Doubly Reflected BSDEs with Call Protection and their Approximation

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New advances in backward SDEs for financial engineering application,
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Convertible bond with underlying stock S

- Coupons from time 0 onwards
- Terminal payoff at $\zeta = \tau \wedge \theta$

$$\mathbf{1}_{\zeta=\tau<T}\ell(\tau, S_\tau) + \mathbf{1}_{\vartheta<\tau}h(\vartheta, S_\vartheta) + \mathbf{1}_{\zeta=T}g(S_T)$$

  - $[0, T]$-valued bond holder put time τ and bond issuer call time θ
  - Cancelable American claim, or game option
- Call protections preventing the issuer from calling the bond on certain random time intervals
  - Typically monitored at discrete monitoring times
  - In a possibly very path-dependent way

Agenda

Mathematical issues

- Doubly reflected backward stochastic differential equations with an intermittent upper barrier, only active on random time intervals (RIBSDE)
- Related variational inequality approach (VI)
  - Highly-dimensional pricing problems (path dependence)
  - Deterministic pricing schemes ruled out by the curse of dimensionality

$\rightarrow$ Simulation methods

Contributions

- A convergence rate for a discrete time approximation scheme by simulation to an RIBSDE
- VI approach
- Practical value of this approach on the benchmark problem of pricing by simulation highly path-dependent convertible bonds
- A demonstration of the real abilities of simulation/regression numerical schemes in high dimension (up to $d = 30$ in this work)

Outline

Diffusion Set-Up with Marker Process

- Diffusion with Lipschitz coefficients in $\mathbb{R}^q$

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$$

- Call protection monitoring times $\mathfrak{T} = \{0 = T_0 < \ldots < T_N = T\}$
- Marker process H keeping track of the path-dependence, in view of 'markovianizing' the model
- $\mathbb{R}^q \times \mathcal{K}$-valued factor process $\mathcal{X} = (X, H)$ (finite set $\mathcal{K}$)
  - $u = u(t, x, k) = u^k(t, x)$
- $\mathcal{K}$-valued pure jump marker process H supposed to be constant except for deterministic jumps at the T_ls

$$H_{T_l} = \kappa_l(X_{T_l}, H_{T_l-})$$

  - Jump functions κ_l^k continuous in x outside $\partial\mathcal{O}$ (constant on $\mathcal{O}$ and on ${}^c\mathcal{O}$) for an open, 'regular' domain $\mathcal{O} \subseteq \mathbb{R}^q$

Call Protection

- Subset K of $\mathcal{K}$
- Call forbidden/possible whenever $H_t \in K$ / $\notin K$
- $\mathfrak{T}$-valued stopping times given as successive times of exit from and entrance to K, so $\vartheta_0 = 0$ and then

$$\vartheta_{2l+1} = \inf\{t > \vartheta_{2l} \,;\, H_t \notin K\} \wedge T \,,\; \vartheta_{2l+2} = \inf\{t > \vartheta_{2l+1} \,;\, H_t \in K\} \wedge T$$

- Call forbidden/possible on the 'even'/'odd' intervals $[\vartheta_l, \vartheta_{l+1})$
  - $H_t \in K$ / $\notin K$

Starting from $H_0 = k \notin K$ ('Call at the beginning')

$0 = \vartheta_0 = \vartheta_1 < \vartheta_2 \leq \ldots \leq \vartheta_{N+1} = T$
Call possible on the first non-void time interval $[\vartheta_1 = 0 = \vartheta_0, \vartheta_2 > 0)$

Starting from $H_0 = k \in K$ ('No Call at the beginning')

$0 = \vartheta_0 < \vartheta_1 \leq \ldots \leq \vartheta_{N+1} = T$
Call forbidden on the first non-void time interval $[\vartheta_0 = 0, \vartheta_1 > 0)$

Markovian RIBSDE

Reflected BSDE ($\mathcal{S}$) with data

$$f(t, X_t, y, z)\,,\ \xi = g(X_T)\,,\ \ell(t, X_t)\,,\ h(t, X_t)\,,\ \vartheta$$

- 'Standard Lipschitz and L^2-integrability assumptions' (if not for ϑ)
- Mokobodski condition
  - Existence of a square-integrable quasimartingale Q between processes $\ell(t, X_t)$ and $h(t, X_t)$
- Doubly reflected BSDE with lower barrier $L_t = \ell(t, X_t)$ and intermittent (the 'I' in RIBSDE) upper barrier given by, for $t \in [0, T]$

$$U_t = \sum_{l=0}^{[N/2]} \mathbf{1}_{[\vartheta_{2l}, \vartheta_{2l+1})} \infty + \sum_{l=1}^{[(N+1)/2]} \mathbf{1}_{[\vartheta_{2l-1}, \vartheta_{2l})} h(t, X_t)$$

