PERTURBATIONS OF EIGENVALUE PROBLEMS IN ELECTROMAGNETICS

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Abstract

This paper considers the effect of two types of perturbations on eigenvalue problems for Maxwell's equations in bounded domains. The first kind occurs in calculating the change in resonant frequencies of a cavity due to absorbing material on its boundary. The second arises in analyzing a cavity containing a small body. For each of these perturbations, the resonant frequencies for the perturbed problem are constructed. In some special cases, the asymptotic results are shown to agree well with exact results.

1 Introduction

The aim of the first part of this paper is to calculate the change in resonant frequencies of a cavity due to absorbing material on its boundary. The second part concerns the effect of a sample of material on the resonant frequencies of a cavity oscillator in which it is placed.
In the discusion of the propagation or excitation of electromagnetic waves in metallic cavities the boundary surfaces are often assumed to be perfect conductors. Our aim in this paper is to take into account the losses occuring in practice. The absorbing material on the boundary of the cavity can be modeled by a Leontovich boundary condition provided that the thickness of the lossy coating exceeds the penetration depth (see for instance [8]). This impedance boundary condition provides an approximate relationship between the electric and magnetic field. Then, we deal with an impedance boundary value problem for the Maxwell equations involving a small parameter, which is the penetration depth of the boundary surface, in the place of

the conductor boundary value problem. We expand the resonant frequencies in terms of the penetration depth and we construct the first corrections to the resonant frequencies due to the losses.
In the second part of this paper, following an idea of Ward and Keller in [9], we construct by the method of matched asymptotic expansions the perturbed resonant frequencies of a cavity oscillator containing a small inclusion. We consider three cases: a highly conducting body, a perfect conductor and a dielectric body with high electric permittivity and high magnetic permeability.
For spherical geometries, we also show that the asymptotic results for the perturbed eigenvalue problems agree well with corresponding exact analytical results. Our appraochs in this paper are formal.

2 Leontovich boundary conditions and perturbation of the eigenvalues of the Maxwell operator

Let us consider the following unperturbed eigenvalue problem for the Maxwell operator in a bounded smooth connected domain Ω of $\mathbf{R}^3$:

$$\begin{cases} \mathbf{curl}\,\mathbf{E} \,=\, i\omega\,\mu\,\mathbf{H} \quad \text{in } \Omega, \\ \mathbf{curl}\,\mathbf{H} \,=\, -i\omega\,\varepsilon\,\mathbf{E} \quad \text{in } \Omega, \\ \mathbf{E} \wedge \mathbf{n} = 0 \quad \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \varepsilon\,|\mathbf{E}|^2 \,=\, 1, \end{cases} \tag{1}$$

where $\mathbf{n}$ is the outward normal. The coefficients ε (called the electric permittivity) and μ (called the magnetic permeability) are strictly positive bounded real functions. We assume that they are in $\mathcal{C}^2(\overline{\Omega})$. It is now well known (see, for example, [4] and [2]) that the spectrum of the Maxwell operator

$$\mathcal{M}(\varepsilon,\mu) = \begin{pmatrix} 0 & -\dfrac{1}{\varepsilon}\,\mathbf{curl} \\ \dfrac{1}{\mu}\,\mathbf{curl} & 0 \end{pmatrix}$$

consists of a countably infinite set of eigenvalues arranged repeatedly according to their multiplicities $\Big\{\omega_n\Big\}_{n\in\mathbf{Z}}$, where $\omega_{-n} = \omega_n, \omega_0 = 0$, and $|\omega_n| \to +\infty, n \to +\infty$ and corresponding orthonormal eigenfields $\Big\{\mathbf{E}_n, \mathbf{H}_n\Big\}, n \in \mathbb{Z}$, such that for any $\mathbf{E} \in H_0(\mathrm{curl}, \Omega)$ and $\mathbf{H} \in H^0(\mathrm{curl}, \Omega)$, where the Sobolev spaces $H_0(\mathrm{curl}, \Omega)$ and $H^0(\mathrm{curl}, \Omega)$ are given by:

$$
\begin{aligned}
H_0(\mathrm{curl}, \Omega) &= \Big\{ \mathbf{e} \in (L^2(\Omega))^3, \mathbf{curl}\, \mathbf{e} \in (L^2(\Omega))^3, \mathbf{e} \wedge \mathbf{n} = 0 \text{ on } \partial\Omega \Big\}, \\
H^0(\mathrm{curl}, \Omega) &= \Big\{ \mathbf{h} \in (L^2(\Omega))^3, \mathbf{curl}\, \mathbf{h} \in (L^2(\Omega))^3, \mathbf{h} \,.\, \mathbf{n} = 0 \text{ on } \partial\Omega \Big\},
\end{aligned}
$$

($L^2(\Omega)$ is the space of complex square integrable functions defined in Ω), we have

$$
\left\{
\begin{aligned}
& \mathbf{E} = \sum_{n \in \mathbf{Z}} \Big(\int_\Omega \varepsilon\, \mathbf{E} \,.\, \mathbf{E}_n \Big)\, \mathbf{E}_n, \\
& \mathbf{H} = \sum_{n \in \mathbf{Z}} \Big(\int_\Omega \mu\, \mathbf{H} \,.\, \mathbf{H}_n \Big)\, \mathbf{H}_n, \\
& \mathbf{curl}\, \mathbf{E} = i\mu \sum_{n \in \mathbf{Z}} \omega_n \, \Big(\int_\Omega \varepsilon\, \mathbf{E} \,.\, \mathbf{E}_n \Big)\, \mathbf{E}_n, \\
& \mathbf{curl}\, \mathbf{H} = -i\varepsilon \sum_{n \in \mathbf{Z}} \omega_n \, \Big(\int_\Omega \mu\, \mathbf{H} \,.\, \mathbf{H}_n \Big)\, \mathbf{H}_n.
\end{aligned}
\right.
$$

The kernel of the operator $\mathcal{M}(\varepsilon, \mu)$ in the Sobolev space

$$
\Big\{ \mathbf{e} \in H_0(\mathrm{curl}, \Omega), \mathrm{div}\, \varepsilon\, \mathbf{e} = 0 \text{ in } \Omega \Big\} \times \Big\{ \mathbf{h} \in H^0(\mathrm{curl}, \Omega), \mathrm{div}\, \mu\, \mathbf{h} = 0 \text{ in } \Omega \Big\}
$$

(which is the eigenspace associated to zero) is equal to $\mathbb{H}_1 \times \mathbb{H}_2$ where $\mathbb{H}_1$ (resp. $\mathbb{H}_2$) characterises electrostatics (resp. magnetostatics) in the bounded set Ω. $\mathbb{H}_1$ (resp. $\mathbb{H}_2$) is also called the space of harmonic Dirichlet (resp. Neumann) vector fields. It is also well-known that $\mathbb{H}_1$ (resp. $\mathbb{H}_2$) is isomorphic to the first (resp. second) cohomology space, that is to say the quotient of the closed 1-(resp. 2-)forms in Ω by the exact differential 1-(resp. 2-)forms in Ω. The dimension of $\mathbb{H}_1$ (called the first Betti number) is equal to the number of cuts needed to render $\partial\Omega$ simply connected. The dimension of $\mathbb{H}_2$ (called the second Betti number) is equal to the number of bounded connected components of the open set $\mathbf{R}^3 \setminus \overline{\Omega}$. These classical results on the kernel of the Maxwell operator $\mathcal{M}(\varepsilon, \mu)$ can be found in [3]. They are shown to be still valid for a large class of domains with not necessarily smooth boundaries (segment property) in [7] by Picard. We shall also remark that it is not evident to realize the operator $\mathcal{M}(\varepsilon, \mu)$ as a self-adjoint operator in $(L^2(\Omega))^3 \times (L^2(\Omega))^3$. The difficulties are connected with the fact that the boundary value problem (1) is not elliptic and is not semi-bounded. To prove correctly the above results the Maxwell operator is included in the operator of an elliptic boundary value problem by making use for example of a Hodge decomposition lemma of the Sobolev space $H_0(\mathrm{curl}, \Omega)$ (see the review paper of Birman and Solomyak [2]). However, for smooth Ω, ε and μ, the realization of the Maxwell operator $\mathcal{M}(\varepsilon, \mu)$ as a self-adjoint operator can be attained by defining it on fields of class $(H^1(\Omega))^3 \times (H^1(\Omega))^3$ where

$$H^1(\Omega) = \Big\{ \varphi \in L^2(\Omega), \mathbf{grad}\, \varphi \in (L^2(\Omega))^3 \Big\}.$$

