Lower bounds for the Stock Price density in a Local-Stochastic Volatility Model

V. Bally and S. De Marco

Vlad Bally

Université de Marne-la-Vallée

and équipe Mathfi

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Local - Stochastic Volatility models

We consider the model

\[ dX_t = -\frac{1}{2} \eta^2(t, X_t) V_t dt + \eta(t, X_t) \sqrt{V_t} (\rho dW^1_t + \rho_* dW^2_t), \]
\[ dV_t = b(t, V_t) V_t dt + \sigma(t, X_t) \sqrt{V_t} dW^1_t \]

where \( \rho_* = \sqrt{1 - \rho^2} \).

**Remark** : \( S_t = S_0 e^{X_t} \) is the stock price which satisfies

\[ dS_t = S_t \eta(t, \ln S_t / S_0) \sqrt{V_t} (\rho dW^1_t + \rho_* dW^2_t). \]

For \( \eta \equiv 1 \), \( b(v) = a - bv \) and constant \( \sigma \), this is the Heston model.

**Problem** : In the Heston model, it is known that the moments blow up:

For some \( p > 1 \) there exists \( T > 0 \) such that

\[ E(S_T^p) = \infty. \]
**Consequences.** Implied volatility: $\sigma(T, k)$ defined as the solution of the equation

$$E(S_0 e^{X_T} - S_0 e^k)_+ = C_{BS}(S_0 e^k, T, \sigma(T, k))$$

with $k = \log(K/S_0) = \text{log-forward moneyness}$.

Critical exponents:

$$p^*_T(X) = \sup\{p : E(S^{p}_T) = E(e^{pX_T}) < \infty\},$$
$$q^*_T(X) = \sup\{p : E(S^{-p}_T) = E(e^{-pX_T}) < \infty\}.$$

**Lee’s moment formula (model free):**

$$\lim_{k \to \infty} \sup_{k} \frac{T \sigma^2(T, k)}{k} = g(p^*_T(X) - 1) \quad \lim_{k \to -\infty} \sup_{k} \frac{T \sigma^2(T, k)}{k} = g(q^*_T(X))$$

$$g(p) = 2 - 4(\sqrt{p^2 + p} - p), \quad g(\infty) = 0.$$
**Calibration**: In Heston model one may compute explicitly, for each \( p \in \mathbb{N} \)

\[
T_p(a, b, \sigma) = \sup \{ t : E(S_t^p) = E(e^{pX_t}) < \infty \} < \infty \quad \rightarrow \quad p_T^*(X) = p_T^*(a, b, \sigma)
\]

One employs the market data to compute

\[
\lim_{k \to \infty} \sup \frac{T \sigma^2(T, k)}{k} = s_+, \quad \lim_{k \to -\infty} \sup \frac{T \sigma^2(T, k)}{k} = s_-
\]

and obtains parameters guesses from

\[
g(p_T^*(a, b, \sigma) - 1) = s_+, \quad g(q_T^*(a, b, \sigma)) = s_-
\]

**Our aim**: Proving that in the previous local-stochastic volatility models we have moment explosion:

\[
p_T^*(X) \leq C_T < \infty \quad \forall T
\]

**Drawback**: \( C_T \) is a rough constant.
**Theorem.** (B - S. De Marco)

**A.** Suppose that

i) $(t, v) \to \sigma(t, v), \ (t, x) \to \eta(t, x)$ Lipschitz continuous and bounded

ii) $0 < \sigma \leq \sigma(t, v), \ 0 < \eta \leq \eta(t, x)$

iii) $v \to b(t, v)$ sub-linear growth

Then

$$P(X_T > x) \geq e^{-c_T x}$$

In particular, for each $x > 0$

$$E(e^{pX_T}) \geq E(e^{pX_T}1_{\{X_T > x\}}) \geq e^{px-c_T x} = e^{(p-c_T)x} \to \infty \ for \ p > c_T.$$

