Uniform Stabilization of Beams by means of Pointwise Feedbacks

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Abstract

The paper considers the uniform stabilization of beams with damping acting at an internal point. We prove the exponentiel decay of the energy of solutions. We also prove that this result is optimal in the sens that we can not get uniform stabilization if we apply dissipation in one point which is at the same time nodal and trinodal.

1 Introduction

We consider in this paper the stabilization of a beam. The stabilization is obtained by a pointwise feedback. We will prove that to get the uniform stabilization of the beam we need only to have the strong stabilization which depends on the support of the pointwise feedback. Mainly, the uniform stabilization holds for any arbitrary internal point unless it is at the same time nodal and trinodal. This is a sufficient and necessary condition for both the uniform and strong stabilization.

The usual model for beams with pointwise dissipation leads to the following equations

$$\partial_t^2 u + \Delta^2 u + F(\partial_t u) = 0, \quad \Omega \times (0,\infty), \tag{1}$$

$$u = \partial_x u = 0, \quad \partial\Omega \times (0,\infty), \tag{2}$$

$$u(x,0) = u^0(x), \partial_t u(x,0) = u^1(x), \quad \Omega, \tag{3}$$

where Ω is the interval $]0,1[$ and F the feedback operator, given by [4]

$$F(\partial_t u)(x,t) = \partial_t u(\xi^+,t)\,\delta(x-\xi^+) + \partial_t u(\xi^+,t)\,\delta(x-\xi^-),$$

$\delta_{\xi^+}, \delta_{\xi^-}$ defined

$$\int_\Omega \phi(x)\, \delta(x-\xi^-)\, dx = \lim_{x \to \xi^-} \phi(x)$$

and

$$\int_\Omega \phi(x)\, \delta(x-\xi^+)\, dx = \lim_{x \to \xi^+} \phi(x)$$

for any ϕ in $C_0^\infty(\Omega \setminus \xi)$.

There is a large number of publications in the mathematical literature close to our present work. In particular, it was shown by Khapalov in [4] that for the one-dimensional wave equation (with two transmission conditions) the uniform stabilization holds by acting at an arbitrary internal point. However, if one considers only one of the transmission conditions the result does not remain true. In this direction, the works [8] and [11] by Lasiecka and Triggiani considered the wave equation in several dimensions. These authors proved that the uniform stabilization with exponential decay does not hold in some cases while the strong (asymptotic) stabilization is possible for a dense set of points.

Now we return to our initial problem (1)-(3). It is easily seen that it can be rewritten on each subdomain as follows

$$\partial_t^2 u + \Delta^2 u = 0, \quad (\Omega \setminus \xi) \times (0, \infty), \tag{4}$$

$$u = \partial_x u = 0, \quad \partial\Omega \times (0, \infty), \tag{5}$$

$$\partial_x^3 u(\xi^+, t) = -\partial_t u(\xi^+, t), (0, \infty), \tag{6}$$

$$\partial_x^3 u(\xi^-, t) = \partial_t u(\xi^-, t), (0, \infty), \tag{7}$$

$$[\partial_x u]_\xi = [\partial_x^2 u]_\xi = 0, \quad (0, \infty), \tag{8}$$

$$u(x, 0) = u^0(x), \partial_t u(x, 0) = u^1(x), \quad \Omega, \tag{9}$$

where $[G]_\xi = G(\xi^+) - G(\xi^-)$ and $\xi \in \Omega$.

In the 2D case, our initial problem can be formulated as follows

$$\partial_t^2 u + \Delta^2 u = 0, \quad (\Omega \setminus \gamma) \times (0, \infty), \tag{10}$$

$$u = \partial_\nu u = 0, \quad \partial\Omega \times (0, \infty), \tag{11}$$

$$\partial_\nu \Delta u = -\partial_t u,\ \gamma^+ \times (0, \infty), \tag{12}$$

$$\partial_\nu \Delta u = \partial_t u, \ \gamma^- \times (0,\infty), \tag{13}$$

$$[\partial_\nu u]_\gamma = [\Delta u]_\gamma = 0, \quad \gamma \times (0,\infty), \tag{14}$$

$$u(x,0) = u^0(x), \partial_t u(x,0) = u^1(x), \quad \Omega, \tag{15}$$

where γ^+ (γ^-) denotes $\gamma_{exterieur}$ ($\gamma_{interieur}$), Ω a bounded locally Lipschitz open set of $\mathbb{R}^2$, ν the outward normal to $\partial\Omega$ and γ a closed curve in the interior of Ω, satisfying the following assumptions: γ is an injective map on $]0,1[$; γ is of class $C^1([0,1],\Omega)$; γ does not across the boundary of Ω.

In what follows, we investigate the one dimensional problem $(1)-(3)$ or equivalently $(4)-(9)$.

