CHIRALITY IN THE MAXWELL EQUATIONS BY THE DIPOLE APPROXIMATION. PART I

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Abstract. We show how chiral materials could be realized by embedding chiral objects (helices) in an isotropic medium.

Key words. Chirality, Maxwell's equations, Drude-Born-Fedorov equations, dipole approximation.

AMS subject classifications. 35Q60, 78A45.

Introduction. Chiral media are examples of media responding with both magnetic and electric polarization to electric or magnetic excitation. They belong to the general class of bianisotropic media. Chiral media can be characterized by a set of constitutive relations in which the electric and magnetic fields are coupled. The coupling strength is given by the magnitude of a quantity known as the chirality admittance. It determines the bulk electromagnetic properties of these materials. Chiral materials are used in many optics applications. Antennas coated with chiral materials have significantly interesting radiation characteristics due to the extra degree of freedom offered by the presence of the chirality admittance. Different expressions exist in the literature for the constitutive relations in chiral media. They are in fact equivalent to *the Drude-Born-Fedorov equations*, after appropriate redefinitions of the field vectors and the material parameters. For interesting explanations of these equations and various physical aspects of the propagation inside chiral media, we refer to [24], [25], [26] and the list of references therein which is representative of the work that is presently being conducted in this area. The original investigations of the effect of chirality on the polarization of light, also date back to the nineteenth century. Arago [9], Biot [12], Cauchy [15], Pasteur [31] and Fresnel [22] all examined optical activity in chiral media. The concept of chirality has also played an important role in chemistry with the works of Prelog [32]. At the best of our knowledge, few mathematical contributions have been published in this relatively young applied mathematical field. Recently, we have obtained some mathematical results on the scattering of electromagnetic waves in chiral media: representation theorems, the radiation condition for the Drude-Born-Fedorov equations, existence and uniqueness of a solution, guided waves, convergence to Maxwell's equations, equivalent boundary conditions for thin chiral coatings, low frequency asymptotics [1]-[6]. We also refer to Artola and Cessenat [10] (*Homogenization of the Drude-Born-Fedorov equations*), Ola [29] and Athanasiadis et Stratis [11] (*Transmission problems by a bounded chiral object*), McDowall [27], [28] and Boutet de Monvel and Shepelsky [13], [14] (*Inverse scattering problems in chiral media*).

The main purpose of the present paper is to provide a mathematical foundation of *the Drude-Born-Fedorov equations* governing the propagation of electromagnetic waves in chiral media. By making use of the *dipole approximation,* we shall show how chiral materials could be realized by embedding chiral objects (helices) in an isotropic medium and derive the Drude-Born-Fedorov equations in the case of ram-

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domly oriented identical helices. To do this, we first replace the helices along with the perfect conductor boundary condition on them by a collection of equivalent electric and magnetic dipoles in the constitutive relations which are then together with the Maxwell equations to hold everywhere. This is *the Born approximation.* Then, we analyze with the help of *the integral equations method* the new constitutive relations with the Maxwell equations when the number of the helices is large and we establish the connextion between the chirality and the electromagnetic parameters. We shall show how *the Drude-Born-Fedorov model* has to be understood as a continuum limit of the Maxwell equations in the presence of a collection of electric dipoles.
Our *dipole approximation,* introduced in this paper, is close to the *Point Interaction Approximation* used for instance by Figari, Papanicolaou, Rubinstein, Teta, Orlandi and Ozawa [18], [19], [20], [21], [30], [33] to analyze boundary value problems in regions with small inclusions. The main idea in this method is to replace the inclusions along with the boundary conditions on them by an inhomogeneous term in the differential equation which is then to hold everywhere.

1. Statement of the problem. Let Ω be a $\mathcal{C}^{2,\alpha}$-bounded domain in $\mathbf{R}^3$, where $0 < \alpha < 1$ and let $N \in \mathbb{N}$. A collection of identical small helices $(\mathcal{S}_j^N)_{j=1}^{N^3}$ with typical size $N^{-3/2}$ are periodically distributed, with period N^{-1}, in the domain Ω. Let $N^{-3/2}\,a$ be the radius of a single helix $\mathcal{S}^N$, $N^{-3/2}\,b$ its width, $N^{-3/2}\,l$ its length and let $\hat{\mathbf{u}}$ a unit vector along its straight part direction. a, b and l are three strictly positive fixed constants. Let Ω^N be the domain in $\mathbf{R}^3$ defined by

$$\Omega^N = \Omega \setminus \cup(\mathcal{S}_j^N)_{j=1}^{N^3}.$$

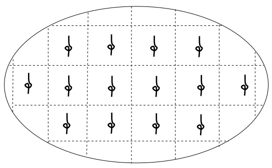

Figure 1 : *The domain* Ω^N

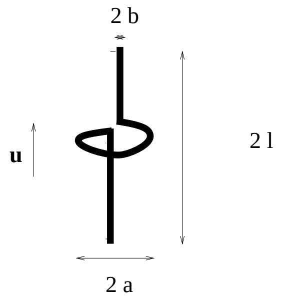

Figure 2 : *A single helix*

We consider the Maxwell equations

$$\left\{ \begin{array}{ll} \mathbf{curl}\,\mathbf{E}^N = i\,\omega\,\mathbf{B}^N & \text{in } \Omega^N, \\ \mathbf{curl}\,\mathbf{H}^N = -i\,\omega\,\mathbf{D}^N + \mathbf{f} & \text{in } \Omega^N, \end{array} \right. \tag{1.1}$$

with the constitutive relations

$$\left\{ \begin{array}{ll} \mathbf{D}^N = \varepsilon_0\,\mathbf{E}^N & \text{in } \Omega^N, \\ \mathbf{B}^N = \mu_0\,\mathbf{H}^N & \text{in } \Omega^N, \end{array} \right. \tag{1.2}$$

and the perfect conductor boundary condition $\mathbf{E}^N \wedge \mathbf{n} = 0$ on $\partial\Omega^N$, where $\mathbf{f} \in (\mathcal{C}^{0,\alpha}(\overline{\Omega}))^3$, $0 < \alpha < 1$, ω the frequency, ε_0 the electric permittivity and μ_0 the magnetic permeability. Throughout this paper, ε_0 and μ_0 are assumed to be two strictly positive constants. We also assume that ω^2 is not an eigenfrequency of the Maxwell operator $\left(\mathbf{curl}\,\mathbf{curl} - \omega^2\,\varepsilon_0\,\mu_0\right)$ in Ω with a perfect conductor boundary condition on $\partial\Omega$ and we introduce the Green kernel $g(\mathbf{x}-\mathbf{y})$ of the Helmholtz equation

$$\begin{cases} (\Delta_{\mathbf{y}} + \omega^2\,\varepsilon_0\,\mu_0)\,g(\mathbf{x}-\mathbf{y}) = -\delta(\mathbf{x}-\mathbf{y}), \quad \mathbf{x},\mathbf{y} \in \Omega, \\ g(\mathbf{x}-\mathbf{y}) = 0, \quad \mathbf{y} \in \partial\Omega, \mathbf{x} \in \Omega. \end{cases}$$

