Ergodic BSDEs and Ergodic Optimal Control

Ying Hu (IRMAR - Université Rennes 1)

joint work with

Arnaud Debussche (IRMAR, ENS Cachan)
Marco Fuhrman (Politecnico Milano)
Gianmario Tessitore (Bicocca Milano)

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Ergodic Control Problem

We address the following optimal control problem with

- State equation

$$dX_t^{x,u} = (AX_t^{x,u} + F(X_t^{x,u}))dt + G\,dW_t + G\,R(u_t), \qquad X_0^{x,u} = x$$

- Cost functional

$$J(x,u) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T L(X_s^{x,u}, u_s)ds.$$

Main features

ergodic cost functional

infinite dimensional equation (Banach space valued)

possibly degenerate G

Very incomplete list of references

BSDEs and infinite horizon stochastic control

- P. Briand and Y. Hu, *J. Funct. Anal.* (1998) (Finite dimensions - all positive discounts)
- M. Fuhrman and G. Tessitore, *Ann. Probab.* (2004) (Infinite dimensions - only large discounts)
- F. Masiero, *A.M.O.* (2007), (Banach spaces)

Ergodic stochastic control

- A. Bensoussan and J. Frehse, *J. Reine Angew. Math.* (1992) (Finite dimensions, classical solutions of HJB)
- M. Arisawa, P. L. Lions, *Comm. Partial Differential Equations* (1998) (Finite dimensions, viscosity solutions of HJB)
- B. Goldys and B. Maslowski, *J. Math. Anal. Appl.*, (1999) (Infinite dimensions, mild solutions of HJB, smoothing of Kolmogorov semigroup)

Forward (state) equation

$$\begin{cases} dX_t = AX_t dt + F(X_t)dt + GdW_t, & t \geq 0, \\ X_0 = x \in E. \end{cases}$$

E Banach, $E \subset H$ Hilbert space H.

- A generates a C_0 semigroup in E that has an extension to H.
- W is a cylindrical Wiener process in the Hilbert space Ξ
- $F : E \to E$ is continuous and has polynomial growth.
- $A + F$ is strictly dissipative (with constant η).
- G is bdd. $\Xi \to H$. The stochastic convolution

$$W_t^A = \int_0^t S(t-s)GdW_s, \quad t \geq 0,$$

has an E-continuous version with $\sup_t \mathbb{E}|W_t^A|_E^2 < \infty$.

Results on the forward (state) equation

$$\begin{cases} dX_t^x = AX_t^x dt + F(X_t^x)dt + GdW_t, & t \geq 0, \\ X_0^x = x \in E. \end{cases}$$

- $\forall x \in E$ there exists a unique E continuous mild solution X^x.
- Moreover $|X_t^{x_1} - X_t^{x_2}| \leq e^{-\eta t} |x_1 - x_2|$, $t \geq 0$, $x_1, x_2 \in E$.
- Finally $\sup_t \mathbb{E}|X_t^x|_E \leq C(1 + |x|)$.

Ergodic BSDEs (EBSDEs)

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_\sigma^x, Z_\sigma^x) - \lambda]\, d\sigma - \int_t^T Z_\sigma^x \, dW_\sigma, \quad 0 \le t \le T < \infty,$$

or equivalently

$$-dY_t^x = [\psi(X_t^x, Z_t^x) - \lambda]\, dt - Z_t^x dW_t$$

A solution is a triple (Y, Z, λ).

- λ is a real number.
- Y is a real continuous prog. meas. process such that $\mathbb{E} \sup_{t\in[0,T]} Y_s^2 < \infty$, $\forall T > 0$
- Z is a prog. meas. process with values in Ξ^* such that $\mathbb{E} \int_0^T |Z_s|_{\Xi^*}^2 < \infty$, $\forall T > 0$.

Main Result

On the function $\psi : E \times \Xi^* \to \mathbb{R}$ we assume:

- $|\psi(x,z) - \psi(x',z')| \leq K_x|x - x'| + K_z|z - z'|$, $x, x' \in E,\ z, z' \in \Xi^*$.
- $\psi(\,\cdot\,, 0)$ is bounded.

