Exponential and power utility maximization problems under partial information: some convergence results.

Marina Santacroce

Politecnico di Torino

Joint work with D. Covello and M. Mania

New advances in Backward SDEs for financial engineering applications
Tamerza, 25th October 2010

Outline

Outline

Utility maximization under partial information: semimartingale setting

Semimartingale model
Expected utility and partial information
Equivalent problem and solution

Exponential utility maximization

Assumptions
Equivalent problem
Value process and BSDE

Outline

Outline

Outline

The model

- Let $S = (S_t, t \in [0, T])$ be a continuous semimartingale which represents the returns process of the traded asset.
- $(\Omega, \mathcal{A}, \mathscr{A} = (\mathscr{A}_t, t \in [0, T]), P)$, where $\mathcal{A} = \mathscr{A}_T$ and $T < \infty$ is a fixed time horizon.
- Assume the interest rate equal to zero.

The model

- Let $S = (S_t, t \in [0, T])$ be a continuous semimartingale which represents the returns process of the traded asset.
- $(\Omega, \mathcal{A}, \mathscr{A} = (\mathscr{A}_t, t \in [0, T]), P)$, where $\mathcal{A} = \mathscr{A}_T$ and $T < \infty$ is a fixed time horizon.
- Assume the interest rate equal to zero.

The process S admits the decomposition

$$S_t = S_0 + N_t + \int_0^t \lambda_u d\langle N \rangle_u, \quad \langle \lambda \cdot N \rangle_T < \infty \quad a.s.,$$

where N is a continuous $\mathscr{A}$-local martingale and λ is a $\mathscr{A}$-predictable process (Structure condition).

Utility maximization and partial information

Denote by $\mathscr{G} = (\mathscr{G}_t, t \in [0, T])$ a filtration smaller than $\mathscr{A}$

$$\mathscr{G}_t \subseteq \mathscr{A}_t, \quad \text{for every} \quad t \in [0, T].$$

$\mathscr{G}$ represents the information available to the investor.

Utility maximization and partial information

Denote by $\mathscr{G} = (\mathscr{G}_t, t \in [0, T])$ a filtration smaller than $\mathscr{A}$

$$\mathscr{G}_t \subseteq \mathscr{A}_t, \quad \text{for every} \quad t \in [0, T].$$

$\mathscr{G}$ represents the information available to the investor.

We consider the utility maximization problem (with random payoff H at time T) when $\mathscr{G}$ is the available information,

$$\text{maximize} \quad E[U(X_T^{x,\pi} - H)] \quad \text{over all} \quad \pi \in \Pi(\mathscr{G}).$$

- $\Pi(\mathscr{G})$ is a certain class of self-financing strategies ($\mathscr{G}$-predictable and S-integrable processes).

We see in some detail the exponential case

- $U(x) = -e^{-\alpha x}$.

Then we will briefly consider the problem when $H = 0$ for

- $U(x) = \frac{x^p}{p}$.

In most papers, under various setups, (see, e.g., Lakner (1998), Pham and Quenez (2001), Zohar (2001)) expected utility maximization problems have been considered for market models where only stock prices are observed, while the drift can not be directly observed.
$\Longrightarrow$ under the hypothesis $\mathscr{F}^S \subseteq \mathscr{G}$.

We consider the case when $\mathscr{G}$ does not necessarily contain all information on the prices of the traded asset i.e.

S is not a $\mathscr{G}$-semimartingale in general!

In most papers, under various setups, (see, e.g., Lakner (1998), Pham and Quenez (2001), Zohar (2001)) expected utility maximization problems have been considered for market models where only stock prices are observed, while the drift can not be directly observed.
$\Longrightarrow$ under the hypothesis $\mathscr{F}^S \subseteq \mathscr{G}$.

We consider the case when $\mathscr{G}$ does not necessarily contain all information on the prices of the traded asset i.e.

S is not a $\mathscr{G}$-semimartingale in general!

$\Longrightarrow$ In this case, we solve the problem in 2 steps:

In most papers, under various setups, (see, e.g., Lakner (1998), Pham and Quenez (2001), Zohar (2001)) expected utility maximization problems have been considered for market models where only stock prices are observed, while the drift can not be directly observed.
$\Longrightarrow$ under the hypothesis $\mathscr{F}^S \subseteq \mathscr{G}$.

We consider the case when $\mathscr{G}$ does not necessarily contain all information on the prices of the traded asset i.e.

S is not a $\mathscr{G}$-semimartingale in general!

$\Longrightarrow$ In this case, we solve the problem in 2 steps:

- Step 1: Prove that the expected utility maximization problem is equivalent to another maximization problem of the filtered terminal net wealth (reduced problem)
- Step 2: Apply the dynamic programming method to the reduced problem.

