Optimal control of stochastic delay equations and time-advanced backward stochastic differential equations

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Abstract

We consider the problem of controlling optimally a delay jump diffusion, i.e. a system described by a stochastic differential equation with delay, driven by Brownian motions and compensated Poisson random measures. Such delay systems may occur in several situations, e.g. in finance and biology where the growth of the state depends not only on the current value of the state but also on previous state values.
We give both a sufficient and a necessary maximum principle for such control problems. These maximum principles involve backward stochastic differential equations (BSDEs) which are "anticipative", in the sense that they have a time-advanced drift coefficient. We prove existence and uniqueness theorems for such time-advanced BSDEs. The results are illustrated by examples.

1 INTRODUCTION

Let $B(t) = B(t, \omega)$ be a Brownian motion and $\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$, where ν is the Lévy measure of the jump measure $N(\cdot, \cdot)$, be an independent compensated Poisson random measure on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P)$.

We consider a controlled stochastic delay equation of the form

$$
\begin{aligned}
dX(t) &= b(t, X(t), Y(t), A(t), u(t), \omega)dt \\
&\quad + \sigma(t, X(t), Y(t), A(t), u(t), \omega)dB(t) \\
&\quad + \int_{\mathbb{R}} \theta(t, X(t), Y(t), A(t), u(t), z, \omega)\tilde{N}(dt, dz) \; ; \; t \in [0, T] \qquad (1.1)
\end{aligned}
$$

$$
X(t) = x_0(t) \; ; \; t \in [-\delta, 0], \qquad (1.2)
$$

where

$$
Y(t) = X(t - \delta), \;\; A(t) = \int_{t-\delta}^{t} e^{-\rho(t-r)} X(r) dr, \qquad (1.3)
$$

and $\delta > 0$, $\rho \geq 0$ and $T > 0$ are given constants.

Here

$$
\begin{aligned}
b :&[0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \Omega \to \mathbb{R} \\
\sigma :&[0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \Omega \to \mathbb{R}
\end{aligned}
$$

and

$$\theta : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$$

are given functions such that, for all t, $b(t,x,y,a,u,\cdot)$, $\sigma(t,x,y,a,u,\cdot)$ and $\theta(t,x,y,a,u,z,\cdot)$ are $\mathcal{F}_t$-measurable for all $x \in \mathbb{R}$, $y \in \mathbb{R}$, $a \in \mathbb{R}$, $u \in \mathcal{U}$ and $z \in \mathbb{R}_0 := \mathbb{R}\backslash\{0\}$. The function $x_0(t)$ is assumed to be continuous, deterministic.

Let $\mathcal{E}_t \subseteq \mathcal{F}_t$; $t \in [0, T]$ be a given subfiltration of $\{\mathcal{F}_t\}_{t \in [0,T]}$, representing the information available to the controller who decides the value of $u(t)$ at time t. For example, we could have $\mathcal{E}_t = \mathcal{F}_{(t-c)^+}$ for some given $c > 0$. Let $\mathcal{U} \subset \mathbb{R}$ be a given set of admissible control values $u(t)$; $t \in [0, T]$ and let $\mathcal{A}_{\mathcal{E}}$ be a given family of admissible control processes $u(\cdot)$, included in the set of càdlàg, $\mathcal{E}$-adapted and $\mathcal{U}$-valued processes $u(t)$; $t \in [0, T]$ such that (1.1)-(1.2) has a unique solution $X(\cdot) \in L^2(\lambda \times P)$ where λ denotes the Lebesgue measure on $[0, T]$.

The performance functional is assumed to have the form
(1.4)

$$J(u) = E\left[\int_0^T f(t, X(t), Y(t), A(t), u(t), \omega)dt + g(X(T), \omega)\right] \; ; \; u \in \mathcal{A}_{\mathcal{E}}$$

where $f = f(t, x, y, a, u, \omega) : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \Omega \to \mathbb{R}$ and $g = g(x, \omega) : \mathbb{R} \times \Omega \to \mathbb{R}$ are given $\mathcal{C}^1$ functions w.r.t. (x, y, a, u) such that

$$E[\int_0^T \{|f(t, X(t), A(t), u(t))| + \left|\frac{\partial f}{\partial x_i}(t, X(t), Y(t), A(t), u(t))\right|^2\}dt$$
$$+|g(X(T))| + |g'(X(T))|^2] < \infty \text{ for } x_i = x, y, a \text{ and } u.$$

Here, and in the following, we suppress the ω, for notational simplicity.

The problem we consider in this paper is the following:

Find $\Phi(x_0)$ and $u^* \in \mathcal{A}_{\mathcal{E}}$ such that

$$\Phi(x_0) := \sup_{u \in \mathcal{A}_{\mathcal{E}}} J(u) = J(u^*). \tag{1.5}$$

Any control $u^* \in \mathcal{A}_{\mathcal{E}}$ satisfying (1.5) is called an *optimal control*.

Variants of this problem have been studied in several papers. Stochastic control of delay systems is a challenging research area, because delay systems have, in general, an infinite-dimensional nature. Hence, the natural general approach to them is infinite-dimensional. For this kind of approach in the context of control problems we refer to Chojnowska-Michalik (1978), Federico (2009), Federico,Goldys and Gozzi (2009, 2009a) [1, 7, 8, 9] in the stochastic Brownian case.

Nevertheless, in some cases systems with delay can be reduced to finite-dimensional systems, in the sense that the information we need from their dynamics can be represented by a finite-dimensional variable evolving in terms of itself. In such a context, the crucial point is to understand when this finite dimensional reduction of the problem is possible and/or to find conditions ensuring that. There are some papers dealing with this subject in the stochastic Brownian case: We refer to Kolmanovski and Shaikhet (1996), Elsanousi,Ø. and Sulem (2000), Larssen (2002), Larssen and Risebro (2003), Ø. and Sulem (2001) [10, 6, 12, 13, 15]. The paper David (2008) [3] represents an extension of Ø. and Sulem (2001)[15] to the case when the equation is driven by a Lévy noise.
We also mention the paper El Karoui and Hamadène (2003) [5], where certain control problems of stochastic functional differential equations are studied by means of the Girsanov transformation. This approach, however, does not work if there is a delay in the noise components.