We are interested in the behavior of the fields $(\mathbf{E}^N, \mathbf{H}^N)$ when $N \to +\infty$. To analyze this problem, we first replace the helices $(\mathcal{S}_j^N)_{j=1}^{N^3}$ along with the perfect conductor boundary condition on them by a collection of equivalent electric and magnetic dipoles in the constitutive relations which are then together with the Maxwell equations to hold everywhere in Ω and then, analyze the new constitutive relations with the Maxwell equations in Ω in the continuum limit $N \to +\infty$ by a relatively simple and direct way based on the integral equations method. No attempt is made in this paper to mathematically justify this approximation known in the literature as *the Born approximation*. The generalization of the present approach to treat non homogeneous constitutive relations (ε_0 and μ_0 variables) is a nontrivial task. We refer to our work [8] for the analysis of this problem under the restrictive assumption $\mu_0 = \mathit{constant}$.

2. Dipole approximation and the new constitutive relations. Placed in an electromagnetic field $(\mathbf{E}, \mathbf{H})$, at a fixed frequency ω (large compared to the small helix), the helix $\mathcal{S}_j^N$ can be replaced according to the *Born approximation* (quasi-static approximation) by an electric dipole p^N and a magnetic dipole m^N. The electric current induced by the variation of the electric field in the straight portion of the helix must flow by continuity in its circular portion and then it contributes to the formation of both the electric and magnetic dipole moments. The magnetic current induced by the variation of the magnetic field in the circular portion of the helix must also flow by continuity in its straight portion and then it contributes as the electric current to the formation of both the electric and magnetic dipole moments. The electric dipole p^N and the magnetic dipole m^N are given by (see for instance [23] and [34])

$$\begin{cases} p^N = \xi_e^N\;(\mathbf{E}\,.\,\hat{\mathbf{u}})\;\hat{\mathbf{u}} + i\,\xi_{em}^N\;(\mathbf{H}\,.\,\hat{\mathbf{u}})\;\hat{\mathbf{u}}, \\ m^N = \xi_m^N\;(\mathbf{H}\,.\,\hat{\mathbf{u}})\;\hat{\mathbf{u}} + i\,\xi_{me}^N\;(\mathbf{E}\,.\,\hat{\mathbf{u}})\;\hat{\mathbf{u}}. \end{cases} \tag{2.1}$$

In (2.1), the coefficients $\xi_e^N, \xi_m^N, \xi_{em}^N$ and ξ_{me}^N, are expressed in terms of the parameters a, b, l and the frequency ω. Since (see once again [23] and [34])

$$\xi_e^N = 0(N^{-9/2}), \xi_m^N = 0(N^{-9/2}), \xi_{me}^N = 0(N^{-6}), \xi_{em}^N = 0(N^{-3}),$$

we make the following approximation by neglecting the terms $\xi_m^N\;(\mathbf{H}\,.\,\hat{\mathbf{u}}), \xi_{me}^N\;(\mathbf{E}\,.\,\hat{\mathbf{u}})$ and $\xi_e^N\;(\mathbf{E}\,.\,\hat{\mathbf{u}})$

$$p^N = i\,\xi_{em}^N\;(\mathbf{H}\,.\,\hat{\mathbf{u}})\;\hat{\mathbf{u}} = i\,N^{-3}\,\beta\,(\mathbf{H}\,.\,\hat{\mathbf{u}})\;\hat{\mathbf{u}} \quad \text{and} \quad m^N = 0,$$

where the parameter β is given by

$$\beta = \pm\,\frac{2\,\pi\,l\,a}{\omega\,(\log(8\,a/b) - 2)}.$$

The upper (lower) sign corresponds to the right-handed (left-handed) helix. A change in the sign of β means taking the mirroir image of the helix.

Now we return to our original problem. Using the Born approximation, we replace the helices $(\mathcal{S}_j^N)_{j=1}^{N^3}$ by a collection of electric dipoles placed at the centers $(\mathbf{x}_j)_{j=1}^{N^3}$ of the cells. The constitutive relations (1.2) become

$$\begin{cases} \mathbf{D}^N = \varepsilon_0\, \mathbf{E}^N + i\,\beta\, N^{-3} \displaystyle\sum_{j=1}^{N^3} (\mathbf{H}^N \,.\, \hat{\mathbf{u}})(\mathbf{x}_j)\, \delta(\mathbf{x}-\mathbf{x}_j)\, \hat{\mathbf{u}} \quad \text{in } \Omega, \\ \mathbf{B}^N = \mu_0\, \mathbf{H}^N \quad \text{in } \Omega, \end{cases} \tag{2.2}$$

and the Maxwell equations (1.1) are now posed in the domain Ω instead of Ω^N. Combining the new constitutive relations (2.2) together with the Maxwell system (1.1), which now holds everywhere in Ω, yields

$$\begin{cases} \mathbf{curl}\, \mathbf{E}^N &= i\,\omega\, \mu_0\, \mathbf{H}^N \quad \text{in } \Omega, \\ \mathbf{curl}\, \mathbf{H}^N &= -i\,\omega\,\varepsilon_0\, \mathbf{E}^N + \mathbf{f} \\ & \quad +\omega\,\beta\, N^{-3} \displaystyle\sum_{j=1}^{N^3} (\mathbf{H}^N \,.\, \hat{\mathbf{u}})(\mathbf{x}_j)\, \delta(\mathbf{x}-\mathbf{x}_j)\, \hat{\mathbf{u}} \quad \text{in } \Omega, \\ \mathbf{E}^N \wedge \mathbf{n} &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

By eliminating the magnetic field $\mathbf{H}^N$ in the above equations, we obtain

$$\begin{cases} \Big(\mathbf{curl}\,\mathbf{curl} - \omega^2\, \varepsilon_0\, \mu_0\Big)\, \mathbf{E}^N \\ \qquad -\omega\,\beta\, N^{-3} \displaystyle\sum_{j=1}^{N^3} (\mathbf{curl}\, \mathbf{E}^N \,.\, \hat{\mathbf{u}})(\mathbf{x}_j)\, \delta(\mathbf{x}-\mathbf{x}_j)\, \hat{\mathbf{u}} = i\,\omega\,\mu_0\, \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{E}^N \wedge \mathbf{n} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

We now show the existence and uniqueness of $\mathbf{E}^N$ solution to the boundary value problem (2.3). We state and prove the following result.