(In Mania et al. (2008) a similar approach is used in the context of mean variance hedging).

Filtration $\mathscr{F}$ and decomposition of S w.r.t. $\mathscr{F}$

- Let $\mathscr{F} = (\mathscr{F}_t, t \in [0, T])$ be the augmented filtration generated by $\mathscr{F}^S$ and $\mathscr{G}$.

Filtration $\mathscr{F}$ and decomposition of S w.r.t. $\mathscr{F}$

- Let $\mathscr{F} = (\mathscr{F}_t, t \in [0, T])$ be the augmented filtration generated by $\mathscr{F}^S$ and $\mathscr{G}$.
- S is a $\mathscr{F}$-semimartingale:

$$S_t = S_0 + \int_0^t \widehat{\lambda}_u^{(\mathscr{F})} d\langle M\rangle_u + M_t,$$

(Decomposition of S with respect to $\mathscr{F}$)

$$M_t = N_t + \int_0^t [\lambda_u - \widehat{\lambda}_u^{(\mathscr{F})}] d\langle N\rangle_u \quad \text{is} \quad \mathscr{F}\text{-local martingale}$$

where we denote by $\widehat{\lambda}^{(\mathscr{F})}$ the $\mathscr{F}$-predictable projection of λ.
- Note that $\langle M\rangle = \langle N\rangle$ are $\mathscr{F}^S$-predictable.

Assumptions

In the sequel we will make the following assumptions:

A) $\langle M \rangle$ is $\mathscr{G}$-predictable and $d\langle M \rangle_t dP$ a.e. $\widehat{\lambda}^{\mathscr{F}} = \widehat{\lambda}^{\mathscr{G}}$, hence for each t

$$E(\lambda_t | \mathscr{F}^S_{t-} \vee \mathscr{G}_t) = E(\lambda_t | \mathscr{G}_t), \quad P - \text{a.s.}$$

B) any $\mathscr{G}$-martingale is a $\mathscr{F}$-local martingale,

C) the filtration $\mathscr{G}$ is continuous,

D) for any $\mathscr{G}$-local martingale $m(g)$ $\langle M, m(g) \rangle$ is $\mathscr{G}$-predictable,

E) H is an $\mathscr{A}_T$-measurable bounded random variable, such that P- a.s.

$$E[e^{\alpha H} | \mathscr{F}_T] = E[e^{\alpha H} | \mathscr{G}_T],$$

Assumptions

In the sequel we will make the following assumptions:

A) $\langle M \rangle$ is $\mathscr{G}$-predictable and $d\langle M \rangle_t dP$ a.e. $\widehat{\lambda}^{\mathscr{F}} = \widehat{\lambda}^{\mathscr{G}}$, hence for each t

$$E(\lambda_t | \mathscr{F}^S_{t-} \vee \mathscr{G}_t) = E(\lambda_t | \mathscr{G}_t), \quad P - \text{a.s.}$$

B) any $\mathscr{G}$-martingale is a $\mathscr{F}$-local martingale,

C) the filtration $\mathscr{G}$ is continuous,

D) for any $\mathscr{G}$-local martingale $m(g)$ $\langle M, m(g) \rangle$ is $\mathscr{G}$-predictable,

E) H is an $\mathscr{A}_T$-measurable bounded random variable, such that P- a.s.

$$E[e^{\alpha H} | \mathscr{F}_T] = E[e^{\alpha H} | \mathscr{G}_T],$$

$\Rightarrow$ If $\mathscr{F}^S \subseteq \mathscr{G}$, then $\langle M \rangle$ is $\mathscr{G}$-predictable. Conditions A), B), D) and the equality in E) are *automatically* satisfied.

Assumptions

In the sequel we will make the following assumptions:

A) $\langle M \rangle$ is $\mathscr{G}$-predictable and $d\langle M \rangle_t dP$ a.e. $\widehat{\lambda}^{\mathscr{F}} = \widehat{\lambda}^{\mathscr{G}}$, hence for each t

$$E(\lambda_t | \mathscr{F}^S_{t-} \vee \mathscr{G}_t) = E(\lambda_t | \mathscr{G}_t), \quad P - \text{a.s.}$$

B) any $\mathscr{G}$-martingale is a $\mathscr{F}$-local martingale,

C) the filtration $\mathscr{G}$ is continuous,

D) for any $\mathscr{G}$-local martingale $m(g)$ $\langle M, m(g) \rangle$ is $\mathscr{G}$-predictable,

E) H is an $\mathscr{A}_T$-measurable bounded random variable, such that P- a.s.

