Limit theorems for BSDE with local time applications to non–linear PDE

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Motivations :

- BSDE and mathematical finance (El Karoui et al. 1997),
- Probabilistic interpretation of PDE Pardoux-Peng,
- Stochastic differential games and stochastic control : Hamadène-Lepeltier 1995 etc
- Quadratic BSDE (Imkeller, CIRM 2006)

BSDE with local time

Consider the following particular BSDE

$$Y_t = \xi + \int_t^T f(Y_s) Z_s^2 ds - \int_t^T Z_s dW_s. \qquad (2.1)$$

From the equality $d\langle Y, Y\rangle_t = Z_t^2 dt$ and from occupation time formula, we have, for any bounded measurable function f

$$\int_0^t f(Y_s) Z_s^2 ds = \int_{-\infty}^{\infty} L_t^a(Y) f(a) da.$$

Set $\nu(da) = f(a)da$, then (2.1) takes the form

$$Y_t = \xi + \int_{\mathbb{R}} \left(L_T^a(Y) - L_t^a(Y)\right) \nu(da) - \int_t^T Z_s dW_s \qquad (2.2)$$

BSDE with local time

The process $L_t^a(Y)$ is the local time of the continuous semi-martingales Y and can be expressed by Tanaka's formula as

$$L_t^a(Y) = |Y_t - a| - |Y_0 - a| - \int_0^t \mathrm{sgn}(Y_s - a)dY_s$$

and

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0. \end{cases}$$

It is proved by Dermoune *et al.* '99 that there exists an adapted couple (Y, Z) solution to equation (2.2) under the following conditions :

- (H1) The r.v. ξ belongs to $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.
- (H2) The measure ν is bounded and $|\nu(\{x\})| < 1$, $\forall\ x$ in $\mathbb{R}$.

BSDE with local time

Our aim in this talk is

- to prove some limit theorems for the class of BSDE of the form (2.2), that are some kind of the stability properties for BSDEs.
- We show that a solution to (2.2) can be obtained as a limit of sequence of solution to (2.1).
- To prove a comparison theorem for the above singular BSDE, As application : limit theorems in the monotone case.
- We deduce limit theorems for a class of non–linear PDEs involving the square of the gradient and a comparison theorem is discussed for this PDEs.

BSDE with local time

The main tool to study the BSDE (2.2) is the Zvonkin's transformation . Let us set

$$f_\nu(x) = \exp\left(2\nu^c((-\infty, x])\right) \prod_{y \le x} \left(\frac{1 + \nu((\{y\}))}{1 - \nu((\{y\}))} \right)$$

where ν^c is the continuous part of the measure ν.
If f is of bounded variation (increasing in our case), $f(x-)$ will denote the left limit of f at a point x and $f'(dx)$ will be the bounded measure associated with f.

BSDE with local time

It is well known that the function (since ν is bounded) that $f_\nu(\cdot)$ is increasing, right continuous and satisfies

$$0 < m \leq f_\nu(x) \leq M \quad \forall\, x \in \mathbb{R}$$

for some constants m, M. Moreover f_ν satisfies

$$f_\nu'(dx) - \{f_\nu(x) + f_\nu(x-)\}\,\nu(dx) = 0.$$

Set

$$F_\nu(x) = \int_0^x f_\nu(y)dy \;\; \text{and} \;\; g_\nu(x) = f_\nu(F_\nu^{-1}(x)).$$

The functions F_ν and F_ν^{-1} are Lipschitz functions.

BSDE with local time

Let $\mathcal{M}^2_T(\mathbb{R}\times\mathbb{R}^d)$ denote the space of $\mathcal{F}_t$–prog. meas. proc. (Y, Z) satisfying (**??**)

Proposition

$(Y, Z) \in \mathcal{M}^2_T(\mathbb{R}\times\mathbb{R}^d)$ solves (2.2) iff

$$\left(\tilde{Y}, \tilde{Z}\right) = \left(F_\nu(Y), \frac{Z}{2}\left\{f_\nu(Y) + f_\nu(Y-)\right\}\right)$$

solves $\tilde{\xi} = F_\nu(\xi)$ the BSDE

$$\tilde{Y}_t = \tilde{\xi} - \int_t^T \tilde{Z}_s dW_s, \tag{2.3}$$

BSDE with local time

Proof. The proof is based on Tanaka's formula to $F_\nu(Y_t)$ with the symmetric derivative of the convex function F_ν instead of its left derivative.