This realization can be established by adding $-\mathbf{grad}\,\mathrm{div}\,(\varepsilon\,\mathbf{E}), -\mathbf{grad}\,\mathrm{div}\,(\mu\,\mathbf{H})$ respectively to the second order system generated by elimination of $\mathbf{E}$ or $\mathbf{H}$ in the Maxwell system (1) (see [4] and the references therein).
The numerical approximation of the eigenvalues $\Big\{ \omega_n \Big\}_{n \in \mathbf{Z}}$ for the Maxwell equations (1) was considered by Levillain in [5] where he proved that this problem leads to a second-order rate of convergence when using mixed finite elements.

Now we introduce a perturbed form of this problem by replacing the perfect conductor boundary condition in (1) by a Leontovich boundary condition on $\partial\Omega$ in the form:

$$-\mathbf{E}^\delta_{\partial\Omega} + i\,\delta\,z\;\mathbf{H}^\delta \wedge \mathbf{n} \;=\; 0 \quad \text{on } \partial\Omega, \tag{2}$$

where δ is a small parameter, $\mathbf{E}^\delta_{\partial\Omega}$ is the tangential component of $\mathbf{E}^\delta$ on $\partial\Omega$ and z is a smooth function defined on $\partial\Omega$. Throughout what follows, we denote by $\mathbf{u}_{\partial\Omega}$ the tangential component of the field $\mathbf{u}$ on $\partial\Omega$. We require that the function z satisfies

$$\Re e\,(z) \geq 0, \Im m(z) \;\leq -z_0, \tag{3}$$

where z_0 is a strictly positive constant. The perturbed problem is to find ω^δ such that there exist $\mathbf{E}^\delta$ and $\mathbf{H}^\delta$ satisfying

$$\begin{cases} \mathbf{curl}\,\mathbf{E}^\delta \;=\; i\omega^\delta\,\mu\,\mathbf{H}^\delta \quad \text{in } \Omega, \\ \mathbf{curl}\;\mathbf{H}^\delta \;=\; -i\omega^\delta\,\varepsilon\,\mathbf{E}^\delta \quad \text{in } \Omega, \\ -\mathbf{E}^\delta_{\partial\Omega} + i\,\delta\,z\;\mathbf{H}^\delta \wedge \mathbf{n} \;=\; 0 \quad \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \varepsilon\,|\mathbf{E}^\delta|^2 \;=\; 1. \end{cases} \tag{4}$$

This problem occurs in calculating the change in resonant frequencies of a cavity due to absorbing material on its boundary. It also occurs in analyzing a cavity oscillator immersed in a highly conducting medium. At high conductivity σ, the exterior domain can be replaced by a Leontovich boundary condition applied directly on the boundary $\partial\Omega$ of the cavity Ω. The parameter δ in the boundary condition (2) is of order $\dfrac{1}{\sqrt{\sigma}}$ which is proportional to the penetration depth of the absorbing material.

It is obvious to see that, under the assumptions (3) on the function z, the frequency $\omega^\delta \neq 0$ can not be on the positive real axis. In what follows, we call such frequencies $\left\{\omega^\delta\right\}$ for which the boundary value problem (4) admits a solution the resonant frequencies of (4). Our analysis below is limited to the resonant frequencies of (4) which tend to the eigenvalues $\left\{\omega_n\right\}_{n\in\mathbf{Z}}$ of the unperturbed boundary value problem for the Maxwell equations (1) as $\delta \to 0$.

We first deal with a simply connected boundary $\partial\Omega$. Under this assumption, the kernel $\mathbb{H}_1 \times \mathbb{H}_2$ of the Maxwell operator $\mathcal{M}(\varepsilon, \mu)$ is reduced to $\left\{0\right\}\times\left\{0\right\}$. By eliminating the magnetic field $\mathbf{H}^\delta$ we turn the boundary value problem (4) into a second order system, with $\mathbf{E}^\delta$ and $\mathbf{H}^\delta$ now separated, given by

$$\left\{\begin{aligned} &\mathbf{curl}\,\frac{1}{\mu}\,\mathbf{curl}\,\mathbf{E}^\delta - (\omega^\delta)^2\,\varepsilon\,\mathbf{E}^\delta \;=\; 0 \quad \text{in } \Omega,\\ &-\mathbf{E}^\delta_{\partial\Omega} + \delta\,\frac{z}{\omega^\delta\,\mu}\,\mathbf{curl}\,\mathbf{E}^\delta \,\wedge\,\mathbf{n} \;=\; 0 \quad \text{on } \partial\Omega,\\ &\int_\Omega \varepsilon\,|\mathbf{E}^\delta|^2 \;=\; 1. \end{aligned}\right. \tag{5}$$

Next let $\mathbf{E}^t$ be a test function belonging to the Sobolev space $\mathbb{M}$ defined by

$$\mathbb{M} = \left\{\mathbf{u} \in H(\mathrm{curl}, \Omega), \mathbf{u}_{\partial\Omega} \in (L^2(\partial\Omega))^3\right\}.$$

Then by applying Green's theorem to the boundary value problem (5), we obtain the variationally posed problem

$$\int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\mathbf{E}^\delta \,.\,\mathbf{curl}\,\mathbf{E}^t - (\omega^\delta)^2 \int_\Omega \varepsilon\,\mathbf{E}^\delta \,.\,\mathbf{E}^t - \frac{\omega^\delta}{\delta}\int_{\partial\Omega} \frac{1}{z}\,\mathbf{E}^\delta_{\partial\Omega} \cdot \mathbf{E}^t_{\partial\Omega} = 0. \tag{6}$$

Let $\omega^0 \in \left\{\omega_n\right\}_{n\in\mathbf{Z}}$ be a strictly positive simple isolated eigenvalue of the boundary value problem for the Maxwell equations (1) and the field $\mathbf{E}^0$ be a normalized eigenvector ($\int_\Omega \varepsilon\,|\mathbf{E}^0|^2 = 1$) corresponding to the eigenvalue ω^0. We seek a solution of the variational problem (6) for which ω^δ tends to ω^0 and $\mathbf{E}^\delta$ tends to $\mathbf{E}^0$ as $\delta \to 0$. We expand ω^δ and $\mathbf{E}^\delta$ as

$$\left\{ \begin{array}{lcl} \omega^\delta & = & \omega^0 + \delta\,\omega^1 + \delta^2\,\omega^2 + \ldots, \\ \mathbf{E}^\delta & = & \mathbf{E}^0 + \delta\,\mathbf{E}^1 + \delta^2\,\mathbf{E}^2 + \ldots. \end{array} \right. \tag{7}$$