**B.** Suppose moreover that

$$x \to \eta(t, x) \quad \text{is in} \ C_b^3.$$  

Then

$$P(X_T \in dx) = p_T(x)dx \quad \text{and} \quad p_T(x) \geq e^{-c_T x}.$$
**Tubes estimates** (B, Fernandez, Meda, 2008) We consider a general Itô process $Y_t \in \mathbb{R}^n$

$$Y_t = Y_0 + \sum_{j=1}^{d} \int_{0}^{t} \phi_j(s, Y_s) dW_s^j + \int_{0}^{t} \phi_0(s, Y_s) ds$$

and a deterministic curve $y_t \in \mathbb{R}^n$. We want to give a lower bound of the form

$$P(|Y_t - y_t| \leq R_t, 0 \leq t \leq T) \geq \exp(-C(1 + \int_{0}^{T} F(t) dt))$$

where $R_t$ is a time depending radius and $F(t)$ is a rate function which is explicit.

**Remark.** i) The coefficients may depend on the trajectory in an adapted way: $\phi_j(s, y) = \phi_j(s, \omega, y)$ which is $\sigma(W_u, u \leq s)$ measurable. In particular, if $\phi_j(s, y) = \phi_j(s, \omega)$ we get a general Itô process.

ii) $Y$ may be some **Non - Markov process.** EX : If $X_t$ is a diffusion process on $\mathbb{R}^m$ and $\Phi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a twice differentiable function then $Y_t = \Phi(X_t)$ is no more a Markov process (options on a basket).
Hypothesis. Consider the exit time from the tube 

\[ \tau_R = \inf \{ t : |Y_t - y_t| \leq R_t \} . \]

We assume that 

\[ i) \textbf{(Bounded)} \quad \sum_{j=1}^{d} \phi_j(t, Y_t)^2 + |\phi_0(t, Y_t)| \leq c_t \quad 0 \leq t \leq \tau_R, \]

\[ ii) \textbf{(Lip)} \quad \sum_{j=1}^{d} E(\left| \phi_j(s, Y_s) - \phi_j(t, Y_t) \right|^2 1_{\{s \vee t < \tau_R\}} | F_{s \wedge t} ) \leq L_t |s - t| , \]

\[ iii) \textbf{(Ellipticity)} \quad \inf_{|\xi|=1} \sum_{j=1}^{d} \left< \phi_j(t, Y_t), \xi \right> \geq \lambda_t \quad 0 \leq t \leq \tau_R \]

We also assume that the deterministic curves \( f_t = y_t, R_t, \lambda_t, c_t, L_t \) satisfy: There exists \( \mu \geq 1 \) and \( h > 0 \) such that 

\[ iv) \textbf{(Growth)} \quad f_t \leq \mu f_s \quad \text{for} \quad |t - s| \leq h \]

or put it otherwise 

\[ |\ln f_t - \ln f_s| \leq \ln \mu \quad \text{for} \quad |t - s| \leq h. \]
And we define the rate function

\[ F(t) = \frac{1}{h} + \frac{|\partial_t y_t|^2}{\lambda_t} + (c_t^2 + L_t^2) \left( \frac{1}{\lambda_t} + \frac{1}{R_t^2} \right). \]

**Theorem.** A. (B-F-M 2008) Under the above hypothesis

\[ P(|Y_t - y_t| \leq R_t, 0 \leq t \leq T) \geq \exp(-C(n)\mu^{p(n)}(1 + \int_0^T F(t)dt))). \]

**B.** (B 2005) Suppose moreover that \( \phi_j(t, Y_t) \in D^{n+3,p}, t \geq 0, j = 0, \ldots, d. \) Then

\[ P(Y_T \in dx) = p_T(x)dx \quad \text{and} \quad p_T(x) \geq \exp(-S(n)\mu^{p(n)}(1 + \int_0^T F(t)dt))). \]

