Optimal profitability of an investment under uncertainty- A backward SDE approach

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Position of the problem

Let Y^1 and Y^2 denote the expected profit and cost yields respectively. The constituants of these cash flows are

(a) Per unit time dt, the profit yield is ψ^1 and the cost yield is ψ^2;

(b) When exiting/abandoning the project at time t, the incurred cost is $a(t)$ and the incurred profit is $b(t)$ (usually $a \neq b$ but often non-negative).

Exit/abandonment strategy:

The decision to exit the project at time t, depends on whether

$$Y_t^1 \leq Y_t^2 - a(t) \text{ or } Y_t^2 \geq Y_t^1 + b(t).$$

A Snell envelop formulation

If $\mathcal{F}_t$ denotes the history of the project up to time t,

The expected profit yield, at time t, is

$$Y_t^1 = \text{ess sup}_{\tau \geq t} E^{\mathcal{F}_t} \left[\int_t^\tau \psi^1(s, Y_s^1) ds + \left(Y_\tau^2 - a(\tau) \right) 1_{[\tau < T]} + \xi^1 1_{[\tau = T]} \right]$$

where, the sup is taken over all exit times τ from the project.

The optimal exit time related to the incurred cost $Y^2 - a$ should be

$$\tau_t^* = \inf\{s \geq t,\ Y_s^1 = Y_s^2 - a(s)\} \wedge T.$$

The expected cost yield at time t, is

$$Y_t^2 = \text{ess inf}_{\sigma \geq t} E^{\mathcal{F}_t}\left[\int_t^\sigma \psi^2(s, Y_s^2)ds + \left(Y_\sigma^1 + b(\sigma)\right) 1_{[\sigma < T]} + \xi^2 1_{[\sigma = T]}\right]$$

where, the inf is taken over all exit times σ from the project.

The optimal exit time related to the incurred profit $Y^1 + b$ should be

$$\sigma_t^* = \inf\{s \geq t,\ \ Y_s^2 = Y_s^1 + b(s)\} \wedge T.$$

Problem I

Establish existence, uniqueness and continuity of (Y^1, Y^2) which solves the coupled system of Snell envelops

$$Y_t^1 = \text{ess sup}_{\tau \geq t} E^{\mathcal{F}_t}\left[\int_t^\tau \psi^1(s, Y_s^1)ds + \left(Y_\tau^2 - a(\tau)\right) 1_{[\tau<T]} + \xi^1 1_{[\tau=T]}\right]$$

$$Y_t^2 = \text{ess inf}_{\sigma \geq t} E^{\mathcal{F}_t}\left[\int_t^\sigma \psi^2(s, Y_s^2)ds + \left(Y_\sigma^1 + b(\sigma)\right) 1_{[\sigma<T]} + \xi^2 1_{[\sigma=T]}\right]$$

where, the sup and inf are taken over $\mathcal{F}_t$-stopping times.

Continuity insures optimality of the stopping times τ^* and σ^*.

Problem II- Extension to optimal swithching

$$
\begin{aligned}
Y_t^{1,i} &= \text{ess sup}_{\tau\geq t}\Big(E^{\mathcal{F}_t}\left[\int_t^\tau \phi^i(s, Y_s^{1,i})ds + \xi^{1,i}1_{[\tau=T]}\right] \\
&\quad +E^{\mathcal{F}_t}\left[\left(\max_{j\neq i}(Y^{1,j} - a_{ij}(\tau)) \vee (Y_\tau^{2,i} - a_i(\tau))\right) 1_{[\tau<T]}\right]\Big), \\
Y_t^{2,i} &= \text{ess inf}_{\sigma\geq t}\Big(E^{\mathcal{F}_t}\left[\int_t^\sigma \psi^i(s, Y_s^{2,i})ds + \xi^i 1_{[\sigma=T]}\right] \\
&\quad +E^{\mathcal{F}_t}\left[\left(\min_{j\neq i}\left(Y_\sigma^{2,j} + b_{ij}(\sigma)\right) \wedge (Y_\sigma^{1,i} + b_i(\sigma))\right) 1_{[\sigma<T]}\right]\Big),
\end{aligned}
$$

where, the sup and inf are taken over $\mathcal{F}_t$-stopping times.