$$E[e^{\alpha H} | \mathscr{F}_T] = E[e^{\alpha H} | \mathscr{G}_T],$$

Let $\widehat{S}_t = E(S_t | \mathscr{G}_t)$ be the $\mathscr{G}$-optional projection of S_t. Since $\widehat{\lambda}^{\mathscr{F}} = \widehat{\lambda}^{\mathscr{G}} = \widehat{\lambda}$

$$\widehat{S}_t = E(S_t | \mathscr{G}_t) = S_0 + \int_0^t \widehat{\lambda}_u d\langle M \rangle_u + \widehat{M}_t$$

where $\widehat{M}_t$ is the $\mathscr{G}$-local martingale $E(M_t | \mathscr{G}_t)$.

Equivalent problem

We consider $U(x) = -e^{-\alpha(x)}$ and we rewrite the related problem as

$$\text{minimize} \quad E[e^{-\alpha(\int_0^T \pi_u dS_u - H)}] \quad \text{over all} \quad \pi \in \Pi(\mathscr{G}). \tag{1}$$

where the class of strategies is defined as

$$\Pi(\mathscr{G}) = \{\pi : \mathscr{G} - \text{predictable}, \pi \cdot M \in BMO(\mathscr{F})\}$$

(w.l.g. we put the initial capital $x = 0$).

Equivalent problem

We consider $U(x) = -e^{-\alpha(x)}$ and we rewrite the related problem as

$$\text{minimize} \quad E[e^{-\alpha(\int_0^T \pi_u dS_u - H)}] \quad \text{over all} \quad \pi \in \Pi(\mathscr{G}). \tag{1}$$

where the class of strategies is defined as

$$\Pi(\mathscr{G}) = \{\pi : \mathscr{G} - \text{predictable}, \pi \cdot M \in BMO(\mathscr{F})\}$$

(w.l.g. we put the initial capital $x = 0$).

PROPOSITION Let conditions A)-E) be satisfied. Then the optimization problem (1) is equivalent to

$$\text{minimize} \quad E[e^{-\alpha(\int_0^T \pi_u d\widehat{S}_u - \widetilde{H}) + \frac{\alpha^2}{2} \int_0^T \pi_u^2 (1-\kappa_u^2) d\langle M\rangle_u}], \text{ over all } \pi \in \Pi(\mathscr{G}) \tag{2}$$

$$\widetilde{H} = \frac{1}{\alpha} \ln E[e^{\alpha H} | \mathscr{G}_T], \quad \kappa_t^2 = \frac{d\langle \widehat{M}\rangle_t}{d\langle M\rangle_t}.$$

Remarks

- The previous proposition says that the optimization problems (1) and (2) are equivalent.

Remarks

- The previous proposition says that the optimization problems (1) and (2) are equivalent.
- It is sufficient to solve problem (2), which is formulated in terms of $\mathscr{G}$-adapted processes.

Remarks

- The previous proposition says that the optimization problems (1) and (2) are equivalent.
- It is sufficient to solve problem (2), which is formulated in terms of $\mathscr{G}$-adapted processes.
- We can see (2) as an exponential hedging problem under complete information with a (multiplicative) correction term and we can solve it using methods for complete information.

Remarks

- The previous proposition says that the optimization problems (1) and (2) are equivalent.
- It is sufficient to solve problem (2), which is formulated in terms of $\mathscr{G}$-adapted processes.
- We can see (2) as an exponential hedging problem under complete information with a (multiplicative) correction term and we can solve it using methods for complete information.

Let

$$V_t = \operatorname*{ess\,inf}_{\pi \in \Pi(\mathscr{G})} E[e^{-\alpha(\int_t^T \pi_u d\widehat{S}_u - \widetilde{H}) + \frac{\alpha^2}{2} \int_t^T \pi_u^2 (1-\kappa_u^2) d\langle M \rangle_u} | \mathscr{G}_t],$$

be the value process related to the *equivalent* problem.