2 Existence

Before stating our existence result, we should introduce some notations. Let $H_0^2(\Omega \setminus \xi)$ be the following Sobolev space

$$H_0^2(\Omega \setminus \xi) = \Big\{ u, u = \left\{ \begin{array}{l} u_1 \in H^2(0,\xi) \\ u_2 \in H^2(\xi,1) \end{array} \right., u = \partial_x u = 0 \text{ on } \partial\Omega \Big\},$$

This space is equipped with the norm given by

$$||u||_{H_0^2(\Omega\setminus\xi)} = \Big(\int_\Omega (\Delta u)^2 \, dx \Big)^{1/2},$$

which is equivalent to

$$||u|| = \Big(||u_1||^2_{H^2(0,\xi)} + ||u_2||^2_{H^2(\xi,1)} \Big)^{1/2}.$$

Let E be the energy space

$$E = H_0^2(\Omega \setminus \xi) \times L^2(\Omega)$$

equipped with the natural norm

$$||u(.,t)||_E = \Big(\int_\Omega \Big\{ (\Delta u)^2 + (\partial_t u)^2 \Big\} \, dx \Big)^{1/2}.$$

We state and prove the following result:

Theorem 2.1 *For any $\xi \in \Omega$ and (u^0, u^1) in E, the problem $(4)-(9)$ has a unique solution in* $\Big(C([0,\infty[, H_0^2(\Omega \setminus \xi)) \ \cap \ C^1([0,\infty[, L^2(\Omega))\Big)$.

Proof. The problem (4)-(9) can be formulated by the semigroup approach:

$$\partial_t U = GU,$$

where $U = (u, \partial_t u)$ in E and

$$G = \begin{pmatrix} 0 & I \\ -\Delta^2 & 0 \end{pmatrix}, \quad \text{for } x \text{ in } \Omega \setminus \xi,$$

with

$$D(G) = \Big\{ \ (v,w) \in (H_0^4(\Omega \setminus \xi) \ \cap \ H_0^2(\Omega \setminus \xi))^2, [\partial_x v]_\xi = [\partial_x^2 v]_\xi, v = \partial_x \ v = 0 \\ \text{on } \partial\Omega, \partial_x^3 v(\xi^+) = -w(\xi^+) \text{ and } \partial_x^3 v(\xi^-) = w(\xi^-)\Big\}.$$

It follows that to prove the existence and uniqueness of a solution of (4)-(9) in $C([0,\infty[, H_0^2(\Omega \setminus \xi)) \ \cap \ C^1([0,\infty[, L^2(\Omega))$, we have only to show that G is the infinitesimal generator of a C_0 semigroup of linear contractions in E. To do this, we show that G is maximal monotone.
Let $U = (v,w) \in D(G)$. We easily have

$$\Re e(GU, U) = -w^2(\xi^+) - w^2(\xi^-).$$

Thus, G is monotone. It remains to show that G is maximal:

$$\Im m(I - G) = E.$$

Let $F = (f,g) \in E$. We have to find $U \in D(G)$ such that

$$(I - G)U = F \text{ i.e. } v - w = f,$$

and

$$w + \Delta^2 v = g.$$

By eliminating w we arrive at

$$\begin{cases} v + \Delta^2 v = f + g, \quad x \neq \xi, \\ [\partial_x v]_\xi = [\partial_x^2 v]_\xi = 0, \\ \partial_x^3 v(\xi^-) = -f(\xi^-) + v(\xi^-), \\ \partial_x^3 v(\xi^+) = f(\xi^+) - v(\xi^+), \\ v(0) = v(1) = v'(0) = v'(1) = 0. \end{cases}$$

Now, we show that the conditions

$$[\partial_x v]_\xi = [\partial_x^2 v]_\xi = 0, \partial_x^3 v(\xi^-) = -f(\xi^-) + v(\xi^-), \partial_x^3 v(\xi^+) = f(\xi^+) - v(\xi^+),$$

uniquely determine the values of

$$\partial_x^2\, v(0), \partial_x^3\, v(0), \partial_x^2\, v(1) \text{ and } \partial_x^3\, v(1).$$

To see this we have to prove that the map which associates to the quantities $(\partial_x^2\, v(0), \partial_x^3\, v(0), \partial_t^2\, v(1), \partial_t^3\, v(1))$ the quantities

$$\Big([\partial_x\, v]_\xi, [\partial_x^2\, v]_\xi, \partial_x^3\, v(\xi^-) - v(\xi^-), \partial_x^3\, v(\xi^+) + v(\xi^+)\Big),$$

is a nonsingular transformation from $\mathbb{R}^4$ into $\mathbb{R}^4$. This is immediately deduced from the fact that it is an injective map. If

$$\begin{cases} v + \Delta^2\, v = 0, \quad x \neq \xi, \\ [\partial_x v]_\xi = [\partial_x^2 v]_\xi = 0, \\ \partial_x^3 v(\xi^-) = v(\xi^-), \\ \partial_x^3 v(\xi^+) = -v(\xi^+), \\ v(0) = v(1) = v'(0) = v'(1) = 0, \end{cases}$$

then $v = 0$. The function v is uniquely determined. The same result holds for $w = v - f$. From this we can deduce that G is maximal and the proof of Theorem 2.1 is complete.

Note that we can prove by similar arguments an existence result for the 2D case.