Theorem (Existence of solutions for EBSDEs)

- $\exists \lambda \in \mathbb{R}$;
- $\exists v : E \to \mathbb{R}$ *Lipschitz* ($v(0) = 0$);
- $\exists \zeta : E \to \Xi^*$ *measurable*

such that if we set $\bar{Y}_t^x := v(X_t^x)$, $\bar{Z}_t^x := \zeta(X_t^x)$

then $(\bar{Y}^x, \bar{Z}^x, \lambda)$ *is a solution of the EBSDE.*

Sketch of the proof

Considering with strictly monotonic drift $\alpha > 0$:

$$Y_t^{x,\alpha} = Y_T^{x,\alpha} + \int_t^T (\psi(X_\sigma^x, Z_\sigma^{x,\alpha}) - \alpha Y_\sigma^{x,\alpha}) d\sigma - \int_t^T Z_\sigma^{x,\alpha} dW_\sigma.$$

Lemma (Briand-Hu 1998, Royer 2004)

$\exists!$ solution $(Y^{x,\alpha}, Z^{x,\alpha})$ $Y^{x,\alpha}$ bounded cont., $Z^{x,\alpha} \in L^2_{\mathcal{P},\mathrm{loc}}$.

Moreover $|Y_t^{x,\alpha}| \leq M/\alpha$, $\mathbb{P}$-a.s. for all $t \geq 0$.

Define $v^\alpha(x) = Y_0^{\alpha,x}$. Clearly, $|v^\alpha(x)| \leq M/\alpha$ and $Y_t^{\alpha,x} = v^\alpha(X_t^x)$

Claim $|v^\alpha(x) - v^\alpha(x')| \leq \frac{K_x}{\eta}|x - x'|, \qquad x, x' \in E.$

Proof of claim Set
$\tilde{Y} = Y^{\alpha,x} - Y^{\alpha,x'}$, $\tilde{Z} = Z^{\alpha,x} - Z^{\alpha,x'}$,
$\beta_t = \frac{\psi(X_t^{x'}, Z_t^{\alpha,x'}) - \psi(X_t^{x'}, Z_t^{\alpha,x})}{|Z_t^{\alpha,x} - Z_t^{\alpha,x'}|_{\Xi^*}^2} \left(Z_t^{\alpha,x} - Z_t^{\alpha,x'}\right)^*$, notice β bdd.
$f_t = \psi(X_t^x, Z_t^{x,\alpha}) - \psi(X_t^{x'}, Z_t^{x,\alpha})$.

$\exists \tilde{\mathbb{P}}$ under which $\tilde{W}_t = \int_0^t \beta_s ds + W_t$ is a Wiener process.

$$\Longrightarrow \tilde{Y}_t = \tilde{Y}_T - \alpha \int_t^T \tilde{Y}_\sigma d\sigma + \int_t^T f_\sigma d\sigma - \int_t^T \tilde{Z}_\sigma d\tilde{W}_\sigma.$$
$$\Longrightarrow \quad |\tilde{Y}_t| \leq e^{-\alpha(T-t)} \tilde{\mathbb{E}}^{\mathcal{F}_t} |\tilde{Y}_T| + \tilde{\mathbb{E}}^{\mathcal{F}_t} \int_t^T e^{-\alpha(s-t)} |f_s| ds$$

Since $\tilde{Y}$ is bdd and $|f_t| \leq K_x e^{-\eta t}|x - x'|$ (by dissip.of forw. equat.) if $T \to \infty$ we get $|\tilde{Y}_t| \leq K_x(\eta + \alpha)^{-1} e^{\alpha t}|x - x'|$. $\square$

Proof of main result

Set $\overline{v}^{\alpha}(x) = v^{\alpha}(x) - v^{\alpha}(0)$,
We know $|\overline{v}^{\alpha}(x)| \leq K_x \eta^{-1}|x|$; $\alpha|v^{\alpha}(0)| \leq M$; $\{\overline{v}^{\alpha}\}$ unif. Lip.

$\Longrightarrow \quad \exists \alpha_n \searrow 0$ such that $\overline{v}^{\alpha_n}(x) \to \overline{v}(x)$, $\forall x$ and $\alpha_n \overline{v}^{\alpha_n}(0) \to \lambda$.