Remark

Stroock and Yor (1981), Le Gall '84) and Rutkowski '90 have already used the transformation F_ν to study the SDE

$$X_t = x + \int_0^t \sigma(X_s)dW_s + \int_{\mathbb{R}} L_t^a(X)\nu(da).$$

BSDE with local time

Theorem

Under the assumptions (H1) and (H2), there exists a unique solution (Y^ν, Z^ν) belonging to $\mathcal{M}^2_T(\mathbb{R}\times\mathbb{R}^d)$ for the equation (2.2). Moreover

$$Y^\nu_t = F^{-1}_\nu\left(\mathbb{E}\left[F_\nu(\xi)\ /\ \mathcal{F}_t\right]\right), \quad 0 \le t \le T.$$

BSDE with local time

Example

- *Let* $\nu = \alpha\delta$, *where* $|\alpha| < 1$. *Then* $f_\nu(x) = 1$ *for* $x < 0$ *and* $f_\nu(x) = \frac{1+\alpha}{1-\alpha}$ *for* $x \geq 0$. *The function* $F_\nu(x) = x$ *for* $x < 0$ *and* $F_\nu(x) = \frac{1+\alpha}{1-\alpha}x$ *for* $x \geq 0$. *The solution of the BSDE*

$$Y_t = \xi + \alpha L_T^0(Y) - \alpha L_t^0(Y) - \int_t^T Z_s dW_s,$$

where $\xi \in]-\infty, 0[$ *or* $\xi \in [0, \infty[$ *is given by*

$$Y_t = \mathbb{E}\left[\xi\,/\mathcal{F}_t\right],$$

and $L_t^0(Y) = 0$ *for all* $0 \leq t \leq T$.

BSDE with local time

Remark

In the case where ν is a non–necessary bounded measure on $\mathbb{R}$ which is diffuse and σ–finite, the associated function $f_\nu(x) = \exp(2\nu((-\infty, x]))$ is positive, continuous and non necessary bounded function. Hence the function $F_\nu(x)$ is only locally Lipschitz, however if ξ and $F_\nu(\xi)$ are square integrable random variables then the BSDE (2.2) has a unique solution which is given by

$$Y_t^\nu = F_\nu^{-1}\left(\mathbb{E}\left[F_\nu(\xi)\,/\mathcal{F}_t\right]\right).$$

Limit theorems for BSDEs

Let $\nu_n(da)$, $n = 1, 2, \ldots$ be a sequence of Radon measures and ξ^n a sequence of random variables in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. Suppose that there exist two positive constants ε, M such that :

$$\begin{array}{rcll} |\nu_n|(\mathbb{R}) & \leq & M & \forall\, n \geq 1, \\ |\nu_n(\{x\})| & \leq & \varepsilon \; < \; 1 & \forall\, n \geq 1,\ \forall\, x \in \mathbb{R}. \end{array}$$

Let (Y^n, Z^n) be the solution of

$$Y_t = \xi^n + \int_{\mathbb{R}} \left(L^a_T(Y) - L^a_t(Y)\right) \nu_n(da) - \int_t^T Z_s dW_s.$$

Limit theorems for BSDEs

Assume that $\xi^n \longrightarrow_{n\to\infty} \xi$ in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.
Assume further that there exist a function f BV such that :

$$\lim_{n\to+\infty} \int_{-L}^{L} |f_{\nu_n} - f|^2(x)\,dx = 0 \quad \text{for all } L > 0,$$

$$\nu(da) = \frac{f'(da)}{f(a) + f(a-)}.$$

Then

$$\lim_{n\to+\infty} \mathbb{E} \sup_{0\le t\le T} |Y_t^n - Y_t^\nu|^2 + \mathbb{E}\int_0^T |Z_s^n - Z_s^\nu|^2\, ds = 0 \qquad (3.4)$$

where (Y^ν, Z^ν) is the unique solution to the BSDE equation :

$$Y_t = \xi + \int_{\mathbb{R}} \left(L_T^a(Y) - L_t^a(Y)\right)\nu(da) - \int_t^T Z_s dW_s.$$

Limit theorems for BSDEs

Proof. We shall use the following notations :

$$f_{\nu_n}(x) = \exp(2\nu_n^c((-\infty, x])) \prod_{y \leq x} \left(\frac{1 + \nu_n((\{y\}))}{1 - \nu_n((\{y\}))} \right)$$

$$F_{\nu_n}(y) = \int_0^y f_{\nu_n}(x)dx \ \text{ and } \ F(y) = \int_0^y f(x)dx.$$

By Theorem 1, it holds

$$Y_t^n = F_{\nu_n}^{-1}\left(\mathbb{E}\left[F_{\nu_n}(\xi^n) / \mathcal{F}_t\right]\right) \quad 0 \leq t \leq T.$$

$$Y_t^\nu = F^{-1}\left(\mathbb{E}\left[F(\xi) / \mathcal{F}_t\right]\right) \quad 0 \leq t \leq T.$$

Limit theorems for BSDEs

The convergence of f_{ν_n} to f in $L^2_{\rm loc}(\mathbb{R})$ implies that F_{ν_n} converges to F uniformly on compact sets and then, using a truncating argument, $F_{\nu_n}(\xi^n)$ converges to $F(\xi)$. It follows that $\overline{Y_t}^n := \mathbb{E}[F_{\nu_n}(\xi^n)/\mathcal{F}_t]$ converges to $\mathbb{E}[F(\xi)/\mathcal{F}_t] =: \overline{Y_t}^\nu$ in $L^2(\Omega)$. It is, trivial to see that $F^{-1}_{\nu_n}$ converges to F^{-1} uniformly on compact sets and so $Y^n_t = F^{-1}_{\nu_n}(\overline{Y_t}^n)$ converges to $F^{-1}(\overline{Y_t}^\nu) = Y^\nu_t$. Hence $\mathbb{E}\sup_{0\leq t\leq T}|Y^n_t - Y^\nu_t|$ tends to zero when n goes to infinity, and using the isometry property, one can see that $\mathbb{E}\int_0^T |Z^n_s - Z^\nu_s|^2\, ds$ converges to zero when n tends to infinity. □

