \frac{1}{\sqrt{l(l+1)}}\, |\alpha_l^m|^2, \\
&\leq\; C \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \frac{1}{\sqrt{l(l+1)}}\, |\alpha_l^m|^2, \\
&\leq\; C\, ||g||^2_{TH^{-1/2}(\mathrm{curl}, S_R)},
\end{aligned}$$

where the constant C is independent of g. The dependence of the constant C on the frequency ω can be explicitly characterized. $\square$

Now, we wish to analyze the asymptotic behavior of the operators T^ω and D^ω as the frequency ω tends to zero. We start by recalling some well-known results on the Logarithmic derivative of the Hankel function $h_l^{(1)}$. Following [18], we have

$$\omega \frac{(h_l^{(1)})'(\omega)}{h_l^{(1)}(\omega)} = -\frac{p_l(\omega)}{q_l(\omega)} + i\frac{\omega}{q_l(\omega)},$$

where

$$\begin{cases} p_l(t) = c_l^0 + 2c_l^1 \dfrac{1}{t^2} + \ldots + (l+1)c_l^l \dfrac{1}{t^{2l}}, \\ q_l(t) = c_l^0 + c_l^1 \dfrac{1}{t^2} + \ldots + c_l^l \dfrac{1}{t^{2l}}, \end{cases}$$

and

$$c_l^m = \frac{(m+l)!(2m)!}{4^m (m!)^2 (l-m)!}.$$

Therefore, we can write $\gamma_l(\omega)$ in the form

$$\gamma_l(\omega) = 1 - \frac{(l+1)c_l^l + \ldots + (\omega)^{2l} c_l^0}{c_l^l + \ldots + (\omega)^{2l} c_l^0} + i\frac{(\omega)^{2l+1}}{c_l^l + \ldots + (\omega)^{2l} c_l^0}. \tag{13}$$

Firstly, we have the following lemmas. The proofs are given in the appendix B.

Lemma 2.3 *The following inequality holds*

$$\frac{(l+1-m)\, c_l^{l-m}}{c_l^l} < 1, \tag{14}$$

for all $l \geq 1$ and $m = 1$ to l.

Lemma 2.4 *There exist ω_0 and a constant C independent of ω such that for $0 < \omega < \omega_0$ the following estimate holds for l large enough*

$$\left| \frac{1}{\gamma_l(\omega)} + \frac{1}{l} \right| \leq \frac{C}{l^2}\, \omega. \tag{15}$$

Next, from (13) and (14), we shall obtain the following results.

Lemma 2.5 *There exists $\omega_0 > 0$ such that*

$$\gamma_l(\omega) \;=\; \sum_{j=0}^{+\infty} \gamma_l^j \,\omega^j, \quad \forall\, 0 < \omega < \omega_0.$$

for all $l \geq 1$ (ω_0 is independent of l). Furthermore, for any $j \geq 1$ there exists a constant C_j independent of l such that

$$|\frac{\gamma_l^j}{\gamma_l^0}| \;\leq\; C_j \quad \text{for all } l \geq 1. \tag{16}$$

Lemma 2.6 *Let $(\delta_l^j)_{j\geq 0, l\geq 1}$ be defined by*

$$\frac{1}{\gamma_l(\omega)} \;=\; \frac{1}{\gamma_l^0} \sum_{j=0}^{+\infty} \delta_l^j \,\omega^j. \tag{17}$$

For any $j \geq 1$ there exists a constant C_j independent of l such that

$$|\delta_l^j| \;\leq\; C_j \quad \text{for all } l \geq 1. \tag{18}$$

With the help of these results, we obtain the following two crucial results for our asymptotic analysis at low frequencies.

Lemma 2.7 *The operator*

$$D^\omega - D : TH^{-1/2}(\mathrm{curl}, S_R) \mapsto H^{-1/2}(S_R),$$

is compact, where D is defined for any $g \in TH^{-1/2}(\mathrm{curl}, S_R)$ in the form (5) by

$$D(g) \;=\; -\sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \sqrt{\frac{l+1}{l}}\, \alpha_l^m \, Y_l^m.$$

Furthermore, there exists ω_0 such that for $0 < \omega < \omega_0$ the following inequality holds

$$||(D^\omega - D)(g)||_{H^{1/2}(S_R)} \leq C\,\omega\, ||g||_{TH^{-1/2}(\mathrm{curl}, S_R)},$$

where the constant C independent of ω.

Proof. Let g be in $TH^{-1/2}(\mathrm{curl}, S_R)$. We have

$$(D^\omega - D)(g) \;=\; \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \sqrt{l(l+1)}\, (\frac{1}{\gamma_l(\omega)} + \frac{1}{l})\, \alpha_l^m \, Y_l^m.$$

Therefore, by (15), we obtain that there existes a constant C independent of ω such that

$$\begin{aligned} \|(D^\omega - D)(g)\|_{H^{1/2}(S_R)} &\leq C\,\omega \sum_{l=1}^{+\infty}\sum_{m=-l}^{l} \sqrt{1+l(l+1)}\,\frac{1}{l^2}\,|\alpha_l^m|^2, \\ &\leq 3C\,\omega \sum_{l=1}^{+\infty}\sum_{m=-l}^{l} \frac{1}{\sqrt{1+l(l+1)}}\,|\alpha_l^m|^2 + \sqrt{1+l(l+1)}\,|\beta_l^m|^2, \\ &\leq 3C\,\omega\,\|g\|_{TH^{-1/2}(\mathrm{curl},S_R)}, \end{aligned}$$

and then we have the claim. □

The operator D is then in some sense the limit of the operator D^ω as ω tends to zero. Now, by using Lemma 2.5, we also verify without any difficulty that we have:

Lemma 2.8 *There exists a constant C such that for any $g \in TH^{-1/2}(\mathrm{curl}, S_R)$ the following inequality holds*

$$\|T_j^\omega(g)\|_{TH^{-1/2}(\mathrm{div},S_R)} \leq C\,\|g\|_{TH^{-1/2}(\mathrm{curl},S_R)}, \tag{19}$$

where the operator T_j^ω is defined for any $g \in TH^{-1/2}(\mathrm{curl}, S_R)$ in the form (5) by

$$\begin{aligned} T_j^\omega(g) &= \sqrt{\frac{\varepsilon_0}{\mu_0}} \sum_{l=1}^{+\infty}\sum_{m=-l}^{l} i\omega\,\frac{\alpha_l^m}{\gamma_l^0} \sum_{p=0}^{j} \delta_l^p\,\omega^p\,G_l^m \\ &\quad + \frac{\beta_l^m \gamma_l^0}{i\omega} \sum_{p=0}^{j} \frac{\gamma_l^p}{\gamma_l^0}\omega^p\,R_l^m. \end{aligned}$$

Proof. Let g be in $TH^{-1/2}(\mathrm{curl}, S_R)$. We have

$$\begin{aligned} \|T_j^\omega(g)\|^2_{TH^{-1/2}(\mathrm{div},S_R)} &= \frac{\varepsilon_0}{\mu_0} \sum_{l=1}^{+\infty}\sum_{m=-l}^{l} \sqrt{1+l(l+1)}\,\omega^2\,|\frac{\alpha_l^m}{\gamma_l^0}|^2\,|\sum_{p=0}^{j}\delta_l^p\,\omega^p|^2 \\ &\quad + \frac{1}{\sqrt{1+l(l+1)}}\,|\frac{\beta_l^m\gamma_l^0}{\omega}|^2|\sum_{p=0}^{j}\frac{\gamma_l^p}{\gamma_l^0}\omega^p\,|^2. \end{aligned}$$