  - 'Nominal' upper obstacle $h(t, X_t)$ only active on the 'odd' random time intervals $[\vartheta_{2l-1}, \vartheta_{2l})$
  - Call protection on the 'even' random time intervals $[\vartheta_{2l}, \vartheta_{2l+1})$

Risk-neutral pricing problems in finance

Driver coefficient function f typically given as

$$f = f(t,x,y) = c(t,x) - \mu(t,x)y$$

- Dividend and interest-rate related functions c and μ
  - Single-name credit risk (counterparty risk)
    - Recovery-adjusted dividend-yields c
    - Credit-spread adjusted interest-rates μ
    - Pre-default factor process X
- Affine in y, does not depend on z
  - Historical rather than RN modeling $\rightarrow$ 'z-dependent' f
  - Market imperfections $\rightarrow$ nonlinear f

Terminal cost functions typically given by

$$\ell(t,x) = \bar{P} \vee S\,,\ h(t,x) = \bar{C} \vee S\,,\ g(x) = \bar{N} \vee S$$

$\bar{P} \le \bar{N} \le \bar{C}$ Constants
$S = x_1$ first component of x

- Mokobodski condition satisfied with $Q = S$ provided S is a square-integrable Itô process

Highly path dependent call protection

Example ('l out of d')

Given a constant trigger level $\bar{S}$ and constants $l \leq d \leq N$, call possible iff S has been $\geq \bar{S}$ on at least l of the last d monitoring times

- $\mathcal{K} = \{0,1\}^d$, $\kappa_l^k(x) = (\mathbf{1}_{S \geq \bar{S}}, k_1, \ldots, k_{d-1})$
- H_t vector of the indicator functions of the events $S_{T_l} \geq \bar{S}$ at the last d monitoring dates preceding time t

Call possible iff $|H_t| \geq l \Leftrightarrow H_t \notin K$ with $|k| = \sum_{1 \leq p \leq d} k_p$ and $K = \{k \in \mathcal{K}\, ;\, |k| < l\}$

Solution of the RIBSDE

Definition

A solution $\mathcal{Y}$ to $(\mathcal{S})$ is a triple $\mathcal{Y} = (Y, Z, A)$ such that:

(i) $Y \in \mathcal{S}^2, Z \in \mathcal{H}_q^2, A \in \mathcal{A}^2$

(ii) $Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)ds + A_T - A_t - \int_t^T Z_s dW_s \; t \in [0, T]$

(iii) $L_t \leq Y_t$ on $[0, T]$, $Y_t \leq U_t$ on $[0, T]$

and $\int_0^T (Y_t - L_t)dA_t^+ = \int_0^T (U_{t-} - Y_{t-})dA_t^- = 0$

(iv) A^+ is continuous, and

$\{(\omega, t)\,;\, \Delta Y \neq 0\} = \{(\omega, t)\,;\, \Delta A^- \neq 0\} \subseteq \bigcup_{l=0}^{[N/2]} [\![\vartheta_{2l}]\!]$

$\Delta Y = \Delta A^-$ on $\bigcup_{l=0}^{[N/2]} [\![\vartheta_{2l}]\!]$

- $\mathcal{S}^2, \mathcal{H}_q^2$ and $\mathcal{A}^2$ 'usual L^2 spaces'
- $A^{\pm}$ Jordan component of A
- Convention that $0 \times \pm\infty = 0$ in (iii)
- Obvious extension to a random terminal time θ

- For l decreasing from N to 0, let us define $\mathcal{Y}^l = (Y^l, Z^l, A^l)$ on $[\vartheta_l, \vartheta_{l+1}]$ as the solution, with A^l continuous, to the stopped RBSDE (for l even) or R2BSDE (for l odd) with data (with $Y^{N+1}_{\vartheta_{N+1}} \equiv g(X_T)$)

$$\begin{cases} f(t, X_t, y, z)\,,\ Y^{l+1}_{\vartheta_{l+1}}\,,\ \ell(t, X_t) & (l \text{ even}) \\ f(t, X_t, y, z)\,,\ \min(Y^{l+1}_{\vartheta_{l+1}}, h(\vartheta_{l+1}, X_{\vartheta_{l+1}}))\,,\ \ell(t, X_{st})\,,\ h(t, X_t) & (l \text{ odd}) \end{cases}$$

- Let us define $\mathcal{Y} = (Y, Z, A)$ on $[0, T]$ by, for every $l = 0, \ldots, N$:
  - $(Y, Z) = (Y^l, Z^l)$ on $[\vartheta_l, \vartheta_{l+1})$, and also at $\vartheta_{N+1} = T$ in case $l = N$. So in particular

$$Y_0 = \begin{cases} Y_0^0, & k \in K \\ Y_0^1, & k \notin K \end{cases}$$

    where k is the initial condition of the marker process H.
  - $dA = dA^l$ on $(\vartheta_l, \vartheta_{l+1})$,

$$\Delta A_{\vartheta_l} = \Delta A^-_{\vartheta_l} = \left(Y^l_{\vartheta_l} - h(\vartheta_l, X_{\vartheta_l})\right)^+ = \Delta Y_{\vartheta_l}\ (= 0 \text{ for l odd })$$

    and $\Delta A_T = \Delta Y_T = 0$.