3 Asymptotic behavior of solutions

In this section we shall study the energy decay of solutions of (4)-(9) when t goes to $+\infty$. It is by now well-known that, by the invariance principle of La Salle the energy decay property is reduced to a unique continuation result. We state and prove the following lemma.

Lemma 3.1 *Assume that the only function satisfying the following problems:*

$$(I) \begin{cases} \partial_t^2\, w + \Delta^2\, w = 0, \ dans\ \Omega \times (0, \infty), \\ w(0) = w(1) = \partial_x\, w(0) = \partial_x\, w(1) = 0, \\ w'(\xi^+) = w'(\xi^-) = 0, \end{cases}$$

and

$$(II)\begin{cases} \partial_t^2\, w + \Delta^2\, w = 0, \;\; dans\ \Omega \times (0, \infty), \\ w(0) = w(1) = \partial_x\, w(0) = \partial_x\, w(1) = 0, \\ \partial_x^3 w(\xi^+, t) = \partial_x^3 w(\xi^-, t) = 0, \end{cases}$$

is the trivial solution. Then, for any (u^0, u^1) *in* E*, the solution of* $(4) - (9)$ *satisfies the energy estimate*

$$\lim_{t\to\infty} ||\, u(.,t)||_E = 0.$$

Proof. Let $T(t)$ be the semigroup of contractions generated by the operator G already introduced in section 2. Let $E(t,u)$ be defined by

$$E(t,u) = \frac{1}{2}\int_\Omega \Big\{ (\Delta\, u)^2 + (\partial_t\, u)^2 \Big\}\, dx.$$

It is easy to see that this quantity decreases for any solution u of (1)-(3). Thus, to prove that $T(t)U$ converges to zero where $U = (u^0, u^1)$, it is enough to prove this result only for $U = (u^0, u^1)$ in a dense subset of E. Since $D(G)$ is dense in E, it is then enough to prove that

$$\lim_{t\to\infty} T(t)U = 0, \text{for any } U \text{ in } D(G).$$

We have that $orb(u^0, u^1) = \cup_{t\geq 0} T(t)(u^0, u^1)$ is precompact in E for any U in $D(G)$. Indeed, the ω-limit set of $U = (u^0, u^1)$ defined by

$$w(u^0, u^1) = \Big\{ y \in E, t_n \to \infty, \quad n \to \infty \Big\},$$

is nonvoid for any $U = (u^0, u^1)$ in $D(G)$ (we refer to [2] for more details). On the other hand, by LaSalle's invariance principle ([2] page 18):

$$if\ (w^0, w^1) \in w(u^0, u^1)$$

then

$$T(t)(w^0, w^1) = (w(t), w'(t)) \in w(u^0, u^1)$$

and

$$||(w(t), w'(t))||_E = ||(w^0, w^1)||_E, \text{for any } t \geq 0.$$

Since

$$\begin{aligned} E'(t,u) &= -(w'(\xi^+, t))^2 - (w'(\xi^-, t))^2 \\ &= -(\partial_x^3 w(\xi^+, t))^2 - (\partial_x^3 w(\xi^-, t))^2 = 0, \end{aligned}$$

w satisfies the system (I) and (II). By hypothesis, w is the trivial solution. Then, $w(u^0, u^1) = (0, 0)$ and we obtain the claim.

We have the following strong stabilization result:

Theorem 3.1 *$\forall\, \xi \in \mathbb{R} \setminus (N \cap \tilde{N}), \forall\, (u^0, u^1) \in E$, we have*

$$\lim_{t \to \infty} ||u(t, .)||_E = 0, \tag{16}$$

where N is the set of nodal points:

$$N = \cup_{n \in \mathbb{N}} \Big\{ x \in \mathbb{R} \text{ such that } \phi_n(x) = 0 \Big\},$$

$\tilde{N}$ is the set of trinodal points:

$$\tilde{N} = \cup_{n \in \mathbb{N}} \Big\{ x \in \mathbb{R} \text{ such that } \partial_x^3 \phi_n(x) = 0 \Big\},$$

and $(\phi_n)_n$ are the eigenfunctions of the operator $(\Delta^2, H^4(\Omega) \cap H_0^2(\Omega))$.

Proof. We first note that since $(\phi_n)_n$ are analytical, the set of zeros of ϕ_n and the set of zeros of $\partial_x^3 \phi_n$ for any fixed n are finite. The sets N and $\tilde{N}$ are countable. This implies that the Lebesgue mesure of $N \cap \tilde{N}$ is nul and then, we have strong stabilization in $\Omega, p.p.$. We refer to the Appendix for more details on the set $\mathbb{R} \setminus (N \cap \tilde{N})$).