Therefore, from Lemma 2.5 and Lemma 2.6, we can show that there exists a constant C_j such that

$$\begin{aligned}
||T_j^\omega(g)||^2_{TH^{-1/2}(\mathrm{div},S_R)} &\leq C_j \sum_{l=1}^{+\infty}\sum_{m=-l}^{l} \frac{\sqrt{1+l(l+1)}}{l^2}\,|\alpha_l^m|^2 + \frac{l^2}{\sqrt{1+l(l+1)}}\,|\beta_l^m|^2,\\
&\leq 3C_j \sum_{l=1}^{+\infty}\sum_{m=-l}^{l} \frac{1}{\sqrt{1+l(l+1)}}\,|\alpha_l^m|^2 + \sqrt{1+l(l+1)}\,|\beta_l^m|^2,\\
&\leq 3C_j\,||g||_{TH^{-1/2}(\mathrm{curl},S_R)},
\end{aligned}$$

and then we have the claim. □

3 The variational problem for Maxwell's equations

Several proofs of the existence of solutions to the scattering problem for nonhomogeneous Maxwell's equations in unbounded domains are now known. The proof of the existence and uniqueness of a solution for the case when ε and μ are in $\mathcal{C}^2$ were first given by Müller [16]. Abboud and Nédélec [2] proved existence and uniqueness for Maxwell's equations under the same regularity assumptions on the dielectric coefficients ε and μ but possibly discontinuous across the interface dielectric medium/exterior domain. They used the standard Sobolev space H^1 as a basis of their variational problem. Costabel [8] used an other modified bilinear form coercive over H^1 to establish the existence of a solution in a bounded domain. Kirsch and Monk [11] obtained a simple variational proof of the existence of a solution based on the use of the space $H(\mathrm{curl},\Omega)$ where Ω is a smooth bounded domain. The crucial points in their analysis are the Hodge decomposition of the space $H(\mathrm{curl},\Omega)$ and the compact embedding of the set $\Big\{u \in H(\mathrm{curl},\Omega) : \int_\Omega \varepsilon\, u \,.\, \overrightarrow{\mathrm{grad}}\,\varphi = 0 \text{ for all } \varphi \in H^1(\Omega)\Big\}$ into $(L^2(\Omega))^3$. Note that the use of the Hodge decomposition of the space $H(\mathrm{curl},\Omega)$ is a well-known idea for the study of Maxwell's equations in bounded domains (see Birman and Solomyak [5] and Leis [15]). Abboud [3] has analyzed the scattering of electromagnetic waves from periodic gratings in three dimensions. His method is very close to [11] but the compactness result that he used is quite different. More recently, Hazard and Lenoir [10] obtained by adding a regularizing

term such as $\overrightarrow{\mathrm{grad}}\,\mathrm{div}$ in the time-harmonic Maxwell equations an elliptic problem similar to the vector Helmholtz equation.
For the uniqueness of a solution to the scattering problem from inhomogeneities, several results are also known, but the list of such results is still incomplete. In fact, under certain regularity assumptions on ε and μ, namely ε and μ are $\mathcal{C}^{1,1}$ in every subdomain of the dielectric medium, it can be shown that a principle of unique continuation is valid (see Vogelsang [26]). As a consequence we would have that any solution of Maxwell's equations is unique. For our purposes, since ε and μ are assumed to be in $L^\infty(B_R)$, we have to suppose that the Maxwell system (1) has at most only one solution. Now, following for instance Kirsch and Monk [11], to obtain a variational formulation of the Maxwell system (1) in $H(\mathrm{curl}, B_R)$ we shall first eliminate the magnetic field H^ω from (1). We simply substitute $H^\omega = \dfrac{1}{i\omega\mu}\overrightarrow{\mathrm{curl}}\,E^\omega$ into $\overrightarrow{\mathrm{curl}}\,H^\omega = -i\omega\varepsilon E^\omega$ to obtain

$$\overrightarrow{\mathrm{curl}}\,\frac{1}{\mu}\,\overrightarrow{\mathrm{curl}}\,E^\omega - \omega^2\varepsilon\,E^\omega \;=\; 0 \quad \text{in } B_R. \tag{20}$$

If we multiply (20) by a test function Φ in $H(\mathrm{curl}, B_R)$, integrate over B_R, and using integration by parts, we obtain

$$\int_{B_R} \frac{1}{\mu}\,\overrightarrow{\mathrm{curl}}\,E^\omega\,.\,\overrightarrow{\mathrm{curl}}\,\Phi - \omega^2 \int_{B_R} \varepsilon\,E^\omega\,.\,\Phi - \Big(\overrightarrow{\mathrm{curl}}\,E^\omega \,\wedge\, n, \Phi_{S_R}\Big) \;=\; 0, \tag{21}$$

where the last term in this equation is to take in the sense of the duality $TH^{-1/2}(\mathrm{div}, S_R)$, $TH^{-1/2}(\mathrm{curl}, S_R)$. Now, using the definition of the electromagnetic operator T^ω, we may write

$$\begin{aligned}
&\int_{B_R} \frac{1}{\mu}\,\overrightarrow{\mathrm{curl}}\,E^\omega\,.\,\overrightarrow{\mathrm{curl}}\,\Phi - \omega^2 \int_{B_R} \varepsilon\,E^\omega\,.\,\Phi \\
&-i\omega\mu_0\Big(T^\omega(E^\omega_{S_R}), \Phi_{S_R}\Big) \;=\; i\omega\mu_0\,\Big(g^{in}_\omega, \Phi_{S_R}\Big),
\end{aligned} \tag{22}$$

where

$$g^{in}_\omega \;=\; T^\omega(E^{in}_{\omega,S_R}) - H^{in}_\omega \,\wedge\, n. \tag{23}$$

We have from [11] or [18]:

Lemma 3.1 *Suppose that the Maxwell system* (24) *admit at most one solution. Then the variational equation* (22) *is uniquely solvable in* $H(\mathrm{curl}, B_R)$.

4 The derivation of the higher-order terms

This section is probably the most important section of this article because essentially all of what follows is either based on or motivated by the results we are about to discuss. We proceed formally to reduce the calculation of higher-order terms to the solutions of certain canonical problems. Then, we shall show that each of these problems admit a unique solution. This will be accomplished with the help of the lemmas given above. The convergence of the electric and magnetic fields with these higher-order terms is the basic aim of the next section. Let us first recall that the electric and magnetic fields are solutions of the following scattering problem

$$\left\{\begin{array}{l} \overrightarrow{\mathrm{curl}}\, H^\omega \;=\; -i\omega\varepsilon\, E^\omega \quad \text{in } B_R, \\ \overrightarrow{\mathrm{curl}}\, E^\omega \;=\; i\omega\mu\, H^\omega \quad \text{in } B_R, \\ T^\omega(E^\omega_{S_R}) - H^\omega \wedge n \;=\; T^\omega(E^{in}_{\omega,S_R}) - H^{in}_\omega \wedge n \quad \text{on } S_R, \end{array}\right. \tag{24}$$

where we have replaced the Silver-Müller radiation condition by the exact nonlocal boundary condition

$$T^\omega(E^\omega_{S_R}) - H^\omega \wedge n \;=\; T^\omega(E^{in}_{\omega,S_R}) - H^{in}_\omega \wedge n \tag{25}$$

on the artificial boundary S_R.