Our approach in the current paper is different from all the above. Note that the presence of the terms $Y(t)$ and $A(t)$ in (1.1) makes the problem non-Markovian and we cannot use a (finite dimensional) dynamic programming approach. However, we will show that it is possible to obtain a (Pontryagin-Bismut-Bensoussan type) maximum principle for the problem. To this end, we define the *Hamiltonian*

$$H : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \Omega \to \mathbb{R}$$

by

$$H(t, x, y, a, u, p, q, r(\cdot), \omega) = H(t, x, y, a, u, p, q, r(\cdot)) = f(t, x, y, a, u)$$

(1.6)

$$+ b(t, x, y, a, u)p + \sigma(t, x, y, a, u)q + \int_{\mathbb{R}_0} \theta(t, x, y, a, u, z)r(z)\nu(dz);$$

where $\mathcal{R}$ is the set of functions $r : \mathbb{R}_0 \to \mathbb{R}$ such that the last term in (1.6) converges.

We assume that b, σ and θ are $\mathcal{C}^1$ functions with respect to (x, y, a, u) and that

$$E\left[\int_0^T \left\{\left|\frac{\partial b}{\partial x_i}(t, X(t), Y(t), A(t), u(t))\right|^2 + \left|\frac{\partial \sigma}{\partial x_i}(t, X(t), Y(t), A(t), u(t\right.\right.\right.$$

(1.7)

$$\left.\left.\left. + \int_{\mathbb{R}_0} \left|\frac{\partial \theta}{\partial x_i}(t, X(t), Y(t), A(t), u(t), z)\right|^2 \nu(dz)\right\} dt\right] < \infty$$

for $x_i = x, y, a$ and u.

Associated to H we define the adjoint processes $p(t), q(t), r(t,z)$; $t \in [0,T]$, $z \in \mathbb{R}_0$, by the following backward stochastic differential equation (BSDE):

(1.8)

$$\begin{cases} dp(t) &= \mu(t)dt + q(t)dB(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz) \; ; \; t \in [0,T] \\ p(T) &= g'(X(T)), \end{cases}$$

where

$$\mu(t) = -\frac{\partial H}{\partial x}(t, X(t), Y(t), A(t), u(t), p(t), q(t), r(t,\cdot))$$
$$-\frac{\partial H}{\partial y}(t+\delta, X(t+\delta), Y(t+\delta), A(t+\delta),$$
$$u(t+\delta), p(t+\delta), q(t+\delta), r(t+\delta,\cdot))\chi_{[0,T-\delta]}(t)$$

(1.9)

$$- e^{\rho t}\left(\int_t^{t+\delta} \frac{\partial H}{\partial a}(s, X(s), Y(s), A(s), u(s), p(s), q(s), r(s,\cdot))e^{-\rho s}\chi_{[0}\right.$$

Note that this BSDE is *anticipative*, or *time-advanced*, in the sense that the driver $\mu(t)$ contains future values of $X(s), u(s), p(s), q(s), r(s, \cdot)\ ;\ s \leq t + \delta$.
In the case when there are no jumps and no integral term in (1.9), anticipative BSDEs (ABSDEs for short) have been studied by Peng and Yang (2009) [17], who prove existence and uniqueness of such equations under certain conditions. They also relate a class of linear ABSDEs to a class of linear stochastic delay control problems with no delay in the noise coefficients. Thus, in our paper we extend this relation to general nonlinear control problems and general nonlinear ABSDEs by means of the maximum principle, and throughout the discussion we include the possibility of delays also in all the noise coefficients, as well as the possibility of jumps.

2 A SUFFICIENT MAXIMUM PRINCIPLE

In this section we establish a maximum principe of sufficient type, i.e. we show that - under some assumptions - maximizing the Hamiltonian leads to an optimal control.

THEOREM 2.1 [Sufficient maximum principle]

Let $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ with corresponding state processes $\hat{X}(t), \hat{Y}(t), \hat{A}(t)$ and adjoint processes $\hat{p}(t), \hat{q}(t), \hat{r}(t,z)$, assumed to satisfy the ABSDE (1.8)-(1.9). Suppose the following hold:

(i) The functions $x \to g(x)$ and

$$(2.1) \qquad (x, y, a, u) \to H(t, x, y, a, u, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot))$$

are concave, for each $t \in [0, T]$, a.s.

(ii)

$$E\left[\int_0^T \left\{\hat{p}(t)^2\left(\sigma^2(t)+\int_{\mathbb{R}_0}\theta^2(t,z)\nu(dz)\right)\right.\right.$$

(2.2) $$\left.\left. +X^2(t)\left(\hat{q}^2(t)+\int_{\mathbb{R}_0}\hat{r}^2(t,z)\nu(dz)\right)\right\} dt\right] < \infty$$

for all $u \in \mathcal{A}_{\mathcal{E}}$.

(iii)

$$\max_{v\in\mathcal{U}} E\left[H(t,\hat{X}(t),\hat{X}(t-\delta),\hat{A}(t),v,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) \mid \mathcal{E}_t\right]$$

(2.3)

$$= E\left[H(t,\hat{X}(t),\hat{X}(t-\delta),\hat{A}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) \mid \mathcal{E}_t\right]$$

for all $t \in [0, T]$, a.s.

Then $\hat{u}(t)$ is an optimal control for the problem (1.5).

Proof. Choose $u \in \mathcal{A}_{\mathcal{E}}$ and consider

(2.4) $$J(u) - J(\hat{u}) = I_1 + I_2$$

where

(2.5)

$$I_1 = E\left[\int_0^T \{f(t, X(t), Y(t), A(t), u(t)) - f(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \hat{u}(t))\}dt\right.$$

(2.6) $$I_2 = E[g(X(T)) - g(\hat{X}(T))].$$

By the definition of H and concavity of H we have

$$I_1 = E\Big[\int_0^T \{H(t, X(t), Y(t), A(t), u(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))$$

$$- H(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))$$

$$- (b(t, X(t), Y(t), A(t), u(t)) - b(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \hat{u}(t)))\hat{p}(t)$$

$$- (\sigma(t, X(t), Y(t), A(t), u(t)) - \sigma(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \hat{u}(t)))\hat{q}(t)$$

$$- \int_{\mathbb{R}} (\theta(t, X(t), Y(t), A(t), u(t), z) - \theta(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \hat{u}(t), z))\hat{r}$$

$$\leq E\Big[\int_0^T \{\frac{\partial \hat{H}}{\partial x}(t)(X(t) - \hat{X}(t)) + \frac{\partial \hat{H}}{\partial y}(t)(Y(t) - \hat{Y}(t)) + \frac{\partial \hat{H}}{\partial a}(t)(A(t$$

$$+ \frac{\partial H}{\partial u}(t)(u(t) - \hat{u}(t)) - (b(t) - \hat{b}(t))\hat{p}(t) - (\sigma(t) - \hat{\sigma}(t))\hat{q}(t)$$