THEOREM Under assumptions A)-E) and $\int_0^T \widehat{\lambda}_t^2 d\langle M\rangle_t \leq C$,

the value process V related to the equivalent problem (2) is the unique bounded strictly positive solution of the following BSDE

$$Y_t = Y_0 + \frac{1}{2}\int_0^t \frac{(\psi_u \kappa_u^2 + \widehat{\lambda}_u Y_u)^2}{Y_u} d\langle M\rangle_u + \int_0^t \psi_u d\widehat{M}_u + L_t \tag{3}$$

$$Y_T = E[e^{\alpha H}|\mathscr{G}_T]$$

Moreover the optimal strategy exists in the class $\Pi(\mathscr{G})$ and is equal to

$$\pi_t^* = \frac{1}{\alpha}(\widehat{\lambda}_t + \frac{\psi_t \kappa_t^2}{Y_t}). \tag{4}$$

THEOREM Under assumptions A)-E) and $\int_0^T \widehat{\lambda}_t^2 d\langle M\rangle_t \leq C$,

the value process V related to the equivalent problem (2) is the unique bounded strictly positive solution of the following BSDE

$$Y_t = Y_0 + \frac{1}{2}\int_0^t \frac{(\psi_u \kappa_u^2 + \widehat{\lambda}_u Y_u)^2}{Y_u} d\langle M\rangle_u + \int_0^t \psi_u d\widehat{M}_u + L_t \tag{3}$$

$$Y_T = E[e^{\alpha H}|\mathscr{G}_T]$$

Moreover the optimal strategy exists in the class $\Pi(\mathscr{G})$ and is equal to

$$\pi_t^* = \frac{1}{\alpha}(\widehat{\lambda}_t + \frac{\psi_t \kappa_t^2}{Y_t}). \tag{4}$$

$\Longrightarrow$ *We prove the existence of a solution using results of Tevzadze (2008) (see also Morlais (2008) for related results) and uniqueness by directly showing that the unique solution of the BSDE is the value of the problem.*

THEOREM Under assumptions A)-E) and $\int_0^T \widehat{\lambda}_t^2 d\langle M\rangle_t \leq C$,

the value process V related to the equivalent problem (2) is the unique bounded strictly positive solution of the following BSDE

$$Y_t = Y_0 + \frac{1}{2}\int_0^t \frac{(\psi_u \kappa_u^2 + \widehat{\lambda}_u Y_u)^2}{Y_u} d\langle M\rangle_u + \int_0^t \psi_u d\widehat{M}_u + L_t \tag{3}$$
$$Y_T = E[e^{\alpha H}|\mathscr{G}_T]$$

Moreover the optimal strategy exists in the class $\Pi(\mathscr{G})$ and is equal to

$$\pi_t^* = \frac{1}{\alpha}(\widehat{\lambda}_t + \frac{\psi_t \kappa_t^2}{Y_t}). \tag{4}$$

$\Longrightarrow$ *We prove the existence of a solution using results of Tevzadze (2008) (see also Morlais (2008) for related results) and uniqueness by directly showing that the unique solution of the BSDE is the value of the problem.*

$\Longrightarrow$ *If* $\mathscr{G}_t = \mathscr{A}_t \Rightarrow \widehat{M}_t = M_t = N_t, \quad \widehat{\lambda}_t = \lambda_t, \; Y_T = e^{\alpha H}$: *the bsde takes on the form*

$$Y_t = Y_0 + \frac{1}{2}\int_0^t \frac{(\psi_u + \lambda_u Y_u)^2}{Y_u} d\langle N\rangle_u + \int_0^t \psi_u dN_u + L_t, \quad Y_T = e^{\alpha H}.$$

Partial information and power utility maximization

Consider the problem of maximizing the *power* utility of terminal wealth when $\mathscr{G}$ is the available information.

$$\text{maximize} \quad E\left[\frac{(X_T^{x,\pi})^p}{p}\right] \quad \text{over all } \pi \in \Pi(G),$$

where $\Pi(\mathscr{G})$ is a certain class of ($\mathscr{G}$-predictable) strategies.

- x represents the initial endowment (we set $x = 1$)
- the strategy π denotes the proportion of wealth invested in the asset

$\Rightarrow$ the wealth process related to the self-financing strategy π is $X_t^\pi = 1 + \int_0^t \pi_u X_{u-}^\pi dS_u$

Partial information and power utility maximization

Consider the problem of maximizing the *power* utility of terminal wealth when $\mathscr{G}$ is the available information.

$$\text{maximize} \quad E\left[\frac{(X_T^{x,\pi})^p}{p}\right] \quad \text{over all } \pi \in \Pi(G),$$

where $\Pi(\mathscr{G})$ is a certain class of ($\mathscr{G}$-predictable) strategies.

- x represents the initial endowment (we set $x = 1$)
- the strategy π denotes the proportion of wealth invested in the asset

$\Rightarrow$ the wealth process related to the self-financing strategy π is $X_t^\pi = 1 + \int_0^t \pi_u X_{u-}^\pi dS_u$

We rewrite the problem in exponential form

$$\text{minimize} \quad E\left[\mathcal{E}_T^p(\pi \cdot S))\right] \quad \text{over all } \pi \in \Pi(G),$$

where $\mathcal{E}(X)$ denotes the Doléans-Dade exponential of X.