If we subsitute the above asymptotic expansions (7) into the variational equation (6) and equate terms of each order in δ, then the term of order δ^{-1} in (6) vanishes since $\mathbf{E}^0 \wedge \mathbf{n} = 0$ on $\partial\Omega$ while the terms of order δ^0 yield

$$\int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\mathbf{E}^0 \,.\, \mathbf{curl}\,\mathbf{E}^t - (\omega^0)^2 \int_\Omega \varepsilon\,\mathbf{E}^0 \,.\, \mathbf{E}^t - \omega^0 \int_{\partial\Omega} \frac{1}{z}\,\mathbf{E}^1_{\partial\Omega} \,.\, \mathbf{E}^t_{\partial\Omega} = 0, \tag{8}$$

$\forall\, \mathbf{E}^t \in \mathbb{M}$. From the terms of order δ^1 in (6), we obtain

$$\left\{ \begin{array}{l} \displaystyle\int_\Omega \frac{1}{\mu}\,\mathbf{curl}\,\mathbf{E}^1 \,.\, \mathbf{curl}\,\mathbf{E}^t - (\omega^0)^2 \int_\Omega \varepsilon\,\mathbf{E}^1 \,.\, \mathbf{E}^t - \omega^1 \int_{\partial\Omega} \frac{1}{z}\,\mathbf{E}^1_{\partial\Omega} \,.\, \mathbf{E}^t_{\partial\Omega} \\ \qquad \displaystyle -\omega^0 \int_{\partial\Omega} \frac{1}{z}\,\mathbf{E}^2_{\partial\Omega} \,.\, \mathbf{E}^t_{\partial\Omega} = 2\,\omega^0\,\omega^1 \int_\Omega \varepsilon\,\mathbf{E}^0 \,.\, \mathbf{E}^t, \forall\, \mathbf{E}^t \in \mathbb{M}. \end{array} \right. \tag{9}$$

To write the boundary condition satisfied by $\mathbf{E}^1$ on $\partial\Omega$, we apply Green's theorem to (8) to obtain

$$\mathbf{E}^1_{\partial\Omega} = -\frac{z}{\omega^0\,\mu}\,\mathbf{curl}\,\mathbf{E}^0 \,\wedge\, \mathbf{n}. \tag{10}$$

We now determine the first correction ω^1 to the resonant frequency ω^δ. Upon using $\overline{\mathbf{E}^0}$ as a test function in (9), $\overline{\mathbf{E}^1}$ as a test function in (8) and the boundary condition (10), we obtain

$$2\omega^0\,\omega^1 \int_\Omega \varepsilon\,|\mathbf{E}^0|^2 = -\frac{1}{\omega^0} \int_{\partial\Omega} z\,|\frac{1}{\mu}\,\mathbf{curl}\,\mathbf{E}^0 \,\wedge\, \mathbf{n}|^2,$$

The integral on the left-hand side is equal to $2\omega^0\,\omega^1$ because $\mathbf{E}^0$ is normalized, and we obtain

$$\omega^1 = -\frac{1}{2(\omega^0)^2} \int_{\partial\Omega} z\,|\frac{1}{\mu}\,\mathbf{curl}\,\mathbf{E}^0 \,\wedge\, \mathbf{n}|^2. \tag{11}$$

The imaginary part of the first correction is then strictly positive. To determine the first correction $\mathbf{E}^1$ to $\mathbf{E}^\delta$, we first observe that from the asymptotic expansion of $\mathbf{E}^\delta$ as $\delta \to 0$, the identity $\displaystyle\int_\Omega \varepsilon\, |\mathbf{E}^\delta|^2 = 1$ gives

$$\Re e \int_\Omega \varepsilon\, \mathbf{E}^1 \,.\, \overline{\mathbf{E}^0} = 0.$$

Next, we can easily prove that the boundary value problem

$$\left\{ \begin{array}{l} \displaystyle \mathbf{curl}\, \frac{1}{\mu}\, \mathbf{curl}\, \mathbf{e}^1 - (\omega^0)^2\, \varepsilon\, \mathbf{e}^1 \; = \; 2\, \omega^0\, \omega^1\, \varepsilon\, \mathbf{E}^0, \quad \text{in } \Omega, \\ \displaystyle \mathbf{e}^1_{\partial\Omega} = - \frac{z}{\omega^0\, \mu}\, \mathbf{curl}\, \mathbf{E}^0 \; \wedge \, \mathbf{n} \quad \text{on } \partial\Omega, \\ \displaystyle \int_\Omega \varepsilon\, \mathbf{e}^1 \,.\, \overline{\mathbf{E}^0} = 0, \end{array} \right. \tag{12}$$

with ω^1 given by (11) is uniquely solvable in the Sobolev space $(H^1(\Omega))^3$ provided that ε, μ are in $\mathcal{C}^1(\overline{\Omega})$, $\partial\Omega$ is of class $\mathcal{C}^2$ and $z \in \mathcal{C}^1(\partial\Omega)$. This gives that the first correction $\mathbf{E}^1$ to the eigenvector $\mathbf{E}^\delta$ which satisfies

$$\left\{ \begin{array}{l} \displaystyle \mathbf{curl}\, \frac{1}{\mu}\, \mathbf{curl}\, \mathbf{E}^1 - (\omega^0)^2\, \varepsilon\, \mathbf{E}^1 \; = \; 2\, \omega^0\, \omega^1\, \varepsilon\, \mathbf{E}^0, \quad \text{in } \Omega, \\ \displaystyle \mathbf{E}^1_{\partial\Omega} = - \frac{z}{\omega^0\, \mu}\, \mathbf{curl}\, \mathbf{E}^0 \; \wedge \, \mathbf{n} \quad \text{on } \partial\Omega, \\ \displaystyle \Re e \int_\Omega \varepsilon\, \mathbf{E}^1 \,.\, \overline{\mathbf{E}^0} = 0, \end{array} \right. \tag{13}$$

is $\lambda\, \mathbf{E}^0 + \mathbf{e}^1$ where the coefficient $\lambda \in i\mathbf{R}$ is unspecified. We will see below that λ does not affect the higher-order terms in the asymptotic expansion of ω^δ.

Now we will extend the above calculations to obtain the second term in the asymptotic expansion of ω^δ. Upon using Green's theorem in (9), we find that

$$\mathbf{E}^2_{\partial\Omega} = - \frac{\omega^1}{\omega^0}\, \mathbf{E}^1_{\partial\Omega} - \frac{z}{\omega^0\, \mu}\, \mathbf{curl}\, \mathbf{E}^1 \; \wedge \, \mathbf{n}.$$

The terms of order δ^2 in (6) yield

$$\mathbf{curl}\, \frac{1}{\mu}\, \mathbf{curl}\, \mathbf{E}^2 - (\omega^0)^2\, \varepsilon\, \mathbf{E}^2 \; = \; 2\, \omega^0\, \omega^1\, \varepsilon\, \mathbf{E}^1 + ((\omega^1)^2 \; + 2\omega^0\, \omega^2)\, \varepsilon\, \mathbf{E}^0. \tag{14}$$

Thus, if we multiply (14) by $\overline{\mathbf{E}^0}$, by integrating by parts, it follows that

$$\int_{\partial\Omega} \frac{1}{\mu} \mathbf{curl}\, \overline{\mathbf{E}^0} \; \wedge \, \mathbf{n} \,.\, \mathbf{E}^2_{\partial\Omega} = 2\omega^0\, \omega^1 \int_\Omega \varepsilon\, \mathbf{E}^1 \,.\, \overline{\mathbf{E}^0} + (\omega^1)^2 + 2\omega^0\, \omega^2.$$