Now to prove the claim it is enough to prove the following unique continuation result: if w satisfies the following problems

$$(I) \begin{cases} \partial_t^2\, w + \Delta^2\, w = 0, \; dans\ \Omega \times (0, \infty), \\ w(0) = w(1) = \partial_x\, w(0) = \partial_x\, w(1) = 0, \\ w'(\xi^+) = w'(\xi^-) = 0, \end{cases}$$

and

$$(II) \begin{cases} \partial_t^2\, w + \Delta^2\, w = 0, \; dans\ \Omega \times (0, \infty), \\ w(0) = w(1) = \partial_x\, w(0) = \partial_x\, w(1) = 0, \\ \partial_x^3 w(\xi^+, t) = \partial_x^3 w(\xi^-, t) = 0, \end{cases}$$

then $w = 0$. If we consider (I) and set $\tilde{w} = w'$ then the above problem becomes

$$\partial_t^2\, \tilde{w} + \Delta^2\, \tilde{w} = 0, \; in\ \Omega \times (0, \infty),$$

$$\tilde{w}(0) = \tilde{w}(1) = \partial_x \tilde{w}(0) = \partial_x \tilde{w}(1) = 0,$$

$$\tilde{w}(\xi^+) = \tilde{w}(\xi^-) = 0.$$

Let (ϕ_n) be a basis of eigenfunctions of Δ^2 in $H_0^2(\Omega)$. For

$$\tilde{w}(x,0) = \Sigma_{n\geq 1} a_n \phi_n(x), \tilde{w}'(x,0) = \Sigma_{n\geq 1} b_n \phi_n(x)$$

we have:

$$\tilde{w}(\xi,t) = \Sigma_{n\geq 1}\Big\{ a_n cos(\lambda_n^{1/2}t) + b_n \lambda_n^{-1/2} sin(\lambda_n^{1/2}t)\Big\}\, \phi_n(\xi),$$

and

$$\partial_x^3\, w(\xi,t) = \Sigma_{n\geq 1}\Big\{ a_n cos(\lambda_n^{1/2}t) + b_n \lambda_n^{-1/2} sin(\lambda_n^{1/2}t)\Big\}\partial_x^3\, \phi_n(\xi).$$

Since

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T e^{\lambda_n^{1/2}t}\, e^{\lambda_m^{1/2}t}\, dt = 0, \,\forall\, m\, \neq\, n$$

we obtain from [3] that

$$\tilde{w}(\xi,t) = 0 \text{ if and only if } a_n \phi_n(\xi) = b_n \phi_n(\xi) = 0,$$

and

$$\partial_x^3\, w(\xi,t) = 0 \text{ if and only if } a_n \partial_x^3\, \phi_n(\xi) = b_n \partial_x^3\, \phi_n(\xi) = 0.$$

Now recalling that $\xi \in \mathbb{R} \setminus (N \cap \tilde{N})$ we obtain that $w = 0$. Lemma 3.1 gives the claim.

Remark 3.1 *We can give a different proof of Theorem 3.2. It is enough from [7] to verify that G is maximal-dissipative, its eigenvalues are purely imaginary and its resolvant is compact. We have already proved the first statement. Let (λ, U) such that $GU = i\lambda U$ where $U = (v,w)$. v is then given by*

$$v = \begin{cases} a\, cos\lambda^{1/2}x - b\, sin\lambda^{1/2}x - a\, ch\lambda^{1/2}x + b\, sh\lambda^{1/2}x, \; 0 \leq x \leq \xi \\ c\, cos\lambda^{1/2}(x-1) - d\, sin\lambda^{1/2}(x-1) - c\, ch\lambda^{1/2}(x-1) \\ \quad +d\, sh\lambda^{1/2}(x-1), \; \xi \leq x \leq 1, \end{cases}$$

where a, b, c and d are such that

$$[\partial_x v]_\xi = [\partial_x^2 v]_\xi = 0$$

and the transmission conditions holds

$$\partial_x^3 v(\xi^+) = -i\lambda v(\xi^+)\;,\; \partial_x^3 v(\xi^-) = i\lambda v(\xi^-).$$

Assume that the eigenvalues of G are purely imaginary implies that λ is real. Then

$$v(\xi^+) = v(\xi^-) = 0.$$

Thus, $v \in H^4(\Omega) \cap H_0^2(\Omega)$ satisfies

$$\Delta^2 v = \lambda^2 v,$$

and so, v an eigenfunction of the operator $(\Delta^2, H^4(\Omega) \cap H_0^2(\Omega))$, and

$$v(\xi) = \partial_x^3 v(\xi) = 0.$$

This is in contradiction with the fact that $\xi \in \mathbb{R} \setminus (N \cap \tilde{N})$.

Finally, it is easy to verify that the resolvant of G is compact.

Now we state the strong stabilization result. We shall prove the following theorem:

Theorem 3.2 *For any $\xi \in \mathbb{R} \setminus (N \cap \tilde{N})$, there exists $C, \gamma > 0$ such that for any solution u of $(4) - (9)$ in $C([0, \infty[, H_0^2(\Omega \setminus \xi)) \cap C^1([0, \infty[, L^2(\Omega))$, we have:*

$$||u(.,t)||_E \leq C\ e^{-\gamma t} ||u(.,0)||_E,\ t > 0.$$

To prove this thoerem, it is enough to prove the following lemma (see [5] and [6]).