Equivalent problem

The problem is

$$\text{minimize} \quad E\left[\mathcal{E}_T^p(\pi\cdot S)\right] \quad \text{over all} \quad \pi\in\Pi(G), \tag{5}$$

where the class of strategies is defined as

$$\Pi(\mathscr{G}) = \{\pi : \mathscr{G}-\text{predictable}, \ \pi\cdot M\in BMO(\mathscr{F})\}$$

PROPOSITION Let conditions A)-D) be satisfied. Then the optimization problem (5) is equivalent to

$$\text{minimize} \quad E[\mathcal{E}_T^p(\pi\cdot\widehat{S})\, e^{\frac{p(p-1)}{2}\int_0^T \pi_u^2(1-\kappa_u^2)d\langle M\rangle_u}] \ \text{over all} \quad \pi\in\Pi(G). \tag{6}$$

where $\kappa_t^2 = \frac{d\langle\widehat{M}\rangle_t}{d\langle M\rangle_t}$.

Equivalent problem

The problem is

$$\text{minimize} \quad E\left[\mathcal{E}_T^p(\pi \cdot S)\right] \quad \text{over all} \quad \pi \in \Pi(G), \tag{5}$$

where the class of strategies is defined as

$$\Pi(\mathscr{G}) = \{\pi : \mathscr{G} - \text{predictable}, \ \pi \cdot M \in BMO(\mathscr{F})\}$$

PROPOSITION Let conditions A)-D) be satisfied. Then the optimization problem (5) is equivalent to

$$\text{minimize} \quad E[\mathcal{E}_T^p(\pi \cdot \widehat{S})\ e^{\frac{p(p-1)}{2}\int_0^T \pi_u^2(1-\kappa_u^2)d\langle M\rangle_u}] \ \text{over all} \quad \pi \in \Pi(G). \tag{6}$$

where $\kappa_t^2 = \frac{d\langle \widehat{M}\rangle_t}{d\langle M\rangle_t}$.

The value process related to the *reduced* problem is

$$V_t(p) = \operatorname*{ess\,inf}_{\pi \in \Pi(G)} E[\mathcal{E}_{tT}^p(\pi \cdot \widehat{S})\ \exp\{\frac{p(p-1)}{2}\int_t^T \pi_u^2(1-\kappa_u^2)d\langle M\rangle_u\}|G_t].$$

BSDE related to power utilities maximization

THEOREM Under assumptions A)-D) and $\int_0^T \widehat{\lambda}_t^2 d\langle M\rangle_t \leq C$,

the value process associated to the power utility maximization problem is characterized as the unique bounded positive solution of

$$Y_t = Y_0 + \frac{p}{2(p-1)} \int_0^t Y_u(\widehat{\lambda}_u + \frac{\psi_u \kappa_u^2}{Y_u})^2 d\langle M\rangle_u + \int_0^t \psi_u d\widehat{M}_u + L_t, \quad Y_T = 1$$

and the optimal strategy is

$$\pi_t^* = \frac{1}{1-p}(\widehat{\lambda}_t + \frac{\psi_t \kappa_t^2}{Y_t})$$

BSDE related to power utilities maximization

THEOREM Under assumptions A)-D) and $\int_0^T \widehat{\lambda}_t^2 d\langle M\rangle_t \leq C$,

the value process associated to the power utility maximization problem is characterized as the unique bounded positive solution of

$$Y_t = Y_0 + \frac{p}{2(p-1)} \int_0^t Y_u(\widehat{\lambda}_u + \frac{\psi_u \kappa_u^2}{Y_u})^2 d\langle M\rangle_u + \int_0^t \psi_u d\widehat{M}_u + L_t, \quad Y_T = 1$$

and the optimal strategy is

$$\pi_t^* = \frac{1}{1-p}(\widehat{\lambda}_t + \frac{\psi_t \kappa_t^2}{Y_t})$$

LET US COMPARE THIS BSDE WITH THE BSDE RELATED TO THE EXPONENTIAL UTILITY MAXIMIZATION FOR $H = 0$ AND $\alpha = 1$

BSDEs and unified characterization

The BSDE related to exponential utility maximization (with $\alpha = 1$ and $H = 0$) is

$$Y_t = Y_0 + \frac{1}{2}\int_0^t Y_u(\widehat{\lambda}_u + \frac{\psi_u \kappa_u^2}{Y_u})^2 d\langle M\rangle_u + \int_0^t \psi_u d\widehat{M}_u + L_t, \quad Y_T = 1$$

and the one related to power utility maximization is

$$Y_t(q) = Y_0(q) + \frac{q}{2}\int_0^t Y_u(q)(\widehat{\lambda}_u + \frac{\psi_u(q)\kappa_u^2}{Y_u(q)})^2 d\langle M\rangle_u + \int_0^t \psi_u(q) d\widehat{M}_u + L_t(q), \; Y_T(q) = 1.$$

where $q = \frac{p}{p-1}$.