Define $\overline{Y}_t^{x,\alpha} = Y_t^{x,\alpha} - v^{\alpha}(0) = \overline{v}^{\alpha}(X_t^x)$ and $\overline{Y}^x = \overline{v}(X^x)$, then

$$\mathbb{E}\int_0^T |\overline{Y}_t^{x,\alpha_n} - \overline{Y}_t^x|^2 dt \to 0 \quad \text{and} \quad \mathbb{E}|\overline{Y}_T^{x,\alpha_n} - \overline{Y}_T^x|^2 \to 0$$

By standard BSDE arguments $\exists \overline{Z}^x \in L^2_{\mathcal{P},\mathrm{loc}}(\Omega; L^2(0,\infty;\Xi))$ s. t.

$$\mathbb{E}\int_0^T |Z_t^{x,\alpha_n} - \overline{Z}_t^x|^2_{\Xi^*} dt \to 0$$

Finally we remark that $\overline{Y}^{x,\alpha}$ verifies

$$\overline{Y}_t^{x,\alpha} = \overline{Y}_T^{x,\alpha} + \int_t^T (\psi(X_\sigma^x, Z_\sigma^{x,\alpha}) - \alpha \overline{Y}_\sigma^{x,\alpha} - \alpha v^\alpha(0))d\sigma - \int_t^T Z_\sigma^{x,\alpha} dW_\sigma.$$

Now we can pass to the limit as $n \to \infty$ to obtain

$$\overline{Y}_t^x = \overline{Y}_T^x + \int_t^T (\psi(X_\sigma^x, \overline{Z}_\sigma^x) - \overline{\lambda})d\sigma - \int_t^T \overline{Z}_\sigma^x dW_\sigma.$$

The construction of $\zeta : E \to \Xi^*$ such that $\overline{Z}_t^x = \zeta(X_t^x)$, exploits the fact that the same holds for $\overline{Z}^{x,\alpha}$. $\square$

Uniqueness of λ

The solution $(\overline{Y}^x, \overline{Z}^x, \lambda)$ we have constructed verifies

$$|\overline{Y}_t^x| \leq c|X_t^x|.$$

If we require similar conditions then we immediately obtain uniqueness of λ.

Theorem

Suppose that, for some $x \in E$, (Y', Z', λ') is a solution of (EBSDE) and verifies

$$|Y_t'| \leq c_x(|X_t^x| + 1), \text{ for all } t \geq 0.$$

Then $\lambda' = \lambda$.

Lack of uniqueness of EBSDEs

Clearly if (Y, Z, λ) is a solution then $(Y + c, Z, \lambda)$ is a solution.

Even if we ask $Y_0^0 = 0$ the solution to EBSDE is, not unique.

If we do not require $Y_t = v(X_t^x)$, $Z_t = \zeta(X_t^x)$ then can construct several solutions of the above EBSDE (with Y and Z bounded).

If we require $Y_t = v(X_t^x)$, $Z_t = \zeta(X_t^x)$ with v and ζ continuous and X^x to be recursive (see [Seidler 1997])
then v can be characterized (as in [Goldys-Maslowski 1999]) by:

$$v(x) = \inf_u \limsup_{r \to 0} \limsup_{T \to \infty} \mathbb{E} \int_0^{\tau_r^T} [\psi(X_s^{x,u}, u(X_s^{x,u})) - \lambda] ds.$$

where $\tau_r^T = \inf\{s \in [0, T] : |X_s^{u,x}| < r\}$.

Optimal Ergodic Control problem

Let X^x be the solution to equation

$$dX_t^x = (AX_t^{x,u} + F(X_t^{x,u}))dt + GdW_t, \qquad X_0^{x,u} = x$$

An admissible control u is a progressively measurable process with values in a Borel subset U of a complete metric space.

The ergodic cost corresponding to u and the starting point $x \in E$ is

$$J(x,u) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^{u,T}\int_0^T L(X_s^x, u_s)ds,$$

where

$$\rho_T^u = \exp\left(\int_0^T R(u_s)dW_s - \frac{1}{2}\int_0^T |R(u_s)|_{\Xi^*}^2 ds\right), \quad \mathbb{P}_T^u = \rho_T^u\mathbb{P}.$$

Where $R : U \to \mathbb{R}$, $L : U \times E \to \mathbb{R}$ with R, L bdd in u; L Lip. in x.