Proposition

$\mathcal{Y} = (Y, Z, A)$ is the unique solution to $(\mathcal{S})$

Verification principle

Risk-neutral pricing problems in finance

Financial interpretation of a solution $\mathcal{Y}$ to $(\mathcal{S})$

Y_0 'NFLVR' Arbitrage price at time 0 for the game option with payoff functions c, l, h, g and call protection ϑ
Bilateral super-hedging price and infimal issuer super-hedging price

up to a local martingale cost process

Z Hedging strategy

Outline

Approximation of the Forward Process

- Time-grid $\mathfrak{t} = \{0 = t_0 < t_1 < \ldots < t_n = T\} \supseteq \mathfrak{T}$
- Euler scheme approximation of $\widehat{X}$

$$\widehat{X}_{t_{i+1}} = \widehat{X}_{t_i} + b(t_i, \widehat{X}_{t_i})(t_{i+1} - t_i) + \sigma(t_i, \widehat{X}_{t_i})(W_{t_{i+1}} - W_{t_i})$$

- Approximation of the marker process H

$$\widehat{H}_{T_l} = \kappa_l(\widehat{X}_{T_l}, \widehat{H}_{T_l-})$$

Approximation of the Call Protection Switching Times

Approximation $\widehat{\vartheta}$ of ϑ obtained by using $\widehat{\mathcal{X}} = (\widehat{X}, \widehat{H})$ instead of $\mathcal{X}$ in the definition of ϑ

Proposition (Assuming σ non-degenerate and 'some regularity of σ and b around $\partial\mathcal{O}$)

For every $l \leq N+1$

$$\mathbb{E}\Big[|\vartheta_l - \widehat{\vartheta}_l|\Big] \leq C_\varepsilon |\mathfrak{t}|^{\frac{1}{2}-\varepsilon}$$

$|\mathfrak{t}| = \max_{i \leq n-1}(t_{i+1} - t_i)$

Approximation of the RIBSDE

- Projection operator $\widehat{\mathcal{P}}$ defined by

$$\widehat{\mathcal{P}}(t,x,y) = y + [\ell(t,x) - y]^+ - [y - h(t,x)]^+ \sum_{l=1}^{[(N+1)/2]} \mathbf{1}_{\{\widehat{\vartheta}_{2l-1} \le t \le \widehat{\vartheta}_{2l}\}}$$

- Reflection operating only on a subset $\mathfrak{r}$ of $\mathfrak{t}$ in the approximation scheme for $\mathcal{Y}$

$$\mathfrak{r} = \{0 = r_0 < r_1 < \cdots < r_\nu = T\} \text{ with } \mathfrak{T} \subseteq \mathfrak{r} \subseteq \mathfrak{t}$$

Components Y and Z of a solution $\mathcal{Y} = (Y, Z, A)$ to $(\mathcal{S})$ approximated by a triplet of processes $(\widehat{Y}, \widetilde{Y}, \bar{Z})$ defined on $\mathfrak{t}$

Terminal condition

$$\widehat{Y}_T = \widetilde{Y}_T = g(\widehat{X}_T)$$

and then for i decreasing from $n-1$ to 0

$$\begin{cases} \bar{Z}_{t_i} &= \mathbb{E}\left[\widehat{Y}_{t_{i+1}}\left(\frac{W_{t_{i+1}} - W_{t_i}}{t_{i+1} - t_i}\right) \mid \mathcal{F}_{t_i}\right] \\ \widetilde{Y}_{t_i} &= \mathbb{E}\left[\widehat{Y}_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + (t_{i+1} - t_i) f(t_i, \widehat{X}_{t_i}, \widetilde{Y}_{t_i}, \bar{Z}_{t_i}) \\ \widehat{Y}_{t_i} &= \widetilde{Y}_{t_i} \mathbf{1}_{\{t_i \notin \mathfrak{t}\}} + \widehat{\mathcal{P}}(t_i, \widehat{X}_{t_i}, \widetilde{Y}_{t_i}) \mathbf{1}_{\{t_i \in \mathfrak{t}\}} \end{cases}$$

Continuous-time extension of the scheme still denoted by $(\widehat{Y}, \widetilde{Y}, \bar{Z})$

$\widehat{Z}$ Integrand in a stochastic integral representation of $\widehat{Y}$

Theorem (No call or no call protection, Chassagneux 08)