Limit theorems for BSDEs

Remark

Let $\xi^n = \xi$ *for all* n, $\nu_n(dx) = f_n(x)\,dx$ *where* $f_n(x) \geq 0$ *;*
$\int f_n(x)\,dx = 1$ *and* $\operatorname{supp}(f_n) = [-\frac{1}{n}, \frac{1}{n}]$.
Let us consider the BSDE

$$Y_t^n = \xi + \int_t^T f_n(Y_s^n)\,(Z_s^n)^2\,ds - \int_t^T Z_s^n dW_s,$$

then the last theorem implies the convergence of $Y_\cdot^n$ *to the unique solution of the BSDE*

$$Y_t = \xi + \frac{1-e^2}{1+e^2}\left(L_T^0(Y) - L_t^0(Y)\right) - \int_t^T Z_s dW_s.$$

Limit theorems for BSDEs

If ν_n converges to a measure ν, then in general Y^{ν_n} does not converges to Y^{ν}. We replace the convergence of measures ν_n by the convergence of its associated function f_{ν_n}.

In the sequel $\mathcal{M}(\mathbb{R})$ will denote the space of all bounded measure on $\mathbb{R}$ such that :

$$|\nu(\{x\})| < 1 \; \forall \, x \in \mathbb{R}.$$

Let ν be in $\mathcal{M}(\mathbb{R})$. We define

$$\|\nu\| = |\nu^c(\mathbb{R})| + \frac{1}{2}\sum_{y} \left| \frac{1+\nu(\{y\})}{1-\nu(\{y\})} \right|.$$

Note that

$$\|\nu\| = \mathrm{var}\left(\frac{1}{2}\log\left(f_\nu\right)\right)$$

where, var, denotes the total variation.

Limit theorems for BSDEs

In the space Let $\mathcal{M}^2_T$ we define the distance $d[.,.]$ given by :

$$d\left[(Y,Z),(Y',Z')\right] = \left(\mathbb{E}\sup_{0\leq t\leq T}\left|Y_t - Y'_t\right|^2 + \mathbb{E}\int_0^T \left|Z_s - Z'_s\right|^2 ds\right)^{\frac{1}{2}}.$$

Theorem

Let C be a fixed constant. Then, $\mathcal{K} = \{(Y^\nu, Z^\nu) : \|\nu\| \leq C\}$ is a compact set for the topology induced by $d[\cdot,\cdot]$.
The set of all (Y^ν, Z^ν) belonging to $\mathcal{K}$ such that ν is absolutely continuous with respect to Lebesgue measure is dense in $\mathcal{K}$.

Limit theorems for BSDEs

Proof. Let ν_n be a sequence in $\mathcal{M}(\mathbb{R})$ such that $\|\nu_n\| \leq C$. Since the total variation of the f_{ν_n}'s are uniformly bounded, we can find a function f of bounded variation and a subsequence $(f_{\nu_{n_k}})$ such that :

$$f_{\nu_{n_k}}(x) \longrightarrow f(x) \ \text{ as } k \longrightarrow +\infty, \ \text{ for all } x \in \mathbb{R} \backslash D_f$$

where, D_f, is at most countable. Set

$$\nu(da) = \frac{f'(da)}{f(a) + f(a-)}.$$

Then the first limit Theorem implies that :

$$d\left[\left(Y^{\nu_{n_k}}, Z^{\nu_{n_k}}\right), (Z^{\nu}, Z^{\nu})\right] \longrightarrow 0 \ \text{ when } \ k \longrightarrow +\infty.$$

Limit theorems for BSDEs

It remains to prove that $\|\nu\| \leq C$.
Note that f satisfies the same equation as f_ν, then, there exist $\lambda > 0$ such that $f = \lambda f_\nu$.
Hence

$$\begin{aligned} \|\nu\| = \mathrm{var}\left(\frac{1}{2}\log\left(f_\nu\right)\right) &= \mathrm{var}\left(\frac{1}{2}\log\left(f\right)\right) \\ &\leq \limsup_{n\to+\infty} \mathrm{var}\left(\frac{1}{2}\log\left(f_{\nu_n}\right)\right) \leq C. \end{aligned}$$

Limit theorems for BSDEs

Let us prove the second point ; Let $\nu \in \mathcal{M}(\mathbb{R})$ and θ_n an approximation of the identity.
Set

$$f_n = f_\nu * \theta_n \quad \text{and} \quad g_n = \frac{f_n'}{2f_n}.$$

Let (Y^n, Z^n) be the unique solution of the following BSDE

$$Y_t^n = \xi + \int_t^T g_n(Y_s^n)\,(Z_s^n)^2\,ds - \int_t^T Z_s^n dW_s.$$

Using Theorem 3.4, it is easy to see that :

$$d\left[(Y^n, Z^n), (Y^\nu, Z^\nu)\right] \longrightarrow 0 \ \text{ as } \ n \longrightarrow +\infty.$$

Comparison theorems for BSDEs

Lepeltier and San Martin consider BSDEs with terminal data $\xi \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P})$, and gave a comparison theorem for BSDE with parameter (f, ξ) i.e.

$$Y_t = \xi + \int_t^T f(Y_s) Z_s^2 ds - \int_t^T Z_s dW_s.$$

In the following theorem, we prove a general comparison theorem, without boundedness of the terminal value of the BSDE. As a byproduct, we obtain the comparison theorem, for the standard BSDE under fairly weak conditions on the coefficients.