Ergodic control and EBSDEs

We first define the Hamiltonian in the usual way

$$\psi(x,z) = \inf_{u\in U}\{L(x,u) + zR(u)\}, \qquad x \in E,\ z \in \Xi^*.$$

Under the present assumptions ψ is a Lipschitz function and $\psi(\cdot, 0)$ is bounded thus the EBSDE

$$-dY_t^x = [\psi(X_t^x, Z_t^x) - \lambda]\,dt - Z_t^x dW_t$$

has at least a solution (Y^x, Z^x, λ)

Synthesis of Optimal control

Theorem

Suppose that, for some $x \in E$, a triple (Y, Z, λ) verifies EBSDE and $|Y_t^x| \leq c_x(|X_t^x| + 1)$, for all $t \geq 0$.

Then the following holds:

(i) *For arbitrary control u we have $J(x, u) \geq \lambda$ and the equality holds if and only if $L(X_t^x, u_t) + Z_t R(u_t) = \psi(X_t^x, Z_t)$.*

(ii) *If the infimum in the definition of ψ is attained at $u = \gamma(x, z)$ then the control $\bar{u}_t = \gamma(X_t^x, Z_t)$ verifies $J(x, \bar{u}) = \lambda$.*

Recall that λ is univocally determined.

Differentiability and identification of Z

We recall that in the proof of the existence of EBSDE we have constructed specific $v : E \to \mathbb{R}$ and $\zeta : E \to \mathbb{R}$ such that if $\bar{Y}_t^x = v(X_t^x)$, $\bar{Z}_t^x = \zeta(X_t^x)$ then

$$-d\bar{Y}_t^x = \left[\psi(X_t^x, \bar{Z}_t^x) - \lambda\right] dt - \bar{Z}_t^x dW_t$$

Theorem

If F and ψ are continuously Gâteaux differentiable then the function v is continuously Gâteaux differentiable.

If $\exists$ a Banach space $\Xi_0 \subset \Xi$, s. t. $G : \Xi_0 \to E$ is bdd. (see [Masiero]) then $\bar{Z}_t^x = \nabla_x v(X_t^x) G$.

Consequently the optimal feedback law for the ergodic control problem becomes $\bar{u}(x) = \gamma(x, \nabla v(x) G)$

Other consequences of Identification

We introduce here the Kolmogorov semigroup corresponding to X:

$$P_t[\phi](x) = \mathbb{E}\phi(X_t^x); \qquad \forall \phi : E \to \mathbb{R} \text{ with polynomial growth.}$$

Definition

The semigroup $(P_t)_{t\geq 0}$ is strongly Feller if

$$|P_t[\phi](x) - P_t[\phi](x')| \leq k_t \|\phi\|_0 |x - x'|.$$

Definition

F is genuinely dissipative if for all $x, x' \in E$, there exists $z^* \in \partial|x - x'|$ such that $< F(x) - F(x'), z^* > \leq c|x - x'|^{1+\epsilon}$.

Corollary

Suppose that F is continuously Gâteaux differentiable and that ψ has linear growth in z with respect to the Ξ_0^* norm.

If the Kolmogorov semigroup (P_t) is strongly Feller then:

$$\lambda = \int_E \psi(x, \zeta(x))\mu(dx),$$

where μ is the unique invariant measure of X.

If, in addition F is genuinely dissipative then v is bounded.

Ergodic H.J.B. Equations

If $\bar{Y}_0^x = v(x)$ is differentiable (v, λ) is a mild solution of the "ergodic" Hamilton-Jacobi-Bellman equation:

$$\mathcal{L}v(x) + \psi\left(x, \nabla v(x) G\right) = \lambda, \quad x \in E,$$

where $\mathcal{L}$ is formally defined by

$$\mathcal{L}f\left(x\right) = \frac{1}{2} Tr\left(GG^* \nabla^2 f\left(x\right)\right) + \langle Ax, \nabla f\left(x\right)\rangle_{E,E^*} + \langle F\left(x\right), \nabla f\left(x\right)\rangle_{E,E^*}.$$