Let us now formally expand the electric and the magnetic fields (E^ω, H^ω) in power series of the frequency ω

$$\left\{\begin{array}{l} E^\omega \;=\; \displaystyle\sum_{j=0}^{+\infty} E^j\,\omega^j, \\ H^\omega \;=\; \displaystyle\sum_{j=0}^{+\infty} H^j\,\omega^j, \end{array}\right. \tag{26}$$

where the fields (E^j, H^j) are in $H(\mathrm{curl}, B_R) \times H(\mathrm{curl}, B_R)$. Therefore, the tangential component of the electric field on S_R can be written in the form

$$E^\omega_{S_R} \;=\; \sum_{l=1}^{+\infty}\sum_{m=-l}^{l} u_l^m(\omega)\, G_l^m + v_l^m(\omega)\, R_l^m,$$

where its expansion coefficients $u_l^{m,j}$ and $v_l^{m,j}$ have the following expansion forms with respect to the frequency ω

$$\left\{ \begin{aligned} u_l^m(\omega) &= \sum_{j=0}^{+\infty} u_l^{m,j}\omega^j \\ v_l^m(\omega) &= \sum_{j=0}^{+\infty} v_l^{m,j}\omega^j . \end{aligned} \right.$$

From the exact nonlocal boundary condition (25) on S_R we obtain that

$$\begin{aligned} &\sqrt{\frac{\varepsilon_0}{\mu_0}} \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} i\omega \frac{1}{\gamma_l^0} \Big(\sum_{j=0}^{+\infty} \delta_l^j \omega^j \Big) \Big(\sum_{j=0}^{+\infty} u_l^{m,j}\omega^j \Big) G_l^m \\ &+ \frac{\gamma_l^0}{i\omega} \Big(\sum_{j=0}^{+\infty} \frac{\gamma_l^j}{\gamma_l^0}\omega^j \Big) \Big(\sum_{j=0}^{+\infty} v_l^{m,j}\omega^j \Big) R_l^m - \sum_{j=0}^{+\infty} \Big(H^j \wedge n \Big) \omega^j = g_\omega^{in}, \end{aligned}$$

where g_ω^{in} is defined by (23). Now, using the fact that

$$\Big(\sum_{j=0}^{+\infty} \delta_l^j \omega^j \Big) \Big(\sum_{j=0}^{+\infty} u_l^{m,j}\omega^j \Big) = \sum_{j=0}^{+\infty} \Big(\sum_{p=0}^{j} \delta^{j-p} u_l^{m,p} \Big) \omega^j ,$$

and

$$\Big(\sum_{j=0}^{+\infty} \frac{\gamma_l^j}{\gamma_l^0}\omega^j \Big) \Big(\sum_{j=0}^{+\infty} v_l^{m,j}\omega^j \Big) = \sum_{j=0}^{+\infty} \Big(\sum_{p=0}^{j} \frac{\gamma^{j-p}}{\gamma_l^0} v_l^{m,p} \Big) \omega^j ,$$

we obtain

$$\begin{aligned} &\sqrt{\frac{\varepsilon_0}{\mu_0}} \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} i\omega \frac{1}{\gamma_l^0} \sum_{j=0}^{+\infty} \Big(\sum_{p=0}^{j} \delta_l^{j-p} u_l^{m,p} \Big) \omega^j \, G_l^m \\ &+ \frac{\gamma_l^0}{i\omega} \sum_{j=0}^{+\infty} \Big(\sum_{p=0}^{j} \frac{\gamma_l^{j-p}}{\gamma_l^0} v_l^{m,p} \Big) \omega^j \, R_l^m - \sum_{j=0}^{+\infty} \Big(H^j \wedge n \Big) \omega^j = g_\omega^{in} . \end{aligned} \tag{27}$$

If we assume that the incident electric field E_ω^{in} is analytic with respect to ω we can write the quantity g_ω^{in} in the form

$$g_\omega^{in} = \frac{1}{\omega} g_{-1}^{in} + \sum_{j=0}^{+\infty} g_j^{in} \omega^j ,$$

since $\gamma_l(\omega)$ is analytic with respect to ω in a real neighborhood of 0.
Now, we shall show how to obtain the fields (E^j, H^j). Upon inserting the expansions (26) of (E^ω, H^ω) in the system (24), we get

$$\left\{\begin{aligned} \overrightarrow{\operatorname{curl}}\, E^0 &= 0 \quad \text{in } B_R,\\ \overrightarrow{\operatorname{curl}}\, H^0 &= 0 \quad \text{in } B_R,\\ \operatorname{div} \varepsilon\, E^0 &= 0 \quad \text{in } B_R,\\ \operatorname{div} \mu\, H^0 &= 0 \quad \text{in } B_R, \end{aligned}\right. \tag{28}$$

and more generally,

$$\left\{\begin{aligned} \overrightarrow{\operatorname{curl}}\, E^{j+1} &= i\mu\, H^j \quad \text{in } B_R,\\ \overrightarrow{\operatorname{curl}}\, H^{j+1} &= -i\varepsilon\, E^j \quad \text{in } B_R,\\ \operatorname{div} \varepsilon\, E^{j+1} &= 0 \quad \text{in } B_R,\\ \operatorname{div} \mu\, H^{j+1} &= 0 \quad \text{in } B_R. \end{aligned}\right. \tag{29}$$

To complete the derivation of the canonical problem satisfied by the field E^j or the field H^j, we need to find a boundary condition satisfied by these fields on S_R. Identifying different terms in the identity (27) following increasing powers of the frequency ω, we obtain

$$\left\{\begin{aligned} -i\sqrt{\frac{\varepsilon_0}{\mu_0}} \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \gamma_l^0 v_l^{m,0}\, R_l^m &= g_{-1}^{in},\\ -i\sqrt{\frac{\varepsilon_0}{\mu_0}} \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \gamma_l^0 \Big(\frac{\gamma_l^1}{\gamma_l^0} v_l^{m,0} + v_l^{m,1}\Big) R_l^m - H^0 \wedge n &= g_0^{in}, \end{aligned}\right. \tag{30}$$

and more generally,

$$\begin{aligned} &i\sqrt{\frac{\varepsilon_0}{\mu_0}} \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \frac{1}{\gamma_l^0} \Big(\sum_{p=0}^{j} \delta_l^{j-p} u_l^{m,p}\Big) G_l^m\\ &-\gamma_l^0 \Big(\sum_{p=0}^{j+2} \frac{\gamma_l^{j+2-p}}{\gamma_l^0} v_l^{m,p}\Big) R_l^m - H^{j+1} \wedge n = g_{j+1}^{in}, \quad j \geq 0. \end{aligned} \tag{31}$$

We first derive a boundary condition on $S_R (R = 1)$ satisfied by the field E^0. Since

$$\left\{\begin{aligned} \operatorname{div}_{S_R} G_l^m &= -\sqrt{l(l+1)}\, Y_l^m,\\ \operatorname{div}_{S_R} R_l^m &= 0, \end{aligned}\right. \tag{32}$$

then by taking the surface divergence of the identity (27) for $j = 0$ we obtain

$$-\sqrt{\frac{\varepsilon_0}{\mu_0}} \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \frac{i}{\gamma_l^0} u_l^{m,0} \sqrt{l(l+1)}\, Y_l^m - \mathrm{div}_{S_R}(H^1 \wedge n) \;=\; \mathrm{div}_{S_R}(g_1^{in}). \quad (33)$$

Here we have used the fact that $\delta_l^0 = 1$. But from (24) we see that

$$\mathrm{div}_{S_R}(H^1 \wedge n) \;=\; \overrightarrow{\mathrm{curl}}\, H^1 . n \;=\; -i\varepsilon_0\, E^0 . n \quad \text{on } S_R.$$

Thus

$$-\sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \frac{1}{\gamma_l^0} u_l^{m,0} \sqrt{l(l+1)}\, Y_l^m + E^0 . n \;=\; -\frac{i}{\varepsilon_0}\, \mathrm{div}_{S_R}(g_1^{in}). \quad (34)$$