Comparison theorems for BSDEs

Theorem

Let ν, μ be in $\mathcal{M}(\mathbb{R})$. Let (Y^ν, Z^ν), (Y^μ, Z^μ) be two processes such that :

$$
\begin{aligned}
Y_t^\nu &= \xi + \int_{\mathbb{R}} \left(L_T^a(Y^\nu) - L_t^a(Y^\nu)\right)\nu(da) - \int_t^T Z_s^\nu \, dW_s, \\
Y_t^\mu &= \xi' + \int_{\mathbb{R}} \left(L_T^a(Y^\mu) - L_t^a(Y^\mu)\right)\mu(da) - \int_t^T Z_s^\mu \, dW_s.
\end{aligned}
$$

Assume that $\xi \geq \xi'$ a.s. and the measure $\nu - \mu$ is positive. Then $Y_t^\nu \geq Y_t^\mu$ for all t $\mathbb{P}$–a.s.

Comparison theorems for BSDEs

Proof. Let us first recall Tanaka's formula. Since F_μ is a convex function, then

$$F_\mu(Y_T^\nu) = F_\mu(Y_t^\nu) + \int_t^T \frac{1}{2}\left(f_\mu(Y_s^\nu) + f_\mu(Y_s^\nu -)\right) dY_s^\nu + \frac{1}{2}\int_{\mathbb{R}} \left(L_T^a(Y^\nu) - L_t^a(Y^\nu)\right) f_\mu^{'}(da)$$

hence

$$F_\mu(\xi) = F_\mu(Y_t^\nu) + (M_T - M_t) - \frac{1}{2}\int_{\mathbb{R}} \{f_\mu(a) + f_\mu(a-)\} \left(L_T^a(Y^\nu) - L_t^a(Y^\nu)\right)(\nu - \mu)(da)$$

where $M.$ is a square integrable martingale.

Comparison theorems for BSDEs

Since the function $a \mapsto [f_\mu(a) + f_\mu(a-)](L^a_T(Y^\nu) - L^a_t(Y^\nu))$ is positive, and F_μ is an increasing function, then

$$F_\mu\left(Y^\nu_t\right) \geq \mathbb{E}\left[F_\mu(\xi')\,/\mathcal{F}_t\right]$$

and

$$Y^\nu_t \geq F^{-1}_\mu\left(\mathbb{E}\left[F_\mu\left(\xi'\right)/\mathcal{F}_t\right]\right) = Y^\mu_t.$$

□

Comparison theorems for BSDEs

An immediate consequence of the above comparison result is the

Corollary

Let $(\nu_n)_{n\geq 1}$ be an sequence of measures such that $\sup_{n\geq 1} \|\nu_n\| < +\infty$ and f_{ν_n} increases to a BV function f. If ξ^n increases to $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ as $n \to \infty$. Then $d[(Y^\nu, Z^\nu) - (Y^n, Z^n)] \to 0$ where

$$\nu(da) = \frac{f'(da)}{f(a) + f(a-)}.$$

and (Y^n, Z^n), (Y^ν, Z^ν) solves the corresponding BSDEs

Comparison theorems for BSDEs

Corollary

Let (f^1, ξ^1) *and* (f^2, ξ^2) *be two parameters of BSDE, and let* $\left(Y^1, Z^1\right)$ *and* $\left(Y^2, Z^2\right)$ *be associated solution.*
Suppose that :
$\xi^1 \leq \xi^2$ *a.s. and* $f^1(y) \leq f^2(y)$ *for almost all* y*.*
Then for all $t \in [0, T]$*, we have* $Y^1_t \leq Y^2_t$ *a.s.*

Comparison theorems for BSDEs

Corollary

Let (f^1, ξ^1) and (f^2, ξ^2) be two parameters of BSDE, and let $\left(Y^1, Z^1\right)$ and $\left(Y^2, Z^2\right)$ be associated solution.
Suppose that :
$\xi^1 \leq \xi^2$ *a.s. and* $f^1(y) \leq f^2(y)$ *for almost all* y.
Then for all $t \in [0, T]$, *we have* $Y_t^1 \leq Y_t^2$ *a.s.*
As a consequence of the above results, we have obtained an interesting limit theorem for generalized BSDE in monotonic case.

Comparison theorems for BSDEs

Theorem

Let $(\nu_n)_{n\geq 1}$ *be an increasing sequence of measures such that* $\sup_{n\geq 1}\|\nu_n\| < +\infty$, *assume* ξ^n *increases to* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ *as* n *tends towards infinity. Then*

$$\lim_{n\to+\infty} \mathbb{E} \sup_{0\leq t\leq T} |Y_t^\nu - Y_t^n|^2 + \int_0^T |Z_s^\nu - Z_s^n|^2\, ds = 0,$$

where (Y^ν, Z^ν) *is the unique solution of the BSDE*

$$Y_t^\nu = \xi + \int_{\mathbb{R}} (L_T^a(Y^\nu) - L_t^a(Y^\nu))\, \nu(da) - \int_t^T Z_s^\nu dW_s,$$

and $\nu = \sup_{n\geq 1} \nu_n$.