This gives that the second correction ω^2 to the resonant frequency ω^δ is

$$\omega^2 = -\omega^1 \int_\Omega \varepsilon\, \mathbf{E}^1 \,.\, \overline{\mathbf{E}^0} - \frac{3(\omega^1)^2}{2\omega^0} - \frac{1}{2(\omega^0)^2} \int_{\partial\Omega} \frac{z}{\mu} (\mathbf{curl}\, \overline{\mathbf{E}^0} \wedge \mathbf{n}) \,.\, (\mathbf{curl}\, \mathbf{E}^1 \wedge \mathbf{n}), \tag{15}$$

while the second correction $\mathbf{E}^2$ to the eigenvector $\mathbf{E}^\delta$ satisfies

$$\left\{ \begin{aligned} & \mathbf{curl}\, \frac{1}{\mu} \mathbf{curl}\, \mathbf{E}^2 - (\omega^0)^2 \varepsilon\, \mathbf{E}^2 = 2\,\omega^0\, \omega^1\, \varepsilon\, \mathbf{E}^1 \\ & \qquad + ((\omega^1)^2 + 2\omega^0\, \omega^2)\, \varepsilon\, \mathbf{E}^0 \quad \text{in } \Omega, \\ & \mathbf{E}^2_{\partial\Omega} = -\frac{\omega^1}{\omega^0} \mathbf{E}^1_{\partial\Omega} + \frac{z}{\omega^0\, \mu} \mathbf{curl}\, \mathbf{E}^1 \wedge \mathbf{n} \quad \text{on } \partial\Omega, \\ & 2\, \Re e \int_\Omega \varepsilon\, \mathbf{E}^0 \,.\, \overline{\mathbf{E}^2} + \int_\Omega \varepsilon\, |\mathbf{E}^1|^2 = 0. \end{aligned} \right.$$

At this stage, we must verify that the second correction ω^2 given by (15) does not depend on the undetermined coefficient λ. From (11), we obtain that

$$\omega^1 \int_\Omega \varepsilon\, |\mathbf{E}^0|^2 + \frac{1}{2(\omega^0)^2} \int_{\partial\Omega} \frac{z}{\mu^2} (\mathbf{curl}\, \overline{\mathbf{E}^0} \wedge \mathbf{n}) \,.\, (\mathbf{curl}\, \mathbf{E}^0 \wedge \mathbf{n}) = 0,$$

and then we have immediately the claim.

We summarize our results in the following statement.

Proposition 2.1 *Assume that ω^δ is a resonant frequency of* (4) *that tends to a simple isolated eigenvalue ω^0 of* (1) *as $\delta \to 0$. Then, for $\delta << 1$,*

$$\begin{aligned} \omega^\delta = {} & \omega^0 - \frac{\delta}{2(\omega^0)^2} \int_{\partial\Omega} z\, |\frac{1}{\mu} \mathbf{curl}\, \mathbf{E}^0 \wedge \mathbf{n}|^2 \\ & + \delta^2 \Big[\frac{1}{2(\omega^0)^2} \int_{\partial\Omega} z\, |\frac{1}{\mu} \mathbf{curl}\, \mathbf{E}^0 \wedge \mathbf{n}|^2 \Big(\int_\Omega \varepsilon\, \mathbf{e}^1 \,.\, \overline{\mathbf{E}^0} \Big) \\ & - \frac{3}{2(\omega^0)^3} \int_{\partial\Omega} z\, |\frac{1}{\mu} \mathbf{curl}\, \mathbf{E}^0 \wedge \mathbf{n}|^2 - \frac{1}{2(\omega^0)^2} \int_{\partial\Omega} \frac{z}{\mu^2} (\mathbf{curl}\, \overline{\mathbf{E}^0} \wedge \mathbf{n}) \,.\, (\mathbf{curl}\, \mathbf{e}^1 \wedge \mathbf{n}) \Big] \\ & + 0(\delta^3). \end{aligned}$$

Here $\mathbf{E}^0$ is a normalized ($\int_{\partial\Omega} \varepsilon\, |\mathbf{E}^0|^2 = 1$) eigenvector of (1) *corresponding to ω^0 and $\mathbf{e}^1$ is the unique solution of the boundary value problem* (12).

To conclude this section we make the following remarks.

Remark 2.1 *If we expand* $\mathbf{H}^\delta$ *as* $\mathbf{H}^\delta = \mathbf{H}^0 + \delta\,\mathbf{H}^1 + \delta^2\,\mathbf{H}^2 + \ldots$, *then the terms* $\mathbf{H}^0$ *and* $\mathbf{H}^1$ *are given by*

$$\left\{\begin{array}{l} \mathbf{H}^0 = -\dfrac{i}{\omega^0\,\mu}\,\mathbf{curl}\,\mathbf{E}^0, \\ \\ \mathbf{H}^1 = -\dfrac{i}{\omega^0\,\mu}\,\mathbf{curl}\,\mathbf{E}^1 - \dfrac{\omega^1}{\omega^0}\,\mathbf{H}^0. \end{array}\right. \tag{16}$$

Remark 2.2 *If* $\omega^0 = 0$, *then* $\omega^\delta = 0$ *for* $\delta << 1$. *The Leontovich approximation does not perturb the space of harmonic Dirichlet vector fields in* Ω.

Remark 2.3 *If* $\omega^0 (\omega^0 \neq 0)$ *is not simple, then we obtain* M *distinct approximate resonant frequencies*

$$\omega_j^\delta = \omega^0 + \delta\,\omega_j^1 + 0(\delta^2),$$

where the first correction ω_j^1 *is given by*

$$\omega_j^1 = -\frac{1}{2(\omega^0)^2}\int_{\partial\Omega} z\,|\frac{1}{\mu}\,\mathbf{curl}\,\mathbf{E}_j^0\,\wedge\,\mathbf{n}|^2.$$

Here M *is the multiplicity of the eigenvalue* ω^0 *and* $(\mathbf{E}_j^0)_{1\leq j\leq M}$ *are normalized* $(\int_\Omega \varepsilon\,\mathbf{E}_j^0\,.\,\overline{\mathbf{E}_l^0} = \delta_{jl})$ *eigenvectors corresponding to* ω^0.

Remark 2.4 *The imaginary part of the first correction* ω^1 *is proportional to the electric current on* $\partial\Omega$ *created at the frequency* ω^0 *which is given by* $\mathbf{H}^0\,\wedge\,\mathbf{n}$. *It is also proportional to the imaginary part of the function* z.

Remark 2.5 *The numerical computations of* $\mathbf{e}^1$ *solution of the boundary value problem* (12) *can be done by extending the analysis of [5] which is based on mixed finite elements and a saddle point formulation for Maxwell's equations.*

Remark 2.6 *The above calculations can be extended without any new difficulty to obtain further terms in the asymptotic expansion of the resonant frequency* ω^δ.