$\Rightarrow$ the value process of the exponential corresponds to $q = 1$.

BSDEs and unified characterization

The BSDE related to exponential utility maximization (with $\alpha = 1$ and $H = 0$) is

$$Y_t = Y_0 + \frac{1}{2}\int_0^t Y_u(\widehat{\lambda}_u + \frac{\psi_u \kappa_u^2}{Y_u})^2 d\langle M\rangle_u + \int_0^t \psi_u d\widehat{M}_u + L_t, \quad Y_T = 1$$

and the one related to power utility maximization is

$$Y_t(q) = Y_0(q) + \frac{q}{2}\int_0^t Y_u(q)(\widehat{\lambda}_u + \frac{\psi_u(q)\kappa_u^2}{Y_u(q)})^2 d\langle M\rangle_u + \int_0^t \psi_u(q) d\widehat{M}_u + L_t(q), \; Y_T(q) = 1.$$

where $q = \frac{p}{p-1}$.

$\Rightarrow$ the value process of the exponential corresponds to $q = 1$.

In the context of full information Mania and Tevzadze (2003) provide a similar *unified characterization* to study the convergence of q-optimal martingale measures to the minimal entropy martingale measure (see also Hobson (2004) for related results for stochastic volatility models).

$\Longrightarrow$ We will use the BSDE characterization to receive the *convergence of the optimal strategies* for the utility optimization problems.

(See Nutz (2010) for related results in full information)

- Aim: Study the *convergence of the optimal strategies* of the power utility maximization problem to the one related to the exponential problem as $p \to -\infty$, hence as $q = \frac{p}{p-1} \to 1$
- Remark: In partial information, we can not resort to duality arguments and we can not receive the convergence of the strategies using the convergence of utility functions.

$\Rightarrow$ Our approach will use the characterization of the optimal strategies through the BSDEs.

- The convergence of strategies in full information can be obtained as a corollary.

- Aim: Study the *convergence of the optimal strategies* of the power utility maximization problem to the one related to the exponential problem as $p \to -\infty$, hence as $q = \frac{p}{p-1} \to 1$
- Remark: In partial information, we can not resort to duality arguments and we can not receive the convergence of the strategies using the convergence of utility functions.

$\Rightarrow$ Our approach will use the characterization of the optimal strategies through the BSDEs.

- The convergence of strategies in full information can be obtained as a corollary.

Recall the optimal strategies are respectively:

$$\pi^*(q) = (1-q)(\widehat{\lambda} + \frac{\psi(q)\kappa^2}{Y(q)}) \quad \text{and} \quad \pi^*(1) = \widehat{\lambda} + \frac{\psi(1)\kappa^2}{Y(1)}$$

taking in mind that $\psi(q)$ and $Y(q)$ are part of the solution of the BSDE(q).

The main point consists in studying the family of BSDE(q) (varying with the parameter q) and in particular find some *estimates* which involves the *martingale part of the solution*.

Idea of the proof

$\Longrightarrow$ The proof can be roughly summarized as follows:

Step 1 Find an *estimate* for a *proper function of* $Y(q)$ *and* $Y(1)$, namely

$$|\ln Y(1) - q \ln Y(q)| \leq c|1-q|.$$

$Y(q)$ ($Y(1)$) stands for the "solution" of the generic (respectively $q = 1$) element of the family of the BSDEs

Idea of the proof

$\Longrightarrow$ The proof can be roughly summarized as follows:

Step 1 Find an *estimate* for a *proper function of* $Y(q)$ *and* $Y(1)$, namely

$$|\ln Y(1) - q \ln Y(q)| \leq c|1-q|.$$

$Y(q)$ ($Y(1)$) stands for the "solution" of the generic (respectively $q = 1$) element of the family of the BSDEs

Step 2 (Main result) *Convergence of the martingale part* of $\ln Y(q)$:
$q\frac{\psi(q)}{Y(q)} \cdot \widehat{M} \to \frac{\psi(1)}{Y(1)} \cdot \widehat{M}$ as $q \to 1$, (in BMO).