By mild solution we mean that for all $0 < t < T$ it holds

$$v(x) = P_{T-t}\left[v\right](x) + \int_t^T \left(P_{\tau-t}\left[\psi(\cdot, \nabla v\left(\cdot\right) G)\right](x) - \lambda\right) d\tau, \qquad x \in E.$$

Example

We consider, for $t \in [0, T]$ and $\xi \in [0,1]$, the equation:

$$\begin{cases} d_t X^u(t,\xi) = \left[\frac{\partial^2}{\partial \xi^2} X^u(t,\xi) + f(\xi, X^u(t,\xi)) + \chi_{[a,b]}(\xi) u(t,\xi) \right] dt \\ \qquad\qquad\qquad + \chi_{[a,b]}(\xi)(\xi) \dot{W}(t,\xi)\, dt, \\ X^u(t,0) = X^u(t,1) = 0, \\ X^u(t,\xi) = x_0(\xi), \end{cases} \tag{1}$$

where $0 \leq a \leq b \leq 1$ and $\dot{W}(t,\xi)$ is a space-time white noise on $[0, T] \times [0,1]$.

We introduce the cost functional

$$J(x,u) = \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \int_0^T \int_0^1 l(\xi, X_s^u(\xi), u_s(\xi)) \mu(d\xi)\, ds \tag{2}$$

Here μ is a finite regular measure on $[0,1]$.

An admissible control $u(\tau, \xi)$ is a predictable process such that for all $\tau \geq 0$, and $\mathbb{P}$-a.s. $u(\tau, \cdot) \in U := \{v \in C([0,1]) : |v(\xi)| \leq \delta\}$

We suppose the following:

- $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and for every $\xi \in [0,1]$, $f(\xi, \cdot)$ is decreasing in x.
  Moreover $|f(\xi, x)| \leq C(1 + |x|)^m$.
- $l : [0,1] \times \mathbb{R} \times \mathcal{U} \rightarrow \mathbb{R}$ is continuous and bounded.
- $x_0 \in C([0,1])$.

Weak dissipative assumption

Let us now suppose that F is Lipschitz, bounded and Gâteaux differentiable (of class $\mathcal{G}^1$) and G is invertible.
We assume that there exists $k > 0$ such that

$$\langle Ax, x\rangle \leq -k|x|_H^2 \quad \forall x \in D(A)$$

Main tool: Coupling estimate (see, e.g. Hairer and Mattingly, Annals of Mathematics 2006).
Recurrence property: Da Prato and Zabczyk 1992.

Basic coupling estimate

Theorem

Let $\Upsilon : H \to H$ be a bounded Lipschitz map $H \to H$ and let $\mathbf{X}^x$ be the **strong** *solution of the equation*

$$\begin{cases} d\mathbf{X}_t^x = A\mathbf{X}_t^x dt + \Upsilon(\mathbf{X}_t^x)dt + GdW_t, & t \geq 0, \\ \mathbf{X}_0^x = x \in H. \end{cases} \tag{3}$$

Then there exist $\hat{c} > 0$ and $\hat{\eta} > 0$ such that for all $\phi \in B_b(H)$ with $\sup_{x\in H} |\phi(x)| \leq 1$

$$\left|\mathcal{P}_t[\phi](x) - \mathcal{P}_t[\phi](x')\right| \leq \hat{c}(1 + |x|^2 + |x'|^2)e^{-\hat{\eta}t} \tag{4}$$

where $\mathcal{P}_t[\phi](x) = \mathbb{E}\phi(\mathbf{X}_t^x)$ is the Kolmogorov semigroup associated to equation (3).
We stress the fact that $\hat{c}$ and $\hat{\eta}$ depend on Υ only through $\sup_{x\in H} |\Upsilon(x)|$.

bounded and measurable drift

Corollary

Relation (4) can be extended to the case in which Υ *is only bounded and measurable, and there exists a uniformly bounded sequence of Lipschitz functions* $\{\Upsilon_n\}_{n\geq 1}$ *(i.e.* $\forall n, \Upsilon_n$ *is Lipschitz and* $\sup_n \sup_x |\Upsilon_n(x)| < \infty$*) such that*

$$\lim_n \Upsilon_n(x) = \Upsilon(x), \quad \forall x \in H$$

(in this case the solution of equation (3) has to be intended the weak sense).