Comparison theorems for BSDEs

Proof. For any measurable set, we have $\nu(A) = \sup_{n\geq 1} \nu_n(A)$, it follows from the bound $\sup_{n\geq 1} \|\nu_n\| < +\infty$, that ν is a bounded measure.
Set

$$F_n(y) := F_{\nu_n}(y) \ \text{ and } \quad F(y) := F_{\nu}(y).$$

Then $F_n(\cdot)$ is increasing and converges to the continuous function $F(\cdot)$, hence by Dini's theorem this convergence is uniform. By the comparison Theorem 3, the sequence Y_t^n is increasing. Set

$$Y_t^{\nu} = \lim_{n\to+\infty} Y_t^n,$$

hence $F_n(Y_t^n)$ converges to $F(Y_t^{\nu})$.

Comparison theorems for BSDEs

But

$$F_n\left(Y_t^n\right) = \mathbb{E}\left[F_n(\xi^n)\,/\mathcal{F}_t\right] \quad 0 \le t \le T.$$

and

$$\left|F_n\left(\xi^n\right)\right| \le \left(\left|\xi^1\right| + |\xi|\right)\exp\left(2|\nu|(\mathbb{R})\right).$$

Then passing to the limit, using dominated convergence theorem for conditional expectation, it holds that

$$Y_t^\nu = F^{-1}\left(\mathbb{E}\left[F(\xi)\,/\mathcal{F}_t\right]\right) \quad 0 \le t \le T.$$

Comparison theorems for BSDEs

By Theorem 1, (Y^ν, Z^ν) is the unique solution of the BSDE

$$Y_t^\nu = \xi + \int_{\mathbb{R}} (L_T^a(Y^\nu) - L_t^a(Y^\nu))\,\nu(da) - \int_t^T Z_s^\nu dW_s.$$

We deduce from Burkholder–Davis–Gundy inequality, that

$$\lim_{n\to+\infty} \mathbb{E} \sup_{0\leq t\leq T} |Y_t^\nu - Y_t^n|^2 = 0,$$

using the transformation F_ν and the isometry property, we get

$$\lim_{n\to+\infty} \mathbb{E} \int_0^T |Z_s^\nu - Z_s^n|^2\, ds = 0.$$

Applications to non–linear PDE

This section is devoted to limit theorems for PDE that can be deduced from the above limit theorems for BSDE using the connection between these different kind of equations.
Let $\{X_s^{x,t} : 0 \leq t \leq s \leq T\}$ be the unique solution of the stochastic differential equation

$$X_s^{x,t} = x + \int_t^s b(X_r^{x,t})dr + \int_t^s \sigma(X_r^{x,t})dW_r,$$

where the coefficients b and σ are globally Lipschitz.

Applications to non–linear PDE

Let ν be a measure on $\mathbb{R}$ and satisfy the assumption (H2), we consider the singular non–linear Cauchy problem

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial t} = Lu - \dfrac{1}{2}\sigma^2(x)\left(\dfrac{\partial F_\nu(u)}{\partial x}\right)^2 F_\nu(u)^* \left(\dfrac{d^2 F_\nu^{-1}}{d^2 x}\right)^2 \\ u(0,x) = g(x),\ x \in \mathbb{R}, \end{array}\right\} \qquad (5.5)$$

where g is a continuous real valued function with polynomial growth and L is the infinitesimal generator of the diffusion process $\{X_s^{x,t} : 0 \leq t \leq s \leq T\}$ and $\pi^*(\phi)$ stands for the pullback of the distribution ϕ by π.

Applications to non–linear PDE

In the case where the convex function F_ν is twice continuously differentiable, the equation (5.5) takes the form

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial t} = Lu + \sigma^2(x)\left(\dfrac{F''_\nu(u)}{2F'_\nu(u)}\right)\left(\dfrac{\partial u}{\partial x}\right)^2 \\ u(0,x) = g(x),\ x \in \mathbb{R}. \end{array}\right\} \tag{5.6}$$

This situation corresponds to the case where $\nu << dx$.
Let $\{Y_s^{x,t} : s \in [t, T]\}$ be the unique solution to BSDE

$$Y_s^{x,t} = g(X_T^{x,t}) + \int_s^T \frac{F''_\nu}{2F'_\nu}(Y_r^{x,t})\left(Z_r^{x,t}\right)^2 ds - \int_s^T Z_r^{x,t} dW_r,\ t \le s$$

Then $Y_t^{x,t}$ is a viscosity solution to the equation (5.6).

Applications to non–linear PDE

Now, let $\{(Y_s^{x,t}, Z_s^{x,t}) : s \in [t, T]\}$ be the unique solution to the singular BSDE

$$Y_s^{x,t} = g(X_T^{x,t}) + \int_{\mathbb{R}} \left(L_T^a\left(Y^{x,t}\right) - L_s^a\left(Y^{x,t}\right)\right) \nu(da) - \int_s^T Z_r^{x,t} dW_r.$$

With the help of the transformation F_ν one can see that $Y_t^{x,t}$ is a viscosity solution to the equation (5.5).
Let us now go back to the corresponding limit theorems.