Here \( C(n) \) is an universal constant depending on the dimension \( n \) only. And \( S(n) \) is a constant which depends on \( n \) but also on the Sobolev norms (in Malliavin sense) of \( \phi_j(t, Y_t) \).
Back to our problem:

\[ dX_t = -\frac{1}{2} \eta^2(t, X_t) V_t dt + \eta(t, X_t) \sqrt{V_t}(\rho dW_t^1 + \rho_* dW_t^2), \]
\[ dV_t = b(t, V_t) V_t dt + \sigma(t, X_t) \sqrt{V_t} dW_t^1. \]

Remark 1.

\[ P(X_T > x) \geq P(X_T \in B_R(x + R)) \geq P(|X_t - x_t| \leq R, t \leq \tau_R). \]
so we need ball estimates for \( X_T \).

Remark 2. The ellipticity condition is

\[ (\rho \land \rho_\ast) \times (\eta \land \sigma) \times V_t \geq \lambda_t, \quad 0 \leq t \leq \tau_R \]
so we need tubes estimates for \( V_t \).

We take two deterministic curves \( x_t \) and \( v_t \) and a deterministic time dependent radius \( R_t \)
and we want to lower bound

\[ P(|X_t - x_t| + |V_t - v_t| \leq R_t, t \leq \tau_R). \]
Rate function

\[ F_{x,v}(t) = \frac{1}{h} + \frac{|\partial_t x_t|^2 + |\partial_t v_t|^2}{\lambda_t} + (c_t^2 + L_t^2)(\frac{1}{\lambda_t} + \frac{1}{R_t^2}) \]

with

\[ \lambda_t = c(\rho, \eta, \sigma) \times v_t, \quad c_t = C \times (1 + v_t), \quad L_t = C \times (1 + v_t) \]

so, up to a constant

\[ F_{x,v}(t) = \frac{1}{h} + \frac{|\partial_t x_t|^2 + |\partial_t v_t|^2}{v_t} + (1 + v_t)^2(\frac{1}{v_t} + \frac{1}{R_t^2}). \]

Optimization

\[ |\partial_t x_t| = |\partial_t v_t| \quad v_t = R_t^2. \]

So we take

\[ x_t = v_t + (X_0 - V_0) \quad and \quad R_t = \sqrt{v_t} \]

and we get

\[ F_{x,v}(t) = \frac{1}{h} + \frac{|\partial_t v_t|^2}{v_t} + \frac{(1 + v_t)^2}{v_t} \sim \frac{1}{h} + \frac{|\partial_t v_t|^2}{v_t} + v_t. \]
We look for $v_t$ which minimizes
\[
\int_0^T \left( \frac{|\partial_t v_t|^2}{v_t} + v_t \right) dt
\]
under the constrained $x = x_T = v_T + (X_0 - V_0) \rightarrow v_T = x - (X_0 - V_0)$.

Solution
\[
v_t = \sqrt{V_0} \times \sqrt{x + V_0} \times \frac{\sinh \frac{t}{2}}{\sinh \frac{T}{2}}.
\]

Final result. A.
\[
P(X_T \geq x) \geq \exp(-c_T(\rho)x), \quad p_*(X) \leq c_T'(\rho), \quad q_*(X) \leq c_T''(\rho).
\]
The constant $c_T(\rho)$ blows up as $|\rho| \to 1$.

Problem : get estimates which are uniform in $\rho$? Andersen & Piterbag '07 - for Heston :
$\rho \to -1 \Rightarrow T(\rho) \to \infty \Rightarrow c_T(\rho) \to \infty \forall T$

B. Density : under regularity assumptions for the coefficients : $p_T(x) \geq \exp(-c_T'(\rho)x)$.

Malliavin calculus with respect to $W^2$, conditionally with respect to $W^1$. 
Back to Itô processes: Idea of the proof. We want to give "tubes estimates" around $y_t$ for

$$Y_t = Y_0 + \sum_{j=1}^{d} \int_0^t \phi_j(s, Y_s) dW_s^j + \int_0^t \phi_0(s, Y_s) ds.$$ 