LEMMA 2.1. *There exists a unique solution* $\mathbf{E}^N \in (\mathcal{C}^{2,\alpha}(\Omega \setminus \cup(\mathbf{x}_j)_{j=1}^{N^3}))^3$ *of the boundary value problem* (2.3).

Proof. If $\mathbf{E}^N$ satisfies the boundary value problem (2.3) then, from the Stratton-Chu representation formula, we get

$$\begin{aligned} \mathbf{E}^N(\mathbf{x}) &= i\,\omega\,\mu_0 \int_\Omega g(\mathbf{x}-\mathbf{y})\, \mathbf{f}(\mathbf{y})\, d\mathbf{y} - \mathbf{grad}_{\mathbf{x}} \int_\Omega g(\mathbf{x}-\mathbf{y})\, \mathrm{div}_{\mathbf{y}}\, \mathbf{E}^N(\mathbf{y})\, d\mathbf{y} \\ & \quad -i\,\mu_0^{-1}\,\beta\, N^{-3} \Big[\sum_{j=1}^{N^3} (\mathbf{curl}\, \mathbf{E}^N \,.\, \hat{\mathbf{u}})(\mathbf{x}_j)\, g(\mathbf{x}-\mathbf{x}_j)\Big]\, \hat{\mathbf{u}}, \quad \mathbf{x} \in \Omega \setminus \cup(\mathbf{x}_j)_{j=1}^{N^3}. \end{aligned}$$

Taking the curl of the above equation and multiplying by $\hat{\mathbf{u}}$, we can see that

$$\mathbf{curl}\, \mathbf{E}^N(\mathbf{x}) \,.\, \hat{\mathbf{u}} = i\,\omega\,\mu_0 \Big(\mathbf{curl} \int_\Omega g(\mathbf{x}-\mathbf{y})\, \mathbf{f}(\mathbf{y})\, d\mathbf{y}\Big) \,.\, \hat{\mathbf{u}}, \quad \mathbf{x} \in \Omega \setminus \cup(\mathbf{x}_j)_{j=1}^{N^3}.$$

Recalling that $\mathbf{f}$ is in $(\mathcal{C}^{0,\alpha}(\overline{\Omega}))^3$, this gives a full meaning to the quantity $(\mathbf{curl}\, \mathbf{E}^N \,.\, \hat{\mathbf{u}})(\mathbf{x}_j)$

$$(\mathbf{curl}\, \mathbf{E}^N \,.\, \hat{\mathbf{u}})(\mathbf{x}_j) = i\,\omega\,\mu_0 \Big(\mathbf{curl} \int_\Omega g(\mathbf{x}-\mathbf{y})\, \mathbf{f}(\mathbf{y})\, d\mathbf{y}\Big)(\mathbf{x}_j) \,.\, \hat{\mathbf{u}}. \tag{2.4}$$

It follows that solving the following integral equation

$$
(2.5)\qquad \begin{aligned}
&\mathbf{E}^N(\mathbf{x}) + \mathbf{grad}_{\mathbf{x}} \int_\Omega g(\mathbf{x}-\mathbf{y})\,\mathrm{div}_{\mathbf{y}}\,\mathbf{E}^N(\mathbf{y})\; d\mathbf{y} = i\,\omega\,\mu_0 \int_\Omega g(\mathbf{x}-\mathbf{y})\,\mathbf{f}(\mathbf{y})\; d\mathbf{y}\\
&+\omega\,\beta\,N^{-3}\Big[\sum_{j=1}^{N^3}\big(\Big(\mathbf{curl}\int_\Omega g(\mathbf{x}-\mathbf{y})\,\mathbf{f}(\mathbf{y})\; d\mathbf{y}\Big)(\mathbf{x}_j)\,.\,\hat{\mathbf{u}}\big)\; g(\mathbf{x}-\mathbf{x}_j)\Big]\;\hat{\mathbf{u}},
\end{aligned}
$$

for $\mathbf{x}\in\Omega\setminus\cup(\mathbf{x}_j)_{j=1}^{N^3}$, gives the claim. Let $\eta>0$ and g^η a regularized version of the kernel g, given by

$$
(2.6)\qquad g^\eta(\mathbf{x}-\mathbf{y}) = \Big(g(\mathbf{x}-\mathbf{y}) - \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\Big) + \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|+\eta}, \quad \mathbf{x},\mathbf{y}\in\Omega.
$$

It is clear that there exists $\mathbf{E}^N_\eta$ solution of the regularized integral equation

$$
\begin{aligned}
&\mathbf{E}^N_\eta(\mathbf{x}) + \mathbf{grad}_{\mathbf{x}} \int_\Omega g^\eta(\mathbf{x}-\mathbf{y})\,\mathrm{div}_{\mathbf{y}}\,\mathbf{E}^N_\eta(\mathbf{y})\; d\mathbf{y} = i\,\omega\,\mu_0 \int_\Omega g^\eta(\mathbf{x}-\mathbf{y})\,\mathbf{f}(\mathbf{y})\; d\mathbf{y}\\
&+\omega\,\beta\,N^{-3}\Big[\sum_{j=1}^{N^3}\big(\Big(\mathbf{curl}\int_\Omega g^\eta(\mathbf{x}-\mathbf{y})\,\mathbf{f}(\mathbf{y})\; d\mathbf{y}\Big)(\mathbf{x}_j)\,.\,\hat{\mathbf{u}}\big)\; g^\eta(\mathbf{x}-\mathbf{x}_j)\Big]\;\hat{\mathbf{u}}, \quad \mathbf{x}\in\Omega.
\end{aligned}
$$