Lemma 3.2 *For any $M > 0$, there exist C_0 and $T > 0$ depending on M such that if*

$$E(0, u) \leq M,$$

then

$$\int_0^T E(t,u)\ dt \leq C_0 \int_0^T \Big\{ (\partial_t\, u(\xi^+, t))^2 + (\partial_t\, u(\xi^-, t))^2 \Big\}\ dt. \qquad (17)$$

Proof. For the proof we take $u_0 \in H^4(\Omega \setminus \xi) \cap H_0^2(\Omega \setminus \xi)$ and $u^1 \in H_0^2(\Omega \setminus \xi)$ where

$$H^4(\Omega \setminus \xi) = \Big\{ u \text{ such that } u_{|(0,\xi)} \in H^4(0,\xi),\ u_{|(\xi,1)} \in H^4(\xi, 1) \Big\},$$

equipped with the norm $H^4(\Omega)$ and then we conclude by a density argument. The proof is divided into three steps.

First step. Multipliyng (4) by $x\,\partial_x u$ and integrating over $Q = \Omega \times (0,T)$ (see [9] and [12]), we obtain

$$\Big[\int_\Omega x\,\partial_t u\,\partial_x u\;dx\Big]_0^T + \frac{1}{2}\int_Q \Big\{(\partial_t\,u)^2 + 3(\partial_x^2\,u)^2\Big\}\;dx\,dt - \frac{1}{2}\int_0^T \xi\,(\partial_t\,u(\xi^-,t))^2\,dt$$
$$+\frac{1}{2}\int_0^T \xi\,(\partial_t\,u(\xi^+,t))^2\,dt + \int_0^T \xi\,\partial_x u(\xi,t)\Big\{\partial_x^3 u(\xi^-,t) - \partial_x^3 u(\xi^+,t)\Big\}\,dt$$
$$= \frac{1}{2}\int_0^T (\partial_x^2\,u(1,t))^2\;dt. \tag{18}$$

If we mulptiply (4) by $u(x,t)$ and we integrate by parts over Q, we arrive at

$$\Big[\int_\Omega u\,\partial_t\,u\;dx\Big]_0^T - \frac{1}{2}\Big[\Big\{(u(\xi^+,t))^2 + (u(\xi^-,t))^2\Big\}\Big]_0^T = \int_Q \Big\{(\partial_t\,u)^2 - (\partial_x^2\,u)^2\Big\}\;dx\;dt. \tag{19}$$

Combining (18) and (19) gives

$$\Big[\int_\Omega \Big\{x\,\partial_t u\,\partial_x u + \delta u \partial_t\,u\Big\}\;dx\Big]_0^T - \frac{1}{2}\Big\{(u(\xi^+,t))^2 + (u(\xi^-,t))^2\Big\}\Big]_0^T$$
$$+(\frac{1}{2} - \delta)\int_Q (\partial_t\,u)^2\;dx\,dt + (\frac{3}{2} + \delta)\int_Q (\partial_x^2\,u)^2\;dx\,dt + \frac{1}{2}\int_0^T \xi\Big\{(\partial_t u(\xi^+,t))^2 -$$
$$(\partial_t u(\xi^-,t))^2\Big\}\,dt\; + \int_0^T \xi \partial_x u(\xi,t)\Big\{\partial_x^3\,u(\xi^-,t) - \partial_x^3\,u(\xi^+,t)\Big\}\;dt = \frac{1}{2}\int_0^T (\partial_x^2\,u(1,t))^2\;dt, \tag{20}$$

where $\delta \in (0, 1/2)$. Thus,

$$\int_0^T E(t,u)\;dt \le C\Big\{\int_0^T (\partial_x^2\,u(1,t))^2\;dt + \int_0^T \Big\{(\partial_t\,u(\xi+,t))^2 + (\partial_t\,u(\xi-,t))^2\Big\}\,dt + \mathcal{X}\Big\},$$

where

$$\mathcal{X} = |\Big[\int_\Omega \Big\{x\,\partial_t u\,\partial_x u + \delta u\,\partial_t\,u +\Big\}\;dx - \delta\Big\{(u(\xi^+,t))^2 + (u(\xi^-,t))^2\Big\}\Big]_0^T|. \tag{21}$$

The second step is devoted to estimate the quantity

$$\int_0^T (\partial_x^2\,u(1,t))^2\;dt.$$

Second step. Let $h \in W^{2,\infty}(\Omega)$ such that $h(1) = 1$, $h(0) = 0$ and $h(\xi) = 0$. If we muptiply (4) by $h\partial_x u$ and we integrate by parts over Q, we obtain

$$\frac{h(1)}{2}\int_0^T (\partial_x^2 u(1,t))^2\ dt = \frac{1}{2}\int_Q h'(\partial_t u)^2\ dx\ dt + \frac{1}{2}\int_Q h'(\partial_x^2 u)^2\ dx\ dt +$$
$$\int_Q h''(\partial_x^2 u)^2\ dx\ dt + \Big[\int_\Omega h\partial_t u \partial_x u\ dx\Big]_0^T - \frac{1}{2}h(\xi)\int_0^T \Big\{(\partial_t u(\xi^-,t))^2 - (\partial_t u(\xi^+,t))^2\Big\}\ dt$$
$$+ \int_0^T h(\xi)(\partial_x^3 u(\xi^-,t) - \partial_x^3 u(\xi^+,t))\partial_x u(\xi,t)\ dt. \tag{22}$$