Theorem

Assume that $\Upsilon : H \to H$ can be approximated (in the sense of poi ntwise convergence) by a uniformly bounded sequence of Lipschitz functions $\{\Upsilon_n\}_{n\geq 1}$.
Then the solution of equation (3) is recurrent in the sense that for all $\Gamma \in H$, Γ open:

$$\lim_{T\to\infty} \hat{\mathbb{P}}\{\exists t \in [0, T] : \hat{X}_t^x \in \Gamma\} = 1.$$

In particular, setting $\tau^x = \inf\{t : |\hat{X}_t^x| < \epsilon\}$, then $\forall \epsilon > 0$, $\lim_{T\to\infty} \hat{\mathbb{P}}\{\tau^x < T\} = 1$.

Proof: Doob's Method.

Approximation

Let now $\psi : H \times \Xi^* \to \mathbb{R}$ continuous, with

$$|\psi(x,0)| \leq \ell; \qquad |\psi(x,z) - \psi(x,z')| \leq \ell |z - z'| \tag{5}$$

and let $\alpha > 0$ be fixed.
We consider the following (decoupled) forward-backward system (with infinite horizon):

$$\begin{cases} dX_t^x = AX_t^x dt + F(X_t^x)dt + GdW_t, \qquad t \geq 0, \\ -dY_t^{\alpha,x} = \psi(X_t^x, Z_t^{\alpha,x})dt - \alpha Y_t^{\alpha,x} dt - Z_t^{\alpha,x} dW_t, \qquad t \geq 0, \\ \hat{X}_0^x = x \in H. \end{cases} \tag{6}$$

As it is well known the BSDE in the above system admits a unique solution with $Y^{\alpha,x}$ bounded. In particular $|Y_t^{\alpha,x}| \leq \ell/\alpha$.

Main Estimates

Theorem

There exists a constant $c(\ell, \hat{c}, \hat{\eta}) > 0$ *such that for all* x, $x' \in H$

$$|v^\alpha(x) - v^\alpha(x')| \leq c(1 + |x|^2 + |x'|^2); \tag{7}$$

and for all $x \in H$,

$$|\nabla v^\alpha(x)| \leq c(1 + |x|^2). \tag{8}$$

We stress the fact that $c > 0$ is independent of α.

Proof of Theorem

Set

$$\tilde{\Upsilon}^\alpha(x) = \begin{cases} \dfrac{\psi(x, \nabla v^\alpha(x)G) - \psi(x,0)}{|\nabla v^\alpha(x)G|^2} (\nabla v^\alpha(x)G)^* & \text{if } \nabla v^\alpha(x)G \neq 0 \\ 0 \text{ if } \nabla v^\alpha(x)G = 0. \end{cases}$$

Then

$$\psi(X_t^x, Z_t^{\alpha,x}) = \psi(X_t^x, 0) + \tilde{\Upsilon}^\alpha(X_t^x) Z_t^{\alpha,x}.$$

$\tilde{\Upsilon}^\alpha$ is the pointwise limit of a uniformly bounded sequence of Lipschitz functions.

For all $T > 0$, the couple of processes $(Y^{\alpha,x}, Z^{\alpha,x})$ is a solution to the following finite horizon linear BSDE, $t \in [0, T]$,

$$\begin{cases} -dY_t^{\alpha,x} = \psi(X_t^x, 0)dt + \tilde{\Upsilon}^\alpha(X_t^x) Z_t^{\alpha,x} dt - \alpha Y_t^{\alpha,x} dt - Z_t^{\alpha,x} dW_t, \\ Y_T^{\alpha,x} = v^\alpha(X_T^x). \end{cases} \tag{9}$$