Now, let the operator

$$K : H^s(S_R) \mapsto H^{s+1}(S_R),$$

(for our purposes $s = -3/2$) be defined by

$$K(\varphi) \;=\; \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} -\frac{1}{\gamma_l^0}\, \varphi_l^m Y_l^m,$$

($\gamma_l^0 = l$) for $\varphi \in H^s(S_R)$ of the form $\varphi = \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \varphi_l^m Y_l^m$. Using the fact that $u_l^{m,0} = 0$ for all $l \geq 1$ and $m = -l$ to l, the definition of the operator K and the boundary condition on S_R (34), we obtain that E^0 satisfies on S_R

$$E^0 . n - K(\mathrm{div}_{S_R} E^0_{S_R}) \;=\; -\frac{i}{\varepsilon_0} \mathrm{div}_{S_R}(g_1^{in}). \quad (35)$$

Let us observe that the operator K is related to the operator D by

$$D(g) = K(\mathrm{div}_{S_R}\, g), \quad \forall\, g \in TH^{-1/2}(\mathrm{curl}, S_R).$$

Now, to derive a boundary condition on S_R satisfied by the field H^0, we carry out the same procedure. Let us first observe that from (30) we can deduce that

$$\begin{aligned}
\mathrm{div}_{S_R}(H^0_{S_R}) \;&=\; -i\sqrt{\frac{\varepsilon_0}{\mu_0}} \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \sqrt{l(l+1)}\gamma_l^0\, v_l^{m,1}\, Y_l^m - curl_{S_R}\, g_0^{in} \\
&\quad -i\sqrt{\frac{\varepsilon_0}{\mu_0}} \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} \sqrt{l(l+1)}\gamma_l^1\, v_l^{m,0}\, Y_l^m.
\end{aligned}$$

But

$$-\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\sqrt{l(l+1)}v_l^{m,1}\,Y_l^m \;=\; \mathrm{div}_{S_R}\,(E^1\wedge n) \;=\; i\mu_0\,H^0\,.\,n.$$

Thus,

$$i\mu_0\,H^0\,.\,n+\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\sqrt{l(l+1)}v_l^{m,1}\,Y_l^m \;=\; 0. \tag{36}$$

Making use of the operator K, we can rewrite (36) in the more suitable form

$$\begin{aligned} H^0\,.\,n-\;K(\mathrm{div}_{S_R}\,H^0_{S_R}) \;&=\; -i\mu_0\,K(\mathrm{curl}_{S_R}\,g_0^{in}) \\ -\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}&\frac{\sqrt{l(l+1)}\gamma_l^1}{\gamma_l^0}\,v_l^{m,0}\,Y_l^m. \end{aligned} \tag{37}$$

More generally, assume that the fields (E^i,H^i) are known for $i=0$ to j, we wish to derive boundary conditions satisfied by the fields (E^{j+1},H^{j+1}) on the fictive boundary S_R. From (31) it follows that

$$\begin{aligned} &\sqrt{\frac{\varepsilon_0}{\mu_0}}\sum_{l=1}^{+\infty}\sum_{m=-l}^{l} i\frac{1}{\gamma_l^0}\Big(\sum_{p}^{j-1}\delta_l^{j-1-p}u_l^{m,p}\Big)\,G_l^m \\ &-i\gamma_l^0\Big(\sum_{p=0}^{j+1}\frac{\gamma_l^{j+1-p}}{\gamma_l^0}v_l^{m,p}\Big)R_l^m - H^j\,\wedge\,n \;=\; g_j^{in}, \end{aligned}$$

and so,

$$\begin{aligned} -i\sqrt{\frac{\varepsilon_0}{\mu_0}}\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\gamma_l^0 v_l^{m,j+1}\,R_l^m \;\;&=\;\; g_j^{in}+H^j\,\wedge\,n \\ &\quad -i\sqrt{\frac{\varepsilon_0}{\mu_0}}\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\frac{1}{\gamma_l^0}\Big(\sum_{p=0}^{j-1}\delta_l^{j-1-p}u_l^{m,p}\Big)\,G_l^m \\ &\quad +i\sqrt{\frac{\varepsilon_0}{\mu_0}}\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\gamma_l^0\Big(\sum_{p=0}^{j}\frac{\gamma_l^{j+1-p}}{\gamma_l^0}v_l^{m,p}\Big)R_l^m. \end{aligned} \tag{38}$$

Thus, the expansion coefficients $(v_l^{m,j+1})_{l\geq 1, -l\leq m\leq l}$ are determined. Furthermore, it is easy to see that

$$\begin{aligned}
&\mathrm{div}_{S_R}\Big\{\sum_{l=1}^{+\infty}\sum_{m=-l}^{l} +i\frac{1}{\gamma_l^0}\Big(\sum_{p=0}^{j+1}\delta_l^{j+1-p}u_l^{m,p}\Big)\,G_l^m\Big\}\\
&= \mathrm{div}_{S_R}\,(H^{j+2}\wedge n) + \mathrm{div}_{S_R}\,(g_{j+2}^{in}),\\
&= -i\varepsilon_0\,E^{j+1}\,.\,n + \mathrm{div}_{S_R}\,(g_{j+2}^{in}).
\end{aligned}$$

Thus,

$$i\varepsilon_0\,E^{j+1}\,.\,n + i\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\frac{\sqrt{l(l+1)}}{\gamma_l^0}\Big(\sum_{p=0}^{j+1}\delta_l^{j+1-p}u_l^{m,p}\Big)\,Y_l^m \;=\; \mathrm{div}_{S_R}\,(g_{j+2}^{in}).$$

Finally, we obtain that E^{j+1} satisfies on S_R the following boundary condition

$$\begin{aligned}
&E^{j+1}\,.\,n - \;K(\mathrm{div}_{S_R}\,(E_{S_R}^{j+1})) \;=\; -\frac{i}{\varepsilon_0}\mathrm{div}_{S_R}\,(g_{j+2}^{in})\\
&+\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\frac{\sqrt{l(l+1)}}{\gamma_l^0}\Big(\sum_{p=0}^{j}\delta_l^{j+1-p}u_l^{m,p}\Big)\,Y_l^m.
\end{aligned}$$

Similarly, the following identity holds for the field H^{j+1}

$$\mathrm{div}_{S_R}\,(H_{S_R}^{j+1}) + i\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\gamma_l^0\sqrt{l(l+1)}\sum_{p=0}^{j+2}\frac{\gamma_l^{j+2-p}}{\gamma_l^0}v_l^{m,p}\,Y_l^m \;=\; \mathrm{curl}_{S_R}\,(g_{j+1}^{in}), \tag{39}$$

which can be rewritten in the form

$$\begin{aligned}
&\mathrm{div}_{S_R}\,(H_{S_R}^{j+1}) + i\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\gamma_l^0\sqrt{l(l+1)}v_l^{m,j+2}\,Y_l^m \;=\; curl_{S_R}\,(g_{j+1}^{in})\\
&+i\sqrt{\frac{\varepsilon_0}{\mu_0}}\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\gamma_l^0\sqrt{l(l+1)}\sum_{p=0}^{j+1}\frac{\gamma_l^{j+2-p}}{\gamma_l^0}v_l^{m,p}\,Y_l^m,
\end{aligned}$$

where the expansion coefficients $(v_l^{m,j+1})_{l\geq 1, -l\leq m\leq l}$ are known from the identity (38). Making use of the operator K, we see that

$$\begin{aligned}
&K(\mathrm{div}_{S_R}\,(H_{S_R}^{j+1})) - i\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\sqrt{l(l+1)}v_l^{m,j+2}\,Y_l^m \;=\; K(\mathrm{curl}_{S_R}\,(g_{j+1}^{in}))\\
&-i\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\sqrt{l(l+1)}\sum_{p=0}^{j+1}\frac{\gamma_l^{j+2-p}}{\gamma_l^0}v_l^{m,p}\,Y_l^m.
\end{aligned}$$

But

$$-\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\sqrt{l(l+1)}v_l^{m,j+2}\,Y_l^m \;=\; \mathrm{div}_{S_R}\,(E^{j+2}\wedge n),$$
$$\;=\; i\mu_0\,H^{j+1}\,.\,n.$$

Thus

$$K(\mathrm{div}_{S_R}\,(H_{S_R}^{j+1})) - H^{j+1}\,.\,n \;=\; i\mu_0\,K(\mathrm{curl}_{S_R}\,(g_{j+1}^{in}))$$
$$-\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\sqrt{l(l+1)}\sum_{p=0}^{j+1}\frac{\gamma_l^{j+2-p}}{\gamma_l^0}v_l^{m,p}\,Y_l^m \quad \text{on } S_R.$$

To summarize, we have the following lemma.

