Quadratic BSDEs Revisited: A Forward Point of View

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- Quadratic semimartingale
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4 Quadratic BSDEs

Motivation

- Quadratic BSDE's appear naturally in a lot of optimization problems
  - Mean variance problems
  - Utility maximization
  - Risk sensitive problem
  - Large deviations....
- From the seminal paper of Kobylanski (2000) on bounded solutions, different extensions are provided, in particular by many people in the audience
  - We use many ideas from Briand and Hu (2006), where bounded assumption are relaxed.
  - See also the very interesting paper of Tevzadze (2008).
- Need of unified point of view on estimates and convergence results.

Definition of Quadratic BSDE

The data

- Probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$ with continuous martingales
- A coefficient $g(t, y, z)$ with good measurability and a terminal condition ξ_T

The equation

- An equation $-dY_t = g(t, Y_t, Z_t)dt - Z_t dW_t, \quad Y_T = \xi_T.$, where W is a d-dimensional BM
- A solution is a pair $(Y, Z) \in \mathbb{R} \times \mathbb{R}^d$ of adapted processes such that the paths of Y are continuous, and $\int_0^T |Z_t|^2 dt < \infty, \quad \int_0^T |g(t, Y_t, Z_t)| dt < \infty$, $\mathbb{P}$-a.s

The quadratic case

- Quadratic BSDE= quadratic coefficient$=d\mathbb{P} \otimes dt-\mathrm{a.s.}$

$$|g(., t, y, z)| \leq Q(t, y, z) \equiv 1/\delta |l_t| + c_t |y| + \frac{\delta}{2} |z|^2,$$

Quadratic semimartingale : Definition

BSDEs : a forward point of view

- More flexible point of view, localization technique may be used

Definition A quadratic semimartingale is a continuous process $Y_. = Y_0 - V + M$ satisfying the constraint :

- Structure condition $\mathcal{Q}(\Lambda, C, \delta)$: There exist two adapted increasing processes (Λ, C) and $\delta > 0$ s.t.

$$dlVl_t \ll 1/\delta d\Lambda_t + |Y_t|\, dC_t + \frac{\delta}{2} d\langle M\rangle_t, \quad \mathbb{P}\text{–a.s.},$$

- *V is a predictable process with finite total variation $|V|$*
- *M is a local martingale with quadratic variation $\langle M\rangle$*
- *$\ll$ stands for the absolute continuity of increasing processes.*

Typical examples

Notation : $\mathcal{Q}(0,0,\delta) = q_\delta$, and $q(M) = -\frac{1}{2}\langle M\rangle = -\underline{q}(M)$, and
$r_t(r_0, M) = r_0 + M_t - \frac{1}{2}\langle M\rangle_t = r_0 + r_t(M)$

Example of q-semimartingale or BSDEs

- Log of exponential martingale is a q-semimartingale
$$\mathcal{E}(M) = \exp(M - \tfrac{1}{2}\langle M\rangle) = \exp(r_.(M))$$
- Entropic process : If $\xi_T \in \mathbb{L}^1_{\exp}$,
$\rho_t(\xi_T) = \ln \mathbb{E}\big[\exp(\xi_T)|\mathcal{F}_t\big] = \rho_0(\xi_T) + r_t(M)$
and $\mathcal{E}(M)$ is a u.i. martingale

Example of $-q$-BSDEs

- $\underline{r}_t(M) = -r_t(-M)$ is a $-q$-semimartingale, and if $-\xi_T \in \mathbb{L}^1_{\exp}$, $\underline{r}_t(\xi_T) = -\rho_t(-\xi_T) = \underline{\rho}_0(\xi_T) - r_t(-M)$ is a solution.
    - $\mathcal{E}(-M)$ is u.i integrable, but in general $\mathcal{E}(M)$ is not
    - True if $|\xi_T| \in \mathbb{L}^1_{\exp}$

Basic properties of quadratic quasimartingale I

Definition A quadratic submartingale is a continuous semimartingale $X = X_0 + M - V$ such that $-V_. + \frac{1}{2}\langle M\rangle_. = A_.$ is a predictable increasing process. Equivalently, $X = X_0 + r_t(M) + A$

Properties Let Y be a $\mathcal{Q}(\Lambda, C, \delta)$-semimartingale.