Indeed,

$$
\forall\,\mathbf{x}\in\Omega\setminus\cup(\mathbf{x}_j)_{j=1}^{N^3}, \quad \mathbf{E}^N_\eta \text{remains bounded when } \eta\to 0.
$$

Let

$$
\mathbf{E}^N(\mathbf{x}) = \lim_{\eta\to 0}\mathbf{E}^N_\eta(\mathbf{x}), \quad \mathbf{x}\in\Omega\setminus\cup(\mathbf{x}_j)_{j=1}^{N^3}.
$$

$\mathbf{E}^N(\mathbf{x})$ satisfies the integral equation (2.5). The uniqueness follows from the uniqueness of a solution to the integral equation (see [16] for a proof)

$$
\mathbf{v}(\mathbf{x}) + \mathbf{grad}_{\mathbf{x}} \int_\Omega g(\mathbf{x}-\mathbf{y})\,\mathrm{div}_{\mathbf{y}}\,\mathbf{v}(\mathbf{y})\; d\mathbf{y} = 0, \quad \mathbf{x}\in\Omega,
$$

if ω^2 is not an eigenfrequency of the Maxwell operator $\Big(\mathbf{curl}\,\mathbf{curl}\; - \omega^2\,\varepsilon_0\,\mu_0\Big)$ in Ω with a perfect conductor boundary condition on $\partial\Omega$. □

3. Convergence to a chiral model. In this section, we shall prove the convergence of $\mathbf{E}^N$, defined by (2.3), to the unique solution $\mathbf{E}$ of the following boundary value problem

$$
(3.1)\qquad \begin{cases} \Big(\mathbf{curl}\,\mathbf{curl}\; - \omega^2\,\varepsilon_0\,\mu_0\Big)\,\mathbf{E} - \omega\,\beta\,|\Omega|^{-1}\,(\mathbf{curl}\,\mathbf{E}\,.\,\hat{\mathbf{u}})\;\hat{\mathbf{u}} = i\,\omega\,\mu_0\,\mathbf{f} \quad \text{in } \Omega,\\ \mathbf{E}\wedge\mathbf{n} = 0 \quad \text{on } \partial\Omega, \end{cases}
$$

where *the chiral term* is

$$
-\omega\,\beta\,|\Omega|^{-1}\,(\mathbf{curl}\,\mathbf{E}\,.\,\hat{\mathbf{u}})\;\hat{\mathbf{u}}.
$$

To do this, we begin by proving the existence and the uniqueness of a solution $\mathbf{E}\in(\mathcal{C}^{2,\alpha}(\overline{\Omega}))^3$ to the boundary value problem (3.1). We have the following existence and uniqueness result.

LEMMA 3.1. *There exists a unique solution* $\mathbf{E} \in (\mathcal{C}^{2,\alpha}(\overline{\Omega}))^3$ *to the boundary value problem* (3.1).

Proof. If $\mathbf{E}$ is a solution to (3.1) then, making use of the Stratton-Chu representation formula yields

$$\begin{aligned}\mathbf{E}(\mathbf{x}) &= i\,\omega\,\mu_0 \int_\Omega g(\mathbf{x}-\mathbf{y})\,\mathbf{f}(\mathbf{y})\,d\mathbf{y} - \mathbf{grad}_\mathbf{x} \int_\Omega g(\mathbf{x}-\mathbf{y})\,\mathrm{div}_\mathbf{y}\,\mathbf{E}(\mathbf{y})\,d\mathbf{y} \\ &\quad -i\,\mu_0^{-1}\,\beta\,|\Omega|^{-1} \int_\Omega \mathbf{curl}\,\mathbf{E}(\mathbf{y})\,.\,\hat{\mathbf{u}}\;g(\mathbf{x}-\mathbf{y})\,d\mathbf{y}\;\hat{\mathbf{u}}, \quad \mathbf{x}\in\Omega.\end{aligned}$$

Since the integral operator $T : (\mathcal{C}^{2,\alpha}(\overline{\Omega}))^3 \to (\mathcal{C}^{2,\alpha}(\overline{\Omega}))^3$, defined by

$$T(\mathbf{v}) = \mathbf{v} + \mathbf{grad}_\mathbf{x} \int_\Omega g(\mathbf{x}-\mathbf{y})\,\mathrm{div}_\mathbf{y}\,\mathbf{v}(\mathbf{y})\,d\mathbf{y} + i\,\mu_0^{-1}\,\beta\,|\Omega|^{-1} \int_\Omega \mathbf{curl}\,\mathbf{v}(\mathbf{y})\,.\,\hat{\mathbf{u}}\;g(\mathbf{x}-\mathbf{y})\,d\mathbf{y}\;\hat{\mathbf{u}},$$

is a Fredholm operator with index zero (identity + compact), then the uniqueness of a solution gives the existence. To show the uniqueness of a solution, let $\mathbf{E}$ be a solution of the homogeneous problem. We have from the Stratton-Chu representation formula

$$\begin{aligned}\mathbf{E}(\mathbf{x}) &= -\mathbf{grad}_\mathbf{x} \int_\Omega g(\mathbf{x}-\mathbf{y})\,\mathrm{div}_\mathbf{y}\,\mathbf{E}(\mathbf{y})\,d\mathbf{y} \\ &\quad -i\,\mu_0^{-1}\,\beta\,|\Omega|^{-1} \int_\Omega \mathbf{curl}\,\mathbf{E}(\mathbf{y})\,.\,\hat{\mathbf{u}}\;g(\mathbf{x}-\mathbf{y})\,d\mathbf{y}\;\hat{\mathbf{u}}, \quad \mathbf{x}\in\Omega.\end{aligned}$$

Then, $\mathbf{curl}\,\mathbf{E}(\mathbf{x})\,.\,\hat{\mathbf{u}} = 0$. This yields, as in the proof of Lemma 2.1, to $\mathbf{E} = 0$ in Ω if ω^2 is not an eigenfrequency of the Maxwell operator $\Big(\mathbf{curl}\,\mathbf{curl} - \omega^2\,\varepsilon_0\,\mu_0\Big)$ in Ω with a perfect conductor boundary condition on $\partial\Omega$. ☐

We are now in position to state and prove our main result in this paper.