In case of Lipschitz barriers and for $|\mathfrak{r}| \sim |\mathfrak{t}|^{\frac{2}{3}}$ *(resp. semi-convex barriers and for* $|\mathfrak{r}| \sim |\mathfrak{t}|^{\frac{1}{2}}$ *) , one has*

$$\max_{i \le n-1} \sup_{t \in [t_i, t_{i+1})} \mathbb{E}\Big[|Y_t - \widetilde{Y}_{t_i}|^2\Big] + \max_{i \le n-1} \sup_{t \in [t_i, t_{i+1})} \mathbb{E}\Big[|Y_{t-} - \widehat{Y}_{t_i}|^2\Big] \le C|\mathfrak{t}|^{\alpha}$$

with $\alpha = \frac{1}{3}$ *(resp.* $\frac{1}{2}$*).*

Theorem (Call protection, this work, assuming f does not depend on z)

In case of Lipschitz barriers and for $|\mathfrak{r}| \sim |\mathfrak{t}|^{\frac{1}{2}}$ *(resp. semi-convex barriers and for* $|\mathfrak{r}| \sim |\mathfrak{t}|$*) , one has*

$$\max_{i \le n-1} \sup_{t \in [t_i, t_{i+1})} \mathbb{E}\Big[|Y_t - \widetilde{Y}_{t_i}|^2\Big] + \max_{i \le n-1} \sup_{t \in [t_i, t_{i+1})} \mathbb{E}\Big[|Y_{t-} - \widehat{Y}_{t_i}|^2\Big] \le C_{\varepsilon}|\mathfrak{t}|^{\alpha - \varepsilon}$$

with $\alpha = \frac{1}{4}$ *(resp.* $\frac{1}{2}$*).*

- Proof of the theorem based on a suitable concept of time-continuous discretely reflected BSDEs
  - Bermudan options
- Possible extension to the case where f depends on z
- Representations of $\widetilde{Y}$ and $\widehat{Z}$ using approximated optimal policies
  - Cf. ‘MC Backward versus Forward’ in the numerical part

Variational Inequality Approach

- Comparing the simulation results them with those of an alternative, deterministic numerical scheme
- Deterministic scheme for $(\mathcal{S})$ based on an analytic characterization of $(\mathcal{S})$
- Let $\mathcal{E} = [0, T] \times \mathbb{R}^q \times \mathcal{K}$ and for $I = 1, \ldots, N$

$$\mathcal{E}_I = [T_{I-1}, T_I] \times \mathbb{R}^q \times \mathcal{K}\,,\ \mathcal{E}_I^* = [T_{I-1}, T_I) \times \mathbb{R}^q \times \mathcal{K}$$

  - The $\mathcal{E}_I^*$s and $\{T\} \times \mathbb{R}^q \times \mathcal{K}$ partition $\mathcal{E}$

Continuity of ϑ with respect to(t, x, i)

- Continuous outside $\mathfrak{T} \times \mathbb{R}^q \times \mathcal{K}$
- Cadlag on $(\mathfrak{T} \times \mathbb{R}^q \times \mathcal{K}) \setminus (\mathfrak{T} \times \partial\mathcal{O} \times \mathcal{K})$
- Cad but not'lag' on $\mathfrak{T} \times \partial\mathcal{O} \times \mathcal{K}$

Cauchy cascade

Definition

(i) Cauchy cascade (g, ν) on $\mathcal{E}$

- Terminal condition g at T
- Sequence $\nu = (u_I)_{1 \le I \le N}$ of functions u_Is on the $\mathcal{E}_I$s
- Jump condition for $x \notin \partial\mathcal{O}$ (with $u_{N+1} \equiv g$):

$$u_I^k(T_I, x) = \begin{cases} \min(u_{I+1}(T_I, x, \kappa_I^k(x)), h(T_I, x)) & \text{if } k \notin K \text{ and } \kappa_I^k(x) \in K \\ u_{I+1}(T_I, x, \kappa_I^k(x)) & \text{else} \end{cases}$$

(ii) Continuous Cauchy cascade

- Cauchy cascade with continuous ingredients g at T and u_Is on the $\mathcal{E}_I$s, except maybe for discontinuities of the u_I^ks on $\mathfrak{T} \times \partial\mathcal{O}$

(iii) Function on $\mathcal{E}$ defined by a Cauchy cascade

- Concatenation on the $\mathcal{E}_I^*$s of the u_Is + terminal condition g at T

Cascade Characterization of $\mathcal{Y}$

Proposition

$Y_t = u(t, \mathcal{X}_t)$, $t \in [0, T]$, for a deterministic pricing function u, defined by a continuous Cauchy cascade $(g, \nu = (u_I)_{1 \le I \le N})$ on $\mathcal{E}$

Analytic characterization of u?