(2.7)

$$- \int_{\mathbb{R}} (\theta(t,z) - \hat{\theta}(t,z))\hat{r}(t,z)\nu(dz)\}dt\Big],$$

where we have used the abbreviated notation

$$\frac{\partial \hat{H}}{\partial x}(t) = \frac{\partial H}{\partial x}(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)),$$

$$b(t) = b(t, X(t), Y(t), A(t), u(t)),$$

$$\hat{b}(t) = b(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \hat{u}(t) \text{ etc.}$$

Since g is concave we have, by (2.2),

$$
\begin{aligned}
I_2 &\le E[g'(\hat{X}(T))(X(T)-\hat{X}(T))] = E[\hat{p}(T)(X(T)-\hat{X}(T))] \\
&= E\Big[\int_0^T \hat{p}(t)(dX(t)-d\hat{X}(t)) + \int_0^T (X(t)-\hat{X}(t))d\hat{p}(t) \\
&\quad + \int_0^T (\sigma(t)-\hat{\sigma}(t))\hat{q}(t)dt + \int_0^T \int_{\mathbb{R}} (\theta(t,z)-\hat{\theta}(t,z))\hat{r}(t,z)\nu(dz)dt\Big] \\
&= E\Big[\int_0^T (b(t)-\hat{b}(t))\hat{p}(t)dt + \int_0^T (X(t)-\hat{X}(t))\mu(t)dt \\
&\quad + \int_0^T (\sigma(t)-\hat{\sigma}(t))\hat{q}(t)dt + \int_0^T \int_{\mathbb{R}} (\theta(t,z)-\hat{\theta}(t,z))\hat{r}(t,z)\nu(dz)dt\Big]
\end{aligned}
\tag{2.8}
$$

Combining (2.4)-(2.8) we get, using that $X(t) = \hat{X}(t) = x_0(t)$ for all $t \in [-\delta, 0]$,

$$J(u) - J(\hat{u}) \leq E\Bigg[\int_0^T \Bigg\{ \frac{\partial H}{\partial x}(t)(X(t) - \hat{X}(t)) + \frac{\partial H}{\partial y}(t)(Y(t) - \hat{Y}(t))$$

$$+ \frac{\partial \hat{H}}{\partial a}(t)(A(t) - \hat{A}(t)) + \frac{\partial \hat{H}}{\partial u}(t)(u(t) - \hat{u}(t)) + \mu(t)(X(t) - \hat{X}(t$$

$$= E\Bigg[\int_\delta^{T+\delta} \Bigg\{ \frac{\partial \hat{H}}{\partial x}(t-\delta) + \frac{\partial \hat{H}}{\partial y}(t)\chi_{[0,T]}(t) + \mu(t-\delta) \Bigg\} (Y(t) - \hat{Y}$$

(2.9)

$$+ \int_0^T \frac{\partial \hat{H}}{\partial a}(t)(A(t) - \hat{A}(t))dt + \int_0^T \frac{\partial \hat{H}}{\partial u}(t)(u(t) - \hat{u}(t))dt \Bigg].$$

Using integration by parts and substituting $r = t - \delta$, we get

$$\int_0^T \frac{\partial \hat{H}}{\partial a}(s)(A(s) - \hat{A}(s))ds$$

$$= \int_0^T \frac{\partial \hat{H}}{\partial a}(s) \int_{s-\delta}^{s} e^{-\rho(s-r)}(X(r) - \hat{X}(r))drds$$

$$= \int_0^T \left(\int_r^{r+\delta} \frac{\partial \hat{H}}{\partial a}(s) e^{-\rho s} \chi_{[0,T]}(s) ds \right) e^{\rho r}(X(r) - \hat{X}(r))dr$$

(2.10)

$$= \int_\delta^{T+\delta} \left(\int_{t-\delta}^{t} \frac{\partial \hat{H}}{\partial a}(s) e^{-\rho s} \chi_{[0,T]}(s) ds \right) e^{\rho(t-s)}(X(t-\delta) - \hat{X}(t-\delta$$

Combining this with (2.9) and using (1.9) we obtain

$$
\begin{aligned}
J(u) - J(\hat{u}) \leq & \Bigg[\int_{\delta}^{T+\delta} \Bigg\{ \frac{\partial \hat{H}}{\partial x}(t-\delta) + \frac{\partial \hat{H}}{\partial y}(t)\chi_{[0,T]}(t) \\
& + \left(\int_{t-\delta}^{t} \frac{\partial \hat{H}}{\partial a}(s) e^{-\rho s} \chi_{[0,T]}(s) ds \right) e^{\rho(t-\delta)} + \mu(t-\delta) \Bigg\} (Y(\\
& + \int_0^T \frac{\partial \hat{H}}{\partial u}(t)(u(t) - \hat{u}(t))dt \Bigg] \\
= & E\left[\int_0^T \frac{\partial \hat{H}}{\partial u}(t)(u(t) - \hat{u}(t))dt \right] \\
= & E\left[\int_0^T E\left[\frac{\partial \hat{H}}{\partial u}(t)(u(t) - \hat{u}(t)) \mid \mathcal{E}_t \right] dt \right] \\
= & E\left[\int_0^T E\left[\frac{\partial \hat{H}}{\partial u}(t) \mid \mathcal{E}_t \right] (u(t) - \hat{u}(t))dt \right] \leq 0.
\end{aligned}
$$

The last inequality holds because $v = \hat{u}(t)$ maximizes $E[H(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), v, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot) \mid \mathcal{E}_t]$ for each $t \in [0, T]$.