- The role of δ : $\forall \lambda \neq 0$, $\lambda Y_.$ is a $\mathcal{Q}(\Lambda, C, \frac{\delta}{|\lambda|})$-semimartingale, and $M_.^{\lambda Y} = \lambda M_.^Y$.
- $\lambda Y_. - \frac{1}{2}\lambda(\lambda - 1)\langle M\rangle_.$ is a $\mathcal{Q}(\lambda\Lambda, C, \delta)$-semimartingale
- Property of $|Y_.|$:
  - Let $\epsilon_.$ be a predictable process such that $|\epsilon_.| = 1, a.s.$. Then the process $Y_.^\epsilon = \epsilon \cdot Y_. = Y_0 + \int_0^. \epsilon_s\, dY_s$ is a $\mathcal{Q}(\Lambda, C, \delta)$-semimartingale.

Basic properties of quadratic quasimartingale II

- In particular, taking $\epsilon^s_. = \mathrm{sign}(Y.)$, and denoting by $L_.(Y)$ the local time of $Y.$ at 0, the process $|Y_.| - L_.(Y) - 2(Y_0)^- = \epsilon^s \cdot Y_.$ is a $\mathcal{Q}(\Lambda, C, \delta)$-semimartingale.
- If $Y.$ is a $\mathcal{Q}$-quasimartingale, then $|Y_.|$ is a $\mathcal{Q}$-submartingale.

▶ Structure simplification Put

$$X^{\Lambda,C}_.(Y) = Y_. + \Lambda_. + |Y_.| * C_., \quad \bar{X}^{\Lambda,C}_.(Y) = e^{C_.}|Y_.| + e^{C_.} * \Lambda_.$$

- The processes $X^{\Lambda,C}_.(\delta Y)$ and $X^{\Lambda,C}_.(-\delta Y)$ are $\mathcal{Q}$-submartingales.
- The process $\bar{X}^{\Lambda,C}_.(|\delta Y_.|)$ is a $\mathcal{Q}$-submartingale.

Characterisation of quadratic semimartingales via exponential transformation

Motivation For any $Y \in \mathcal{Q}(\Lambda, C)$, $Y = X^{\Lambda,C}_{\cdot}(Y) + X^{\Lambda,C}_{\cdot}(-Y)$ where $X^{\Lambda,C}_{\cdot}(Y)$ and $X^{\Lambda,C}_{\cdot}(-Y)$ are $\mathcal{Q}$- submartingales.

Main result : Converse Property

Let $K_{\cdot}$ be a continuous increasing process

- If there exist two ladlag processes with $\underline{X} + \overline{X} = K$ such that $\exp(\underline{X})_{\cdot}$ and $\exp(\overline{X})_{\cdot}$ are submartingales

the both processes $(\underline{X}, \overline{X})$ are continuous processes, and the process $Y = \overline{X}_{\cdot} - K_{\cdot}$ is a $\mathcal{Q}(K)$-semimartingale.

- Not sufficient to show that $K_{\cdot} = \Lambda_{\cdot} + Y_{\cdot} + |Y_{\cdot}| * C_{\cdot}$.
- In place, use the processes

$$U^{\Lambda,C}_t(e^{Y}) = e^{Y_t} + \int_0^t e^{Y_s} d\Lambda_s + \int_0^t e^{Y_s} |Y_s|\, dC_s$$