Problem III. When pension schemes are also considered

The constituants of the cash flows Y^1 and Y^2 also include the *prospective bonus reserve (or bonus potential)* i.e. future pension payments that are not guaranteed (see e.g. Møller and Steffensen (2007)).

The amount to be maximized (or minimized) in each time interval $[t_j, t_{j+1}]$ is

$$g(t_{j+1})(B(t_{j+1}) - B(t_j)),$$

where,

- $B(t_{j+1}) - B(t_j)$ is the retun that should match the prospective reserve (bonus),
- $g(t)$ is some coefficient that should reflect the distribution of bonuses at the end of the period. It should be adapted to the "backward" history $\mathcal{F}^B_{t,T}$ generated by $(B(T) - B(r),\, t \le r \le T)$.

The accumulated bonus potential during $[0, T]$ is then

$$\sum_{i=0}^{n-1} g(t_{j+1})(B(t_{j+1}) - B(t_j))$$

where, $t_0 = 0 < t_1 < \ldots < t_n = T$.

Taking the limit, we obtain the backward stochastic integral

$$\int_0^T g(s)\overleftarrow{dB}(s).$$

An extended Snell envelop formulation

Given two independent Brownian motions W and B, establish existence, uniqueness and continuity of (Y^1, Y^2), adapted to $\mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$, which solves the coupled system of Snell envelops.

$$
\begin{aligned}
Y_t^1 = \ & \text{ess sup}_{\tau \geq t}\Big(E^{\mathcal{G}_t}\Big[\int_t^\tau \psi^1(s, Y_s^1)ds + \int_t^\tau g^1(s, Y_s^1)\overleftarrow{dB}(s)\Big] \\
& + E^{\mathcal{G}_t}\big[\big(Y_\tau^2 - a(\tau)\big)1_{[\tau<T]} + \xi^1 1_{[\tau=T]}\big]\Big), \\
Y_t^2 = \ & \text{ess inf}_{\sigma \geq t}\Big(E^{\mathcal{G}_t}\Big[\int_t^\sigma \psi^2(s, Y_s^2)ds + \int_t^\sigma g^2(s, Y_s^2)\overleftarrow{dB}(s)\Big] \\
& + E_t^{\mathcal{G}}\big[\big(Y_\sigma^1 + b(\sigma)\big)1_{[\sigma<T]} + \xi^2 1_{[\sigma=T]}\big]\Big).
\end{aligned}
$$

where, $\mathcal{G}_t = \mathcal{F}_t^W \vee \mathcal{F}_{0,T}^B$, and the sup and inf are taken over $\mathcal{G}_t$-stopping times.

A solution of Problem I

The set up

- $W := (W_t)_{0 \le t \le T}$ a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$.
- $(\mathcal{F}_t^W)_{0 \le t \le T}$ the completed natural filtration of W.
- $X := (X_t)_{0 \le t \le T}$ a diffusion process which stands for factors which determine the price of the underlying commodity we wish to control such as e.g. the price of electricity in the energy market.

The Snell envelop versus reflected BSDEs

- $\mathcal{S}^2$ denotes the set of all right-continuous with left limits processes Y satisfying $E\left(\sup_{t\in[0,T]}|Y_t^2|\right) < \infty$.
- $\mathcal{M}^{d,2}$ denotes the set of $\mathcal{F}$-adapted and d-dimensional processes Z such that $E\left(\int_0^T |Z_s|^2 ds\right) < \infty$.
- $\mathcal{A}^+$ denotes the set of right-continuous with left limits and increasing processes K.
- $\mathcal{A}^{+,2}$ the subset of $\mathcal{A}^+$ consisting of all the processes K satisfying, in addition, $E(K_T^2) < \infty$.

Cash-flows: A system of reflected BSDEs formulation

By El-Karoui *et al.* '97, (Y^1, Y^2) should solve the following system of RBSDEs:

$$\begin{cases} Y_t^1 = \xi^1 + \int_t^T \psi^1(s, Y_s^1)ds + (K_T^1 - K_t^1) - \int_t^T Z_s^1 dW_s, \\ Y_t^2 = \xi^2 + \int_t^T \psi^2(s, Y_s^2)ds - (K_T^2 - K_t^2) - \int_t^T Z_s^2 dW_s, \\ Y_t^1 \ \geq Y_t^2 - a(t), \quad Y_t^2 \leq Y_t^1 + b(t), \ 0 \leq t \leq T, \\ \int_0^T \left(Y_t^1 - (Y_t^2 - a(t))\right) dK_t^1 = 0, \ \int_0^T (Y_t^1 + b(t) - Y_t^2) dK_t^2 = 0. \end{cases}$$