Applications to non–linear PDE

Theorem Let $\nu_n(da)$, $n = 1, 2, \ldots$ be a sequence of Radon measures. Suppose that there exist two positive constants ε, M such that :

$$\begin{aligned} |\nu_n|(\mathbb{R}) &\leq M \quad && \forall\, n \geq 1, \\ |\nu_n(\{x\})| &\leq \varepsilon < 1 && \forall\, n \geq 1,\ \forall\, x \in \mathbb{R}. \end{aligned}$$

and there exist a function f BV such that :

$$\int_{-L}^{L} |f_{\nu_n} - f|^2 (x)\, dx \longrightarrow 0 \ \text{ as } \ n \to +\infty \ \text{ for all } \ L > 0,$$

Applications to non–linear PDE

Set

$$\nu(da) = \frac{f'(da)}{f(a) + f(a-)} \quad \text{and} \quad F(x) := \int_0^x f(y)dy \cdot$$

If $u^n(t,x)$ and $u(t,x)$ denote respectively the unique solution of the PDE (5.5) with ν_n respectively with ν.
Then $u^n(t,x)$ converges to $u(t,x)$ as n tends to ∞ for any $(t,x) \in [0,T] \times \mathbb{R}$.

Applications to non–linear PDE

Proof. For any $t \in \mathbb{R}_+$, we let $\{Y_s^{x,t,n} : t \leq s \leq T\}$ and $\{Y_s^{x,t} : t \leq s \leq T\}$ be respectively the solution of the BSDE

$$Y_s^{x,t,n} = g(X_T^{x,t}) + \int_{\mathbb{R}} \left(L_T^a\left(Y^{x,t,n}\right) - L_s^a\left(Y^{x,t,n}\right)\right)\nu_n(da) - \int_s^T Z_r^{x,t,n} dW_r$$

and

$$Y_s^{x,t} = g(X_T^{x,t}) + \int_{\mathbb{R}} \left(L_T^a\left(Y^{x,t}\right) - L_s^a\left(Y^{x,t}\right)\right)\nu(da) - \int_s^T Z_r^{x,t} dW_r.$$

If we set $\tilde{u} := F_\nu(u)$, then equation (5.5) becomes

$$\left.\begin{array}{l} \dfrac{\partial \tilde{u}}{\partial t}(t,x) = L\tilde{u}(t,x) \\ \tilde{u}(0,x) = F_\nu(g(x)), \ x \in \mathbb{R}. \end{array}\right\}$$

Applications to non–linear PDE

Therefore the process $\{\tilde{u}(s, X_s^{x,t}) : t \leq s \leq T\}$ is the unique solution to the BSDE

$$\tilde{Y}_s^{x,t} = F_\nu\left(g(X_T^{x,t})\right) - \int_s^T \tilde{Z}_r^{x,t} dW_r,$$

hence $Y_s^{x,t} = F_\nu^{-1}(\tilde{Y}_s^{x,t}) = F_\nu^{-1}(\tilde{u}(s, X_s^{x,t})) = u(s, X_s^{x,t})$, in particular $u^n(t,x) = Y_t^{x,t,n}$ and $Y_t^{x,t} := u^\nu(t,x)$, then by virtue of the previous results, $u^n(t,x)$ and $u^\nu(t,x)$ are respectively the unique viscosity solution to (5.5) with ν_n respectively ν. So Theorem 3.4 implies that

$$\lim_{n\to+\infty} \mathbb{E} \sup_{t\leq s\leq T} \left| Y_s^{x,t,n} - Y_s^{x,t} \right| = 0$$

which implies that $u^n(t,x)$ converges to $u(t,x)$ as n tends to ∞. The convergence is uniform on compacts by continuity. □

Applications to non–linear PDE

Let u^{h_n} be the unique solution of the following PDE

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial t}(t,x) = Lu(t,x) + \sigma^2(x) h_n(u(t,x)) \left(\dfrac{\partial u}{\partial x}(t,x)\right)^2 \\ u(0,x) = g(x),\ x \in \mathbb{R}. \end{array}\right\} \quad (5.7)$$

The following theorem gives the relative compactness of the family $\{u^\nu : \|\nu\| \leq C\}$ and states that a solution to equation (5.5) is a limit of sequence of solution to the equation (5.7).

Applications to non–linear PDE

Theorem

Let C be a fixed constant. Then, $\mathcal{K} = \{u^{\nu} : \|\nu\| \leq C\}$ is a compact set for the topology induced by uniform convergence. The set of all u^{ν} belonging to $\mathcal{K}$ such that ν is absolutely continuous with respect to Lebesgue measure is dense in $\mathcal{K}$.

Applications to non–linear PDE

Theorem

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Proof. The proof of the first part is an immediate consequence of the connection between BSDEs and PDEs and Theorem 2.
Let us prove the second part ; Let ν be in $\mathcal{M}(\mathbb{R})$ and θ_n be an approximation of the identity, we set $f_n = f_\nu * \theta_n$ and $g_n = \frac{f_n'}{2f_n}$.