Proof

- From the quadratic submartingale decomposition
$$\overline{X}_{.} = \overline{X}_0 + \overline{M}_{.} - \tfrac{1}{2}\langle \overline{M}\rangle_{.} + \overline{A}_{.} \quad \text{and} \quad \underline{X}_{.} = \underline{X}_0 + \underline{M}_{.} - \tfrac{1}{2}\langle \underline{M}\rangle_{.} + \underline{A}_{.}$$
- By uniqueness of the predictable decomposition of X and $-X$,
$$\underline{M}_{.} = -\overline{M}_{.} \text{ and so } \langle \underline{M}\rangle = \langle \overline{M}\rangle, \text{ and } \overline{A}_{.} + \underline{A}_{.} = \langle M\rangle_{.} + 2K_{.}.$$
Since $\langle M\rangle$ and $K_{.}$ are continuous, both increasing processes $\overline{A}$ and $\underline{A}$ are also continuous and then $\overline{X}$ and $\underline{X}$.
- From Radon-Nikodym's Theorem, $d\overline{A}_t = \alpha_t d(\frac{1}{2}\langle M\rangle_t + K_t)$ with $0 \leq \alpha_t \leq 2$.
Substituting $\overline{A}$ into the decomposition of Y, we get
$$dY_t = -(1-\alpha_t)d(\tfrac{1}{2}\langle M\rangle_t + K_t) + dM_t \text{ . with } |1-\alpha_t| \leq 1$$
Therefore, Y is a $\mathcal{Q}(K, 0, 1)$-semimartingale.

Plan

The class $(\mathcal{D}_{\exp})$ and Inequalities I

Definition of the class $(\mathcal{D}_{\exp})$

- The "class (D)", ((D) for Doob)=Optional process X, s.t $|X|$ is dominated by a u.i martingale.
- X such that $\exp(|X|)$ is in class $(\mathcal{D})$ is in the class $(\mathcal{D}_{\exp})$.
- $(\mathcal{D})$-submartingales S are characterized by "submartingale inequalities"

$$\text{for } \sigma \leq \tau \leq T, \quad S_\sigma \leq \mathbb{E}[S_\tau | \mathcal{F}_\sigma], \text{ a.s.}.$$

- If S is positive, $X_. = \ln S_.$ verifies the so-called *entropic inequalities* : $X_\sigma \leq \rho_\sigma(X_\tau)$ *a.s.*

Submartingale Inequalities and Characterization I

$\mathcal{Q}$ semimartingales characterization.

An optional process X with $\exp(|X_T|) \in \mathbb{L}^1$ is a $\mathcal{Q}$-semimartingale in $(\mathcal{D}_{\exp})$ if and only if $\underline{\rho}_\sigma(X_\tau) \leq X_\sigma \leq \rho_\sigma(X_\tau)$ *a.s.*

$\mathcal{Q}(\Lambda, C)$ semimartingales characterization. : (Briand and Hu)

Assume $\bar{X}_T^{\Lambda,C}(|Y_T|) = e^{C_T}|Y_T| + \int_0^T e^{C_s} d\Lambda_s$ in $\mathbb{L}^1_{\exp}$ and

- $|Y_t| \leq \rho_t(e^{C_{t,T}}|Y_T| + \int_t^T e^{C_{t,s}} d\Lambda_s) = \ln \Phi_t(|Y_T|)$
- or equivalently $\exp(e^{C_t}|Y_t| + \int_0^t e^{C_s} d\Lambda_s)$ is a $(\mathcal{D})$ submartingale

Submartingale Inequalities and Characterization II

- $\Phi_{\cdot}(|Y_T|)$ and $U_{\cdot}^{\Lambda,C}(\Phi(|Y_T|))$ are $(\mathcal{D})$-supermartingales.
- any Y s.t $e^{|Y_{\cdot}|} \leq \Phi_t(|Y_T|)$ is a $\mathcal{Q}(\Lambda, C)$ semimartingale if and only if $U_{\cdot}^{\Lambda,C}(e^{Y})$ and $U_{\cdot}^{\Lambda,C}(e^{-Y})$ satisfies submartingale inequalities.
- Y is said to be in $\mathcal{S}_Q(|\mathcal{Y}_T|, \Lambda, C)$

Plan

Quadratic variation estimates I

Theorem

Let $Y_\cdot \in \mathcal{S}_Q(|\eta_T|, \Lambda, C)$, *and* $\bar{X}_T^{\Lambda,C}(|Y_T|) = e^{C_T}|\eta_T| + \int_0^T e^{C_s} d\Lambda_s$.