THEOREM 3.2. *Let* $\mathbf{e}^N(\mathbf{x}) = \mathbf{E}(\mathbf{x}) - \mathbf{E}^N(\mathbf{x})$ *and* $B_{\rho_j}(\mathbf{x}_j)$ *a ball of center* $\mathbf{x}_j$ *and radius* $\rho_j > 0$, *for* $j = 1\dots N^3$, *where* $(\mathbf{x}_j)_{j=1}^{N^3}$ *are the centers of the cells. There exists* $N_0 \in \mathbb{N}$ *such that for* $N \geq N_0$ *and* $\mathbf{x} \in \Omega \setminus \cup_j B_{\rho_j}(\mathbf{x}_j)$, *the following estimate holds*

$$|\mathbf{e}^N(\mathbf{x})| \leq C\,N^{-1}, \tag{3.2}$$

where C *is a constant independent of* N *and* $\mathbf{x}$.

Proof. Making use of the Stratton-Chu representation formula yields

$$\begin{aligned}\mathbf{E}(\mathbf{x}) &= i\,\omega\,\mu_0 \int_\Omega g(\mathbf{x}-\mathbf{y})\,\mathbf{f}(\mathbf{y})\,d\mathbf{y} - \mathbf{grad}_\mathbf{x} \int_\Omega g(\mathbf{x}-\mathbf{y})\,\mathrm{div}_\mathbf{y}\,\mathbf{E}(\mathbf{y})\,d\mathbf{y} \\ &\quad -i\,\mu_0^{-1}\,\beta\,|\Omega|^{-1} \int_\Omega \mathbf{curl}\,\mathbf{E}(\mathbf{y})\,.\,\hat{\mathbf{u}}\;g(\mathbf{x}-\mathbf{y})\,d\mathbf{y}\;\hat{\mathbf{u}}, \quad \mathbf{x}\in\Omega,\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\mathbf{E}^N(\mathbf{x}) &= i\,\omega\,\mu_0 \int_\Omega g(\mathbf{x}-\mathbf{y})\,\mathbf{f}(\mathbf{y})\,d\mathbf{y} - \mathbf{grad}_\mathbf{x} \int_\Omega g(\mathbf{x}-\mathbf{y})\,\mathrm{div}_\mathbf{y}\,\mathbf{E}^N(\mathbf{y})\,d\mathbf{y} \\ &\quad -i\,\mu_0^{-1}\,\beta\,N^{-3}\Big[\sum_{j=1}^{N^3} (\mathbf{curl}\,\mathbf{E}^N\,.\,\hat{\mathbf{u}})(\mathbf{x}_j)\;g(\mathbf{x}-\mathbf{x}_j)\Big]\;\hat{\mathbf{u}},\end{aligned} \tag{3.4}$$

for $\mathbf{x} \in \Omega \setminus \cup(\mathbf{x}_j)_{j=1}^{N^3}$, where the quantity $(\mathbf{curl}\,\mathbf{E}^N\,.\,\hat{\mathbf{u}})(\mathbf{x}_j)$ is defined by (2.4).
By substracting (3.4) from (3.3), it follows that

$$\mathbf{e}^N(\mathbf{x}) - \omega\,\beta\,N^{-3}\sum_{j=1}^{N^3}(\mathbf{curl}\,\mathbf{e}^N(\mathbf{x}_j)\,.\,\hat{\mathbf{u}})\;g(\mathbf{x}-\mathbf{x}_j)\;\hat{\mathbf{u}} + \mathbf{grad}_{\mathbf{x}}\int_\Omega g(\mathbf{x}-\mathbf{y})\,\mathrm{div}_{\mathbf{y}}\,\mathbf{e}^N(\mathbf{y})\;d\mathbf{y}$$
$$= -i\,\mu_0^{-1}\,\beta\left[|\Omega|^{-1}\int_\Omega \mathbf{curl}\,\mathbf{E}(\mathbf{y})\,.\,\hat{\mathbf{u}}\;g(\mathbf{x}-\mathbf{y})\;d\mathbf{y} - N^{-3}\sum_{j=1}^{N^3}(\mathbf{curl}\,\mathbf{E}\,.\,\hat{\mathbf{u}})(\mathbf{x}_j)\;g(\mathbf{x}-\mathbf{x}_j)\right]\hat{\mathbf{u}}.$$

Now, let $\eta > 0$ and g^η a regularized version of the kernel g given by (2.6). Let $\mathbf{e}_\eta^N$ be defined in Ω as the solution of the integral equation

$$\mathbf{e}_\eta^N(\mathbf{x}) - \omega\,\beta\,N^{-3}\sum_{j=1}^{N^3}(\mathbf{curl}\,\mathbf{e}_\eta^N(\mathbf{x}_j)\,.\,\hat{\mathbf{u}})\;g^\eta(\mathbf{x}-\mathbf{x}_j)\;\hat{\mathbf{u}} + \mathbf{grad}_{\mathbf{x}}\int_\Omega g^\eta(\mathbf{x}-\mathbf{y})\,\mathrm{div}_{\mathbf{y}}\,\mathbf{e}_\eta^N(\mathbf{y})\;d\mathbf{y}$$
$$= -i\,\mu_0^{-1}\,\beta\left[|\Omega|^{-1}\int_\Omega \mathbf{curl}\,\mathbf{E}(\mathbf{y})\,.\,\hat{\mathbf{u}}\;g^\eta(\mathbf{x}-\mathbf{y})\;d\mathbf{y} - N^{-3}\sum_{j=1}^{N^3}(\mathbf{curl}\,\mathbf{E}(\mathbf{x}_j)\,.\,\hat{\mathbf{u}})\;g^\eta(\mathbf{x}-\mathbf{x}_j)\right]\hat{\mathbf{u}}.$$