Minimal and maximal solutions

We solve a more general problem and make the following assumptions:

(B1) For each $i = 1, 2$, the process ψ^i depends on $(t, \omega, Y_t^i, Z_t^i)$. Moreover, $(t, \omega, y, z) \to \psi^i(t, \omega, y, z)$'s are Lipschitz continuous with respect to y and z and satisfy,

$$E\left(\int_0^T |\psi^i(t, 0, 0, 0)|^2 ds\right) < \infty.$$

(B2) The obstacles a and b are continuous and in $\mathcal{S}^2$. Moreover, they admit a semimartingale decomposition:

$$a(t) = a(0) + \int_0^t U_s^1 ds + \int_0^t V_s^1 dB_s,$$

(to insure continuity of the minimal solution!)

$$b(t) = b(0) + \int_0^t U_s^2 ds + \int_0^t V_s^2 dB_s,$$

(to insure continuity of the maximal solution!)

for some $\mathcal{F}^W$-prog. meas. processes U^1, V^1, U^2 and V^2.

(B3) ξ^i's are in $L^2(\mathcal{F}_T^W)$ and satisfy

$$\xi^1 - \xi^2 \geq max\{-a(T), -b(T)\}, \quad P - a.s.$$

The main result

Let the coefficients $(\psi^1, \psi^2, a, b, \xi^1, \xi^2)$ satisfy Assumptions (**B1**)-(**B3**). Then the system of RBSDEs admits a minimal and a maximal $\mathcal{F}^W$-prog. meas. solutions $(Y^1, Y^2, Z^1, Z^2, K^1, K^2)$ and $(\bar{Y}^1, \bar{Y}^2, \bar{Z}^1, \bar{Z}^2, \bar{K}^1, \bar{K}^2)$, respectively, which are in $(\mathcal{S}^2)^2 \times (\mathcal{M}^{d,2})^2 \times (\mathcal{A}^{+,2})^2$.

Moreover,

- the processes Y^i and $\bar{Y}^i$, $i = 1, 2$ are $P-$a.s. continuous and admit the above Snell representations.
- the random times τ^* and σ^* defined above and associated with either Y^i or $\bar{Y}^i$, are optimal stopping times.

A minimal solution through the increasing sequences scheme

Start with the pair $(Y^{1,0},\ Z^{1,0})$ that solves uniquely the BSDE

$$Y_t^{1,0} = \xi^1 + \int_t^T \psi^1(s, Y_s^{1,0}, Z_s^{1,0})ds - \int_t^T Z_s^{1,0} dW_s.$$

and introduce the following system of RBSDEs

$$\begin{cases} dY_s^{2,n+1} = \psi^2(s, Y_s^{2,n+1}, Z_s^{2,n+1})ds - dK_s^{2,n+1} - Z_s^{2,n+1} dW_s, \\ dY_s^{1,n+1} = \psi^1(s, Y_s^{1,n+1}, Z_s^{1,n+1})ds + dK_s^{1,n+1} - Z_s^{1,n+1} dW_s, \\ Y_s^{2,n+1} \le Y_s^{1,n} + b(s), \quad Y_s^{1,n+1} \ge Y_s^{2,n+1} - a(s),\ 0 \le s \le T, \\ \int_0^T (Y_t^{1,n+1} - (Y_t^{2,n+1} - a(t))dK_t^{1,n+1} = 0, \quad Y_T^{1,n+1} = \xi^1; \\ \int_0^T (Y_t^{1,n} + b(t) - Y_t^{2,n+1})dK_t^{2,n+1} = 0, \quad Y_T^{2,n+1} = \xi^2. \end{cases}$$

This sequence of solutions satisfies the following properties:

- For any $n \geq 0$, both $(Y^{1,n}, Z^{1,n}, K^{1,n})$ and $(Y^{2,n+1}, Z^{2,n+1}, K^{2,n+1})$ exist and are in $\mathcal{S}^2 \times \mathcal{M}^{d,2} \times \mathcal{A}^{+,2}$.
- The two sequences $(Y^{1,n})_{n\geq 0}$ and $(Y^{2,n})_{n\geq 1}$ are increasing in n, meaning that for all $n \geq 0$,

$$Y_t^{1,n} \leq Y_t^{1,n+1} \quad \text{and} \quad Y_t^{2,n+1} \leq Y_t^{2,n+2} \;\; P\text{-a.s. and for all } t.$$

- the limit process (Y^1, Y^2) of $(Y_t^{1,n}, Y_t^{2,n})$ is continuous, a minimal solution of our system of RBSDEs and admits a Snell envelop representation.