- $\frac{1}{2}\mathbb{E}[\langle M\rangle_{S,T}|\mathcal{F}_S] \leq \Phi_{S,T}(|\eta_T|)\mathbf{1}_{\{S<T\}}$
- *Let* $p^\eta = \sup\{p;\ \mathbb{E}\big[\exp(p\bar{X}_T^C)\big] < +\infty\}$. *Then* $p^\eta \geq 1$ *and* $\forall p \in [1, p^\eta[$, $\mathbb{E}[\langle M\rangle_T^p] \leq (2p)^p\, \mathbb{E}\big[\exp(p\bar{X}_T^C)\big]$.
- *If for any* $S \leq T$, $\Phi_{S,T}(|\eta_T|)$ *is bounded, then the martingale* M *is in BMO.*

Sketch proof

- From Kobylanski, if $v(x) = e^x - 1 - x$, and $V_t^{\Lambda,C}(e^{|Y|}) = v(|Y_t|) + \int_0^t v'(|Y_s|)(d\Lambda_s + |Y_s|dC_s)$, then $V_t^{\Lambda,C}(e^{|Y|}) - \frac{1}{2}\langle M\rangle_t$ is a $(\mathcal{D})$-submartingale.
- Neveu-Garsia Lemma

Neveu-Garsia Lemma I

The Lemma Let $A_.$ a predictable increasing process and U a random variable, positive and integrable. Assume that for $S \leq T$, $\mathbb{E}[A_T - A_S \mathbf{1}_{\{0<S\leq T\}}|\mathcal{F}_S] \leq \mathbb{E}[U\mathbf{1}_{\{S<T\}}|\mathcal{F}_S]$,

- $\forall r \geq 1, \quad \mathbb{E}[A_T^r] \leq r^r \mathbb{E}[U^r)]$
- More generally, for any convex function F s.t $p = \sup_{x>0}(x(\ln F)'(x)) < +\infty$,
$$\mathbb{E}[F(A_T)] \leq \mathbb{E}[F(pU)].$$

Total Variation Estimates The total variation of $V_.$ s.t. $Y_. = Y_0 + M_. - V_.$ satisfies for $1 \leq p < p^\eta$
$$\mathbb{E}[|V|_T^p] \leq (2p)^p \, \mathbb{E}\big[\exp(p\bar{X}_T^C)\big],$$

Strong convergence of martingale parts I

Theorem

Assume the sequence $(Y_.^n)$ of $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$-quasimartingales to be a Cauchy sequence for the a.s. uniform convergence, i.e. $\sup_{t \leq T} |Y_t^n - Y_t^{n+p}|$ tends to 0 almost surely when $n \to \infty$. Different types of convergence hold true for the processes $(M_.^n, V_.^n)$ of the decomposition $Y_.^n = Y_0^n + M_.^n - V_.^n$

Martingales convergence

- *The sequence of martingales $(M_.^n)$ converges to a continuous martingale $M_.$ in $\mathbb{H}^1$.*
- *If, for some $p > 1$, $\bar{X}_T^{\Lambda,C}(|\eta_T|) \in \mathbb{L}^p_{\exp}$, the sequence of martingales $(M_.^n)$ converges to a continuous martingale $M_.$ in $\mathbb{H}^{2p}$.*
- *If $\Phi(|\eta_T|)$ is bounded, the sequence of martingales $(M_.^n)$ converges to a continuous martingale $M_.$ in the BMO-space.*

Theorem

Semimartingale convergence

- *The sequence* $(V^n_.)$ *converges uniformly to a finite variation process* $V_.$ *satisfying the structure condition* $\mathcal{Q}(\Lambda, C)$ *at least in* $\mathbb{L}^1$.
- *The sequence of* $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$*-quasimartingales*$(Y^n_.)$ *converges to the continuous quadratic* $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$*-quasimartingale* $Y_. = Y_0 + M_. - V_.$