3 Perturbation of the eigenvalues of the Maxwell operator in a domain with a small inclusion

We now perturb the eigenvalue problem (1) by removing from Ω a small subdomain Ω^1_δ of "radius" $0(\delta)$, centered at some point $\mathbf{x}_0$ in Ω. The domain Ω^1_δ is obtained from of a fixed domain Ω^1 by shrinking the distance from every point of its boundary $\partial\Omega^1$ to the point $\mathbf{x}_0$ by the factor δ. If we impose a Leontovich boundary condition on the resulting hole, the perturbed problem is

$$\left\{\begin{array}{l} \mathbf{curl}\,\mathbf{E}^\delta \;=\; i\omega^\delta\,\mu\,\mathbf{H}^\delta \quad \text{in } \Omega\setminus\overline{\Omega^1_\delta}, \\ \mathbf{curl}\;\mathbf{H}^\delta \;=\; -i\omega^\delta\,\varepsilon\,\mathbf{E}^\delta \quad \text{in } \Omega\setminus\overline{\Omega^1_\delta}, \\ \mathbf{E}^\delta\wedge\mathbf{n} = 0 \quad \text{on } \partial\Omega, \\ -\mathbf{E}^\delta_{\partial\Omega} + i\,\delta^\alpha\,z\;\mathbf{H}^\delta\wedge\mathbf{n} \;=\; 0 \quad \text{on } \partial\Omega^1_\delta, \\ \displaystyle\int_{\Omega\setminus\overline{\Omega^1_\delta}} \varepsilon\,|\mathbf{E}^\delta|^2 \;=\; 1, \end{array}\right. \tag{17}$$

where $\alpha > 1$ or $\alpha = 0$. On the assumption $\alpha > 1$, equations (17) model the propagation of electromagnetic fields in a cavity containing a highly conducting body Ω^1_δ. δ^α is the penetration depth of Ω^1_δ and it is assumed to be much smaller than the characteristic length δ. Physically, the assumption $\alpha = 0$ corresponds to a dielectric body Ω^1_δ having a high electric permittivity and a high magnetic permeability. These models are discussed in [8].

Our aim in this section is to calculate the effect as $\delta \to 0$ of Ω^1_δ on the eigenvalues of the cavity oscillator Ω in which it is placed. We also treat the case where Ω^1_δ is a perfect conductor. We first note that the boundary value problem (17) admits only the trivial solution if the frequency ω^δ is on the real axis. A mathematical formulation of this kind of boundary value problems is established in [1].

Following [9], we will construct the resonant frequencies and the eigenvectors for the perturbed problem by the method of matched asymptotic expansions for δ small enough. An inner expansion will describe the changes in the solution in a neighborhood of the small subdomain Ω^1_δ. An outer expansion, valid in the region away from the inclusion Ω^1_δ, will account for the relatively small effects that the subdomain Ω^1_δ produces there. These two expansions will be matched to determine the undertermined coefficients in both of them and in the expansion of the resonant frequency. We first deal with $\alpha > 1$.

To write the inner expansion, we introduce the local variable $\mathbf{y} = (\mathbf{x}-\mathbf{x}_0)/\delta$ and we expand $\mathbf{E}^\delta$ and $\mathbf{H}^\delta$ as

$$\begin{cases} \mathbf{E}^\delta(\mathbf{x}) = \mathbf{e}^0(\mathbf{y}) + \delta\, \mathbf{e}^1(\mathbf{y}) + \delta^\alpha\, \mathbf{e}^2(\mathbf{y}) + \dots, \\ \mathbf{H}^\delta(\mathbf{x}) = \mathbf{h}^0(\mathbf{y}) + \delta\, \mathbf{h}^1(\mathbf{y}) + \delta^\alpha\, \mathbf{h}^2(\mathbf{y}) + \dots, \end{cases} \tag{18}$$

for $|\mathbf{x}-\mathbf{x}_0| = 0(\delta)$. The expansion of ω^δ must begin with $\omega^0 \in \left\{\omega_n\right\}_{n\in\mathbf{Z}}$ and the outer expansions of $\mathbf{E}^\delta$ and $\mathbf{H}^\delta$ must begin with $\mathbf{E}^0$ and $\mathbf{H}^0$ the normalized eigenvectors corresponding to ω^0; so we write for $|\mathbf{x}-\mathbf{x}_0| >> 0(\delta)$

$$\begin{cases} \omega^\delta = \omega^0 + \nu_1(\delta)\,\omega^1 + \nu_2(\delta)\,\omega^2 + \dots, \\ \mathbf{E}^\delta = \mathbf{E}^0 + \nu_1(\delta)\mathbf{E}^1 + \nu_2(\delta)\,\mathbf{E}^2 + \dots, \\ \mathbf{H}^\delta = \mathbf{H}^0 + \nu_1(\delta)\,\mathbf{H}^1 + \nu_2(\delta)\,\mathbf{H}^2 + \dots, \end{cases} \tag{19}$$

where the functions $\nu_i(\delta)$, which satisfy $\nu_i(\delta) << 1$ and $\nu_i(\delta) >> \nu_{i+1}(\delta)$ as $\delta \to 0$, are to be found.
By using the outer asymptotic expansions (19), we obtain

$$\begin{cases} \mathbf{curl}\,\dfrac{1}{\mu}\,\mathbf{curl}\,\mathbf{E}^0 - (\omega^0)^2\,\varepsilon\,\mathbf{E}^0 \;=\; 0, \quad |\mathbf{x}-\mathbf{x}_0| >> 0(\delta), \\ \mathbf{E}^0_{\partial\Omega} = 0 \quad \text{on } \partial\Omega, \\ \Re e \displaystyle\int_\Omega \varepsilon\,|\mathbf{E}^0|^2 = 0, \end{cases}$$

$$\begin{cases} \mathbf{curl}\,\dfrac{1}{\mu}\,\mathbf{curl}\,\mathbf{E}^1 - (\omega^0)^2\,\varepsilon\,\mathbf{E}^1 \;=\; 2\,\omega^0\,\omega^1\,\varepsilon\,\mathbf{E}^0, \quad |\mathbf{x}-\mathbf{x}_0| >> 0(\delta), \\ \mathbf{E}^1_{\partial\Omega} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

In the inner region, by using the asymptotic expansions (18) in the place of (19), we obtain

$$\begin{cases} \mathbf{curl}\,\mathbf{e}^0 = 0, \quad \mathbf{y} \notin \Omega^1, \\ \mathbf{curl}\,\mathbf{h}^0 = 0, \quad \mathbf{y} \notin \Omega^1, \\ \mathbf{e}^0_{\partial\Omega^1} = 0, \quad \text{on } \partial\Omega^1, \end{cases} \qquad \begin{cases} \mathbf{curl}\,\mathbf{e}^1 = i\omega^0\,\mu(\mathbf{x}_0)\,\mathbf{h}^0, \quad \mathbf{y} \notin \Omega^1, \\ \mathbf{curl}\,\mathbf{h}^1 = -i\omega^0\,\varepsilon(\mathbf{x}_0)\,\mathbf{e}^0, \quad \mathbf{y} \notin \Omega^1, \\ \mathbf{e}^1_{\partial\Omega^1} = 0, \quad \text{on } \partial\Omega^1. \end{cases}$$

The inner and the outer expansions must asymptotically equal when $|\mathbf{y}| >> 1$ and $|\mathbf{x}-\mathbf{x}_0| << 1$. In this region the matching condition is

$$\mathbf{E}^0(\mathbf{x}) + \nu_1(\delta)\,\mathbf{E}^1(\mathbf{x}) + \nu_2(\delta)\,\mathbf{E}^2(\mathbf{x}) + \dots \sim \mathbf{e}^0(\mathbf{y}) + \delta\,\mathbf{e}^1(\mathbf{y}) + \delta^\alpha\,\mathbf{e}^2(\mathbf{y}) + \dots \tag{20}$$

From the terms of order δ^0 in (20), we obtain the first matching condition

$$\mathbf{e}^0(\mathbf{y}) \to \mathbf{E}^0(\mathbf{x}_0), \quad |\mathbf{y}| \to +\infty. \tag{21}$$