This proves that $\hat{u}$ is an optimal control. $\square$

3 A NECESSARY MAXIMUM PRINCIPLE

A drawback with the sufficient maximum principle in Section 2 is the condition of concavity, which does not always hold in the applications. In this section we will prove a result going in the other direction. More precisely, we will prove the equivalence between being a directional critical point for $J(u)$ and a critical point for the conditional Hamiltonian. To this end, we need to make the following assumptions:

(i) For all $u \in \mathcal{A}_{\mathcal{E}}$ and all bounded $\beta \in \mathcal{A}_{\mathcal{E}}$ there exists $\varepsilon > 0$ such that

$$u + s\beta \in \mathcal{A}_{\mathcal{E}} \text{ for all } s \in (-\varepsilon, \varepsilon).$$

(ii) For all $t_0 \in [0, T]$ and all bounded $\mathcal{E}_{t_0}$-measurable random variables α the control process $\beta(t)$ defined by

$$\beta(t) = \alpha \chi_{[t_0, T]}(t)\ ;\ t \in [0, T] \tag{3.1}$$

belongs to $\mathcal{A}_{\mathcal{E}}$.

(iii) For all bounded $\beta \in \mathcal{A}_{\mathcal{E}}$ the derivative process

$$\xi(t) := \frac{d}{ds} X^{u+s\beta}(t) \mid_{s=0} \tag{3.2}$$

exists and belongs to $L^2(\lambda \times P)$.

It follows from (1.1) that

(3.3)

$$d\xi(t) =$$

$$\left\{ \frac{\partial b}{\partial x}(t)\xi(t) + \frac{\partial b}{\partial y}(t)\xi(t-\delta) + \frac{\partial b}{\partial a}(t) \int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \frac{\partial b}{\partial u}(t)\beta(t) \right\}$$

$$+ \left\{ \frac{\partial \sigma}{\partial x}(t)\xi(t) + \frac{\partial \sigma}{\partial y}(t)\xi(t-\delta) + \frac{\partial \sigma}{\partial a}(t) \int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \frac{\partial \sigma}{\partial u}(t)\beta(\right.$$

$$+ \int_{\mathbb{R}_0} \left\{ \frac{\partial \theta}{\partial x}(t,z)\xi(t) + \frac{\partial \theta}{\partial y}(t,z)\xi(t-\delta) \right.$$

(3.4)

$$\left. + \frac{\partial \theta}{\partial a}(t) \int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \frac{\partial \theta}{\partial u}(t)\beta(t) \right\} \tilde{N}(dt,dz)$$

where we for simplicity of notation have put

$$\frac{\partial b}{\partial x}(t) = \frac{\partial b}{\partial x}(t, X(t), X(t-\delta), A(t), u(t)) \text{ etc } \ldots$$

and we have used that

$$\frac{d}{ds} Y^{u+s\beta}(t) \mid_{s=0} = \frac{d}{ds} X^{u+s\beta}(t-\delta) \mid_{s=0} = \xi(t-\delta) \tag{3.5}$$

and

$$\frac{d}{ds} A^{u+s\beta}(t) \mid_{s=0} = \frac{d}{ds}\left(\int_{t-\delta}^{t} e^{-\rho(t-r)} X^{u+s\beta}(r)dr\right) \mid_{s=0}$$

$$= \int_{t-\delta}^{t} e^{-\rho(t-r)} \frac{d}{ds} X^{u+s\beta}(r) \mid_{s=0} dt = \int_{t-\delta}^{t} e^{-\rho(t-r)} \xi(r)dr. \tag{3.6}$$

Note that

$$\xi(t) = 0 \text{ for } t \in [-\delta, 0]. \tag{3.7}$$

THEOREM 3.1[Necessary maximum principle]

Suppose $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ with corresponding solutions $\hat{X}(t)$ of (1.1)-(1.2) and $\hat{p}(t)$, $\hat{q}(t)$, $\hat{r}(t,z)$ of (1.7)-(1.8) and corresponding derivative process $\hat{\xi}(t)$ given by (3.2).

Assume that

$$E\left[\int_0^T \hat{p}^2(t)\left\{\left(\frac{\partial\sigma}{\partial x}\right)^2(t)\hat{\xi}^2(t) + \left(\frac{\partial\sigma}{\partial y}\right)^2(t)\xi^2(t-\delta)\right.\right.$$

$$+\left(\frac{\partial\sigma}{\partial a}\right)^2(t)\left(\int_{t-\delta}^t e^{-\rho(t-r)}\hat{\xi}(r)dr\right)^2 + \left(\frac{\partial\sigma}{\partial u}\right)^2(t)$$

$$+\int_{\mathbb{R}_0}\left\{\left(\frac{\partial\theta}{\partial x}\right)^2(t,z)\hat{\xi}^2(t) + \left(\frac{\partial\theta}{\partial y}\right)^2(t,z)\hat{\xi}^2(t-\delta)\right.$$

$$\left.\left.+\left(\frac{\partial\theta}{\partial a}\right)^2(t,z)\left(\int_{t-\delta}^t e^{-\rho(t-r)}\hat{\xi}(r)dr\right)^2 + \left(\frac{\partial\theta}{\partial u}\right)^2(t,z)\right\}\nu(dz)\right\}dt$$

(3.8)

$$\left.+\int_0^T \hat{\xi}^2(t)\left\{\hat{q}^2(t) + \int_{\mathbb{R}_0}\hat{r}^2(t,z)\nu(dz)\right\}dt\right] < \infty.$$

Then the following are equivalent:

$$(i)\frac{d}{ds}J(\hat{u}+s\beta)\mid_{s=0}=0 \text{ for all bounded } \beta\in\mathcal{A}_{\mathcal{E}}.$$

$$(ii)E\left[\frac{\partial H}{\partial u}(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),u,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\mid\mathcal{E}_t\right]_{u=\hat{u}(t)}=0$$

a.s. for all $t\in[0,T]$.

Proof. For simplicity of notation we write $\hat{u} = u$, $\hat{X} = X$, $\hat{p} = p$, $\hat{q} = q$ and $\hat{r} = r$ in the following.