From the assumptions on h , we conclude that

$$\int_0^T (\partial_x^2 u(1,t))^2\ dt \ \ \leq\ \ C\Big\{\int_Q \Big((\partial_t u)^2 + (\partial_x^2 u)^2\Big)\ dx\ dt + |\Big[\int_\Omega h\, \partial_t\, u\, \partial_x\, u\ dx\Big]_0^T|\Big\}. \tag{23}$$

Let $\eta \in W^{2,\infty}(\Omega)$ be such that

$$\begin{cases} 0 < \varepsilon \leq \eta(x) << 1, & \forall\, x \in \Omega, \\ \eta'(\xi) = 0. \end{cases}$$

Multiplying (4) by $\eta(x)u(x,t)$ gives

$$\int_Q \eta\, (\partial_x^2 u)^2\ dx\, dt\ -\ \int_Q \eta\, (\partial_t u)^2\ dx\, dt\ +\ 2\int_Q \eta'\, \partial_x u\, \partial_x^2 u\ dx\, dt\ +$$
$$\int_Q \eta''\, u\, \partial_x^2 u\ dx\, dt + \Big[\int_\Omega \eta\, u\, \partial_t u\ dx\Big]_0^T + \int_0^T \eta(\xi)\Big\{u(\xi^-,t)\, \partial_x^3 u(\xi^-,t)\ -$$
$$u(\xi^+,t)\, \partial_x^3 u(\xi^+,t)\Big\}\ dt\ -\ \int_0^T \eta'(\xi)\ \Big\{u(\xi^-,t) - u(\xi^+,t)\Big\}\, \partial_x^2 u(\xi,t)\ dt\ = 0. \tag{24}$$

From (6) and (9), we deduce by using the conditions satisfied by η that

$$\int_Q \eta\, (\partial_x^2\, u)^2\ dx\ dt \leq C\Big\{\int_Q \eta\, (\partial_t\, u)^2\ dx\ dt + \int_Q (\partial_x\, u)^2\ dx\ dt + \mathcal{Y}\Big\} \tag{25}$$

where

$$\mathcal{Y} = |\Big[\int_\Omega \eta\, u\, \partial_t\, u\, dx\, dt + \frac{1}{2}\eta(\xi)\Big\{(u(\xi^-,t))^2 + (u(\xi^+,t))^2\Big\}\Big]_0^T|. \tag{26}$$

By the same manner, we can obtain that

$$\int_Q \eta(\partial_t\, u)^2\ dx\ dt\ \leq\ C\Big\{\int_Q \eta\, (\partial_x^2\, u)^2\ dx\ dt + \int_Q (\partial_x\, u)^2\ dx\ dt + \mathcal{Y}\Big\}. \tag{27}$$

This implies by combining (25) , (27) together with the fact that $0 < \eta << 1$

$$\int_Q (\partial_t u)^2 \, dx \, dt \, + \int_Q (\partial_x^2 u)^2 \, dx \, dt \leq \; C \Big\{ \int_Q (\partial_x u)^2 \, dx \, dt + \mathcal{Y} \Big\}. \qquad (28)$$

Thus,

$$\int_0^T (\partial_x^2 u)^2(1,t) \, dt \leq \; C \Big\{ \int_Q (\partial_x u)^2 \, dx \, dt + \big| \Big[\int_\Omega h \, \partial_t u \, \partial_x u \, dx \Big]_0^T \big| + \mathcal{Y} \Big\}, \; (29)$$

and

$$\begin{aligned} T \, E(T) \leq \int_0^T E(t) \, dt \;\; &\leq \;\; C \Big\{ \int_0^T \Big\{ (\partial_t u(\xi^-,t))^2 + (\partial_t u(\xi^+,t))^2 \; \Big\} dt \\ &+ \int_Q (\partial_x u)^2 \, dx \, dt + \Big[\int_\Omega h \, \partial_t u \, \partial_x u \, dx \Big]_0^T \\ &+ \mathcal{Y} + \mathcal{X} \Big\}. \end{aligned} \qquad (30)$$

But

$$\big| \Big[\int_\Omega h \, \partial_t u \, \partial_x u \, dx \Big]_0^T \big| + \mathcal{Y} + \mathcal{X} \leq C(E(0)+E(T)) = C \Big\{ 2E(t) + (\partial_t u(\xi^-,t))^2 + (\partial_t u(\xi^+,t))^2 \Big\}.$$

Then, for T large enough

$$E(T) \;\; \leq \;\; C \Big\{ \int_0^T \Big\{ (\partial_t u(\xi^-,t))^2 + (\partial_t u(\xi^+,t))^2 \; \Big\} dt + \int_Q (\partial_x u)^2 \, dx \, dt \Big\}. \qquad (31)$$

To get the inequality (17), it is enough to prove that there exists a constant C such that

$$\int_Q (\partial_x u)^2 \, dx \, dt \leq C \int_0^T \Big\{ \partial_t u(\xi^-)^2 + (\partial_t u(\xi^+)^2 \Big\} \, dt. \qquad (32)$$

This is the aim of the final step.