Lemma 4.1 *Assume that the electric and magnetic fields* (E^ω, H^ω) *admit the asymptotic expansions* (26)*. Then we have*

$$\left\{\begin{array}{l} \overrightarrow{\mathrm{curl}}\,E^0 \;=\; 0 \quad \text{in } B_R,\\ \mathrm{div}\,\varepsilon\,E^0 \;=\; 0 \quad \text{in } B_R,\\ E^0\,.\,n - K(\mathrm{div}_{S_R}\,(E_{S_R}^0)) \;=\; -\dfrac{i}{\varepsilon_0}\mathrm{div}_{S_R}\,(g_1^{in}) \quad \text{on } S_R, \end{array}\right. \tag{40}$$

$$\left\{\begin{array}{l} \overrightarrow{\mathrm{curl}}\,H^0 \;=\; 0 \quad \text{in } B_R,\\ \mathrm{div}\,\mu\,H^0 \;=\; 0 \quad \text{in } B_R.\\ H^0\,.\,n - K(\mathrm{div}_{S_R}\,(H_{S_R}^0)) \;=\; -i\mu_0\,K(\mathrm{curl}_{S_R}\,(g_0^{in}))\\ -\displaystyle\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\sqrt{l(l+1)}\frac{\gamma_l^1}{\gamma_l^0}v_l^{m,0}\,Y_l^m \quad \text{on } S_R, \end{array}\right. \tag{41}$$

$$\left\{\begin{array}{l} \overrightarrow{\mathrm{curl}}\,E^{j+1} \;=\; i\mu\,H^j \quad \text{in } B_R,\\ \mathrm{div}\,\varepsilon\,E^{j+1} \;=\; 0 \quad \text{in } B_R,\\ E^{j+1}\,.\,n - K(\mathrm{div}_{S_R}\,(E_{S_R}^{j+1})) \;=\; -\dfrac{i}{\varepsilon_0}\,\mathrm{div}_{S_R}\,(g_{j+2}^{in})\\ +\displaystyle\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\frac{\sqrt{l(l+1)}}{\gamma_l^0}\Big(\sum_{p=0}^{j}\delta_l^{j+1-p}u_l^{m,p}\Big)\,Y_l^m \quad \text{on } S_R, \end{array}\right. \tag{42}$$

and

$$\left\{\begin{array}{l}\overrightarrow{\mathrm{curl}}\, H^{j+1} \;=\; -i\varepsilon\, E^j \quad \text{in } B_R,\\ \mathrm{div}\,\mu\, H^{j+1} \;=\; 0 \quad \text{in } B_R.\\ H^{j+1}\,.\,n - K(\mathrm{div}_{S_R}\,(H^{j+1}_{S_R})) \;=\; -i\mu_0\, K(\mathrm{curl}_{S_R}\,(g^{in}_{j+1}))\\ -\displaystyle\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\sqrt{l(l+1)}\sum_{p=0}^{j+1}\frac{\gamma_l^{j+2-p}}{\gamma_l^0} v_l^{m,p}\, Y_l^m \quad \text{on } S_R.\end{array}\right. \tag{43}$$

Now, in order to prove the existence of a solution to the boundary-value problems (42) and (43) we must analyze in some detail the boundary conditions satisfied by the fields (E^j, H^j) on S_R. We shall prove the following lemma which is essential to reduce the calculation of higher-order terms $(E^j, H^j)_{j\geq 1}$ to a certain canonical problem.

Lemma 4.2 *a) Assume that E^i is in $H(\mathrm{curl}, B_R)$ for $i = 1$ to j. Then the quantity*

$$-\frac{i}{\varepsilon_0}\mathrm{div}_{S_R}\,(g^{in}_{j+2}) + \sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\frac{\sqrt{l(l+1)}}{\gamma_l^0}\Big(\sum_{p=0}^{j}\delta_l^{j+1-p}u_l^{m,p}\Big)\, Y_l^m$$

is in $H^{-1/2}(S_R)$.
b) Assume that E^i is in $H(\mathrm{curl}, B_R)$ for $i = 1$ to $j+1$. Then the quantity

$$-i\mu_0\, K(\mathrm{curl}_{S_R}\,(g^{in}_{j+1})) - \sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\sqrt{l(l+1)}\sum_{p=0}^{j+1}\frac{\gamma_l^{j+2-p}}{\gamma_l^0}v_l^{m,p}\, Y_l^m$$

is in $H^{-1/2}(S_R)$.

Proof. This lemma follows immediately from Lemma 2.8. □
We also need the following results. □

Lemma 4.3 *The boundary-value problem*

$$\left\{\begin{array}{l}\mathrm{div}\,\varepsilon\,\overrightarrow{\mathrm{grad}}\,\psi \;=\; 0 \quad \text{in } B_R,\\ \partial_n\psi - K(\Delta_{S_R}\psi) \;=\; f \in H^{-1/2}(S_R),\end{array}\right. \tag{44}$$

is uniquely solvable in $H^1(B_R)$ within an additive constant.

Proof. Let us first prove that the homogeneous problem: find $\psi \in H^1(B_R)$ satisfying

$$\begin{cases} \operatorname{div} \varepsilon \, \overrightarrow{\operatorname{grad}} \, \psi \; = \; 0 \quad \text{in } B_R, \\ \partial_n \psi - \; K(\Delta_{S_R} \psi) \; = \; 0 \quad \text{on } S_R, \end{cases} \tag{45}$$

admits only the constant functions as solutions. Multiplying

$$\operatorname{div} \varepsilon \, \overrightarrow{\operatorname{grad}} \, \psi \; = \; 0$$

by $\overline{\psi}$, integrating by parts over B_R and making use of the boundary condition on S_R, we obtain

$$\int_{B_R} \varepsilon \, |\overrightarrow{\operatorname{grad}} \, \psi|^2 + \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} (l+1) \, |\psi_l^m|^2 \; = \; 0,$$

where ψ_l^m are the expansion coefficients of ψ on S_R. Hence the problem (45) admits only the constant functions as solutions. We now show that a solution to (44) does exist. It is easy to see that if $\psi \in H^1(B_R)$ satifies (44) then it is solution of the variational equation

$$\int_{B_R} \varepsilon \, \overrightarrow{\operatorname{grad}} \, \psi \, \overrightarrow{\operatorname{grad}} \psi_t + \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} (l+1) \, \psi_l^m \, (\psi_t)_l^m \; = \; \int_{S_R} f \, \psi_t, \quad \forall \, \psi_t \in H^1(B_R),$$

where ψ_l^m and $(\psi_t)_l^m$ are the expansion coefficents of ψ and ψ_t on S_R. Since

$$|\sum_{l=1}^{+\infty} \sum_{m=-l}^{l} (l+1) \, \psi_l^m \, (\psi_t)_l^m| \; \leq \; ||\psi||_{H^{1/2}(S_R)} \, ||\psi_t||_{H^{1/2}(S_R)},$$

the existence follows then from the Lax-Milgram theorem. □

Next, we have to prove the following lemma.

