Lower bounds for the Stock Price density in a Local-Stochastic Volatility Model

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Local - Stochastic Volatility models

We consider the model

$$dX_t = -\frac{1}{2}\eta^2(t, X_t)V_t dt + \eta(t, X_t)\sqrt{V_t}(\rho dW_t^1 + \rho_* dW_t^2),$$
$$dV_t = b(t, V_t)V_t dt + \sigma(t, X_t)\sqrt{V_t} dW_t^1$$

where $\rho_* = \sqrt{1-\rho^2}$.

Remark : $S_t = S_0 e^{X_t}$ is the stock price which satisfies

$$dS_t = S_t \eta(t, \ln S_t/S_0)\sqrt{V_t}(\rho dW_t^1 + \rho_* dW_t^2).$$

For $\eta \equiv 1$, $b(v) = a - bv$ and constant σ, this is the Heston model.

Problem : In the Heston model, it is known that the moments blow up:

For some $p > 1$ there exists $T > 0$ such that

$$E(S_T^p) = \infty.$$

Consequences. Implied volatility : $\sigma(T,k)$ defined as the solution of the equation

$$E(S_0 e^{X_T} - S_0 e^k)_+ = C_{BS}(S_0 e^k, T, \sigma(T,k))$$

with $k = \log(K/S_0)$ = log-forward moneyness.

Critical exponents :

$$p_T^*(X) = \sup\{p : E(S_T^p) = E(e^{pX_T}) < \infty\},$$
$$q_T^*(X) = \sup\{p : E(S_T^{-p}) = E(e^{-pX_T}) < \infty\}.$$

Lee's moment formula (model free) :

$$\limsup_{k\to\infty} \frac{T\sigma^2(T,k)}{k} = g(p_T^*(X) - 1) \quad \limsup_{k\to-\infty} \frac{T\sigma^2(T,k)}{k} = g(q_T^*(X))$$
$$g(p) = 2 - 4(\sqrt{p^2+p} - p), \quad g(\infty) = 0.$$

Calibration : In Heston model one may compute explicitly, for each $p \in N$

$$T_p(a,b,\sigma) = \mathsf{sup}\{t : E(S_t^p) = E(e^{pX_t}) < \infty\} < \infty \quad \to \quad p_T^*(X) = p_T^*(a,b,\sigma)$$

One employs the market data to compute

$$\mathsf{lim}\ \sup_{k\to\infty} \frac{T\sigma^2(T,k)}{k} = s_+, \quad \mathsf{lim}\ \sup_{k\to-\infty} \frac{T\sigma^2(T,k)}{k} = s_-$$

and obtains parameters guesses from

$$g(p_T^*(a,b,\sigma) - 1) = s_+, \quad g(q_T^*(a,b,\sigma)) = s_-$$

Our aim : Proving that in the previous local-stochastic volatility models we have moment explosion :

$$p_T^*(X) \leq C_T < \infty \quad \forall T$$

Drawback : C_T is a rough constant.

Theorem. (B - S. De Marco)

A. Suppose that

$i)$ $(t,v) \to \sigma(t,v)$, $(t,x) \to \eta(t,x)$ Lipschitz continuous and bounded

$ii)$ $0 < \underline{\sigma} \le \sigma(t,v)$, $0 < \underline{\eta} \le \eta(t,x)$

$iii)$ $v \to b(t,v)$ sub-linear growth

Then

$$P(X_T > x) \ge e^{-c_T x}$$

In particular, for each $x > 0$

$$E(e^{pX_T}) \ge E(e^{pX_T} 1_{\{X_T > x\}}) \ge e^{px - c_T x} = e^{(p-c_T)x} \to \infty \quad for \quad p > c_T.$$

B. Suppose moreover that

$$x \to \eta(t,x) \quad \text{is in } C_b^3.$$

Then

$$P(X_T \in dx) = p_T(x)dx \quad and \quad p_T(x) \ge e^{-c'_T x}.$$

Tubes estimates (B, Fernandez, Meda, 2008) We consider a general Itô process $Y_t \in R^n$

$$Y_t = Y_0 + \sum_{j=1}^{d} \int_0^t \phi_j(s, Y_s) dW_s^j + \int_0^t \phi_0(s, Y_s) ds$$

and a deterministic curve $y_t \in R^n$. We want to give a lower bound of the form

$$P(|Y_t - y_t| \leq R_t, 0 \leq t \leq T) \geq \exp(-C(1 + \int_0^T F(t)dt))$$

where R_t is a time depending radius and $F(t)$ is a rate function which is explicit.

Remark. i) The coefficients may depend on the trajectory in an adapted way : $\phi_j(s, y) = \phi_j(s, \omega, y)$ which is $\sigma(W_u, u \leq s)$ measurable. In particular, if $\phi_j(s, y) = \phi_j(s, \omega)$ we get a general Itô process.

ii) Y may be some **Non - Markov process**. **EX** : If X_t is a diffusion process on R^m and $\Phi : R^m \to R^n$ is a twice differentiable function then $Y_t = \Phi(X_t)$ is no more a Markov process (options on a basket).