Third step. To prove (32), we assume that there exists a sequence (u_n) of solutions to $(1) - (3)$ satisfying $u_n(x,0) = u_n^0, \partial_t u_n(x,0) = u_n^1$ such that

$$\lim_{n \to \infty} \frac{||u_n||_{L^2([0,T],H_0^1(\Omega \setminus \xi))}}{\displaystyle\int_0^T (\partial_t u_n(\xi^+,t)^2 + (\partial_t u_n(\xi^-,t)^2 \, dt} = +\infty, \qquad (33)$$

where

$$H_0^1(\Omega\backslash\xi) = \Big\{ u \text{ such that } u_{|(0,\xi)} \in H^1(0,\xi),\ u_{|(\xi,1)} \in H^1(\xi,1) \text{ and } u = 0 \text{ on } \partial\Omega \Big\}.$$

If we set

$$\delta_n = ||u_n||_{L^2([0,T],H_0^1(\Omega\backslash\xi))}, \tag{34}$$

and

$$v_n(x,t) = \frac{u_n(x,t)}{\delta_n}, \tag{35}$$

then, v_n satisfies

$$\begin{aligned} &\partial_t^2 v_n + \Delta^2 v_n + F(\partial_t v_n) = 0 \quad \text{in } Q, \\ &v_n = \partial_x v_n = 0 \quad \text{on } \partial\Omega \times (0,T). \end{aligned} \tag{36}$$

We obtain from (35) and (33) that

$$||v_n||_{L^2([0,T],H_0^1(\Omega\backslash\xi))} = 1,$$

and

$$\int_0^T \Big((\partial_t v_n(\xi^-,t))^2 + (\partial_t v_n(\xi^+,t))^2 \Big)\, dt \to 0, \quad n \to +\infty.$$

Thus, from the identity

$$E'(t,v_n) = -(\partial_t v_n(\xi^+,t))^2 - (\partial_t v_n(\xi^-,t))^2$$

and (31) and (34)-(36), we deduce that the sequence (v_n) is bounded in $\Big(L^\infty([0,T],H^2(\Omega\setminus\xi)) \cap W^{1,\infty}([0,T],L^2(\Omega))\Big)$ and δ_n is bounded in $\mathbb{R}$. It follows that we can extract a subsequence still denoted by (v_n,δ_n) such that

$$\begin{cases} v_n \to v, \quad L^\infty([0,T],H^2(\Omega\setminus\xi)) \text{ weakly} *, \\ v_n' \to v', \quad L^\infty([0,T],L^2(\Omega)) \text{ weakly} *, \\ \delta_n \to \delta \geq 0, \quad \text{in } \mathbb{R}. \end{cases}$$

The compactness result in [10] gives

$$v_n \to v, \quad L^\infty([0,T],H_0^1(\Omega\setminus\xi)) \text{ strongly}.$$

In particular,

$$\lim_{n\to\infty} ||v_n||_{L^2([0,T],H_0^1(\Omega\backslash\xi))} = ||v||_{L^2([0,T],H_0^1(\Omega\backslash\xi))} = 1.$$

Passing to the limit in (36), we see that v satisfies

$$\begin{cases} \partial_t^2 v + \Delta^2 v = 0 \quad \text{in } Q, \\ v = \partial_x v = 0 \quad \text{on } \partial\Omega \times (0,T), \\ \partial_t v(\xi, t) = 0 \quad \text{on } [0,T]. \end{cases}$$

The proof of Theorem 3.1 insures that $v = 0$ in Q. This is in contradiction with the fact that $||v||_{L^2([0,T],H_0^1(\Omega\setminus\xi))} = 1$ and concludes the proof of Lemma 3.2.

Now, for the proof of Theorem 3.2, it is enough to write

$$\begin{aligned} E(T) &\le c_0 \int_0^T \Big((\partial_t u(\xi^-, t))^2 + (\partial_t u(\xi^+, t))^2 \Big)\, dt, \\ &\le c_0 \Big(E(0) - E(T) \Big), \\ &\le \frac{c_0}{1 + c_0} E(0). \end{aligned}$$

Combining this estimate together with the properties of the semigroup gives the desired inequality where

$$\begin{cases} \gamma = \dfrac{1}{T} \ln(1 + \dfrac{1}{c_0}), \\ C = 1 + \dfrac{1}{c_0}. \end{cases}$$

The proof of Theorem 3.2 is now complete.

Remark 3.2 *We note that in order to get the strong stabilization result for* $(10) - (15)$ *the curve* γ *should not be at the same time nodale and trinodal. This means that we have* $\phi_n|_\gamma \neq 0$ *or* $\partial_x^3 \phi_n|_\gamma \neq 0$ *for any* $n \in \mathbb{N}$. *Indeed, we have to assume that the eigenvalues of the operator* $(\Delta^2, H^4(\Omega) \cap H_0^2(\Omega))$ *are simple. However, it is well-known that genericaly this assumption holds true [1].*
To get the uniform stabilization result for the same problem, it should be enough to proceed as in [12] but with a more precise choice of the functions η *and* h *in the proof of Lemma* 3.2.