Idea of the proof

$\Longrightarrow$ The proof can be roughly summarized as follows:

Step 1 Find an *estimate* for a *proper function of $Y(q)$ and $Y(1)$*, namely

$$|\ln Y(1) - q \ln Y(q)| \leq c|1-q|.$$

$Y(q)$ ($Y(1)$) stands for the "solution" of the generic (respectively $q = 1$) element of the family of the BSDEs

Step 2 (Main result) *Convergence of the martingale part* of $\ln Y(q)$:

$q\frac{\psi(q)}{Y(q)} \cdot \widehat{M} \to \frac{\psi(1)}{Y(1)} \cdot \widehat{M}$ as $q \to 1$, (in BMO).

Step 3 *Convergence of the strategies*:

let $\pi^*(q)$ and $\pi^*(1)$ denote respectively the optimal strategies for the power and for the exponential utility maximization problem, we prove

$\frac{q}{1-q}\pi^*(q) \cdot \widehat{M} \to \pi^*(1) \cdot \widehat{M}$ as $q \to 1$, (in BMO) .

Diffusion model with stochastic correlation

We consider a diffusion market model consisting of two correlated risky assets one of which has no liquid market.

The price of the two risky assets follow the dynamics

$$dS_t = \mu(t,\eta)dt + \sigma(t,\eta)dW_t^1, \tag{7}$$

$$d\eta_t = b(t,\eta)dt + a(t,\eta)dW_t. \tag{8}$$

subjected to initial conditions.

- W^1 and W are two Brownian motions with stochastic correlation $\rho_t dt = d\langle W^1, W\rangle_t$
- η represents the price of a nontraded asset
- In Frei and Schweizer (2008) a case like this has been considered in the context of *exponential indifference evaluation*.

Assumptions

Assume that the coefficients μ, σ, a and b are non anticipative functionals such that:

1) $\int_0^T \frac{\mu^2(t,\eta)}{\sigma^2(t,\eta)} dt$ is bounded,

2) $\sigma^2 > 0$, $a^2 > 0$

3) the SDE (8) admits a unique strong solution (η).

4) ρ is $\mathscr{F}^\eta$ adapted.

Under conditions 2), 3) we have $\mathscr{F}^{S,\eta} = \mathscr{F}^{W^1,W}$ and $\mathscr{F}^\eta = \mathscr{F}^W$.

Problem: An agent is trading with the liquid asset S using only observations coming from η in order to

$$\text{minimize} \quad E\left[\mathcal{E}_T^p(\pi \cdot S))\right] \quad \text{over all} \quad \pi \in \Pi(\mathscr{F}^\eta), \tag{9}$$

where π represents the proportion of wealth the agent invests in the stock which depends only on η.

$$\mathscr{F}_t = \mathscr{F}_t^{S,\eta} \subseteq \mathscr{A}_t \quad \text{and} \quad \mathscr{G}_t = \mathscr{F}_t^\eta.$$

Problem: An agent is trading with the liquid asset S using only observations coming from η in order to

$$\text{minimize} \quad E\left[\mathcal{E}_T^p(\pi \cdot S))\right] \quad \text{over all} \quad \pi \in \Pi(\mathscr{F}^\eta), \tag{9}$$

where π represents the proportion of wealth the agent invests in the stock which depends only on η.

$$\mathscr{F}_t = \mathscr{F}_t^{S,\eta} \subseteq \mathscr{A}_t \quad \text{and} \quad \mathscr{G}_t = \mathscr{F}_t^\eta.$$

Under conditions 1)–4) the value process related to (9) is the unique bounded positive solution of the BSDE

$$Y_t = Y_0 + \frac{q}{2}\int_0^t \frac{(\theta_u Y_u + \psi_u \rho_u)^2}{Y_u} du + \int_0^t \psi_u dW_u, \quad Y_T = 1 \tag{10}$$

where $\theta = \frac{\mu}{\sigma}$ is the market price of risk.

Problem: An agent is trading with the liquid asset S using only observations coming from η in order to

$$\text{minimize} \quad E\left[\mathcal{E}_T^p(\pi \cdot S))\right] \quad \text{over all} \quad \pi \in \Pi(\mathscr{F}^\eta), \tag{9}$$

where π represents the proportion of wealth the agent invests in the stock which depends only on η.

$$\mathscr{F}_t = \mathscr{F}_t^{S,\eta} \subseteq \mathscr{A}_t \quad \text{and} \quad \mathscr{G}_t = \mathscr{F}_t^\eta.$$

Under conditions 1)–4) the value process related to (9) is the unique bounded positive solution of the BSDE

$$Y_t = Y_0 + \frac{q}{2}\int_0^t \frac{(\theta_u Y_u + \psi_u \rho_u)^2}{Y_u} du + \int_0^t \psi_u dW_u, \quad Y_T = 1 \tag{10}$$

where $\theta = \frac{\mu}{\sigma}$ is the market price of risk.