Lemma 4.4 *Let f be in $H(\operatorname{curl}, B_R)$ such that $\operatorname{div} f \; = \; 0$ in B_R, $q \in L^\infty(B_R)$, and $\varphi \in H^{-1/2}(S_R)$. The boundary-value problem*

$$\begin{cases} \overrightarrow{\operatorname{curl}} \, u \; = \; f \quad \text{in } B_R, \\ \operatorname{div} q \, u \; = \; 0 \quad \text{in } B_R, \\ u \, . \, n - \; K(\operatorname{div}_{S_R} u_{S_R}) \; = \; \varphi \quad \text{on } S_R, \end{cases} \tag{46}$$

is uniquely solvable in $H(\operatorname{curl}, B_R)$.

Proof. Since $f \in H(\mathrm{curl}, B_R)$ satisfies $\mathrm{div}\, f = 0$ in B_R, we can find $v \in H(\mathrm{curl}, B_R)$ such that $\mathrm{div}\, q\, v = 0$ and $f = \overrightarrow{\mathrm{curl}}\, v$ in B_R. Put $\tilde{u} = u - v$; then $\tilde{u}$ satisfies

$$\begin{cases} \overrightarrow{\mathrm{curl}}\, \tilde{u} = 0 \quad \text{in } B_R, \\ \mathrm{div}\, q\, \tilde{u} = 0 \quad \text{in } B_R, \\ \tilde{u}\,.\, n - K(\mathrm{div}_{S_R} \tilde{u}_{S_R}) = \tilde{\varphi} \quad \text{on } S_R, \end{cases}$$

where

$$\tilde{\varphi} = \varphi - v\,.\, n + K(\mathrm{div}_{S_R} v_{S_R})$$

is in $H^{-1/2}(S_R)$. But $\overrightarrow{\mathrm{curl}}\, \tilde{u} = 0$ in B_R implies that $\tilde{u} = \overrightarrow{\mathrm{grad}}\, \psi$ where ψ is in $H^1(B_R)$. Lemma 4.4 follows then immediately from Lemma 4.3. □

We note that the operator $K(\Delta_{S_R})$ is the Dirichlet-Neumann operator associated to the Laplace equation in $\mathbb{R}^3 \setminus \overline{B_R}$.
Finally, we can then state:

Theorem 4.5 *Assume that the electric and magnetic fields* (E^ω, H^ω) *admit the asymptotic expansions* (26). *Then the fields* (E^j, H^j) *are determined as the unique solutions of the boundary-value problems* $(40) - (43)$ *in* $H(\mathrm{curl}, B_R)$.

5 Convergence result

Now let M be an integer. Let (E^j, H^j) for $j = 0$ to M be defined recursively as the unique solutions of the boundary-value problems $(40) - (43)$. Our aim in this section is to prove that the quantities

$$||E^\omega - \sum_{j=0}^{M} E^j\, \omega^j||_{(L^2(B_R))^3}, ||H^\omega - \sum_{j=0}^{M} H^j\, \omega^j||_{(L^2(B_R))^3}$$

are of order $0(\omega^{M+1})$ which justifies the asymptotic expansions (26) of the electric and magnetic fields with respect to ω. The next theorem provides a new and simple proof of the convergence of the electric and magnetic fields solutions of scattering problems for Maxwell's equations as frequency tends to zero. In our proof of the convergence we shall need the following lemmas.

Lemma 5.1 *The following boundary value problem*

$$\begin{cases} \overrightarrow{\mathrm{curl}}\, v \;=\; 0 \text{ in } B_R, \\ \mathrm{div}\, \varepsilon\, v \;=\; 0 \quad \text{in } B_R, \\ v\,.\,n - \; D(v_{S_R}) \;=\; 0 \quad \text{on } S_R, \end{cases} \tag{47}$$

has only the trivial solution in $H(\mathrm{curl}, B_R)$.

Proof. $\overrightarrow{\mathrm{curl}}\, v \;=\; 0$ in B_R implies that $v = \overrightarrow{\mathrm{grad}}\, \psi$ where $\psi \in H^1(B_R)$ must satisfy

$$\begin{cases} \mathrm{div}\, \varepsilon\, \overrightarrow{\mathrm{grad}}\, \psi \;=\; 0 \quad \text{in } B_R, \\ \partial_n \psi - \; D(\overrightarrow{\mathrm{grad}}_{S_R}\, \psi) \;=\; 0 \quad \text{on } S_R. \end{cases}$$

We can easily verify that the following identity holds

$$\int_{B_R} \varepsilon\, |\overrightarrow{\mathrm{grad}}\, \psi|^2 + \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} (l+1)\, |\alpha_l^m|^2 \;=\; 0$$

where α_l^m are the expansion coefficients of ψ on S_R . Hence, the function ψ is constant in B_R and Lemma 5.1 is then obvious. □

Lemma 5.2 *Assume that* $q \in L^\infty(B_R)$ *and* $q = 1$ *in a neighborhood of* B_R*. Let*

$$V = \Big\{ w \in H(\mathrm{curl}, B_R), \mathrm{div}\, q\, w \;=\; 0 \text{ in } B_R, w\,.\,n - \; D(w_{S_R}) \in H^{1/2}(S_R) \Big\}.$$

Then the embedding $V \hookrightarrow (L^2(B_R))^3$ *is compact.*

Proof. This result holds from the fact that the embeddings $(H^1(B_R))^3 \hookrightarrow (L^2(B_R))^3$ and

$$\Big\{ w \in H(\mathrm{curl}, B_R), \mathrm{div}\, q\, w \;=\; 0 \text{ in } B_R, w\,.\,n - \; D(w_{S_R}) = 0 \Big\} \hookrightarrow (L^2(B_R))^3,$$

are compact (see Abboud [1] for a proof). □

We can state the main result of this section:

Theorem 5.3 *Let M be an integer. Let (E^j, H^j) for $j = 0$ to M be defined recursively as the unique solutions of the boundary-value problems* (40) − (43)*. There exists $\omega_0 > 0$ such that for $0 < \omega < \omega_0$ the following estimates hold*

$$\left\{\begin{array}{l} ||E^\omega - \displaystyle\sum_{j=0}^{M} E^j\,\omega^j||_{(L^2(B_R))^3} \;\leq\; C\,\omega^{M+1}, \\ ||H^\omega - \displaystyle\sum_{j=0}^{M} H^j\,\omega^j||_{(L^2(B_R))^3} \;\leq\; C\,\omega^{M+1}, \end{array}\right. \tag{48}$$

where the constant C is independent of ω.