Applications to non–linear PDE

Let u^{g_n} be the unique solution of the PDE (5.7). Using BSDE representation we get :

$$\lim_{n\to+\infty} \left\| u^{g_n} - u^{\nu} \right\|_{\infty} = 0. \qquad \square$$

We use the connection between BSDEs and PDEs to give a probabilistic proof to a comparison theorem for non–linear PDE.

Theorem

Let g_1 and g_2 be two functions such that $g_1(x) \leq g_2(x), \forall\, x \in \mathbb{R}$. Let ν_1 and ν_2 be in $\mathcal{M}(\mathbb{R})$ such that the measure $\nu_2 - \nu_1 \geq 0$. If u^1 and u^2 are the solutions to the PDE (5.5) corresponding to ν_1 and ν_2. Then $u^1(t,x) \leq u^2(t,x)$.

Comparison theorem for PDEs.

Proof. We can write $u^1(t,x) = Y_t^{x,t,1}$ and $u^2(t,x) = Y_t^{x,t,2}$, where $\{(Y_s^{x,t,i}, Z_r^{x,t,i}) : t \leq s \leq T\}$ is the unique solution to the BSDE

$$Y_s = g_i(X_T^{x,t}) + \int_{\mathbb{R}} (L_T^a(Y) - L_s^a(Y))\, \nu_i(da) - \int_s^T Z_r dW_r,\ i = 1, 2.$$

Now, from Theorem 3, we have $Y_s^{x,t,1} \leq Y_s^{x,t,2}$ for all $t \leq s \leq T$, in particular $Y_t^{x,t,1} \leq Y_t^{x,t,2}$.

Comparison theorem for PDEs.

Remark

Using the same argument as in Corollary 1 and the comparison Theorem 6 one can obtain the corresponding limit theorem in the monotone case for PDE.

Theorem 6 implies the uniqueness property for a class of non–linear PDEs (at least of the form (5.5)).

Comparison theorem for PDEs.

Example

Consider the non-linear Cauchy problem

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) - \frac{1}{2}\left(\frac{\partial u}{\partial x}(t,x)\right)^2 + k(x), \tag{5.8}$$

where $u(0,x) = g(x), x \in \mathbb{R}$ and $k : \mathbb{R} \to [0,+\infty)$ is continuous. If $g = 0$, the unique solution, is given for all $(t,x) \in [0,+\infty[\times\mathbb{R}$ by :

$$u(t,x) = -\log \mathbb{E}\left[\exp\left\{-\int_0^t k(x+W_s)ds\right\}\right].$$

Comparison theorem for PDEs.

In the case where $k = 0$ the equation (5.8) is a particular equation of (5.7) which corresponds to the case where $\sigma = 1$, $b = 0$ and $h = -\frac{1}{2}$, hence we can use BSDE to construct a solution to the equation (5.8), more precisely, let $\{Y_s^{x,t} : t \leq s \leq T\}$ be the unique solution to the BSDE

$$Y_s^{x,t} = g(X_T^{x,t}) - \int_s^T \frac{1}{2}\left(Z_r^{x,t}\right)^2 dr - \int_s^T Z_r^{x,t} dW_r \qquad (5.9)$$

where $X_T^{x,t} = x + \frac{1}{2}(B_T - B_t)$.

Comparison theorem for PDEs.

It is clear that the hypotheses of the Remark 2 are satisfied since the Brownian motion has exponential finite moment, and hence the equation (5.9) has a unique solution and $u(t,x) = Y_t^{x,t}$ is the viscosity solution to the equation (5.8) with $u(0,\cdot) = g(\cdot)$.
In the case where $k \neq 0$, the BSDE associated to the equation (5.8) is the following

$$Y_s^{x,t} = g(X_T^{x,t}) + \int_s^T k(X_r^{x,t})dr - \int_s^T \frac{1}{2}\left(Z_r^{x,t}\right)^2 dr - \int_s^T Z_r^{x,t} dW_r,$$

consequently the function $u(t,x) = Y_t^{x,t}$ is the unique viscosity solution to the equation (5.8).

BSDEs with reflexion

Let us now assume that ν be nonnegative σ–finite measure on $\mathbb{R}$. We want to solve

$$Y_t = \xi + \int_{\mathbb{R}} (L_T^a(Y) - L_t^a(Y))\, \nu(da) - \int_t^T Z_s \cdot dW_s. \tag{5.10}$$

Set $\alpha = \nu(\{x_0\})$, such that $|\alpha| \geq 1$. The equation (5.10) is equivalent to

$$Y_t = \xi + \int_{\mathbb{R}} (L_T^a(Y) - L_t^a(Y))\, \mu(da) \tag{5.11}$$

$$+\alpha \left(L_T^{x_0}(Y) - L_t^{x_0}(Y)\right) - \int_t^T Z_s \cdot dW_s \tag{5.12}$$

where $\mu(da) = \nu(da) - \alpha\delta_{x_0}(da)$, $|\mu(\{x\})| < 1$ for all $x \in \mathbb{R}$.