Since $\mathbf{e}_\eta^N(\mathbf{x})$ is smooth in Ω and $T^\eta : (\mathcal{C}^{1,\alpha}(\overline{\Omega}))^3 \to (\mathcal{C}^{1,\alpha}(\overline{\Omega}))^3$, defined by

$$\mathbf{v} \mapsto \mathbf{v} + \mathbf{grad}_{\mathbf{x}}\int_\Omega g^\eta(\mathbf{x}-\mathbf{y})\,\mathrm{div}_{\mathbf{y}}\,\mathbf{v}(\mathbf{y})\;d\mathbf{y} - \omega\,\beta\,N^{-3}\sum_{j=1}^{N^3}(\mathbf{curl}\,\mathbf{v}(\mathbf{x}_j)\,.\,\hat{\mathbf{u}})\;g^\eta(\mathbf{x}-\mathbf{x}_j)\;\hat{\mathbf{u}},$$

is of type (identity + compact) and it is invertible, the estimate

$$|\Omega|^{-1}\int_\Omega(\mathbf{curl}\,\mathbf{E}(\mathbf{y})\,.\,\hat{\mathbf{u}})\;g^\eta(\mathbf{x}-\mathbf{y})\;d\mathbf{y} - N^{-3}\sum_{j=1}^{N^3}(\mathbf{curl}\,\mathbf{E}(\mathbf{x}_j)\,.\,\hat{\mathbf{u}})\;g^\eta(\mathbf{x}-\mathbf{x}_j) = 0(N^{-1}),$$

$N \to +\infty, \mathbf{x} \in \Omega$, yields by standard methods familiar from numerical analysis to

$$\mathbf{e}_\eta^N(\mathbf{x}) = 0(N^{-1}), \quad N \to +\infty, \quad \mathbf{x} \in \Omega.$$

The estimate (3.2) for $\mathbf{x} \in \Omega\setminus\cup(\mathbf{x}_j)_{j=1}^{N^3}$ is then obtained by passing to the limit $\eta \to 0$, since, for fixed N and $\mathbf{x} \in \Omega \setminus \cup(\mathbf{x}_j)_{j=1}^{N^3}$, $\mathbf{e}_\eta^N(\mathbf{x})$ converges to $\mathbf{e}^N(\mathbf{x})$ as $\eta \to 0$.
The estimate $\mathbf{e}^N(\mathbf{x}) = 0(N^{-1})$ holds then uniformly in $\Omega \setminus \cup_j B_{\rho_j}(\mathbf{x}_j)$ where, for $j = 1 \ldots N^3$, $B_{\rho_j}(\mathbf{x}_j)$ is a ball of center $\mathbf{x}_j$ and radius $\rho_j > 0$. □

Finally, we note that if we assume that the sequence $(\mathbf{x}_j)_{j=1}^{N^3}$ tends to a continuum: that is for every function $\varphi(\mathbf{x})$

$$N^{-3}\sum_{j=1}^{N^3}\varphi(\mathbf{x}_j) \to \int_\Omega V(\mathbf{y})\;\varphi(\mathbf{y})\;d\mathbf{y}, \quad \text{as } N \to +\infty,$$

where $V(\mathbf{y})$ is the continuum density, then we can easily show that the limiting problem is

$$\begin{cases} \left(\mathbf{curl}\,\mathbf{curl} - \omega^2\,\varepsilon_0\,\mu_0\right)\mathbf{E} - \omega\,\beta\,V(\mathbf{x})\,(\mathbf{curl}\,\mathbf{E}\,.\,\hat{\mathbf{u}})\;\hat{\mathbf{u}} = i\,\omega\,\mu_0\,\mathbf{f} \quad \text{in } \Omega, \\ \mathbf{E} \wedge \mathbf{n} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where the chiral term is here

$$-\omega\,\beta\,V(\mathbf{x})\,(\mathbf{curl}\,\mathbf{E}\,.\,\hat{\mathbf{u}})\;\hat{\mathbf{u}}.$$

4. Collection of randomly oriented identical helices and the Drude-Born-Fedorov equations. If we assume that the helices $\mathcal{S}_j^N$ are randomly oriented then the equations (2.3) are replaced by

$$(4.1)\qquad \begin{cases} \Big(\mathbf{curl}\,\mathbf{curl}\, - \omega^2\,\varepsilon_0\,\mu_0\Big)\,\mathbf{E}^N \\ \qquad -\omega\,\beta\,N^{-3}\displaystyle\sum_{j=1}^{N^3}(\mathbf{curl}\,\mathbf{E}^N\,.\,\hat{\mathbf{u}}_{\mathbf{j}})(\mathbf{x}_j)\;\delta(\mathbf{x}-\mathbf{x}_j)\;\hat{\mathbf{u}}_{\mathbf{j}} = i\,\omega\,\mu_0\,\mathbf{f} \quad \text{in } \Omega, \\ \mathbf{E}^N\,\wedge\,\mathbf{n} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where the unit vectors $(\hat{\mathbf{u}}_{\mathbf{j}})_{j=1}^{N^3}$ are randomly oriented.

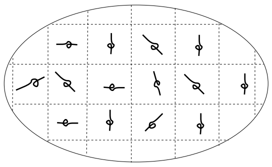

Figure 3 : *Randomly oriented helices*

Since

$$N^{-3}\sum_{j=1}^{N^3}(\mathbf{v}(\mathbf{x}_j)\,.\,\hat{\mathbf{u}}_{\mathbf{j}})\;\hat{\mathbf{u}}_{\mathbf{j}} \to \frac{1}{3|\Omega|}\int_\Omega \mathbf{v}(\mathbf{y})\;d\mathbf{y},$$

for any smooth vector $\mathbf{v}$, the limit equations are

$$(4.2)\qquad \begin{cases} \Big(\mathbf{curl}\,\mathbf{curl}\, - \omega^2\,\varepsilon_0\,\mu_0\Big)\,\mathbf{E} - \omega\,\beta\,\dfrac{1}{3|\Omega|}\,\mathbf{curl}\,\mathbf{E} = i\,\omega\,\mu_0\,\mathbf{f} \quad \text{in } \Omega, \\ \mathbf{E}\,\wedge\,\mathbf{n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