Hypothesis. Consider the exit time from the tube

$$\tau_R = \inf\{t : |Y_t - y_t| \le R_t\}.$$

We assume that

$$i) \ \textbf{(Bounded)} \quad \sum_{j=1}^{d} \left|\phi_j(t, Y_t)\right|^2 + |\phi_0(t, Y_t)| \le c_t \quad 0 \le t \le \tau_R,$$

$$ii) \ \textbf{(Lip)} \quad \sum_{j=1}^{d} E(\left|\phi_j(s, Y_s) - \phi_j(t, Y_t\right|^2 1_{\{s \vee t < \tau_R\}} \mid F_{s \wedge t}) \le L_t \left|s - t\right|,$$

$$iii) \ \textbf{(Ellipticity)} \quad \inf_{|\xi|=1} \sum_{j=1}^{d} \left\langle \phi_j(t, Y_t), \xi \right\rangle \ge \lambda_t \quad 0 \le t \le \tau_R$$

We also assume that the deterministic curves $f_t = y_t, R_t, \lambda_t, c_t, L_t$ satisfy : There exists $\mu \ge 1$ and $h > 0$ such that

$$iv) \ \textbf{(Growth)} \quad f_t \le \mu f_s \quad for \quad |t - s| \le h$$

or put it otherwise

$$|\ln f_t - \ln f_s| \le \ln \mu \quad for \quad |t - s| \le h.$$

And we define the rate function

$$F(t) = \frac{1}{h} + \frac{|\partial_t y_t|^2}{\lambda_t} + (c_t^2 + L_t^2)(\frac{1}{\lambda_t} + \frac{1}{R_t^2}).$$

Theorem. **A**. (B-F-M 2008) Under the above hypothesis

$$P(|Y_t - y_t| \leq R_t, 0 \leq t \leq T) \geq \exp(-C(n)\mu^{p(n)}(1 + \int_0^T F(t)dt))).$$

B. (B 2005) Suppose moreover that $\phi_j(t, Y_t) \in D^{n+3,p}, t \geq 0, j = 0, ..., d$. Then

$$P(Y_T \in dx) = p_T(x)dx \quad and \quad p_T(x) \geq \exp(-S(n)\mu^{p(n)}(1 + \int_0^T F(t)dt)).$$

Here $C(n)$ is an universal constant depending on the dimension n only. And $S(n)$ is a constant which depends on n but also on the Sobolev norms (in Malliavin sense) of $\phi_j(t, Y_t)$.

Back to our problem :

$$dX_t = -\frac{1}{2}\eta^2(t, X_t)V_t dt + \eta(t, X_t)\sqrt{V_t}(\rho dW_t^1 + \rho_* dW_t^2),$$
$$dV_t = b(t, V_t)V_t dt + \sigma(t, X_t)\sqrt{V_t} dW_t^1.$$

Remark 1.

$$P(X_T > x) \geq P(X_T \in B_R(x + R)) \geq P(|X_t - x_t| \leq R, t \leq \tau_R).$$

so we need ball estimates for X_T.

Remark 2. The ellipticity condition is

$$(\rho \wedge \rho_*) \times (\underline{\eta} \wedge \underline{\sigma}) \times V_t \geq \lambda_t, \quad 0 \leq t \leq \tau_R$$

so **we need** tubes estimates for V_t.

We take two deterministic curves x_t and v_t and a deterministic time dependent radius R_t and we want to lower bound

$$P(|X_t - x_t| + |V_t - v_t| \leq R_t, t \leq \tau_R).$$

Rate function

$$F_{x,v}(t) = \frac{1}{h} + \frac{|\partial_t x_t|^2 + |\partial_t v_t|^2}{\lambda_t} + (c_t^2 + L_t^2)(\frac{1}{\lambda_t} + \frac{1}{R_t^2})$$

with

$$\lambda_t = c(\rho, \underline{\eta}, \underline{\sigma}) \times v_t, \quad c_t = C \times (1 + v_t), \quad L_t = C \times (1 + v_t)$$

so, up to a constant

$$F_{x,v}(t) = \frac{1}{h} + \frac{|\partial_t x_t|^2 + |\partial_t v_t|^2}{v_t} + (1 + v_t)^2(\frac{1}{v_t} + \frac{1}{R_t^2}).$$

Optimization

$$|\partial_t x_t| = |\partial_t v_t| \quad v_t = R_t^2.$$

So we take

$$x_t = v_t + (X_0 - V_0) \quad and \quad R_t = \sqrt{v_t}$$

and we get

$$F_{x,v}(t) = \frac{1}{h} + \frac{|\partial_t v_t|^2}{v_t} + \frac{(1 + v_t)^2}{v_t} \sim \frac{1}{h} + \frac{|\partial_t v_t|^2}{v_t} + v_t.$$

We look for v_t which minimizes

$$\int_0^T \left(\frac{|\partial_t v_t|^2}{v_t} + v_t\right)dt$$

under the constrained $x = x_T = v_T + (X_0 - V_0) \to v_T = x - (X_0 - V_0)$.