Generator of X

$$\mathcal{G}\phi(t,x) = \partial_t \phi(t,x) + \partial\phi(t,x) b(t,x) + \frac{1}{2}\mathrm{Tr}[a(t,x)\mathcal{H}\phi(t,x)]$$

$a(t,x)$ $\sigma(t,x)\sigma(t,x)^{\mathsf{T}}$

$\partial\phi$, $\mathcal{H}\phi$ Row-gradient and Hessian of ϕ with respect to x

Cauchy cascade $(\mathcal{VI})$

For I decreasing from N to 1,

- At $t = T_I$ for every $k \in \mathcal{K}$ and $x \notin \partial\mathcal{O}$

$$u_I^k(T_I, x) = \begin{cases} \min(u_{I+1}(T_I, x, \kappa_I^k(x)), h(T_I, x)), & k \notin K \text{ and } \kappa_I^k(x) \in K \\ u_{I+1}(T_I, x, \kappa_I^k(x)), & \text{else} \end{cases} \quad (0)$$

  with $u_{N+1} \equiv g$

- On the time interval $[T_{I-1}, T_I)$ for every $k \in \mathcal{K}$,

$$\begin{cases} \min\Big(-\mathcal{G}u_I^k - f^{u_I^k}, u_I^k - \ell\Big) = 0\,, \ k \in K \\ \max\Big(\min\Big(-\mathcal{G}u_I^k - f^{u_I^k}, u_I^k - \ell\Big), u_I^k - h\Big) = 0\,, \ k \notin K \end{cases} \quad (1)$$

  with for any function $\phi = \phi(t, x)$

$$f^{\phi} = f^{\phi}(t, x) = f(t, x, \phi(t, x))$$

- Technical difficulty due to the potential discontinuity in x of the functions u_l^ks on $\partial\mathcal{O}$
  - Characterizing ν in terms of a suitable notion of discontinuous viscosity solution of $(\mathcal{VI})$?
    - Convergence results? for deterministic approximation schemes to u

Assumption (Call protection non-decreasing with respect to $\mathcal{O}$)

$\mathcal{O}_\varepsilon = \{x \in \mathbb{R}^q \mid \mathrm{d}(x) < \varepsilon\}$.
Index $\varepsilon \leftrightarrow$ *dilatated domain* $\mathcal{O}_\varepsilon$
For every $\varepsilon \geq 0$
(i) $u \leq u_\varepsilon$
(ii) $u_\varepsilon(T_l, x, \kappa^k_{l,+}) \leq u_\varepsilon(T_l, x, k) \leq u_\varepsilon(T_l, x, \kappa^k_{l,-})$

Example

'*l* out of *d*'

$\overline{u}_{\varepsilon,I}$ and $\underline{u}_{\varepsilon,I}$ 'upper and lower semi-continuous envelope' of $u_{\varepsilon,I}$ on $\mathcal{E}_I$

Proposition

- *Every $u_I \equiv u_{0,I}$ is a continuous viscosity solution of (1) on $\mathcal{E}_I^*$*
- *For I decreasing from N to 1*

$$\overline{u}_{0,I} = \lim \searrow_{\varepsilon \searrow 0+} \underline{u}_{\varepsilon,I}$$

which is also the largest subsolution of (0)-(1) on $\mathcal{E}_I$

- *Convergence of 'double' deterministic schemes (u_ε^h) to u*
  - *$h(\varepsilon)$??*

Curse of dimensionality

- $(\mathcal{VI}) = \text{Card}(\mathcal{K})$ equations in the u^ks
- $\sim (q+d)$ – dimensional pricing problem with $d = \log(\text{Card}(\mathcal{K}))$
- Simulation schemes the only viable alternative for d greater than few units

Outline

Benchmark Model

Local drift and volatility pre-default model for a stock $X = S$

$$\frac{dS_t}{S_t} = b(t, S_t)dt + \sigma(t, S_t)dW_t$$

$$b(t, S) = r(t) - q(t) + \eta\gamma(t, S)\,,\ \gamma(t, S) = \gamma_0(S_0/S)^\alpha\,,\ \sigma(t, S) = \sigma$$

$r(t)$ Riskless short interest rate

$q(t)$ Dividend yield

$\gamma(t, S)$ Local default intensity ($\gamma_0,\ \alpha \geq 0$)

$0 \leq \eta \leq 100\%$ Loss Given Default of the firm issuing the bond

Coupon rate function

$$c(t, S) = \bar{c}(t) + \gamma(t, S)\left((1 - \eta)S \vee \bar{R}\right)$$

$\bar{c}$ Nominal coupon rate function

$\bar{R}$ Nominal recovery on the bond upon default

Discounting

$\mu(t, S) = r(t) + \gamma(t, S)$ Credit-risk adjusted interest rate

$\beta_t = e^{-\int_0^t \mu(s, S_s)ds}$ Risk-neutral credit-risk adjusted discount factor

General Conditions for the Numerical Experiments

General Data

$\bar{P}$	$\bar{N}$	$\bar{C}$	η	σ	r	q	γ_0	α	m
0	100	103	1	0.2	0.05	0	0.02	1.2	10^4

m number of Monte Carlo trajectories

Time-step $t_{i+1} - t_i = h$

six hours (four time steps per day) in the case of simulation methods

one day in the case of deterministic schemes

Space-steps in the S variable

$S^{j+1} - S^j = 0.5$ in the case of the (fully implicit) deterministic schemes

Cells of diameter one (segments of length one) in the case of simulation/regression methods involving a method of cells in S