Suppose **(i)** holds. Then

$$0 = \frac{d}{ds} J(u + s\beta) \mid_{s=0}$$

$$= \frac{d}{ds} E \left[\int_0^T f(t, X^{u+s\beta}(t), Y^{u+s\beta}(t), A^{u+s\beta}(t), u(t) + s\beta(t))dt + g(X^{u}\right.$$

$$= E \left[\int_0^T \left\{ \frac{\partial f}{\partial x}(t)\xi(t) + \frac{\partial f}{\partial y}(t)\xi(t-\delta) + \frac{\partial f}{\partial a}(t) \int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dt + \frac{\partial}{\partial}\right.\right.$$

(3.9)

$$
\begin{aligned}
= E\Bigg[&\int_0^T \left\{ \frac{\partial H}{\partial x}(t) - \frac{\partial b}{\partial x}(t)p(t) - \frac{\partial \sigma}{\partial x}(t)q(t) - \int_{\mathbb{R}} \frac{\partial \theta}{\partial x}(t,z)r(t,z)\nu(dz) \right\} \\
&+ \int_0^T \left\{ \frac{\partial H}{\partial y}(t) - \frac{\partial b}{\partial y}(t)p(t) - \frac{\partial \sigma}{\partial y}(t)q(t) - \int_{\mathbb{R}} \frac{\partial \theta}{\partial y}(t,z)r(t,z)\nu(dz) \right\} \\
&+ \int_0^T \left\{ \frac{\partial H}{\partial a}(t) - \frac{\partial b}{\partial a}(t)p(t) - \frac{\partial \sigma}{\partial a}(t)q(t) - \int_{\mathbb{R}} \frac{\partial \theta}{\partial a}(t,z)r(t,z)\nu(dz) \right\}
\end{aligned}
$$

(3.10)

$$
+ \int_0^T \frac{\partial f}{\partial u}(t)\beta(t)dt + g'(X(T))\xi(T)\Bigg].
$$

By (3.3),

$$
\begin{aligned}
&E[g'(X(T))\xi(T)] = E[p(T)\xi(T)] = \\
&E\Bigg[\int_0^T p(t)d\xi(t) + \int_0^T \xi(t)dp(t) \\
&+ \int_0^T q(t)\left\{\frac{\partial\sigma}{\partial x}(t)\xi(t) + \frac{\partial\sigma}{\partial y}(t)\xi(t-\delta)\right. \\
&\left.+\frac{\partial\sigma}{\partial a}(t)\int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dr + \frac{\partial\sigma}{\partial u}(t)\beta(t)\right\} dt \\
&+ \int_0^T \int_{\mathbb{R}} r(t,z)\left\{\frac{\partial\theta}{\partial x}(t,z)\xi(t) + \frac{\partial\theta}{\partial y}(t,z)\xi(t-\delta)\right. \\
&+ \frac{\partial\theta}{\partial a}(t,z)\int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dr \\
&\left.+\frac{\partial\theta}{\partial u}(t)\beta(t)\right\} \nu(dz)dt\Bigg]
\end{aligned}
$$

(3.11)

$$
\begin{aligned}
= E\Bigg[\int_0^T p(t) \Big\{ & \frac{\partial b}{\partial x}(t)\xi(t) + \frac{\partial b}{\partial y}(t)\xi(t-\delta) + \frac{\partial b}{\partial a}(t) \int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr \\
& + \int_0^T \xi(t)\mu(t)dt \\
& + \int_0^T q(t) \Big\{ \frac{\partial \sigma}{\partial x}(t)\xi(t) + \frac{\partial \sigma}{\partial y}(t)\xi(t-\delta) + \frac{\partial \sigma}{\partial a}(t) \int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr \\
& + \int_0^T \int_{\mathbb{R}} r(t,z) \Big\{ \frac{\partial \theta}{\partial x}(t,z)\xi(t) + \frac{\partial \theta}{\partial y}(t,z)\xi(t-\delta) + \frac{\partial \theta}{\partial a}(t,z) \int_{t-\delta}^{t} e^{-\rho} \\
& \quad + \frac{\partial \theta}{\partial u}(t,z)\beta(t) \Big\} \nu(dz)dt \Bigg]
\end{aligned}
$$

(3.12)

Combining (3.9) and (3.12) we get

$$\begin{aligned}
0 = & E\left[\int_0^T \xi(t)\left\{\frac{\partial H}{\partial x}(t)+\mu(t)\right\}dt+\int_0^T \xi(t-\delta)\frac{\partial H}{\partial y}(t)dt\right.\\
& \left.+\int_0^T\left(\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr\right)\frac{\partial H}{\partial a}(t)dt+\int_0^T \frac{\partial H}{\partial u}(t)\beta(t)dt\right]\\
= & E\left[\int_0^T \xi(t)\left\{\frac{\partial H}{\partial x}(t)-\frac{\partial H}{\partial x}(t)-\frac{\partial H}{\partial y}(t+\delta)\chi_{[0,T-\delta]}(t)\right.\right.\\
& \left.-e^{\rho t}\left(\int_t^{t+\delta}\frac{\partial H}{\partial a}(s)e^{-\rho s}\chi_{[0,T]}(s)ds\right)\right\}dt+\int_0^T \xi(t-\delta)\frac{\partial H}{\partial y}(t)dt\\
& \left.+\int_0^T\left(\int_{s-\delta}^{s} e^{-\rho(s-t)}\xi(t)dt\right)\frac{\partial H}{\partial a}(s)ds+\int_0^T \frac{\partial H}{\partial u}(t)\beta(t)dt\right]
\end{aligned}$$

(3.13)

$$
\begin{aligned}
&= E\left[\int_0^T \xi(t)\left\{-\frac{\partial H}{\partial y}(t+\delta)\chi_{[0,T-\delta]}(t) - e^{\rho t}\left(\int_t^{t+\delta}\frac{\partial H}{\partial a}(s)e^{-\rho s}\chi_{[0,T]}(s\right.\right.\right.\\
&\quad + \int_0^T \xi(t-\delta)\frac{\partial H}{\partial y}(t)dt\\
&\quad + e^{\rho t}\int_0^T\left(\int_t^{t+\delta}\frac{\partial H}{\partial a}(s)e^{-\rho s}\chi_{[0,T]}(s)ds\right)\xi(t)dt + \int_0^T\frac{\partial H}{\partial u}(t)\beta(t)dt\right]
\end{aligned}
$$

(3.14)

$$
= E\left[\int_0^T \frac{\partial H}{\partial u}(t)\beta(t)dt\right],
$$

where we again have used integration by parts.

If we apply (3.14) to

$$\beta(t) = \alpha(\omega)\chi_{[s,T]}(t)$$

where $\alpha(\omega)$ bounded and $\mathcal{E}_{t_0}$-measurable, $s \geq t_0$, we get

$$E\left[\int_s^T \frac{\partial H}{\partial u}(t)dt\ \alpha\right] = 0.$$

Differentiating with respect to s we obtain

$$E\left[\frac{\partial H}{\partial u}(s)\alpha\right] = 0.$$

Since this holds for all $s \geq t_0$ and all α we conclude that

$$E\left[\frac{\partial H}{\partial u}(t_0) \mid \mathcal{E}_{t_0}\right] = 0.$$

This shows that **(i)** $\Rightarrow$ **(ii)**.