Proof. It is easy to see that the field $E^\omega - E^0$ satisfies

$$\left\{\begin{array}{l} \overrightarrow{\operatorname{curl}}\,\dfrac{1}{\mu}\,\overrightarrow{\operatorname{curl}}\,(E^\omega - E^0) - \omega^2\varepsilon\,(E^\omega - E^0) \;=\; \omega^2\varepsilon E^0 \quad \text{in } B_R, \\ \operatorname{div}\varepsilon\,(E^\omega - E^0) \;=\; 0 \quad \text{in } B_R, \\ \overrightarrow{\operatorname{curl}}\,(E^\omega - E^0)\,\wedge\,n \;=\; i\omega\mu_0\left(T^\omega(E^\omega_{S_R} - E^0_{S_R}) - g^{in}_\omega\right) \\ +i\omega\mu_0\,T^\omega(E^0_{S_R}) \quad \text{on } S_R. \end{array}\right. \tag{49}$$

If we multiply the Maxwell equations by $\overline{(E^\omega - E^0)}$, integrate over B_R, and use integration by parts, we obtain

$$\begin{array}{l} \displaystyle\int_{B_R} \frac{1}{\mu}\,|\overrightarrow{\operatorname{curl}}\,(E^\omega - E^0)|^2 - \omega^2 \int_{B_R} \varepsilon\,|E^\omega - E^0|^2 \\ -i\omega\mu_0\Big(T^\omega(E^\omega_{S_R} - E^0_{S_R}), E^\omega_{S_R} - E^0_{S_R}\Big) \\ = \omega^2 \displaystyle\int_{B_R} \varepsilon\,E^0\,.\,\overline{(E^\omega - E^0)} + i\omega\mu_0\left(g^{in}_\omega - T^\omega(E^0_{S_R}), E^\omega_{S_R} - E^0_{S_R}\right). \end{array} \tag{50}$$

Next, using the operator D^ω, we may write

$$\begin{array}{rcl} (E^\omega - E^0)\,.\,n &=& D^\omega(E^\omega_{S_R} - E^0_{S_R}) + D^\omega(E^0_{S_R}) + K(\operatorname{div}_{S_R} E^0_{S_R}) \\ && -\dfrac{i}{\varepsilon_0}\operatorname{div}_{S_R}(g^{in}_1) + E^{in}_\omega\,.\,n - D^\omega(E^{in}_{\omega,S_R}). \end{array} \tag{51}$$

We are now in position to prove Theorem 5.3. Let us first observe that

$$T^\omega(E^0_{S_R}) - \frac{1}{\omega}g^{in}_{-1} \;=\; \sqrt{\frac{\varepsilon_0}{\mu_0}}\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\frac{1}{\gamma^0_l}\,v^{m,0}_l\,G^m_l + 0(\omega),$$

where $0(\omega)$ is in $TH^{-1/2}(\mathrm{div}, S_R)$. If we assume that the quantity

$$\frac{1}{\omega}||E^\omega - E^0||_{(L^2(B_R))^3}$$

is not bounded then from (50) we can deduce that

$$v^\omega \;=\; \frac{E^\omega - E^0}{||E^\omega - E^0||_{(L^2(B_R))^3}}$$

converges weakly in $\Big\{w \in H(\mathrm{curl}, B_R), \mathrm{div}\varepsilon w = 0 \text{ in } B_R\Big\}$ to a certain v. It is easy to see that $\overrightarrow{\mathrm{curl}} v = 0$, $\mathrm{div}\varepsilon v = 0$ in B_R and

$$v^\omega\,.\,n - D(v^\omega_{S_R}) = (D^\omega - D)(\frac{E^\omega}{||E^\omega - E^0||_{(L^2(B_R))^3}}).$$

From Lemma 2.7, it follows that the following identity holds

$$||(D^\omega - D)(\frac{E^\omega}{||E^\omega - E^0||_{(L^2(B_R))^3}})||_{H^{1/2}(S_R)} \;\leq C\,\omega\,||v^\omega||_{H(\mathrm{curl}, B_R)},$$

where C is a constant independent of ω and then, by Lemma 5.1 we obtain that $v = 0$ and then Lemma 5.2 implies that $v^\omega \to 0$ in $(L^2(B_R))^3$, strongly. This leads to a contradiction. The quantity $||E^\omega - E^0||_{(L^2(B_R))^3}$ is then of order ω. Now, in order to prove that $||H^\omega - H^0||_{(L^2(B_R))^3}$ is of order ω we observe that

$$\begin{aligned}
&\int_{B_R}\frac{1}{\mu}\,|\overrightarrow{\mathrm{curl}}\,(E^\omega - E^0 - \omega E^1)|^2 - \omega^2\int_{B_R}\varepsilon\,|E^\omega - E^0 - \omega E^1|^2\\
&-i\omega\mu_0\Big(T^\omega(E^\omega_{S_R} - E^0_{S_R} - \omega E^1_{S_R}), E^\omega_{S_R} - E^0_{S_R} - \omega E^1_{S_R}\Big)\\
&= \omega^2\int_{B_R}\varepsilon\,E^0\,.\,\overline{(E^\omega - E^0 - \omega E^1)}\\
&+i\omega\mu_0\left(g^{in}_\omega - T^\omega(E^0_{S_R}) - \omega T^\omega(E^1_{S_R}), E^\omega_{S_R} - E^0_{S_R} - \omega E^1_{S_R}\right).
\end{aligned} \tag{52}$$

Since the quantity

$$\begin{aligned}
&|\Big(g^{in}_\omega - T^\omega(E^0_{S_R}) - \omega T^\omega(E^1_{S_R}), E^\omega_{S_R} - E^0_{S_R} - \omega E^1_{S_R}\Big)|\\
&= 0(\omega)\,||E^\omega_{S_R} - E^0_{S_R} - \omega E^1_{S_R}||_{TH^{-1/2}(\mathrm{curl}, S_R)},
\end{aligned}$$

it follows by the same argument that $||E^\omega - E^0 - \omega E^1||_{(L^2(B_R))^3}$ is of order ω^2. But

$$\frac{1}{\mu}\,\overrightarrow{\text{curl}}\,(E^\omega - E^0 - \omega E^1) \;=\; i\omega(H^\omega - H^0).$$

Thus, from (52), we can deduce that $\omega^2\,||H^\omega - H^0||^2_{(L^2(B_R))^3}$ is of order ω^4 and then, $||H^\omega - H^0||_{(L^2(B_R))^3}$ is of order ω. Now, since

$$\overrightarrow{\text{curl}}\,\frac{1}{\mu}\,\overrightarrow{\text{curl}}\left(E^\omega - \sum_{j=0}^{M} E^j\,\omega^j\right) - \omega^2\varepsilon\left(E^\omega - \sum_{j=0}^{M} E^j\,\omega^j\right) \;=\; \omega^{M+1}\varepsilon\Big(E^{M-1} + \omega E^M\Big)$$

in B_R, proceeding in a similar way, we obtain that the estimate

$$||E^\omega - \sum_{j=0}^{M} E^j\,\omega^j||_{(L^2(B_R))^3} \;=\; 0(\omega^{M+1}),$$

holds for any integer M. To complete the proof of Theorem 5.3, we see that from

$$||E^\omega - \sum_{j=0}^{M+1} E^j\,\omega^j||_{(L^2(B_R))^3} \;=\; 0(\omega^{M+2}),$$

the variational equation

$$\begin{aligned}
&\int_{B_R} \frac{1}{\mu}\,|\overrightarrow{\text{curl}}\,(E^\omega - \sum_{j=0}^{M+1} E^j\,\omega^j)|^2 - \omega^2 \int_{B_R} \varepsilon\,|E^\omega - \sum_{j=0}^{M+1} E^j\,\omega^j|^2 \\
&-i\omega\mu_0\Big(T^\omega(E^\omega_{S_R} - \sum_{j=0}^{M+1} E^j\,\omega^j), E^\omega_{S_R} - \sum_{j=0}^{M+1} E^j_{S_R}\,\omega^j\Big) \\
&= \omega^{M+2}\int_{B_R} \varepsilon\,(E^M + \omega E^{M+1})\,.\,(\overline{E}^\omega - \sum_{j=0}^{M+1} \overline{E}^j\,\omega^j) \\
&+i\omega\mu_0\left(g^{in}_\omega - T^\omega(\sum_{j=0}^{M+1} E^j_{S_R}\,\omega^j), E^\omega_{S_R} - \sum_{j=0}^{M+1} E^j_{S_R}\,\omega^j\right),
\end{aligned}$$

and the fact that the quantity

$$|\Big(g^{in}_\omega - \sum_{j=0}^{M} T^\omega(E^j_{S_R})\omega^j, E^\omega_{S_R} - \sum_{j=0}^{M} E^j_{S_R}\,\omega^j\Big)|$$

is of order $0(\omega^M)\,||E^\omega_{S_R} - \sum_{j=0}^{M} E^j_{S_R}\,\omega^j||_{TH^{-1/2}(\text{curl},S_R)}$, we have

$$\omega^2 \int_{B_R} \mu\,|H^\omega - \sum_{j=0}^{M} H^j\,\omega^j|^2 \;=\; 0(\omega^{2M+4}).$$