Since $\mathbf{e}^0$ satisfies $\operatorname{div} \mathbf{e}^0 = 0, \mathbf{curl}\, \mathbf{e}^0 = 0$ then $\mathbf{e}^0 = \mathbf{grad}\, \varphi^0$ where φ^0 is a solution of the Laplace equation for $|\mathbf{y}| \to +\infty$. Let φ_i^0 be defined as the unique solution of the Laplace equation:

$$\begin{cases} \Delta\, \varphi_i^0 = 0, \quad \text{in } \mathbf{R}^3 \setminus \overline{\Omega^1}, \\ \varphi_i^0 = -y_i, \quad \text{on } \partial\Omega^1, \\ \dfrac{1}{|\mathbf{y}|}\, \varphi_i^0 \in L^2(\mathbf{R}^3 \setminus \overline{\Omega^1}), \quad \mathbf{grad}\, \varphi_i^0 \in (L^2(\mathbf{R}^3 \setminus \overline{\Omega^1}))^2. \end{cases} \tag{22}$$

Then, from

$$|\mathbf{grad}\, \varphi_i^0| \to 0, \quad |\mathbf{y}| \to +\infty,$$

and (21), it follows that

$$\mathbf{e}^0(\mathbf{y}) = \sum E_i^0(\mathbf{x}_0)\, \mathbf{grad}\, \tilde{\varphi}_i^0(\mathbf{y}),$$

where $\tilde{\varphi}_i^0 = \varphi_i^0 + y_i$. To proceed further with the matching condition (20), we need more terms in the asymptotic behavior of $\mathbf{grad}\, \tilde{\varphi}_i^0$ for $|\mathbf{y}| \to +\infty$. From the behavior of the solution φ_i^0 of the Laplace equation (22) as $|\mathbf{y}| \to +\infty$, we deduce that there exist four constants $C, (C_j)_{j=1}^3$ such that

$$\mathbf{grad}\, \tilde{\varphi}_i^0 = \mathbf{e}_i - C \frac{1}{|\mathbf{y}|^3}\, \mathbf{y} + +C_j \frac{1}{|\mathbf{y}|^3}\, \mathbf{e}_j - 3C_j \frac{y_j}{|\mathbf{y}|^5}\, \mathbf{y} + \ldots, \quad |\mathbf{y}| \to +\infty.$$

Therefore, from (20), we obtain $\nu_1(\delta) = \delta^2$ and

$$\mathbf{E}^1(\mathbf{x}) \sim -C \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|^3} \sum E_i^0(\mathbf{x}_0), \quad \text{as } \mathbf{x} \to \mathbf{x}_0. \tag{23}$$

This condition determines the behavior of $\mathbf{E}^1(\mathbf{x})$ as $\mathbf{x}$ tends to $\mathbf{x}_0$.
To derive the first correction ω^1, we multiply

$$\mathbf{curl}\, \frac{1}{\mu}\, \mathbf{curl}\, \mathbf{E}^1 - (\omega^0)^2\, \varepsilon\, \mathbf{E}^1 \;=\; 2\, \omega^0\, \omega^1\, \varepsilon\, \mathbf{E}^0,$$

by $\overline{\mathbf{E}^0}$ and integrate the resulting equation over $\Omega \setminus \overline{B_\rho}$ where

$$B_\rho = \Big\{ \mathbf{x}, |\mathbf{x} - \mathbf{x}_0| < \rho \Big\}$$

is a small ball of radius ρ centered at $\mathbf{x}_0$. Upon using Green's theorem and the boundary condition $\mathbf{E}^1_{\partial\Omega} = 0$, we obtain

$$\int_{\partial B_\rho} \frac{1}{\mu}\Big[-\mathbf{curl}\,\mathbf{E}^1 \wedge \mathbf{n}\,.\,\overline{\mathbf{E}^0} + \mathbf{curl}\,\overline{\mathbf{E}^0} \wedge \mathbf{n}\,.\,\mathbf{E}^1\Big] = 2\,\omega^0\,\omega^1 \int_{\Omega\setminus\overline{B_\rho}} \varepsilon\,|\mathbf{E}^0|^2. \quad (24)$$

Now, using the behavior of $\mathbf{E}^1$ near $\mathbf{x}_0$, we can evaluate the limit of the left-hand side of (24) as ρ tends to zero. The integral on the right-hand side tends to $2\,\omega^0\,\omega^1$ because $\mathbf{E}^0$ is normalized. It follows from (23) that the first correction to the resonant frequency ω^δ vanishes. We must determine the unknown function $\nu_2(\delta)$. To do so, we first determine the next term in $\mathbf{e}^0$ for $|\mathbf{y}| \to +\infty$. We find that

$$\mathbf{e}^0(\mathbf{y}) \sim \mathbf{E}^0(\mathbf{x}_0) - C\frac{\sum E^0_i(\mathbf{x}_0)}{|\mathbf{y}|^3}\,\mathbf{y} + C_j\left(\frac{1}{|\mathbf{y}|^3}\,\mathbf{e}_j - 3\frac{y_j}{|\mathbf{y}|^5}\,\mathbf{y}\right)\sum E^0_i(\mathbf{x}_0), \quad |\mathbf{y}| \to +\infty.$$

Using the matching condition (20), we obtain $\nu_2(\delta) = \delta^3$, provided that we require that

$$|\mathbf{e}^1(\mathbf{y})| \le \frac{Constant}{|\mathbf{y}|}, \quad |\mathbf{y}| \to +\infty.$$

The next task is to consider the effect of a small dielectric body Ω^δ_1 having a high electric permittivity and a high magnetic permeability placed in the cavity oscillator Ω. On the assumption that $\alpha = 0$ in (17), we find that in the inner region

$$\begin{cases} \mathbf{curl}\,\mathbf{e}^0 = 0, \quad \mathbf{y} \notin \Omega^1, \\ \mathrm{div}\,\mathbf{e}^0 = 0, \quad \mathbf{y} \notin \Omega^1, \\ \mathbf{curl}\,\mathbf{h}^0 = 0, \quad \mathbf{y} \notin \Omega^1, \\ \mathrm{div}\,\mathbf{h}^0 = 0, \quad \mathbf{y} \notin \Omega^1, \\ -\mathbf{e}^0_{\partial\Omega^1} + iz\,\mathbf{h}^0 \wedge \mathbf{n} = 0, \quad \text{on } \partial\Omega^1. \end{cases}$$

This gives that $\mathbf{e}^0 = \mathbf{grad}\,\varphi^0$ where φ^0 is a solution of the Laplace equation for $|\mathbf{y}| \to +\infty$ and satisfies

$$\mathrm{div}_{\partial\Omega^1}\,\frac{1}{z}\,\mathbf{grad}_{\partial\Omega^1}\,\varphi^0 = 0, \quad \text{on } \partial\Omega^1,$$

where $\mathrm{div}_{\partial\Omega^1}$ and $\mathbf{grad}_{\partial\Omega^1}$ are the surface divergence and the tangential gradient on $\partial\Omega^1$, respectively. Under the assumption (3) on the function z,

we obtain $\varphi^0 = 0$ on $\partial\Omega^1$ and then we arrive at the same results. The same conclusion also holds for a small perfectly conducting body placed in the cavity oscillator Ω.
The conclusion from the above calculations is that the effect of a small highly conducting subdomain or a small dielectric body having high electric permittivity and high magnetic permeability or a small perfect conductor on the resonant frequencies of a cavity oscillator is of order three. We have the following result.

Proposition 3.1 *Assume that ω^δ is a resonant frequency of* (17) *that tends to a simple isolated eigenvalue ω^0 of* (1) *as $\delta \to 0$. Then, for $\delta << 1$,*

$$\omega^\delta = \omega^0 + 0(\delta^3).$$

4 Examples for spherical geometries

In this section, we consider some examples of perturbed eigenvalue problems in spherical geometries. For these simple geometries, our previous asymptotic results will be compared with exact analytical results. To construct explicit solutions, we assume below that Ω is a ball of radius 1 centered at the origin of a spherical coordinate system (r, θ, φ) and the functions z, ε, μ are constants.