Conversely, since every bounded $\beta \in \mathcal{A}_{\mathcal{E}}$ can be approximated by linear combinations of controls β of the form (3.2), we can prove that **(ii)** $\Rightarrow$ **(i)** by reversing the above argument. $\square$

4 TIME-ADVANCED BSDE'S WITH JUMPS

We now study time-advanced backward stochastic differential equations driven both by Brownian motion $B(t)$ and compensated Poisson random measures $\tilde{N}(dt, dz)$.

4.1 Framework

Given a positive constant δ, denote by $D([0,\delta],\mathbb{R})$ the space of all càdlàg paths from $[0,\delta]$ into $\mathbb{R}$. For a path $X(\cdot):\mathbb{R}_+\to\mathbb{R}$, X_t will denote the function defined by $X_t(s)=X(t+s)$ for $s\in[0,\delta]$. Put $\mathcal{H}=L^2(\nu)$. Consider the L^2 spaces $V_1:=L^2([0,\delta],ds)$ and $V_2:=L^2([0,\delta]\to\mathcal{H},ds)$. Let

$$F:\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\times V_1\times\mathbb{R}\times\mathbb{R}\times V_1\times\mathcal{H}\times\mathcal{H}\times V_2\times\Omega\to\mathbb{R}$$

be a predictable function. Introduce the following Lipschitz condition: There exists a constant C such that

$$\begin{aligned}&|F(t,p_1,p_2,p,q_1,q_2,q,r_1,r_2,r,\omega)-F(t,\bar{p}_1,\bar{p}_2,\bar{p},\bar{q}_1,\bar{q}_2,\bar{q},\bar{r}_1,\bar{r}_2,\bar{r},\omega)|\\&\quad\leq C(|p_1-\bar{p}_1|+|p_2-\bar{p}_2|+|p-\bar{p}|_{V_1}+|q_1-\bar{q}_1|+|q_2-\bar{q}_2|+|q-\bar{q}\\&\quad+|r_1-\bar{r}_1|_{\mathcal{H}}+|r_2-\bar{r}_2|_{\mathcal{H}}+|r-\bar{r}|_{V_2}.\end{aligned} \tag{4.1}$$

4.2 Existence and uniqueness (part 1)

We first consider the following time-advanced backward stochastic differential equation in the unknown $\mathcal{F}_t$ adapted processes $(p(t), q(t), r(t,z))$:

$$dp(t) = F\big(t, p(t), p(t+\delta)\chi_{[0,T-\delta]}(t), p_t\chi_{[0,T-\delta]}(t), q(t), q(t+\delta)\chi_{[0,T-\delta]}(t),$$
$$q_t\chi_{[0,T-\delta]}(t), r(t), r(t+\delta)\chi_{[0,T-\delta]}(t), r_t\chi_{[0,T-\delta]}(t)\big)\, dt$$

(4.2)

$$+ q(t)dB(t) + \int_{\mathbb{R}} r(t,z)\tilde{N}(dt,dz)\ ; \ t \in [0,T]$$

(4.3) $$p(T) = G,$$

where G is a given $\mathcal{F}_t$-measurable random variable.
Note that the time-advanced BSDE (1.8)-(1.9) for the adjoint processes of the Hamiltonian is of this form.

For this type of time-advanced BSDEs we have the following result:

Theorem

Assume that $E[G^2] < \infty$ *and that condition* (4.1) *is satisfied. Then the BSDE* (4.2)-(4.3) *has a unique solution* $p(t), q(t), r(t,z))$ *such that*

$$E\left[\int_0^T \left\{p^2(t) + q^2(t) + \int_{\mathbb{R}} r^2(t,z)\nu(dz)\right\} dt\right] < \infty. \tag{4.4}$$

Moreover, the solution can be found by inductively solving a sequence of BSDEs backwards as follows:

STEP 0:

In the interval $[T-\delta, T]$ we let $p(t), q(t)$ and $r(t,z)$ be defined as the solution of the classical BSDE

$$
\begin{aligned}
dp(t) &= F\left(t, p(t), 0, 0, q(t), 0, 0, r(t,z), 0, 0\right) dt \\
&\quad + q(t) dB(t) + \int_{\mathbb{R}} r(t,z) \tilde{N}(dt, dz)\ ; \ t \in [T-\delta, T]
\end{aligned} \tag{4.5}
$$

$$
p(T) = G. \tag{4.6}
$$

STEP k ; $k \geq 1$:

If the values of $(p(t), q(t), r(t,z))$ have been found for $t \in [T - k\delta, T - (k-1)\delta]$, then if $t \in [T - (k+1)\delta, T - k\delta]$ the values of $p(t+\delta), p_t, q(t+\delta), q_t, r(t+\delta, z)$ and r_t are known and hence the BSDE

$$dp(t) = F\left(t, p(t), p(t+\delta), p_t, q(t), q(t+\delta), q_t, r(t), r(t+\delta), r_t\right) dt$$

(4.7)

$$+ q(t) dB(t) + \int_{\mathbb{R}} r(t,z) \tilde{N}(dt, dz) \; ; \; t \in [T - (k+1)\delta, T - k\delta]$$

(4.8) $\qquad p(T - k\delta) = \text{ the value found in Step } k - 1$

has a unique solution in $[T - (k+1)\delta, T - k\delta]$.

We proceed like this until k is such that $T-(k+1)\delta \leq 0 < T-k\delta$ and then we solve the corresponding BSDE on the interval $[0, T-k\delta]$.

Proof. The proof follows directly from the above inductive procedure. The estimate (4.4) is a consequence of known estimates for classical BSDEs (Pardoux-Peng). $\square$

4.3 Existence and uniqueness (part 2)

Next, consider the following backward stochastic differential equation in the unknown $\mathcal{F}_t$-adapted processes $(p(t), q(t), r(t,x))$:

$$dp(t) = F(t, p(t), p(t+\delta), p_t, q(t), q(t+\delta), q_t, r(t), r(t+\delta), r_t)dt$$

(4.9)

$$+ q(t)dB_t + \int_{\mathbb{R}} r(t,z)\tilde{N}(dt,dz), \qquad t \in [0,T]$$

(4.10) $$p(t) = G(t), \qquad t \in [T, T+\delta].$$

where G is a given $\mathcal{F}_T$-measurable stochastic process.