**Step 1.** We choose a time grid $0 = t_0 < t_1 < \ldots < t_N = T$ and we write

$$Y_{t_{k+1}} = Y_{t_k} + I_k + R_k$$

with

$$I_k = \sum_{j=1}^{d} \phi_j(t_k, Y_{t_k})(W_{t_{k+1}}^j - W_{t_k}^j)$$

$$R_k = \sum_{j=1}^{d} \int_{t_k}^{t_{k+1}} (\phi_j(s, Y_s) - \phi_j(t_k, Y_{t_k})) dW_s^j + \int_{t_k}^{t_{k+1}} \phi_0(s, Y_s) ds.$$

**Remark 1.** Let $\delta_k = t_{k+1} - t_k$. Then

$$I_k \sim \sqrt{\delta_k} \quad \text{and} \quad R_k \sim \delta_k$$
Remark 2. \( Y_{t_k} + I_k \) is Gaussian conditionally to \( \sigma(W_s, s \leq t_k) \) and

\[
P(Y_{t_k} + I_k \in B_{r_k}(y_{t_{k+1}})) \geq \frac{1}{\lambda_{t_k}^d} \int_{B_{r_k}(y_{t_{k+1}})} \exp(-\frac{|y - Y_{t_k}|^2}{2\lambda_{t_k}}) dy.
\]

So if we take \( r_k \leq \sqrt{\lambda_{t_k}} \)

\[
|y_{t_{k+1}} - Y_{t_k}| + r_k \leq \sqrt{\lambda_{t_k}} \quad \rightarrow \quad P(Y_{t_k} + I_k \in B_{r_k}(y_{t_{k+1}})) \geq \frac{r_k^d}{\lambda_{t_k}^d} \times e^{-c} \geq e^{-c}.
\]

Consequence: then

\[
P(\bigcap_{k=0}^{N-1} \{Y_{t_{k+1}} \in B_{r_k}(y_{t_{k+1}})\}) \sim p(\bigcap_{k=0}^{N-1} \{Y_{t_k} + I_k \in B_{r_k}(y_{t_{k+1}})\}) \geq e^{-cN}.
\]

Problems:

1. Compute \( N \) (this gives \( \int_0^T F(t)dt \) (Do not \( N \to \infty \))
2. How to det rid of $R_k$??

Taylor expansion:

$$E(\phi_\varepsilon(Y_{t_k} + I_k + R_k - y_{t_{k+1}})) = E(\phi_\varepsilon(Y_{t_k} - y_{t_{k+1}} + I_k))$$

$$+ \int_0^1 E(\phi'_\varepsilon(Y_{t_k} - y_{t_{k+1}} + I_k + \theta R_k)R_k)d\theta$$

1. Tubes - stochastic calculus. If $\varepsilon \sim \delta_k^{n/4}$ then

$$\left| \int_0^1 E(\phi'_\varepsilon(Y_{t_k} - y_{t_{k+1}} + I_k + \theta R_k)R_k)d\theta \right| \leq \frac{1}{2} E(\phi_\varepsilon(Y_{t_k} - y_{t_{k+1}} + I_k))$$

2. Density: We need to let $\varepsilon \to 0$. We take $\Phi_\varepsilon$ s.t. $\Phi'_\varepsilon = \phi_\varepsilon$ and we write

$$\int_0^1 E(\phi'_\varepsilon(Y_{t_k} - y_{t_{k+1}} + I_k + \theta R_k)R_k)d\theta = \int_0^1 E(\Phi''_\varepsilon(Y_{t_k} - y_{t_{k+1}} + I_k + \theta R_k)R_k)d\theta$$

$$= \int_0^1 E(\Phi_\varepsilon(Y_{t_k} - y_{t_{k+1}} + I_k + \theta R_k)H(2))d\theta.$$