Solution

$$v_t = \sqrt{V_0} \times \sqrt{x + V_0} \times \frac{\sinh \frac{t}{2}}{\sinh \frac{T}{2}}.$$

Final result. A.

$$P(X_T \geq x) \geq \exp(-c_T(\rho)x), \quad p_*(X) \leq c'_T(\rho), \quad q_*(X) \leq c''_T(\rho).$$

The constant $c_T(\rho)$ blows up as $|\rho| \to 1$.

Problem : get estimates which are uniform in ρ? Andersen & Piterbag '07 - for Heston :
$\rho \to -1 \Rightarrow T(\rho) \to \infty \Rightarrow c_T(\rho) \to \infty \forall T$

B. Density : under regularity assumptions for the coefficients : $p_T(x) \geq \exp(-c'_T(\rho)x)$.

Malliavin calculus with respect to W^2, conditionally with respect to W^1.

Back to Itô processes : Idea of the proof. We want to give "tubes estimates" around y_t for

$$Y_t = Y_0 + \sum_{j=1}^{d} \int_0^t \phi_j(s, Y_s) dW_s^j + \int_0^t \phi_0(s, Y_s) ds.$$

Step 1. We **choose a time grid** $0 = t_0 < t_1 < \ldots < t_N = T$ and we write

$$Y_{t_{k+1}} = Y_{t_k} + I_k + R_k$$

with

$$I_k = \sum_{j=1}^{d} \phi_j(t_k, Y_{t_k})(W_{t_{k+1}}^j - W_{t_k}^j)$$

$$R_k = \sum_{j=1}^{d} \int_{t_k}^{t_{k+1}} (\phi_j(s, Y_s) - \phi_j(t_k, Y_{t_k})) dW_s^j + \int_{t_k}^{t_{k+1}} \phi_0(s, Y_s) ds.$$

Remark 1. Let $\delta_k = t_{k+1} - t_k$. Then

$$I_k \sim \sqrt{\delta_k} \quad and \quad R_k \sim \delta_k$$

Remark 2. $Y_{t_k} + I_k$ is Gaussian conditionally to $\sigma(W_s, s \le t_k)$ and

$$P(Y_{t_k} + I_k \in B_{r_k}(y_{t_{k+1}})) \ge \frac{1}{\lambda_{t_k}^d} \int_{B_{r_k}(y_{t_{k+1}})} \exp(-\frac{\left|y - Y_{t_k}\right|^2}{2\lambda_{t_k}})dy.$$

So if we take $r_k \le \sqrt{\lambda_{t_k}}$

$$\left|y_{t_{k+1}} - Y_{t_k})\right| + r_k \le \sqrt{\lambda_{t_k}} \quad \rightarrow \quad P(Y_{t_k} + I_k \in B_{r_k}(y_{t_{k+1}})) \ge \frac{r_k^d}{\lambda_{t_k}^d} \times e^{-c} \ge e^{-c}.$$

Consequence : then

$$P(\cap_{k=0}^{N-1}\{Y_{t_{k+1}} \in B_{r_k}(y_{t_{k+1}})\}) \sim P(\cap_{k=0}^{N-1}\{Y_{t_k} + I_k \in B_{r_k}(y_{t_{k+1}})\}) \ge e^{-cN}.$$

Problems :

1. Compute N (this gives $\int_0^T F(t)dt$) (**Do not** $N \rightarrow \infty$)

2. How to det rid of R_k???

Taylor expansion :

$$E(\phi_\varepsilon(Y_{t_k} + I_k + R_k - y_{t_{k+1}})) = E(\phi_\varepsilon(Y_{t_k} - y_{t_{k+1}} + I_k)) + \int_0^1 E(\phi'_\varepsilon(Y_{t_k} - y_{t_{k+1}} + I_k + \theta R_k)R_k)d\theta$$

1. Tubes - stochastic calculus. If $\varepsilon \sim \delta_k^{n/4}$ then

$$\left| \int_0^1 E(\phi'_\varepsilon(Y_{t_k} - y_{t_{k+1}} + I_k + \theta R_k)R_k)d\theta \right| \leq \frac{1}{2} E(\phi_\varepsilon(Y_{t_k} - y_{t_{k+1}} + I_k))$$

2. Density : We need to let $\varepsilon \to 0$. We take Φ_ε s.t. $\Phi'_\varepsilon = \phi_\varepsilon$ and we write

$$\begin{aligned}\int_0^1 E(\phi'_\varepsilon(Y_{t_k} - y_{t_{k+1}} + I_k + \theta R_k)R_k)d\theta &= \int_0^1 E(\Phi''_\varepsilon(Y_{t_k} - y_{t_{k+1}} + I_k + \theta R_k)R_k)d\theta \\ &= \int_0^1 E(\Phi_\varepsilon(Y_{t_k} - y_{t_{k+1}} + I_k + \theta R_k)H_{(2)})d\theta.\end{aligned}$$