Therefore, the quantity $||H^\omega - \sum_{j=0}^{M} H^j\,\omega^j||_{(L^2(B_R))^3}$ is of order $0(\omega^{M+1})$. □

Appendix A. In this appendix, we give the explicit expression of the operator T^ω. Let us recall the standard vector basis functions for Maxwell's equations

$$\begin{cases} M^m_l(x) \;=\; \overrightarrow{\text{curl}}\Big(x\,h^{(1)}_l(\omega\sqrt{\varepsilon_0\mu_0})\,Y^m_l(x)\Big), \\ N^m_l(x) \;=\; \dfrac{1}{i\omega\sqrt{\varepsilon_0\mu_0}}\,\overrightarrow{\text{curl}}\,M^m_l(x), \end{cases}$$

where $h^{(1)}_l$ denotes the spherical Hankel functions of the first kind and order l. From the definitions of G^m_l and R^m_l, it is easy to see that

$$-n\wedge\Big(n\,\wedge\,N^m_l(x)\Big) \;=$$
$$-\frac{i}{\omega\sqrt{\varepsilon_0\mu_0}}\sqrt{l(l+1)}\,\Big(h^{(1)}_l(\omega\sqrt{\varepsilon_0\mu_0}) + \omega\sqrt{\varepsilon_0\mu_0}(h^{(1)}_l)'(\omega\sqrt{\varepsilon_0\mu_0})\Big)\,G^m_l(x),$$

and

$$-n\wedge\Big(n\,\wedge\,M^m_l(x)\Big) \;=\; -\sqrt{l(l+1)}\,h^{(1)}_l(\omega\sqrt{\varepsilon_0\mu_0})\,R^m_l(x),$$

on $S_R(R=1)$. Now, if we expand the tangential vector field g^ω in the form

$$g^\omega \;=\; \sum_{l=1}^{+\infty}\sum_{m=-l}^{l} \alpha^m_l(\omega)\,G^m_l + \beta^m_l(\omega)\,R^m_l,$$

the solution of Maxwell's equations (3) satisfying the outgoing radiation condition can be written for $|x| > R = 1$ in the form (see for instance [7])

$$\mathcal{E}^\omega(x) \;=\; \sum_{l\geq 1}\sum_{m=-l}^{l}\Big(\delta^m_l(\omega)\,M^m_l(x) + \theta^m_l(\omega)\,N^m_l(x)\Big),$$

with uniform convergence on compact subsets of $|x| > R$, where

$$\delta^m_l(\omega) \;=\; -\frac{\beta^m_l(\omega)}{\sqrt{l(l+1)}\,h^{(1)}_l(\omega\sqrt{\varepsilon_0\mu_0})},$$

and

$$\theta_l^m(\omega) = \frac{i\omega\sqrt{\varepsilon_0\mu_0}\,\alpha_l^m(\omega)}{\sqrt{l(l+1)}\Big(h_l^{(1)}(\omega\sqrt{\varepsilon_0\mu_0}) + \omega\sqrt{\varepsilon_0\mu_0}(h_l^{(1)})'(\omega\sqrt{\varepsilon_0\mu_0})\Big)}.$$

Since

$$\begin{cases} \overrightarrow{\mathrm{curl}}\, M_l^m(x) = i\omega\sqrt{\varepsilon_0\mu_0}\, N_l^m(x), \\ \overrightarrow{\mathrm{curl}}\, N_l^m(x) = -i\omega\sqrt{\varepsilon_0\mu_0}\, M_l^m(x), \end{cases}$$

the magnetic field $\mathcal{H}^\omega$ is given by

$$\mathcal{H}^\omega(x) = \frac{1}{\mu_0}\sum_{l\geq 1}\sum_{m=-l}^{l}\Big(\delta_l^m(\omega)\, N_l^m(x) - \theta_l^m(\omega)\, M_l^m(x)\Big).$$

Therefore, we have the following explicit representation for the operator T^ω

$$T^\omega(g^\omega) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\sum_{l=1}^{+\infty}\sum_{m=-l}^{l} i\omega\frac{\alpha_l^m(\omega)}{\gamma_l(\omega)}\, G_l^m + \frac{\beta_l^m(\omega)\gamma_l(\omega)}{i\omega}\, R_l^m,$$

where

$$\gamma_l(\omega) = 1 + \omega\frac{(h_l^{(1)})'(\omega)}{h_l^{(1)}(\omega)}.$$

Appendix B. In this appendix we shall give the proofs of Lemmas 2.3-2.6.

Proof of Lemma 2.3. It is easy to see that

$$\frac{l\, c_l^{l-1}}{c_l^l} < 1 \text{ and } \frac{c_l^m}{c_l^{m+1}} < 1 \quad m = 0 \text{ to } l-1.$$

By rewriting the quantity

$$\frac{(l+1-m)\, c_l^{l-m}}{c_l^l} = \frac{(l+1-m)\, c_l^{l-1}}{c_l^l}\frac{c_l^{l-2}}{c_l^{l-1}}\cdots\frac{c_l^{l-m}}{c_l^{l-m+1}},$$

we obtain the claim.

Proof of Lemma 2.4. Combining the following two identities

$$\Big|\frac{1}{\gamma_l(\omega)} + \frac{1}{l}\Big| \leq \omega \sup_{0<\omega<\omega_0}\Big|\frac{\gamma_l'(\omega)}{\gamma_l^2(\omega)}\Big|,$$

and

$$\sup_{0<\omega<\omega_0} \left|\frac{\gamma_l'(\omega)}{\gamma_l^2(\omega)}\right| \leq \frac{C}{l^2}\,\omega,$$

where l is large enough and the constant C is independent of ω, gives the claim.

Next, from (13) and (14), we shall prove Lemma 2.5.

Proof of Lemma 2.5. Let

$$Q_l(t) = t^{2l} q_l(t) = c_l^l\Big(1 + \ldots + \frac{c_l^0}{c_l^l} t^{2l}\Big).$$

From Lemma 2.3, it follows that there exists a complex neighborhood V of zero such that for any $l \geq 1$, $Q_l(t)$ does not have a zero in V. We can deduce that there exists a ball $B_r (r > 0)$ such that the function $\omega \mapsto \gamma_l(\omega)$ is analytic in B_r. (16) follows from the fact that there exists a constant C independent of ω such that

$$|\gamma_l(\omega)| \leq C|\gamma_l^0|, \quad \forall\, \omega \in \partial B_r.$$

Proof of Lemma 2.6. By using the fact that there exists a complex neighorhood V of zero such that

$$P_l(t) = t^{2l} p_l(t) + i t^{2l}$$

does not have a zero in V, the identity (17) holds. (18) is easily deduced from (15).

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