4.1 Perturbation of the boundary condition

In this subsection, we analyze the perturbed eigenvalue problem (4). We first characterize the set $\left\{\omega_n\right\}_{n\in\mathbb{Z}}$ of eigenvalues of (1).
Let $(Y_l^m)_{l\geq 0, -l\leq m\leq l}$ be an orthonormal sequence of spherical harmonics of order l on the unit sphere S $(r = 1)$ normalized such that

$$\int_S Y_l^m \, Y_{l'}^{m'} = \delta_{l,l'}^{m,m'}.$$

An orthonormal basis of vector spherical harmonics is

$$\begin{cases} \mathbf{I}_l^m(\theta,\varphi) = \mathbf{grad}\, H_{l+1}^m(\mathbf{x})|_S, \quad l \geq 0, -l \leq m \leq l, \\ \mathbf{T}_l^m(\theta,\varphi) = \mathbf{grad}\, H_l^m(\mathbf{x}) \,\wedge\, \mathbf{x}|_S, \quad l \geq 1, -l \leq m \leq l, \\ \mathbf{N}_l^m(\theta,\varphi) = (2l-1)\, H_{l-1}^m(\mathbf{x})\, \mathbf{x}|_S - \mathbf{grad}\, H_{l-1}^m(\mathbf{x})|_S, \\ \qquad l \geq 1, -(l-1) \leq m \leq (l-1), \end{cases}$$

where
$$H_l^m(\mathbf{x}) = r^l\, Y_l^m(\frac{\mathbf{x}}{r}).$$
The transverse electric and magnetic mutipole solutions for the Maxwell equations
$$\begin{cases} \mathbf{curl}\,\mathbf{E} = i\omega\,\mu\,\mathbf{H}, \\ \mathbf{curl}\,\mathbf{H} = -i\omega\,\varepsilon\,\mathbf{E}, \end{cases}$$
are (see for instance [6])
$$\begin{cases} \mathbf{E}(\mathbf{x}) = j_l(\omega\,\sqrt{\varepsilon\mu}\,r)\,\mathbf{T}_l^m(\theta,\varphi), \\ \mathbf{H}(\mathbf{x}) = i\sqrt{\frac{\varepsilon}{\mu}}\,\frac{1}{2l+1}\Big[(l+1)\,j_{l-1}(\omega\,\sqrt{\varepsilon\mu}\,r)\,\mathbf{I}_{l-1}^m(\theta,\varphi) + l\,j_{l+1}(\omega\,\sqrt{\varepsilon\mu}\,r)\,\mathbf{N}_{l+1}^m(\theta,\varphi)\Big], \end{cases} \tag{25}$$
and
$$\begin{cases} \mathbf{E}(\mathbf{x}) = i\sqrt{\frac{\mu}{\varepsilon}}\,\frac{1}{2l+1}\Big[(l+1)\,j_{l-1}(\omega\,\sqrt{\varepsilon\mu}\,r)\,\mathbf{I}_{l-1}^m(\theta,\varphi) + l\,j_{l+1}(\omega\,\sqrt{\varepsilon\mu}\,r)\,\mathbf{N}_{l+1}^m(\theta,\varphi)\Big], \\ \mathbf{H}(\mathbf{x}) = j_l(\omega\,\sqrt{\varepsilon\mu}\,r)\,\mathbf{T}_l^m(\theta,\varphi). \end{cases} \tag{26}$$
where h_l denotes the spherical Hankel function of the first kind and order l and $j_l = \Re e\,(h_l)$ its real part. Taking the tangential components on S of the transverse electric mutipole solutions (25) and the transverse magnetic mutipole solutions (26) yields
$$\begin{aligned} \mathbf{H}\wedge\mathbf{n} &= \frac{i}{\omega\mu}\Big[\omega\,\sqrt{\varepsilon\mu}\,j_l'(\omega\,\sqrt{\varepsilon\mu}) + j_l(\omega\,\sqrt{\varepsilon\mu})\Big]\,\mathbf{curl}_S\,Y_l^m, \\ -\mathbf{E}_S &= -j_l(\omega\,\sqrt{\varepsilon\mu})\,\mathbf{curl}_S\,Y_l^m, \end{aligned} \tag{27}$$
and
$$\begin{aligned} \mathbf{E}\wedge\mathbf{n} &= \frac{i}{\omega\varepsilon}\Big[\omega\,\sqrt{\varepsilon\mu}\,j_l'(\omega\,\sqrt{\varepsilon\mu}) + j_l(\omega\,\sqrt{\varepsilon\mu})\Big]\,\mathbf{curl}_S\,Y_l^m, \\ -\mathbf{H}_S &= -j_l(\omega\,\sqrt{\varepsilon\mu})\,\mathbf{curl}_S\,Y_l^m, \end{aligned} \tag{28}$$
where $\mathbf{curl}_S$ is the tangential vector rotational on S. Since $\mathbf{E}\wedge\mathbf{n}$ is given by (28) if $\mathbf{E}$ is in the form (26) then it follows from the explicit expressions of the mutipole solutions (25) and (26) that the eigenvalues $\Big\{\omega_n\Big\}_{n\in\mathbf{Z}}$ of the homogeneous Maxwell operator in the unit ball ($r<1$) with a perfect conductor boundary condition on S are such that
$$j_l(\omega_n\,\sqrt{\varepsilon\mu}) = 0, \tag{29}$$

or

$$\omega_n \sqrt{\varepsilon\mu}\, j_l'(\omega_n \sqrt{\varepsilon\mu}) + j_l(\omega_n \sqrt{\varepsilon\mu}) = 0. \tag{30}$$

Now if we perturb this eigenvalue problem by replacing the perfect conductor boundary condition on S by the Leontovich boundary condition (2), from (27) and (28), we obtain that if ω^δ is a resonant frequency for the impedance boundary value problem for the Maxwell equations then it is such that

$$j_l(\omega^\delta \sqrt{\varepsilon\mu}) + \delta \frac{z}{\omega^\delta \mu}\Big[\omega^\delta \sqrt{\varepsilon\mu}\, j_l'(\omega^\delta \sqrt{\varepsilon\mu}) + j_l(\omega^\delta \sqrt{\varepsilon\mu})\Big] = 0, \tag{31}$$

or

$$\frac{1}{\omega^\delta\, \varepsilon}\Big[\omega^\delta \sqrt{\varepsilon\mu}\, j_l'(\omega^\delta \sqrt{\varepsilon\mu}) + j_l(\omega^\delta \sqrt{\varepsilon\mu})\Big] - \delta\, z\, j_l(\omega^\delta \sqrt{\varepsilon\mu}) = 0. \tag{32}$$

We seek for ω^δ satisfying (31) (resp. (32)) which tends to ω^0 satisfying (29) (resp. (30)). Expanding ω^δ as $\omega^\delta = \omega^0 + \delta\, \omega^1 + \ldots$, we find that the first correction to ω^δ is

$$\omega^1 = -\frac{z}{\mu}, \tag{33}$$

if ω^0 satisfies (29) and, by using the identity

$$\frac{d}{dk}(k\, j_l'(k) + j_l(k)) = (\frac{l(l+1)}{k} - k)\, j_l(k),$$

that it is given by

$$\omega^1 = -\frac{z}{\mu}\, \frac{1}{1 - \dfrac{l(l+1)}{\varepsilon\, \mu\, (\omega^0)^2}}, \tag{34}$$

if ω^0 satisfies (30) where l is such that

$$\omega^0 \sqrt{\varepsilon\mu}\, j_l'(\omega^0 \sqrt{\varepsilon\mu}) + j_l(\omega^0 \sqrt{\varepsilon\mu}) = 0. \tag{35}$$