A maximal solution through the decreasing sequences scheme

Start with the pair $(\bar{Y}^{2,0},\ \bar{Z}^{2,0})$ that solves the standard BSDE

$$\bar{Y}_t^{2,0} = \xi^2 + \int_t^T \psi^2(s, \bar{Y}_s^{2,0}, \bar{Z}_s^{2,0})ds - \int_t^T \bar{Z}_s^{2,0} dW_s,$$

and introduce the following system of RBSDEs

$$\begin{cases} d\bar{Y}_s^{1,n+1} = \psi^1(s, \bar{Y}_s^{1,n+1}, \bar{Z}_s^{1,n+1})ds + d\bar{K}_s^{1,n+1} - \bar{Z}_s^{1,n+1} dW_s, \\ d\bar{Y}_t^{2,n+1} = \psi^2(s, \bar{Y}_s^{2,n+1}, \bar{Z}_s^{2,n+1})ds - d\bar{K}_s^{2,n+1} - \bar{Z}_s^{2,n+1} dW_s, \\ \bar{Y}_s^{1,n+1} \geq \bar{Y}_s^{2,n} - a(s), \quad \bar{Y}_s^{2,n+1} \leq \bar{Y}_s^{1,n+1} + b(s),\ 0 \leq s \leq T, \\ \int_0^T (\bar{Y}_t^{1,n+1} - (\bar{Y}_t^{2,n} - a(t))d\bar{K}_t^{1,n+1} = 0, \quad \bar{Y}_T^{1,n+1} = \xi^1, \\ \int_0^T (\bar{Y}_t^{1,n+1} + b(t) - \bar{Y}_t^{2,n+1})d\bar{K}_t^{2,n+1} = 0, \quad \bar{Y}_T^{2,n+1} = \xi^2. \end{cases}$$

This sequence of solutions satisfies the following properties.

- For any $n \geq 0$, both $(\bar{Y}^{2,n}, \bar{Z}^{2,n}, \bar{K}^{2,n})$ and $(\bar{Y}^{1,n+1}, \bar{Z}^{1,n+1}, \bar{K}^{1,n+1})$ exist and are in $\mathcal{S}^2 \times \mathcal{M}^{d,2} \times \mathcal{A}^{+,2}$.
- The two sequences $(\bar{Y}^{1,n})_{n\geq 1}$ and $(Y^{2,n})_{n\geq 0}$ are decreasing in n, meaning that for all $n \geq 0$,

$$\bar{Y}_t^{1,n+1} \geq \bar{Y}_t^{1,n+2} \quad \text{and} \quad \bar{Y}_t^{2,n} \geq \bar{Y}_t^{2,n+1} \;\; P\text{-a.s. and for all } t.$$

- the limit process $(\bar{Y}^1, \bar{Y}^2)$ of $(\bar{Y}_t^{1,n}, \bar{Y}_t^{2,n})$ is continuous, a maximal solution of our system of RBSDEs and admits a Snell envelop representation.

Non-uniqueness: A counter example

Assume

- $\psi^1(t,\omega,y) = y$ and $\psi^2(t,\omega,y) = 2y$,
- $a = b = 0$ and $\xi^1 = \xi^2 = 1$.

The corresponding system of BSDEs is

$$\begin{cases} Y_t^1 = 1 + \int_t^T Y_s^1 ds - \int_t^T Z_s^1 dW_s + \left(K_T^1 - K_t^1\right), \\ Y_t^2 = 1 + 2\int_t^T Y_s^2 ds - \int_t^T Z_s^2 dW_s - \left(K_T^2 - K_t^2\right), \\ Y_t^1 \geq Y_t^2, \quad t \leq T, \\ \int_0^T \left(Y_s^1 - Y_s^2\right) d(K_s^1 + K_s^2) = 0. \end{cases}$$

It can be ckecked that

$$\left(e^{T-t}, e^{T-t}, 0, 0, 0, e^{T}(1-e^{-t})\right)$$

and

$$\left(e^{2(T-t)}, e^{2(T-t)}, 0, 0, \frac{1}{2}e^{2T}(1-e^{-2t}), 0)\right)$$

are solutions of the system of BSDEs.