No Call Protection

Standard Game Option
$\vartheta_1 = 0, \vartheta_2 = T$

Simulated mesh $(S_i^j)_{0\le i\le n}^{1\le j\le m} \to$ Estimate $(u_i^j) = u(t_i, S_i^j)_{0\le i\le n}^{1\le j\le m}$

$u_n = g$, then for $i = n-1\ldots 0$, for $j = 1\ldots m$,

$$u_i^j = \min\left(h_i(S_i^j)\,,\ \max\left(\ell_i(S_i^j),\ e^{-\mu_i^j h}\mathbb{E}_i^j(u_{i+1} + hc_{i+1})\right)\right)$$

$\mathbb{E}_i^j(u_{i+1} + hc_{i+1})$ Conditional expectation given $t = t_i, S_i = S_i^j$

- Computed by non-linear regression of $(u_{i+1} + hc_{i+1})_{1\le j\le m}$ against $(S_i)_{1\le j\le m}$, using a global parametric regression basis $1, S, S^2$ in S

Regression estimate of the delta

$$\delta_i^j = \frac{\mathbb{E}_i^j\left\{u_{i+1}(W_{i+1} - W_i)\right\}}{\sigma_i(S_i^j)S_i^j h}$$

Alternative MC forward estimates of price and delta at time 0

Backward vs Forward MC

Maturity $T = 125$ days, Nominal coupon rate $\bar{c} = 0$

MC Fd less volatile than MC Bd (Deviations over 50 trials, $S_0 = 100.55$)

	Value VI	Dev MC Bd	Dev MC Fd
Price	102.049	0.821	0.010
Delta	0.416	0.071	0.019

MC Fd more accurate than MC Bd (%Err=1 $\leftrightarrow$ relative difference of 1% between MC and VI)

S_0	VI Price	%Err Bd	%Err Fd	VI delta	%Err Bd	%Err Fd
98.55	101.246	1.90	0.04	0.376	1.07	0.07
99.55	101.637	1.92	0.01	0.396	0.95	0.50
100.55	102.049	1.99	0.01	0.416	2.77	0.67
101.55	102.479	1.65	0.07	0.435	3.97	3.47

In the sequel always use MC forward estimates

Call Protection

Non-decreasing sequence of $[0, T]$-valued stopping times $\vartheta = (\vartheta_l)_{l \geq 0}$
Effective call payoff process

$$U_t = \Omega_t^c \infty + \Omega_t h(t, X_t) = U(t, S_t, H_t)$$

$\Omega_t = \mathbf{1}_{\{H_t \notin K\}} = \mathbf{1}_{\{l_t \text{ odd}\}}$ with $\vartheta_{l_t} \leq t < \vartheta_{l_t+1}$

Simulated mesh $(S_i^j, H_i^j)_{0 \leq i \leq n}^{1 \leq j \leq m} \to$ Estimate $(u_i^j) = u(t_i, S_i^j, H_i^j)_{0 \leq i \leq n}^{1 \leq j \leq m}$

$u_n = g$, then for $i = n-1 \ldots 0$, for $j = 1 \ldots m$

$$u_i^j = \min\left(U_i\left(S_i^j, H_i^j\right), \max\left(\ell_i\left(S_i^j\right), e^{-rh}\mathbb{E}_i^j(u_{i+1} + hc_{i+1})\right)\right)$$

min plays no role outside the support of U_i, where $U_i(S, H)$ is equal to $+\infty$

$\mathbb{E}_i^j(u_{i+1} + hc_{i+1})$ Conditional expectation given $t = t_i, S_i = S_i^j,\ H_i = H_i^j$

- computed by non-linear regression of $(u_{i+1} + hc_{i+1})_{1 \le j \le m}$ against $(S_i, H_i)_{1 \le j \le m}$, using for example a method of cells in (S, H)

Numerical Data

'l out of d' with $\bar{S} = 103$
Maturity $T = 180$ days, Nominal coupon rate $\bar{c} = 1.2$/month
Other data unchanged

Reducible Case

In case $l = d$ one can reduce the problem to two space dimensions instead of $d+1$

- S and the number N of consecutive monitoring dates T_ls with $S_{T_l} \geq \bar{S}$ from time t backwards (capped at l)

Two simulation schemes

MC_d a method of cells in (S, H)

MC^1 a method of cells in (S, N)

MC_d more accurate then MC^1 ($S_0 = 100$)

l	1	5	10	20	30
VI^1 price	103.91	105.10	106.03	107.22	108.01
MC^1 %Err	0.04	0.16	0.47	0.88	1.34
MC_d %Err	0.04	0.15	0.03	0.04	0.24