It is easy to see that (11) yields

$$\omega^1 = -\frac{z}{\mu}\, \Big[\frac{1}{2(\omega^0)^2}\frac{1}{\varepsilon\mu}\frac{\displaystyle\int_S |\mathbf{curl}\, \mathbf{E} \wedge \mathbf{x}|^2}{\displaystyle\int_B |\mathbf{E}|^2}\Big],$$

where $\mathbf{E} = (\mathbf{x}) = j_l(\omega \sqrt{\varepsilon\mu}\, r)\, \mathbf{T}_l^m(\theta, \varphi)$. The following identities

$$\int_B |\mathbf{E}|^2 = 4\,\pi\, l(l+1)\, \int_0^1 |j_l(\omega \sqrt{\varepsilon\mu}\, r)|^2\, r^2\, dr,$$

and

$$\int_S |\mathbf{curl}\,\mathbf{E} \wedge \mathbf{x}|^2 = 4\,\pi\, l(l+1)\,\varepsilon\,\mu\,(\omega^0)^2\,|j_l'(\omega\,\sqrt{\varepsilon\mu})|^2,$$

together with

$$|j_l'(\omega\,\sqrt{\varepsilon\mu})|^2 = 2\int_0^1 |j_l(\omega\,\sqrt{\varepsilon\mu}\,r)|^2\,r^2\,dr \quad \text{if } j_l(\omega\,\sqrt{\varepsilon\mu}) = 0,$$

give that the formula (33) agrees well with (11). The same conclusion holds for (34).

4.2 Perturbation of the domain

We now perturb the eigenvalue problem by removing from the unit ball B a small ball B_δ, of radius δ, centered at the origin, and we impose a perfect conductor boundary condition on the resulting hole. The transverse electric and magnetic mutipole solutions corresponding to $B \setminus \overline{B_\delta}$ are

$$\begin{cases} \mathbf{E}(\mathbf{x}) = \Big[j_l(\omega\,\sqrt{\varepsilon\mu}\,r) \;-\; \dfrac{j_l(\omega\,\sqrt{\varepsilon\mu}\,\delta)}{h_l(\omega\,\sqrt{\varepsilon\mu}\,\delta)} h_l(\omega\,\sqrt{\varepsilon\mu}\,r)\Big]\,\mathbf{T}_l^m(\theta,\varphi), \\ \mathbf{H}(\mathbf{x}) = i\sqrt{\dfrac{\varepsilon}{\mu}}\,\dfrac{1}{2l+1}\Big[(l+1)\,j_{l-1}(\omega\,\sqrt{\varepsilon\mu}\,r)\,\mathbf{I}_{l-1}^m(\theta,\varphi) + l\,j_{l+1}(\omega\,\sqrt{\varepsilon\mu}\,r)\,\mathbf{N}_{l+1}^m(\theta,\varphi)\Big] \\ \quad -\dfrac{j_l(\omega\,\sqrt{\varepsilon\mu}\,\delta)}{h_l(\omega\,\sqrt{\varepsilon\mu}\,\delta)}\Big((l+1)\,h_{l-1}(\omega\,\sqrt{\varepsilon\mu}\,r)\,\mathbf{I}_{l-1}^m(\theta,\varphi) + l\,h_{l+1}(\omega\,\sqrt{\varepsilon\mu}\,r)\,\mathbf{N}_{l+1}^m(\theta,\varphi)\Big)\Big], \end{cases} \tag{36}$$

and

$$\begin{cases} \mathbf{E}(\mathbf{x}) = i\sqrt{\dfrac{\varepsilon}{\mu}}\,\dfrac{1}{2l+1}\Big[(l+1)\,j_{l-1}(\omega\,\sqrt{\varepsilon\mu}\,r)\,\mathbf{I}_{l-1}^m(\theta,\varphi) + l\,j_{l+1}(\omega\,\sqrt{\varepsilon\mu}\,r)\,\mathbf{N}_{l+1}^m(\theta,\varphi)\Big] \\ \quad -\dfrac{j_l(\omega\,\sqrt{\varepsilon\mu}\,\delta)}{h_l(\omega\,\sqrt{\varepsilon\mu}\,\delta)}\Big((l+1)\,h_{l-1}(\omega\,\sqrt{\varepsilon\mu}\,r)\,\mathbf{I}_{l-1}^m(\theta,\varphi) + l\,h_{l+1}(\omega\,\sqrt{\varepsilon\mu}\,r)\,\mathbf{N}_{l+1}^m(\theta,\varphi)\Big)\Big], \\ \mathbf{H}(\mathbf{x}) = \Big[j_l(\omega\,\sqrt{\varepsilon\mu}\,r) \;-\; \dfrac{j_l(\omega\,\sqrt{\varepsilon\mu}\,\delta)}{h_l(\omega\,\sqrt{\varepsilon\mu}\,\delta)}\, h_l(\omega\,\sqrt{\varepsilon\mu}\,r)\Big]\,\mathbf{T}_l^m(\theta,\varphi). \end{cases} \tag{37}$$

Imposing $\mathbf{E} \wedge \mathbf{n} = 0$ on S in (36) and (37), we find that the eigenvalues of the Maxwell operator in the cavity $B \setminus \overline{B_\delta}$ are determined by

$$j_l(\omega^\delta\,\sqrt{\varepsilon\mu}) \;-\; \frac{j_l(\omega^\delta\,\sqrt{\varepsilon\mu}\,\delta)}{h_l(\omega^\delta\,\sqrt{\varepsilon\mu}\,\delta)}\,h_l(\omega^\delta\,\sqrt{\varepsilon\mu}) = 0, \quad l \geq 1,$$

and

$$\omega^\delta \sqrt{\varepsilon\mu}\, j_l'(\omega^\delta \sqrt{\varepsilon\mu}) + j_l(\omega^\delta \sqrt{\varepsilon\mu})$$
$$- \frac{j_l(\omega^\delta \sqrt{\varepsilon\mu}\,\delta)}{h_l(\omega^\delta \sqrt{\varepsilon\mu}\,\delta)} \Big(\omega^\delta \sqrt{\varepsilon\mu}\, h_l'(\omega^\delta \sqrt{\varepsilon\mu}) + h_l(\omega^\delta \sqrt{\varepsilon\mu})\Big) = 0, \quad l \geq 1.$$

From

$$h_l(\omega^\delta \sqrt{\varepsilon\mu}\,\delta) \sim \frac{(2l)!}{l!2^l(\omega^0\sqrt{\varepsilon\mu})^{l+1}}\,\delta^{-(l+1)}, \quad \delta \to 0, \quad l \geq 1,$$

and

$$j_1(0) = 0,$$

we obtain

$$\frac{j_l(\omega^\delta \sqrt{\varepsilon\mu}\,\delta)}{h_l(\omega^\delta \sqrt{\varepsilon\mu}\,\delta)} = \begin{cases} 0(\delta^{l+1}), & \text{if } l \geq 2, \\ 0(\delta^3), & \text{if } l = 1. \end{cases}$$

The effect of removing the small ball B_δ is then of order three.

5 Conclusion

In this paper, we have studied two types of perturbations on the eigenvalue problem for the Maxwell operator in a cavity oscillator. The first kind occurs in calculating the change in resonant frequencies of a cavity due to the losses of its boundary. We have obtained for this problem the first corrections to the eigenvalues due to the losses of the metallic boundaries. The second arises in analyzing the effect of a small body on the resonant frequencies of a cavity in which it is placed. We have found that the effect is of order three. We claim that to measure the electrical properties of a sample of material, by the change it produces in the resonant frequencies of a cavity oscillator in which it is placed, the small body must not be "too small".

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