Theorem

Assume $E[\sup_{T \leq t \leq T+\delta} |G(t)|^2] < \infty$ *and that the condition* (4.1) *is satisfied. Then the backward stochastic differential equation* (4.9) *admits a unique solution* $(p(t), q(t), r(t,z))$ *such that*

$$E[\int_0^T \{p^2(t) + q^2(t) + \int_{\mathbb{R}} r^2(t,z)\nu(dz)\}dt] < \infty.$$

Proof

Step 1.

Assume F is independent of p_1, p_2 and p. Set $q^0(t) := 0, r^0(t,x) = 0$. For $n \geq 1$, define $(p^n(t), q^n(t), r^n(t,x))$ to be the unique solution to the following backward stochastic differential equation equation:

$$dp^n(t) = F(t, q^{n-1}(t), q^{n-1}(t+\delta), q_t^{n-1}, r^{n-1}(t,\cdot), r^{n-1}(t+\delta,\cdot), r_t^{n-1}(\cdot)$$

(4.11)

$$+ q^n(t)dB_t + r^n(t,z)\tilde{N}(dt,dz), \qquad t \in [0,T]$$

$$p^n(t) = G(t) \qquad t \in [T, T+\delta].$$

The existence and uniqueness of the solution of above equation is well known.

We then show that $(p^n(t), q^n(t), r^n(t,x))$ forms a Cauchy sequence.

Step 2. General case

Let $p^0(t) = 0$. For $n \geq 1$, define $(p^n(t), q^n(t), r^n(t,z))$ to be the unique solution to the following BSDE:

$$dp^n(t) = F(t, p^{n-1}(t), p^{n-1}(t+\delta), p_t^{n-1}, q^n(t), q^n(t+\delta), q_t^n, r^n(t,\cdot), r^n($$

(4.12)

$$+ q^n(t)dB_t + r^n(t,z)\tilde{N}(dt, dz),$$

$$p^n(t) = G(t); \qquad t \in [T, T+\delta].$$

The existence of $(p^n(t), q^n(t), r^n(t,z))$ is proved in Step 1. By similar arguments as above, we deduce after some computations that

$$E[\int_0^T |p^{n+1}(s) - p^n(s)|^2 ds] \leq \frac{e^{CNT} T^n}{n!}.$$

By this inequality and a similar argument as in Step 1, it can be shown that $(p^n(t), q^n(t), r^n(t,z))$ converges to some limit $(p(t), q(t), r(t,z))$, which is the unique solution of equation (4.9).

$\square$

Theorem 4.3

Assume $E\left[\sup_{T\leq t\leq T+\delta}|G(t)|^{2\alpha}\right]<\infty$ for some $\alpha>1$ and that the following condition holds:

$$
\begin{aligned}
&|F(t,p_1,p_2,p,q_1,q_2,q,r_1,r_2,r)-F(t,\bar{p}_1,\bar{p}_2,\bar{p},\bar{q}_1,\bar{q}_2,\bar{q},\bar{r}_1,\bar{r}_2,\bar{r})| \\
&\quad\leq C(|p_1-\bar{p}_1|+|p_2-\bar{p}_2|+\sup_{0\leq s\leq\delta}|p(s)-\bar{p}(s)|+|q_1-\bar{q}_1|
\end{aligned}
$$

(4.13)

$$
+|q_2-\bar{q}_2|+|q-\bar{q}|_{V_1}+|r_1-\bar{r}_1|_{\mathcal{H}}+|r_2-\bar{r}_2|_{\mathcal{H}}+|r-\bar{r}|_{V_2}.
$$

Then the BSDE (4.9) admits a unique solution $(p(t),q(t),r(t,z))$ such that

$$
E\left[\sup_{0\leq t\leq T}|p(t)|^{2\alpha}+\int_0^T\{q^2(t)+\int_{\mathbb{R}}r^2(t,z)\nu(dz)\}dt\right]<\infty.
$$

Proof

Step 1.

Assume F is independent of p_1, p_2 and p. In this case the condition above reduces to assumption (4.1). By the Step 1 in the proof of Theorem 4.1, there is a unique solution $(p(t), q(t), r(t,z))$ to equation (4.9).

Step 2. General case

Let $p^0(t) = 0$. For $n \geq 1$, define $(p^n(t), q^n(t), r^n(t,z))$ to be the unique solution to the following BSDE:

$$dp^n(t) = F(t, p^{n-1}(t), p^{n-1}(t+\delta), p_t^{n-1}, q^n(t), q^n(t+\delta), q_t^n, r^n(t,\cdot), r^n(t$$

(4.14)

$$+ q^n(t)dB_t + r^n(t,z)\tilde{N}(dt,dz),$$

$$p^n(t) = G(t), \qquad t \in [T, T+\delta].$$

By Step 1, $(p^n(t), q^n(t), r^n(t,z))$ exists. We then proceed to show that $(p^n(t), q^n(t), r^n(t,z))$ forms a Cauchy sequence.

After some computations we end up with the estimate

$$E\left[\int_0^T \sup_{t\leq s\leq T} |p^{n+1}(s) - p^n(s)|^{2\alpha} ds\right] \leq \frac{e^{CNT}T^n}{n!}.$$

Using this inequality and a similar argument as in Step 1, we can show that $(p^n(t), q^n(t), r^n(t,z))$ converges to some limit $(p(t), q(t), r(t,z))$, which is the unique solution of equation (4.9). $\square$

Finally we give a result when the coefficient f is independent of z and r.

Theorem 4.4

Assume that $E\left[\sup_{T\leq t\leq T+\delta}|G(t)|^2\right]<\infty$ and that F satisfies

(4.15)

$$|F(t,y_1,y_2,p)-F(t,\bar{y}_1,\bar{y}_2,\bar{p})|\leq C(|y_1-\bar{y}_1|+|y_2-\bar{y}_2|+\sup_{0\leq s\leq\delta}|p(s)-\bar{p}(s)|)$$

Then the backward stochastic differential equation (4.9) admits a unique solution.

Example 5.1 (Optimal consumption from a cash flow with delay)

Let $\alpha(t), \beta(t)$ and $\gamma(t,z)$ be given bounded adapted processes, $\alpha(t)$ deterministic. Assume that $\int_{\mathbb{R}} \gamma^2(t,z)\nu(dz) < \infty$. Consider a cash flow $X^0(t)$ with the dynamics, for $t \in [0,T]$,

(5.1)

$$dX^0(t) = X^0(t-\delta)\left[\alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz)\right]$$

and such that

$$X^0(t) = x_0(t) > 0 \; ; \; t \in [-\delta, 0], \tag{5.2}$$

where $x_0(t)$ is a given bounded deterministic function.