General Case

Relative Errors and Computation Times (MC vs VIs) for various '0 out of d' examples

d	1	5	10	20	30
Err $MC_{0,d}$	0.05%	0.05%	0.01%	0.05%	0.05%
CPU $MC_{0,d}$	0.5s	0.6s	0.9s	1.4s	1.9s
CPU $VI_{0,d}$	1.0s	16.1s	465s	—	—

Will use two methods for the computation of the conditional expectations in MC_d:

MC_d a method of cells in (S, H),

$MC_d^\sharp$ a method of cells in $(S, |H|^\sharp)$

Approximate $MC_d^\sharp$ Algorithm

$|H|^\sharp$ number of ones in H starting from the $(l-|H|)^{th}$ zero

- $|H|^\sharp = N$ in case $l = d$

Example ($d = 10$, $l = 8$)

- $H = (1, 1, 1, 1, \mathbf{0}, 1, 1, 1, 0, 0)$
  $l - |H| = 8 - 7 = 1$, $|H|^\sharp = 3$ (number of ones on the right of the first zero, in bold in H),
- $H = (1, 1, 1, 0, 1, 1, 1, \mathbf{0}, 0, 0)$ $l - |H| = 8 - 6 = 2$, $|H|^\sharp = 0$ (number of ones on the right of the second zero, in bold in H)

Rationale Entries of H preceding its $(l-|H|)^{th}$ zero irrelevant to the price

- Necessarily superseded by new ones before the bond may become callable
- Approximate algorithm $\sim$ reducible case based on the 'good regressor' $|H|^\sharp$ for estimating highly path-dependent conditional expectations

MC_d good , $\mathrm{MC}_d^\sharp$ 'rather good' ($d = 5, S_0 = 100$)

l	2	3	5
VI_d price	104.07	104.43	105.10
MC_d %Err	0.21	0.15	0.15
$\mathrm{MC}_d^\sharp$ %Err	0.19	0.23	0.18

MC_d good , $\mathrm{MC}_d^\sharp$ 'OK' ($d = 10, S_0 = 100$)

l	2	5	10
VI_d price	104.27	104.87	106.03
MC_d %Err	0.01	0.15	0.03
$\mathrm{MC}_d^\sharp$ %Err	0.04	0.26	0.38

Deviations over 50 trials and relative difference ($d = 30, S_0 = 102.55$)

l	5	10	20	30
Dev MC_d	0.056	0.061	0.086	0.152
Dev $MC_d^\sharp$	0.060	0.069	0.092	0.175
% Err	0.09	0.24	0.72	1.06

'Good regressor' algorithm $MC_d^\sharp$ rather accurate in practice
Ability to work with a 'good' (as opposed to exact), low-dimensional regressor

- An interesting feature of simulation as opposed to deterministic numerical schemes

Contents

Markov Families Embedding

- Everything implicitly parameterized by the initial condition $(t=0,x,k)$ of $\mathcal{X}$
- Superscript ^{t} in reference to an initial condition (t,x,k) of $\mathcal{X}$ (with in particular $t \in [0,T]$ rather than $t=0$ implicitly above)
    - $Y^t = Y^{t,l}$ on $[\vartheta_l^t, \vartheta_{l+1}^t)$, and in particular

$$Y_t^t = \begin{cases} Y_t^{0,t}, & k \in K \\ Y_t^{1,t}, & k \notin K \end{cases}$$

    -

$$\begin{cases} Y_{\vartheta_1^t}^{0,t} = Y_{\vartheta_1^t}^{1,t} \\ Y_{\vartheta_2^t}^{1,t} = \min\left(Y_{\vartheta_2^t}^{2,t}, h(\vartheta_2^t, X_{\vartheta_2^t}^t)\right) \end{cases}$$

Markovianity of Y

Standard semi-group properties of $\mathcal{X}$ and $\mathcal{Y}$ (SDEs uniqueness results) yield, for every $I = 1, \ldots, N$ and $T_{I-1} \le t \le r < T_I$,

$$Y^t_r = u_I(r, \mathcal{X}^t_r)\,,\ \mathbb{Q}\text{−}a.s.$$

for a deterministic function u_I on $\mathcal{E}^*_I$. In particular,

$$Y^t_t = u^k(t, x)\ ,\ \text{for any } (t, x, k) \in \mathcal{E}$$

where u is the function defined on $\mathcal{E}$ by the concatenation of the u_Is and the terminal condition g at T.