A uniqueness result

Theorem. Assume that

(i) the mappings ψ^1 and ψ^2 do not depend on (y, z), i.e., $\psi_i := (\psi_i(t, \omega))$, $i = 1, 2$,

(ii) the barriers a and b satisfy

$$P - a.s. \int_0^T 1_{[a(s)=b(s)]} ds = 0.$$

Then, the solution of the system of BSDE's is unique.

The Markovian framework. A PDE formulation

When the dependence of (Y^1, Y^2) on the sources of uncertainty (the diffusion process $X^{t,x}$) is explicit, we can show that there exists two deterministic functions u^1 and u^2 such that

$$Y^1_s = u^1(s, X^{t,x}_s), \quad Y^2_s = u^2(s, X^{t,x}_s),$$

and are viscosity solutions of the following system of variational inequalities:

$$\begin{cases} \min\{u^1(t,x) - u^2(t,x) + a(t,x), -\mathcal{G}u^1(t,x) - \psi^1(t,x,u^1(t,x))\} = 0 \\ \max\{u^2(t,x) - u^1(t,x) - b(t,x), -\mathcal{G}u^2(t,x) - \psi^2(t,x,u^2(t,x))\} = 0 \\ u^1(T,x) = g^1(x), \quad u^2(T,x) = g^2(x). \end{cases}$$

Through a counter-example, we can show that the system may have infinitely many solutions.

An SPDE related to Problem III

Within the framework in Matoussi & Stoica (2010), solutions $(Y^1, Z^1, K^1, Y^2, Z^2, K^2)$ of Problem II are related to weak solutions $(u^1(t,\omega,x), \nu^1(\omega, dt, dx), u^2(t,\omega,x), \nu^2(\omega, dt, dx))$ of the following SPDE with inter-connected obstacles: ($\phi_t(x)$ being a test function)

$$
\begin{cases}
(u_t^i, \phi_t) - (\xi^i, \phi_T) + \int_t^T \left[(u_s^i, \partial_s \phi_s) + (\nabla u_s^i, \nabla \phi_s)\right] ds, \quad i = 1,2, \\
= \int_t^T (\psi_s^i, \phi_s) ds + \int_t^T (g_s^i, \phi_s) \overleftarrow{dB}(s) + \int_{[t,T]\times R} \phi_s \nu^i(ds, dx), \\
\\
\int_{[0,T]\times R} (\bar{u}_s^1 - (\bar{u}_s^2 - a(s)) \nu^1(ds, dx) = 0, \quad a.s. \\
\\
\int_{[0,T]\times R} (\bar{u}_s^1 + b(s) - \bar{u}_s^2) \nu^2(ds, dx) = 0, \quad a.s. \\
\\
u_t^1 \geq u_t^2 - a(t), \quad u_t^2 \leq u_t^1 + b(t), \quad dP \otimes dt \otimes dx. \\
\\
u^i(T,x) = \xi^i(x), \quad i = 1,2.
\end{cases}
$$

with, $(\bar{u}^1, \bar{u}^2)$ being a quasi-continuous version of (u^1, u^2), where,

- (u^1, u^2) belongs to some appropriate space and is predictable w.r.t . $(\mathcal{F}^B_{t,T})_t$.
- $\nu^i(ds, dx)$'s are random regular measures on $(0, T) \times R$.

Essentially, the regular random measures $\nu^i, i = 1, 2$ are obtained through the relation

$$\int_0^T \int_R \varphi(t, x)\nu^i(dtdx) = E \int_0^T \varphi(t, W_t)dK^i_t,$$

for all test functions φ. The expectation is taken w.r.t. the probability measure carrying the Browinan motion W.

Some references

- BD, S. Hamadène and M-E. Morlais (2010): Optimal stopping of expected profit and cost yields in an investment under uncertainty (*To appear in Stochastics*).
- R. Ben Abdellah, BD, S. Hamadène (2010): Backward SPDEs with inter-connected obstacles (in preparation).