An analogous error estimate to (3.2) can be proved by the same techniques that we used in the above section. The only minor difference compared with Theorem 3.2 arises in the fact that the quantities $(\mathbf{curl}\,\mathbf{E}^N\,.\,\hat{\mathbf{u}}_{\mathbf{j}})(\mathbf{x}_j)$ are here obtained by solving a linear system, instead of making use of the identity (2.4). It is easy to see that

$$\begin{aligned}(\mathbf{curl}\,\mathbf{E}^N\,.\,\hat{\mathbf{u}}_j)(\mathbf{x}_j) \quad &= \quad i\,\omega\,\mu_0\,\Big(\mathbf{curl}\int_\Omega g(\mathbf{x}-\mathbf{y})\,\mathbf{f}(\mathbf{y})\;d\mathbf{y}\Big)(\mathbf{x}_j)\,.\,\hat{\mathbf{u}}_j \\ &\quad -i\,\mu_0^{-1}\,\beta\,N^{-3}\Big[\sum_{l\neq j}^{N^3}(\mathbf{curl}\,\mathbf{E}^N\,.\,\hat{\mathbf{u}}_l)(\mathbf{x}_l)\;(\mathbf{grad}\,g)(\mathbf{x}_j-\mathbf{x}_l)\,\wedge\,\hat{\mathbf{u}}_l\Big]\,.\,\hat{\mathbf{u}}_j,\end{aligned}$$

and then, the quantities $(\mathbf{curl}\,\mathbf{E}^N\,.\,\hat{\mathbf{u}}_{\mathbf{j}})(\mathbf{x}_j)$ are obtained by solving a linear system in the form $I+\beta\,M$, where I denotes the identity matrix and the matrix M is independent of β. This system is invertible outside the roots of the polynomial equation

$$det\,(I+\beta\,M) = 0.$$

Now, we verify that (4.2) are in fact the well-known *Drude-Born-Fedorov equations,* after appropriate redefinitions of the field vectors and the material parameters. Let γ_1 and γ_2 be the roots of the equation

$$\gamma^2 - \frac{\omega\,\beta}{3\,|\Omega|}\,\gamma - \omega^2\,\varepsilon_0\,\mu_0 = 0,$$

$$\mathbf{E}^{DBF} = \mathbf{E}$$

and

$$\mathbf{H}^{DBF} = \mathbf{curl}\,\mathbf{E}^{DBF} - \frac{\gamma_1 + \gamma_2}{2}\,\mathbf{E}^{DBF}.$$

It is easy to verify that

$$\mathbf{curl}\,\mathbf{H}^{DBF} = \frac{\gamma_1 + \gamma_2}{2}\,\mathbf{H}^{DBF} + \frac{\gamma_1^2 + \gamma_2^2}{4}\,\mathbf{E}^{DBF} + i\,\omega\,\mu_0\,\mathbf{f}.$$

This yields to the Drude-Born-Fedorov system

$$(Eq.\ DBF)\begin{cases} \mathbf{curl}\,\mathbf{E}^{DBF} = \dfrac{(\gamma_1 + \gamma_2)}{2}\mathbf{E}^{DBF} + \mathbf{H}^{DBF} \quad \text{in } \Omega, \\ \mathbf{curl}\,\mathbf{H}^{DBF} = \dfrac{(\gamma_1 + \gamma_2)}{2}\,\mathbf{H}^{DBF} + \dfrac{(\gamma_1^2 + \gamma_2^2)}{4}\,\mathbf{E}^{DBF} + i\,\omega\,\mu_0\,\mathbf{f} \quad \text{in } \Omega, \\ \mathbf{E}^{DBF} \wedge \mathbf{n} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

satisfied by the fields $\mathbf{E}^{DBF}$ and $\mathbf{H}^{DBF}$ and where the effectif chirality admittance is given by

$$\frac{(\gamma_1 + \gamma_2)}{2} = \frac{\omega\,\beta}{6\,|\Omega|}.$$

The connextion between the chirality of Ω^N and its electromagnetic parameters is then established.

5. Some extensions of our approach. In this final section, we shall discuss two interesting extensions of the approach developped in this paper. The complete treatement of the first one will be presented in a forthcoming publication [8].

5.1. Periodic coefficients with period N^{-1}. If the electric permittivity and the magnetic permeability ε_0 and μ_0 are assumed to be periodic functions in Ω with period N^{-1}, then we are led to the homogenization of the following boundary value problem

$$(5.1)\quad \begin{cases} \mathbf{curl}\,\dfrac{1}{\mu_0(N\,\mathbf{x})}\mathbf{curl}\,\mathbf{E}^N - \omega^2\,\varepsilon_0(N\,\mathbf{x})\,\mathbf{E}^N \\ \quad -\omega\,\beta\,\dfrac{1}{\mu_0(N\,\mathbf{x})}N^{-3}\displaystyle\sum_{j=1}^{N^3}(\mathbf{curl}\,\mathbf{E}^N\,.\,\hat{\mathbf{u}})(\mathbf{x}_j)\;\delta(\mathbf{x} - \mathbf{x}_j)\;\hat{\mathbf{u}} = i\,\omega\,\mathbf{f} \quad \text{in } \Omega, \\ \mathbf{E}^N \wedge \mathbf{n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Formally, we expand $\mathbf{E}^N$ in the form

$$\mathbf{E}^N(\mathbf{x}) = \mathbf{E}^0_N(\mathbf{x},\mathbf{y}) + N^{-1}\,\mathbf{E}^1_N(\mathbf{x},\mathbf{y}) + \dots,$$

where the new variable $\mathbf{y}$ is defined by $\mathbf{y} = N\,\mathbf{x}$ and $\mathbf{E}^j_N$ are assumed to be periodic in $\mathbf{y}$ with period 1 in each component. We can show that the term $\mathbf{E}^0_N(\mathbf{x},\mathbf{y})$ is in the form

$$\mathbf{E}^0_N(\mathbf{x},\mathbf{y}) = \sum_{l=1}^{3} \langle \mathbf{E}^0_{N,l}\rangle(\mathbf{x})\;\mathbf{grad}_{\mathbf{y}}(\varphi_l + y_l),$$

where $\langle\psi\rangle$ denotes the average of ψ with respect to the rapid variable $\mathbf{y} = (y_1, y_2, y_3)$ and φ_l is defined as the unique solution of

$$\begin{cases} \mathrm{div}_{\mathbf{y}}\varepsilon_0(\mathbf{y})\,\mathbf{grad}_{\mathbf{y}}\,(\varphi_l + y_l) = 0, \\ \varphi_l \text{ is periodic in } \mathbf{y} \text{ with period } (1,\,1,\,1). \end{cases}$$