Sketch of the proof$(B^{i,j} = |V^i| + |V^j|)$

$$\mathbb{E}\big[\langle M^{i,j}_{\sigma,T}\rangle \mid \mathcal{F}_\sigma\big] \leq \mathbb{E}\big[\max|Y^{i,j}_{\sigma,T}|^2\, \mathbf{1}_{\{\sigma<T\}} | \,\mathcal{F}_\sigma\big] + \mathbb{E}\big[\int_\sigma^T \max|Y^{i,j}_{\sigma,s}|\, dB$$

$$\leq \mathbb{E}\big[(\max|Y^{i,j}_T|^2 + \max|Y^{i,j}_T|B^{i,j}_T)\, \mathbf{1}_{\{\sigma<T\}} | \,\mathcal{F}_\sigma\big]$$

$+$ Neveu-Garsia Lemma

Monotone convergence I

Theorem

Let assume the sequence $(Y^n_\cdot)$ to be a monotone sequence of $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$-quasimartingales converging almost surely to a process $Y_\cdot$. Then, $Y_\cdot$ is continuous, the convergence is uniform and all properties given in previous Theorem hold true.

Plan

Quadratic BSDEs I

- An equation $-dY_t = g(t, Y_t, Z_t)dt - Z_t dW_t, \quad Y_T = \xi_T.$, where W is a d-dimensional BM
- Quadratic BSDE= quadratic coefficient=$d\mathbb{P} \otimes dt-$a.s.

$$|g(., t, y, z)| \leq Q(t, y, z) \equiv 1/\delta|l_t| + c_t|y| + \frac{\delta}{2}|z|^2,$$

The "linear" case$|g(., t, y, z)| \leq |l_t| + c_t|y| + k_t|z|$

- Main observation

Linear $\Rightarrow$ quadratic

- $k_t|z| \leq \frac{1}{2}(\frac{1}{\varepsilon}|k_t|^2 + \varepsilon(|z|^2)$

Strongly Quadratic BSDEs I

- Let g be a coefficient satisfying $g(t,y,z) = f(t,y,z) + \frac{\delta}{2}|z|^2$, where $f(t,y,z) \leq l_t + c_t|y| + k|y|$.
- Assume that $\bar{X}_T = exp((\delta+\epsilon)(e^{C}_{T}|\xi_T| + \int_0^T e^{C_s}(l_s + \frac{1}{2}k_\epsilon^2)ds$ is finite,

  and let $\phi_t^\epsilon = \rho_{\delta+\epsilon,t}(e^{C}_{t,T}|\xi_T| + \int_t^T e^{C_{t,s}}(l_s + \frac{1}{2}k_\epsilon^2)ds)$

Then, there exists a maximal solution of the BSDEs satisfying $Y \leq \phi^\epsilon$ obtained from the linear growth BSDEs with generator

$$g_n = f(t,y,z) + \frac{\delta}{2}|z| \inf(|z|, n)$$

The General Existence result I

Theorem

Assume that $\mathbb{E}[\bar{X}_T = exp(p(\delta(e^C_T|\xi_T| + \int_0^T e^{C_s}(l_s ds)ds]$ *is finite, and let* $\phi_t = \rho_{\delta,t}(e^C_{t,T}|\xi_T| + \int_t^T e^{C_{t,s}} ss(l_s ds)$

- *The strongly quadratic growth coefficient* g^n *defined as :* $g^n(t,y,z) = g(t,y,z) \vee (-c_l + ac - \delta\, n\,|z| + \frac{\delta}{2}|z|^2)$ *are decreasing to* g
- *There exists a minimal solution* (Y^n, Z^n) *dominated by* ϕ_t $-dY^n_t = g^n(t, Y^n_t, Z^n_t)dt - Z^n_t dW_t$ *and the sequence* Y^n *is decreasing.*
- *There exists a minimal solution* (Y, Z) *dominated by* ϕ *to the BSDE,* $-dY_t = g(t, Y_t, Z_t)dt - Z_t dW_t.$

Perspectives

- With Anis and his PhD-student, Quadratic BSDEs with jumps
- Different extensions and BMO case
- Optimisation problems

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Thank you for your attention