Since $\tilde{\Upsilon}^\alpha$ is bounded for all $T > 0$ there exists a unique probability $\hat{\mathbb{P}}^{\alpha,x,T}$ such that

$$\hat{W}_t^{\alpha,x} = \int_0^t \hat{\gamma}^\alpha(X_s^x)ds + W_t$$

is a $\hat{\mathbb{P}}^{\alpha,x,T}$-Wiener process for $t \in [0, T]$. Consequently we have

$$v^\alpha(x) = \hat{\mathbb{E}}^{\alpha,x,T}\left[e^{-\alpha T}v^\alpha(X_T^x) + \int_0^T e^{-\alpha s}\psi(X_s^x, 0)ds\right]$$

where $\hat{\mathbb{E}}^{\alpha,x,T}$ denotes the expectation with respect to $\hat{\mathbb{P}}^{\alpha,x,T}$. Letting $T \to \infty$, as $|v^\alpha(x)| \le \frac{l}{\alpha}$, we get

$$v^\alpha(x) = \lim_{T\to\infty} \hat{\mathbb{E}}^{\alpha,x,T}\left[\int_0^T e^{-\alpha s}\psi(X_s^x, 0)ds\right].$$

Key Idea

We rewrite the forward equation (3) with respect to $\hat{W}^{\alpha,x}$ it turns out that X^x verifies

$$\begin{cases} dX_t^x = AX_t^x dt + F(X_t^x)dt + G\tilde{\Upsilon}^\alpha(X_t^x)dt + G\hat{W}_t^{\alpha,x}, \\ \hat{X}_0^x = x \in H. \end{cases} \tag{10}$$

We denote by $\mathcal{P}^\alpha$ the associated Kolmogorov semigroup, i.e.,

$$\mathcal{P}_t^\alpha[\phi](x) = \hat{\mathbb{E}}^{\alpha,x,t}\phi(X_t^x).$$

Applying Theorem with $\Upsilon^\alpha = F + G\tilde{\Upsilon}^\alpha$ (which is also the pointwise limit of a sequence of Lipschitz functions), we obtain

$$\begin{aligned} |v^\alpha(x) - v^\alpha(x')| &\leq \int_0^\infty e^{-\alpha t}\left|\mathcal{P}_t^\alpha[\psi(\cdot,0)](x) - \mathcal{P}_t^\alpha[\psi(\cdot,0)](x')\right| dt \\ &\leq \frac{\hat{c}l}{\hat{\eta}}(1 + |x^2| + |x'|^2) \end{aligned}$$

To prove (ii), let us set

$$\bar{v}^{\alpha}(x) = v^{\alpha}(x) - v^{\alpha}(0).$$

Then, $\bar{Y}_t^{\alpha,x} = Y_t^{\alpha,x} - Y_0^{\alpha,0} = \bar{v}^{\alpha}(X_t^x)$ is the unique solution of the finite horizon BSDE

$$\begin{cases} -d\bar{Y}_t^{\alpha,x} = \psi(X_t^x, Z_t^{\alpha,x})dt - \alpha \bar{Y}_t^{\alpha,x} - \alpha v^{\alpha}(0)dt - Z_t^{\alpha,x} dW_t, \\ Y_1^{\alpha,x} = \bar{v}^{\alpha}(X_1^x). \end{cases}$$

Note that in particular, in the above equation, $|\alpha v^{\alpha}(0)| \leq l$. By Bismut-Elworthy's formula, $\bar{v}^{\alpha}$ is of class $\mathcal{G}^1$ and there exists a constant $c(l, \hat{c}, \hat{\eta}) > 0$ independent of α such that $|\nabla v^{\alpha}(x)| \leq c(1 + |x|^2)$, and the conclusion follows.

Existence of solutions for EBSDEs

Theorem

- $\exists \lambda \in \mathbb{R}$;
- $\exists v : E \to \mathbb{R}$ *locally Lipschitz* ($v(0) = 0$);
- $\exists \zeta : E \to \Xi^*$ *measurable*

such that if we set $\bar{Y}_t^x := v(X_t^x)$, $\bar{Z}_t^x := \zeta(X_t^x)$

then $(\bar{Y}^x, \bar{Z}^x, \lambda)$ *is a solution of the EBSDE.*

Uniqueness of Markovian solution

We prove that the Markovian solution is unique.