Continuity of the u^ks outside $\mathfrak{T} \times \partial\mathcal{O}$

Case $t \notin \mathfrak{T}$

- $u^k(t,x)$ identified 'in the vincinity of (t,x)' to
  - $Y_t^{0,t}$ if $k \in K$ (no call at the beginning)
  - $Y_t^{1,t}$ if $k \notin K$ (call at the beginning)
- $+$ stability estimates on the $Y^{t,l}$'s $\rightarrow$ u^k continuous at (t,x)

Same arguments also show that u^k is 'cad' at every $(t = T_I, x)$
Remains to show that

- the u_Is can be extended by continuity over the $\mathcal{E}_I$s, except maybe at the 'boundary points' $(T_I, x \in \partial\mathcal{O}, k)$
- the cascade jump condition is satisfied

'Left-continuity' of the u_Is and Jump Condition on $\mathfrak{T}$

Given $\mathcal{E}_I^* \ni (t_n, x_n, k) \to (t = T_I, x, k)$ with $x \notin \partial\mathcal{O}$

- Needs to show that $u_I^k(t_n, x_n) = u^k(t_n, x_n) \to u_I^k(T_I, x)$, with $u_I^k(T_I, x)$ defined by the jump condition
- Note ϑ 'cadlag' at $(t = T_I, x)$
    - $\vartheta^{t_n} \to \widetilde{\vartheta}^t$ as $n \to \infty$, where '$\widetilde{H}^t$ may jump at $t = T_I$'
- Intuition '$\widetilde{\mathcal{Y}} = \mathcal{Y} \circ \kappa$', and so '$\lim_{n\infty} u^k(t_n, x_n) = u(t, x, \kappa_I^k(x))$'
- Obviously misses some point since, in case for instance $k \notin K$ and $\kappa_I^k(x) \in K$, one has $\lim_{n\infty} u^k(t_n, x_n) \leq h(t, x)$, whereas $u(t, x, \kappa_I^k(x))$ may be greater than $h(t, x)$.

In fact in case $k \notin K$ and $\kappa_I^k(x) \in K$ one has consistently with the jump condition that $\lim_{n\infty} u^k(t_n, x_n) = \min\left(u(t, x, \kappa_I^k(x)), h(t, x)\right)$, as we now prove. The other three cases can be proven similarly.

Denoting $\widetilde{u}^j(s,y) = \min\left(u(s,y,\kappa_l^j(y)), h(s,y)\right)$ and $\widehat{u}^j(s,y) = \min\left(u^j(s,y), h(s,y)\right)$

$$|\widetilde{u}^k(t,x) - u^k(t_n,x_n)|^2 = |\widetilde{u}^k(t,x) - Y^{1,t_n}_{t_n}|^2 \leq$$
$$2\mathbb{E}|\widetilde{u}^k(t,x) - \widehat{u}(t,\mathcal{X}^{t_n}_t)|^2 + 2|\mathbb{E}(\widehat{u}(t,\mathcal{X}^{t_n}_t) - Y^{1,t_n}_{t_n})|^2$$

- $(t,\mathcal{X}^{t_n}_t) \in \mathcal{E}^*_{l+1}$ and 'close to $(t,x,\kappa^k_l(x))$' $\rightarrow$ first term goes to 0 by continuity of u already established over $\mathcal{E}^*_{l+1}$
- $u(t,\mathcal{X}^{t_n}_t) = Y^{t_n}_t$ and $t \sim \vartheta^{t_n}_2$ so

$$\widehat{u}(t,\mathcal{X}^{t_n}_t) = \min\left(u(\vartheta^{t_n}_2, \mathcal{X}^{t_n}_{\vartheta^{t_n}_2}), h(\vartheta^{t_n}_2, X^{t_n}_{\vartheta^{t_n}_2})\right)$$
$$= \min\left(Y^{2,t_n}_{\vartheta^{t_n}_2}, h(\vartheta^{t_n}_2, X^{t_n}_{\vartheta^{t_n}_2})\right) = Y^{1,t_n}_{\vartheta^{t_n}_2} = Y^{1,t_n}_t$$

$\rightarrow$ second term goes to zero by $(\mathcal{E}^{t_n})$ and convergence of $\mathcal{Y}_{t_n}$ (to $\widetilde{\mathcal{Y}}_t$)

Contents

Non-linear Regressions

Non-linear simulation/regression approaches for computing by regression functions (conditional expectations)

$$x \mapsto \rho(x) = \mathbb{E}(\xi | X = x)$$

ξ, X Real- and $\mathbb{R}^q$-valued square integrable random variables

Pairs $(X^j, \xi^j)^{1 \le j \le m}$ simulated independently according to the law of $(X, \xi) \to$ Estimate the conditional expectation $\mathbb{E}(\xi|X)$

Linear regression of the ξ^js against the $(\varphi^l(X^j))^{1 \le j \le m}_{1 \le l \le p}$, where (φ^l) is a well chosen 'basis' of functions from $\mathbb{R}^q$ to $\mathbb{R}$

Regression basis

- parametric vs non-parametric
- global vs local

Typically

- parametric and global
  - few monomials parameterized by their coefficients
- or non-parametric and local
  - indicator functions of the cells of a grid of hyperrectangles partitioning the state space

Preferred

- Global basis in case of a ‘regular’ regression function $\rho(x)$
  - Case where a good guess is available as for the shape, used to define the regression basis, of ρ
- Local basis otherwise
  - Often simpler and more robust in terms of implementation