BSDEs with reflexion

Lemma

The equation (5.11) is equivalent to

$$\widetilde{Y}_t = \widetilde{\xi} - \int_t^T \widetilde{Z}_s \cdot dW_s + \alpha \left(L_T^{F_\mu(x_0)}(\widetilde{Y}) - L_t^{F_\mu(x_0)}(\widetilde{Y}) \right). \quad (5.13)$$

BSDEs with reflexion

Proof. Let us now use the transformation F_μ, using Tanaka formula we obtain

$$\begin{aligned} F_\mu(Y_T) &= F_\mu(Y_t) + \int_t^T \frac{1}{2}\left(f_\mu(Y_s) + f_\mu(Y_s-)\right) dY_s \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} \left(L_T^a(Y) - L_t^a(Y)\right) f_\mu'(da) \\ &= F_\mu(Y_t) + \int_t^T \frac{1}{2}\left(f_\mu(Y_s) + f_\mu(Y_s-)\right) Z_s \cdot dW_s \\ &\quad + L_T^{F_\mu(x_0)}(F_\mu(Y)) - L_t^{F_\mu(x_0)}(F_\mu(Y)). \end{aligned}$$

BSDEs with reflexion

Set $\alpha_i = \nu(\{x_i\})$, such that $|\alpha_i| \geq 1$ for $i \in I$, where I is at most a countably subset of indices. The equation (5.10) is equivalent to

$$\begin{aligned} Y_t &= \xi + \int_{\mathbb{R}} (L_T^a(Y) - L_t^a(Y))\, \mu(da) \\ &\quad + \sum_{i\in I} \alpha_i \left(L_T^{x_i}(Y) - L_t^{x_i}(Y)\right) - \int_t^T Z_s \cdot dW_s \end{aligned}$$

where $\mu(da) = \nu(da) - \sum_{i\in I} \alpha_i \delta_{x_i}(da)$. Now we may assume $|\mu(\{x\})| < 1$ for all $x \in \mathbb{R}$.

BSDEs with reflexion

Lemma

The equation (5.11) is equivalent to

$$Y_t = F_\mu(\xi) - \int_t^T Z_s \cdot dW_s + \sum_{i \in I} \alpha_i \left(L_T^{F_\mu(x_i)}(Y) - L_t^{F_\mu(x_i)}(Y) \right).$$

BSDEs with reflexion

To study the BSDE of the form equation (5.11) it suffices to solve the following BSDE

$$Y_t = \xi - \int_t^T Z_s \cdot dW_s + \alpha \left(L_T^{x_0}(Y) - L_t^{x_0}(Y) \right). \qquad (5.14)$$

Without loss of generality we may assume that $\alpha \geq 1$. The problem (5.14) is a classical reflected BSDE at the deterministic point x_0.

Proposition

If $\xi \leq x_0$ a.s. or $\xi \geq x_0$ a.s. such that $\nu(\{x_0\}) = \alpha$. Then the equation (5.14) has a unique solution.

BSDEs with reflexion

Proposition

Let x_1 *and* x_2 *be two real numbers such that* $\nu(\{x_1\}) = \alpha_1$, $\nu(\{x_2\}) = \alpha_2$ *and* $x_1 < x_2$. *If* $x_1 \leq \xi \leq x_2$ *a.s. Then the equation*

$$Y_t = \xi - \int_t^T Z_s \cdot dW_s + \alpha_1 \left(L_T^{x_1}(Y) - L_t^{x_1}(Y)\right) + \alpha_2 \left(L_T^{x_2}(Y) - L_t^{x_2}(Y)\right)$$

has a unique solution. Moreover $x_1 \leq Y_t \leq x_2$ *a.s. and* $L_t^{x_1}(Y) = L_t^{x_2}(Y) = 0$, *for all* $t \in [0, T]$.

BSDEs with reflexion

Proof of Proposition 2. Consider the following reflected BSDE

$$Y_t = \xi - \int_t^T Z_s \cdot dW_s + K_T - K_t,$$

where $\{K_t : t \in [0, T]\}$ is an increasing such that $K_0 = 0$. The reflected BSDE has a unique solution. $(Y_t = E(\xi/\mathcal{F}_t), Z_t, 0)$ is a solution. $\xi \leq x_0$ a.s. implies that $Y_t \leq x_0$ a.s. Consequently $L_t^{x_0}(Y) = 0$ for all $t \in [0, T]$, indeed by Tanaka's formula

$$(\xi - x_0)^+ = (Y_t - x_0)^+ + \int_t^T \mathbf{1}_{\{Y_s > x_0\}} dY_s + \frac{1}{2}\left(L_T^{x_0}(Y) - L_t^{x_0}(Y)\right)$$

But $\int_t^T \mathbf{1}_{\{Y_s > x_0\}} dY_s = 0$, hence $L_T^{x_0}(Y) = L_t^{x_0}(Y) = L_0^{x_0}(Y) = 0$.

BSDEs with reflexion

The Proposition 3 corresponds to a two barrier reflected BSDE whose proof is similar to that of Proposition 2.

Proposition

OPEN PROBLEM : If $P(\xi \leq x_0)\, P(\xi \geq x_0) > 0$ *such that* $\nu(\{x_0\}) = \alpha$. *Then equation (5.14) has no unique solution.*

Thank you for your attention