Remark 3.3 *If we study the following control problem by the HUM method [9]:*

$$\begin{cases} \partial_t^2 u + \Delta^2 u = 0, \quad (\Omega \setminus \xi) \times (0,T), \\ u = \partial_x u = 0, \quad \partial\Omega \times (0,T), \\ [\partial_x^3 u]_\xi = v_1(t), \ (0,T), \\ [u]_\xi = v_2(t), \ (0,T) \\ [\partial_x u]_\xi = [\partial_x^2 u]_\xi = 0, \quad (0,T), \\ u(x,0) = u^0(x), \partial_t u(x,0) = u^1(x), \quad \Omega, \end{cases}$$

where $v_i, i = 1,2$ are the controls, we know that the crucial point to get the exact contrability is the following uniqueness result: if ϕ satisfies the system:

$$\begin{cases} \partial_t^2 \phi + \Delta^2 \phi = 0, \quad \Omega \times (0,T), \\ \phi = \partial_x \phi = 0, \quad \partial\Omega \times (0,T), \\ \phi(x,0) = \phi^0(x), \partial_t \phi(x,0) = \phi^1(x), \quad \Omega, \end{cases}$$

and the boundary conditions: $\phi(\xi,t) = 0$ and $\partial_x^3\phi(\xi,t) = 0$ in $(0,T)$ then $\phi = 0$ on $\Omega \times (0,T)$. Since the uniform stabilization of such system implies the exact contrability, we easily see that $\xi \in \mathbb{R} \setminus (N \cap \tilde{N})$ is a necessary and sufficient condition for the exact control.

Appendix

This appendix is devoted to prove the following Lemma.

Lemma 3.3 *The set $\mathbb{R} \setminus (N \cap \tilde{N})$ is residual.*

We first recall the definition of a residual set and then we establish a technical result useful for our purpose.

Definition 3.1 *If X is a topological space, A a subset of X then, A is residual in X if A is a countable intersection of open dense sets in X.*
In particular, such set is dense in X.

Lemma 3.4 *Let $(\phi_i)_i$ be the eigenfunctions of the operator $(\Delta^2, H_0^2(\Omega) \cap H^4(\Omega))$. Then the set of $x \in \Omega$ such that*

$$\frac{d}{dx}\phi_i(x) \neq 0, \ for\ any\ i \geq 1$$

is dense in Ω.

Proof. Since ϕ_i is of class $C^1(\Omega)$ then the set of $x \in \Omega$ such that

$$\frac{d}{dx}\phi_i(x) \neq 0,$$

is an open set in Ω for any $i \geq 1$. On the other hand, if

$$\frac{d}{dx}\phi_i(x) = 0,$$

in a connected open set in Ω then since ϕ_i is an eigenfunction of the operator Δ^2 and since ϕ_i is analytic in Ω then necessary $\phi_i = 0$ in Ω. This is in contradiction with the fact that ϕ_i is an eigenfunction. The interior set of $x \in \Omega$ such that $\frac{d}{dx}\phi_i(x) = 0$, is is void for any $i \geq 1$. Baire's Lemma gives the claim.

Now we are in position to prove that the set $\mathbb{R} \setminus (N \cap \tilde{N})$ is residual.
Proof. From the above definition, it is enough to prove that:
(1) A_n,B_n are open sets of Ω,
(2) A_{n+1} is dense in A_n and B_{n+1} is dense in B_n, for any n,
where

$$A_n = \Big\{\xi \in \Omega \; such \; that \; \phi_i(\xi) \neq 0, \; i = 1, 2.., n\Big\},$$

and

$$B_n = \Big\{\xi \in \Omega \; suchthat \; \partial_x^3\phi_i(\xi) \neq 0, \; i = 1, 2.., n\Big\}.$$

Let $\xi \in A_n/A_{n+1}$, from the above lemma, for $\varepsilon > 0$, there exists $\xi' \in A_n$ such that: $|\xi - \xi'| < \varepsilon/2$ and $\frac{d}{dx}\phi_{n+1}(x) > 0$ for any $x \in]\xi', \xi' + \eta[$ where $0 < \eta < \varepsilon/2$. Thus, if $\phi_{n+1}(\xi') \neq 0$, then we have nothing to prove, else $\phi_{n+1}(\xi') = 0$. If we write:

$$\phi_{n+1}(\xi' + \eta) = \int_{\xi'}^{\xi' + \eta} \frac{d}{dx}\phi_{n+1}(x)\, dx$$

we conclude that $\xi' + \eta \in A_{n+1}$ and $|\xi' + \eta - \xi| < \varepsilon$. This implies that A_{n+1} is dense in A_n. To conclude the proof, we note that A_n is an open set of Ω. By the same manner we prove that the sets B_n are open sets of Ω and B_{n+1} are dense in B_n. The proof is then complete.

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