Theorem

Let (v, ζ), $(\tilde{v}, \tilde{\zeta})$ two couples of functions with $v, \tilde{v} : H \to \mathbb{R}$, continuous, with $|v(x)| \le c(1 + |x|^2)$, $|\tilde{v}(x)| \le c(1 + |x|^2)$, $v(0) = \tilde{v}(0) = 0$ and ζ, $\tilde{\zeta}$ continuous from H to Ξ^* endowed with the weak* topology verifying $|\zeta(x)| \le c(1 + |x|^2)$, $|\tilde{\zeta}(x)| \le c(1 + |x|^2)$.
Assume that for some constants λ, $\tilde{\lambda}$ and all $x \in H$, $(v(X_t^x), \zeta(X_t^x), \lambda)$, $(\tilde{v}(X_t^x), \tilde{\zeta}(X_t^x), \tilde{\lambda})$ verify the EBSDE, then $\lambda = \tilde{\lambda}$, $v = \tilde{v}$, $\zeta = \tilde{\zeta}$.

Proof: Part 1

The equality $\lambda = \tilde{\lambda}$ comes from Girsanov's transformation. Then let $\bar{Y}_t^x = v(X_t^x) - \tilde{v}(X_t^x)$, $\bar{Z}_t^x = \zeta(X_t^x) - \tilde{\zeta}(X_t^x)$ and $\tilde{\Upsilon}$ be defined by linearization. We have

$$-d\bar{Y}_t^x = \tilde{\Upsilon}(X_t^x)\bar{Z}_t^x dt - \bar{Z}_t^x dW_t = -\bar{Z}_t^x dW_t'$$

where $W_t' = -\int_0^t \Upsilon(X_s^x)ds + W_t$ is a Wiener process in [0,T] under the probability $\bar{\mathbb{P}}^{x,T}$.
Moreover, under $\bar{\mathbb{P}}^{x,T}$, X^x satisfies equation (3), in $[0, T]$, with, as before $\Upsilon = G\widetilde{\Upsilon} + F$. Thus, it holds that for all $p \geq 1$, and all $x \in H$

$$\bar{\mathbb{E}}^{x,T}|X_t^x|^p \leq c(1 + |x|^p), \forall 0 \leq t \leq T,$$

where $c > 0$ depends on p, γ, M and $l|G| + \sup_x |F(x)|$, and is independent of T. Thus the growth conditions on ζ and $\tilde{\zeta}$ implies that, for all $T > 0$, $\bar{\mathbb{E}}^{x,T}\int_0^T |\bar{Z}_t^x|^2 dt < \infty$.

Proof: Part 2: Recurrence property

Let $\tau = \inf\{t : |X_t^x| < \epsilon\}$ then for all $T > 0$

$$\bar{Y}_0^x = \bar{\mathbb{E}}^{x,T}\bar{Y}_{T\wedge\tau}^x.$$

For any $\delta > 0$, there exists $\epsilon > 0$ such that $|v(x) - \tilde{v}(x)| \leq \delta$ if $|x| \leq \epsilon$. Then for a constant $c > 0$,

$$\begin{aligned}
|\bar{Y}_0^x| = |\bar{\mathbb{E}}^{x,T}\bar{Y}_{T\wedge\tau}^x| &\leq \bar{\mathbb{E}}^{x,T}|\bar{Y}_\tau^x|1_{\{\tau<T\}} + \bar{\mathbb{E}}^{x,T}|\bar{Y}_T^x|1_{\{\tau\geq T\}} \\
&\leq \delta + \left(\bar{\mathbb{P}}^{x,T}\{\tau \geq T\}\right)^{1/2}\left(\bar{\mathbb{E}}^{x,T}\{|\bar{Y}_T^x|^2\}\right)^{1/2} \\
&\leq \delta + \left(\bar{\mathbb{P}}^{x,T}\{\tau \geq T\}\right)^{1/2}\left(\bar{\mathbb{E}}^{x,T}\{1 + |X_T^x|^4\}\right)^{1/2}.
\end{aligned}$$

Noting that, by recurrence, $\lim_{T\to\infty}\bar{\mathbb{P}}^{x,T}\{\tau \geq T\} = 0$ and sending T to ∞ in the last inequality, we obtain that $|\bar{Y}_0^x| \leq \delta$ an d the claim follows from the arbitrarity of δ.