- If ρ is *constant* the BSDE can be solved explicitly

- If ρ is *stochastic* (using the BSDEs characterization) $\Rightarrow$ we find an *upper* and *lower bounds* for the value process.

PROPOSITION Assume conditions 1) $-$ 4) hold true. Then, the value process V related to problem (9) satisfies

$$\left(E^{\widetilde{Q}}[e^{-\frac{q(1-q\overline{\rho}^2)}{2}\int_t^T \theta_u^2 du)}|\mathcal{F}_t^\eta]\right)^{\frac{1}{1-q\overline{\rho}^2}} \leq V_t \leq \left(E^{\widetilde{Q}}[e^{-\frac{q(1-q\underline{\rho}^2)}{2}\int_t^T \theta_u^2 du)}|\mathcal{F}_t^\eta]\right)^{\frac{1}{1-q\underline{\rho}^2}},$$

where

- $\overline{\rho} = \sup_{s\geq t}\|\rho_s\|_{L^\infty}$ and $\underline{\rho} = \inf_{s\geq t}\|\rho_s\|_{L^\infty}$
- $\widetilde{Q}$ is defined by $\frac{d\widetilde{Q}}{dP} = \mathcal{E}_T(-\theta\, q \cdot W^1)$

ρ constant

Corollary: Assume conditions 1) $-$ 3) and suppose ρ is constant. Then, the value process V is equal to

$$V_t = \left(E^{\widetilde{Q}}[e^{-\frac{q(1-q\rho^2)}{2}\int_t^T \theta_u^2 du)}|\mathcal{F}_t^\eta]\right)^{\frac{1}{1-q\rho^2}}.$$

Moreover, the optimal strategy π^* is identified by

$$\pi_t^* = \frac{(1-q)}{\sigma(t,\eta)}(\theta_t + \frac{\rho h_t}{(1-q\rho^2)(c+\int_0^t h_u d\widetilde{W}_u)}),$$

where h_t is the integrand of the integral representation

$$e^{-\frac{q(1-q\rho^2)}{2}\int_0^T \theta_t^2 dt} = c + \int_0^T h_t d\widetilde{W}_t.$$

Following Theorem 1 of Frei and Schweizer (2008), we can find

THEOREM Under assumptions 1) $-$ 4), there exists a $\mathcal{F}_t^\eta$ measurable random variable $\hat{\rho}_t$ taking values in the interval $[\underline{\rho}, \overline{\rho}]$, such that

$$V_t(\omega) = \left(E^{\widetilde{Q}}[e^{-\frac{q(1-q\rho^2)}{2} \int_t^T \theta_u^2 du)} | \mathcal{F}_t^\eta] \right)^{\frac{1}{1-q\rho^2}} \Big|_{\rho=\hat{\rho}_t(\omega)}. \tag{11}$$

Remark: *In the case of stochastic correlation we can find an explicit expression for the value process but we do not find an explicit expression for the optimal strategy.*

Thank you.

References

• D. Covello and M. Santacroce, Power Utility Maximization under Partial Information: some convergence results. *Stochastic Proccess. Appl.* **120** 2010, 2016–2036. .

• D. Covello, M. Santacroce and E. Sasso, Explicit Formulae for Power Utility Maximization Problems (work in progress).

• C. Frei, M. Schweizer, Exponential Utility Indifference Valuation in Two Brownian Settings with Stochastic Correlation. *Advances in Applied Probability* **40**, 2008, 401–423

• P. Lakner, Optimal trading strategy for an investor: the case of partial information. *Stochastic Proccess. Appl.* **76**, 1998, 77–97.

• M. Mania and M. Santacroce, Exponential utility maximization under partial information. *Finance Stochast.* **14**, 2010, 419–448.

• M. Mania and R. Tevzadze, A unified characterization of q-optimal and minimal entropy martingale measures by semimartingale backward equations. *Georgian Math. J.* **10**, N.2, 2003, 289–310.

• M. Mania, R. Tevzadze and T. Toronjadze, Mean-variance Hedging Under Partial Information. *SIAM J.Control Optim.* **47**, N. 5, 2008, 2381–2409.

• M. Nutz, Risk Aversion Asymptotics for Power Utility Maximization Preprint, 2010

• H. Pham and M. C. Quenez, Optimal portfolio in partially observed stochastic volatility models.*Ann. Appl. Probab.* **11**, N.1, 2001, 210–238.

• R. Tevzadze, Solvability of backward stochastic differential equations with quadratic growth. *Stochastic Proccess. Appl.* **118**, 2008, 503–515.