Averaging over the variable $\mathbf{y}$ and then, passing to the limit $N \to +\infty$, we obtain the following homogenized problem: find $\mathbf{E}^0 = (\mathbf{E}^0_1, \mathbf{E}^0_2, \mathbf{E}^0_3)$ such that

$$\begin{cases} \mathbf{curl}_{\mathbf{x}}\,\mathbf{curl}_{\mathbf{x}}\, \displaystyle\sum_{l=1}^{3} \Big(\langle \frac{1}{\mu_0(\mathbf{y})}\mathbf{grad}_{\mathbf{y}}(\varphi_l + y_l)\rangle\Big)\,\mathbf{E}^0_l - \omega^2 \displaystyle\sum_{l=1}^{3} \Big(\langle \varepsilon_0\,(\mathbf{y})\mathbf{grad}_{\mathbf{y}}(\varphi_l + y_l)\rangle\Big)\,\mathbf{E}^0_l \\ \quad -\omega\,\beta\,|\Omega|^{-1}\Big[\mathbf{curl}_{\mathbf{x}} \displaystyle\sum_{l=1}^{3} \Big(\langle \frac{1}{\mu_0(\mathbf{y})}\mathbf{grad}_{\mathbf{y}}(\varphi_l + y_l)\rangle\Big)\,\mathbf{E}^0_l\Big]\,.\,\hat{\mathbf{u}}\,\hat{\mathbf{u}} = i\,\omega\,\mathbf{f} \quad \text{in } \Omega, \\ \mathbf{E}^0 \wedge \mathbf{n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

We refer to our work [8] for the proof of the existence and uniqueness of a solution to the boundary value problem (5.1) and also for the analysis of the convergence to the homogenized problem, under the restrictive assumption $\mu_0 = \mathit{constant}$ in the domain Ω.

5.2. Helices with typical size of order $N^{-3/2-\eta}, \eta > 0$. If the typical size of the helices is of order $N^{-3/2-\eta}$, where $\eta > 0$, then from $\xi^N_{em} = i\,\beta\,N^{-3-2\eta}$ the Maxwell equations (2.3) become

$$\begin{cases} \Big(\mathbf{curl}\,\mathbf{curl}\; - \omega^2\,\varepsilon_0\,\mu_0\Big)\,\mathbf{E}^N \\ \qquad -\omega\,\beta\,N^{-3-2\eta} \displaystyle\sum_{j=1}^{N^3} (\mathbf{curl}\,\mathbf{E}^N\,.\,\hat{\mathbf{u}})(\mathbf{x}_j)\;\delta(\mathbf{x} - \mathbf{x}_j)\;\hat{\mathbf{u}} = i\,\omega\,\mu_0\,\mathbf{f} \quad \text{in } \Omega, \\ \mathbf{E}^N \wedge \mathbf{n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Expanding $\mathbf{E}^N$ in the form

$$\mathbf{E}^N(\mathbf{x}) = \mathbf{E}^0(\mathbf{x}) + N^{-2\eta}\,\mathbf{E}^1_N(\mathbf{x}) + \dots,$$

we arrive at

$$\begin{cases} \Big(\mathbf{curl}\,\mathbf{curl}\; - \omega^2\,\varepsilon_0\,\mu_0\Big)\,\mathbf{E}^0 = i\,\omega\,\mu_0\,\mathbf{f} \quad \text{in } \Omega, \\ \mathbf{E}^0 \wedge \mathbf{n} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} \Big(\mathbf{curl}\,\mathbf{curl}\; - \omega^2\,\varepsilon_0\,\mu_0\Big)\,\mathbf{E}^1_N - \omega\,\beta\;N^{-3}\displaystyle\sum_{j=1}^{N^3}(\mathbf{curl}\,\mathbf{E}^0\,.\,\hat{\mathbf{u}})(\mathbf{x}_j)\;\delta(\mathbf{x}-\mathbf{x}_j)\;\hat{\mathbf{u}} = 0 \quad \text{in } \Omega, \\ \mathbf{E}^1_N\;\wedge\,\mathbf{n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Following the arguments explained in the third section of the present work, it is easy now to see that $\mathbf{E}^1_N$ converge to $\mathbf{E}^1$ defined as the unique solution of the following boundary value problem

$$\begin{cases} \Big(\mathbf{curl}\,\mathbf{curl}\; - \omega^2\,\varepsilon_0\,\mu_0\Big)\,\mathbf{E}^1 - \omega\,\beta\,|\Omega|^{-1}\;(\mathbf{curl}\,\mathbf{E}^0\,.\,\hat{\mathbf{u}})\;\hat{\mathbf{u}} = 0 \quad \text{in } \Omega, \\ \mathbf{E}^1\;\wedge\,\mathbf{n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Without any difficulty, we can show that the field $\mathbf{E}^{1,N}$ defined by

$$\mathbf{E}^{1,N} = \mathbf{E}^0 + N^{-1}\,\mathbf{E}^1$$

satisfies

$$\begin{cases} \Big(\mathbf{curl}\,\mathbf{curl}\; - \omega^2\,\varepsilon_0\,\mu_0\Big)\,\mathbf{E}^{1,N} \\ \qquad -\omega\,N^{-2\eta}\beta\,|\Omega|^{-1}\;(\mathbf{curl}\,\mathbf{E}^{1,N}\,.\,\hat{\mathbf{u}})\;\hat{\mathbf{u}} = i\,\omega\,\mu_0\,\mathbf{f} + 0(N^{-4\eta}) \quad \text{in } \Omega, \\ \mathbf{E}^{1,N}\;\wedge\,\mathbf{n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Neglecting the terms in $0(N^{-4\eta})$ in the above equations, we obtain the same chiral model as with helices of caracteristic lenght of order $N^{-3/2}$ but now with an effectif chirality admittance divided by $N^{2\eta}$. Once again, by the the same techniques that we used for the proof of our main Theorem 3.2, we can easily show the desired approximation result.

THEOREM 5.1. *Assume that* $\eta \geq \dfrac{1}{4}$. *Let* $\mathbf{e}^N_1(\mathbf{x}) = \mathbf{E}^{1,N}(\mathbf{x}) - \mathbf{E}^N(\mathbf{x})$. *The estimate*

$$\mathbf{e}^N_1(\mathbf{x}) = 0(N^{-\inf(4\eta, 2\eta+1)}), \quad N \to +\infty,$$

holds uniformly in $\Omega \setminus \cup_j B_{\rho_j}(\mathbf{x}_j)$.

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