Suppose that at time $t \in [0, T]$ we consume at the rate $c(t) \geq 0$, a càdlàg adapted process. Then the dynamics of the corresponding net cash flow $X(t) = X^c(t)$ is

$$dX(t) = [X(t-\delta)\alpha(t) - c(t)]dt + X(t-\delta)\beta(t)dB(t) \tag{5.3}$$

$$+ X(t-\delta)\int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz) \; ; \; t \in [0, T] \tag{5.4}$$

$$X(t) = x_0(t) \; ; \; t \in [-\delta, 0]. \tag{5.5}$$

Let $U_1(t,c,\omega) : [0,T] \times \mathbb{R}^+ \times \Omega \to \mathbb{R}$ be a given stochastic utility function satisfying the following conditions

$$
\begin{aligned}
t \to U_1(t,c,\omega) &\text{ is } \mathcal{F}_t\text{-adapted for each } c \geq 0, \\
c \to U_1(t,c,\omega) &\text{ is } \mathcal{C}^1,\ \frac{\partial U_1}{\partial c}(t,c,\omega) > 0, \\
c \to \frac{\partial U_1}{\partial c}(t,c,\omega) &\text{ is strictly decreasing} \\
\lim_{c\to\infty} \frac{\partial U_1}{\partial c}(t,c,\omega) &= 0 \text{ for all } t,\omega \in [0,T] \times \Omega. \qquad (5.6)
\end{aligned}
$$

Put $v_0(t,\omega) = \dfrac{\partial U_1}{\partial c}(t,0,\omega)$ and define

$$(5.7) \quad I(t,v,\omega) = \begin{cases} 0 & \text{if } v \geq v_0(t,\omega) \\ \left(\dfrac{\partial U_1}{\partial c}(t,\cdot,\omega)\right)^{-1}(v) & \text{if } 0 \leq v < v_0(t,\omega) \end{cases}$$

Suppose we want to find a consumption rate $\hat{c}(t)$ such that

$$J(\hat{c}) = \sup\{J(c)\ ;\ c \in \mathcal{A}\} \tag{5.8}$$

where

$$J(c) = E\left[\int_0^T U_1(t, c(t), \omega)dt + kX(T)\right].$$

Here $k > 0$ is constant and $\mathcal{A}$ is the family of all càdlàg, $\mathcal{F}_t$-adapted processes $c(t) \geq 0$ such that $E[|X(T)|] < \infty$.

In this case the Hamiltonian given by (1.6) gets the form

$$
\begin{aligned}
H(t, x, y, a, u, p, q, r(\cdot), \omega) &= U_1(t, c, \omega) + (\alpha(t)y - c)p \\
&\quad + y\beta(t)q + y \int_{\mathbb{R}} \gamma(t, z) r(z) \nu(dz). \qquad (5.9)
\end{aligned}
$$

Maximizing H with respect to c gives the following first order condition for an optimal $\hat{c}(t)$:

$$
\frac{\partial U_1}{\partial c}(t, \hat{c}(t), \omega) = p(t). \qquad (5.10)
$$

The time-advanced BSDE for $p(t), q(t), r(t,z)$ is, by (1.8)-(1.9),

$$dp(t) =$$

$$-\left\{\alpha(t)p(t+\delta) + \beta(t)q(t+\delta) + \int_{\mathbb{R}} \gamma(t,z)r(t+\delta,z)\nu(dz)\right\}\chi_{[0,T-\delta]}(t$$

(5.11)

$$+\, q(t)dB(t) + \int_{\mathbb{R}} r(t,z)\tilde{N}(dt,dz)\ ;\ t \in [0,T].$$

(5.12) $$p(T) = k.$$

Since k is deterministic, we can choose $q = r = 0$ and (5.11)-(5.12) becomes

$$dp(t) = -\alpha(t)p(t+\delta)\chi_{[0,T-\delta]}(t)dt \; ; \; t < T \tag{5.13}$$

$$p(t) = k \text{ for } t \in [T-\delta, T+\delta]. \tag{5.14}$$

To solve this we introduce

$$h(t) := p(T-t) \; ; \; t \in [-\delta, T].$$

Then

$$\begin{aligned} dh(t) &= -dp(T-t) = \alpha(T-t)p(T-t+\delta)dt \\ &= \alpha(T-t)p(T-(t-\delta))dt = \alpha(T-t)h(t-\delta)dt \end{aligned} \tag{5.15}$$

for $t \in [0, T]$, and

$$h(t) = p(T-t) = k \text{ for } t \in [-\delta, 0]. \tag{5.16}$$

This determines $h(t)$ inductively on each interval $[j\delta,(j+1)\delta]$; $j=1,2,\ldots,$ as follows:

If $h(s)$ is known on $[(j-1)\delta,j\delta]$, then for $t\in[j\delta,(j+1)\delta]$ we have

$$h(t)=h(j\delta)+\int_{j\delta}^{t}h'(s)ds=h(j\delta)+\int_{j\delta}^{t}\alpha(T-s)h(s-\delta)ds. \tag{5.17}$$

We have proved:

PROPOSITION 5.1

(Optimal consumption rate in a stochastic system with delay)

The optimal consumption rate $\hat{c}_\delta(t)$ for the problem (5.3)-(5.5), (5.8) is given by

$$\hat{c}_\delta(t) = I(t, h_\delta(T-t), \omega), \tag{5.18}$$

where $h_\delta(\cdot) = h(\cdot)$ is determined by (5.16)-(5.17).

REMARK 5.2

Assume that $\alpha(t) = \alpha > 0$ for all $t \in [0, T]$. Then we see by induction on (5.17) that

$$0 \leq \delta_1 < \delta_2 \Rightarrow h_{\delta_1}(t) > h_{\delta_{2(t)}} \text{ for all } t \in (0, T]$$

and hence, perhaps suprisingly,

$$0 \leq \delta_1 < \delta_2 \Rightarrow \hat{c}_{\delta_1}(t) < \hat{c}_{\delta_2}(t) \text{ for all } t \in [0, T).$$

Thus the optimal consumption rate *increases* if the delay increases. The explanation for this may be that the delay postpones the negative effect on the growth of the cash flow caused by the consumption.

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