An approach to fully coupled FBSDEs via functional differential equations

Matteo Casserini*
joint work with Gechun Liang†

*Department of Mathematics
ETH Zürich

†Oxford-Man Institute

Workshop "New advances in BSDEs for financial engineering applications", Tamerza
October 26, 2010

Outline

Outline

Introduction

Aim: Introduce a forward approach for a general class of fully coupled FBSDEs

Result: System of forward equations where the coefficients depend also on the terminal values of the solution

- Conflict between forward and backward components partly avoided
- Purely probabilistic (random coefficients)
- Allows to treat other types of non–classical forward–backward equations

Motivating observation

- $(Y_t)_{0\le t\le T}$ a semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ with known terminal value $Y_T = \xi \in L^1(\mathcal{F}_T)$.
- Doob-Meyer decomposition:

$$Y_t = M_t - V_t,$$

M martingale, V cont. adapted process of finite variation.

- If V_T is integrable, then:

$$M_t = M(V,\xi)_t = E[\xi + V_T|\mathcal{F}_t] \quad \forall\ t \in [0,T],$$
$$Y_t = Y(V,\xi)_t = E[\xi + V_T|\mathcal{F}_t] - V_t \quad \forall\ t \in [0,T]. \quad (1.1)$$

Motivating observation

- $(Y_t)_{0\leq t\leq T}$ a semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ with known terminal value $Y_T = \xi \in L^1(\mathcal{F}_T)$.
- Doob-Meyer decomposition:

$$Y_t = M_t - V_t,$$

M martingale, V cont. adapted process of finite variation.

- If V_T is integrable, then:

$$M_t = M(V,\xi)_t = E[\xi + V_T|\mathcal{F}_t] \quad \forall\, t \in [0,T],$$
$$Y_t = Y(V,\xi)_t = E[\xi + V_T|\mathcal{F}_t] - V_t \quad \forall\, t \in [0,T]. \quad (1.1)$$

Motivating observation

- $(Y_t)_{0\leq t\leq T}$ a semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ with known terminal value $Y_T = \xi \in L^1(\mathcal{F}_T)$.
- Doob-Meyer decomposition:

$$Y_t = M_t - V_t,$$

M martingale, V cont. adapted process of finite variation.

- If V_T is integrable, then:

$$\begin{aligned} M_t &= M(V,\xi)_t = E[\xi + V_T|\mathcal{F}_t] \quad \forall\ t \in [0,T], \\ Y_t &= Y(V,\xi)_t = E[\xi + V_T|\mathcal{F}_t] - V_t \quad \forall\ t \in [0,T]. \end{aligned} \tag{1.1}$$

Formally: alternative formulation of Brownian FBSDEs

- Probability space $(\Omega, \mathcal{F}, P)$ with a m-dim. BM W
  $(\mathcal{F}_t)_{0 \leq t \leq T}$ corresponding augmented filtration
- Classical fully coupled FBSDE of the form

$$\begin{cases} dY_t &= -f(t, X_t, Y_t, Z_t)dt + Z_t dW_t, \quad Y_T = \Phi(X_T), \\ dX_t &= \mu(t, X_t, Y_t, Z_t)dt + \sigma(t, X_t, Y_t)dW_t, \quad X_0 = x, \end{cases} \tag{1.2}$$

where $f: \ \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^{d \times m} \to \mathbb{R}^d$,
$\mu: \ \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^{d \times m} \to \mathbb{R}^n$,
$\sigma: \ \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^{n \times m}$, $\Phi: \ \Omega \times \mathbb{R}^n \to \mathbb{R}^d$.

Formally: alternative formulation of Brownian FBSDEs

- Probability space $(\Omega, \mathcal{F}, P)$ with a m-dim. BM W
  $(\mathcal{F}_t)_{0\leq t\leq T}$ corresponding augmented filtration
- Classical fully coupled FBSDE of the form

$$\begin{cases} dY_t &= -f(t, X_t, Y_t, Z_t)dt + Z_t dW_t, \quad Y_T = \Phi(X_T), \\ dX_t &= \mu(t, X_t, Y_t, Z_t)dt + \sigma(t, X_t, Y_t)dW_t, \quad X_0 = x, \end{cases} \tag{1.2}$$

where $f: \ \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^{d\times m} \to \mathbb{R}^d$,
$\mu: \ \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^{d\times m} \to \mathbb{R}^n$,
$\sigma: \ \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^{n\times m}$, $\Phi: \ \Omega \times \mathbb{R}^n \to \mathbb{R}^d$.

Formally: alternative formulation of Brownian FBSDEs

Define an associated system of functional differential equations:

$$\begin{cases} dV_t = f(t, X_t, Y(V,X)_t, Z(V,X)_t)dt, \\ dX_t = \mu(t, X_t, Y(V,X)_t, Z(V,X)_t)dt + \sigma(t, X_t, Y(V,X)_t)dW_t \end{cases} \tag{1.3}$$

with initial conditions $V_0 = 0$, $X_0 = x$, where

$$\begin{aligned} M(V,X)_t &:= E[\Phi(X_T) + V_T|\mathcal{F}_t], \\ Y(V,X)_t &:= E[\Phi(X_T) + V_T|\mathcal{F}_t] - V_t, \\ Z(V,X)_t &:= D_t M(V,X)_T = D_t(\Phi(X_T) + V_T) \quad \forall\ t \in [0,T]. \end{aligned} \tag{1.4}$$

Outline

Setting

- $(\Omega, \mathcal{F}, P)$ probability space with a m-dim. BM W, $(\mathcal{F}_t)_{0\leq t\leq T}$ with usual assumptions

- $$\mathcal{C}([0,T],\mathbb{R}^d) := \{V : \Omega \times [0,T] \to \mathbb{R}^d | V \text{ continuous and adapted, } E[\max_j \sup_t |V_t^j|^2] < \infty\}$$

- $\mathcal{C}_0([0,T],\mathbb{R}^d) := \mathcal{C}([0,T],\mathbb{R}^d) \cap \{V | V_0 = 0\}$

- $$\mathcal{M}^2([0,T],\mathbb{R}^d) := \{M : \Omega \times [0,T] \to \mathbb{R}^d | M \text{ square integrable martingale on } [0,T]\}$$

Setting

- $\|V\|_{\mathcal{C}[0,T]} := \sqrt{E[\sup_{0\leq t\leq T}|V_t|^2]}$
- $\mathcal{S}([0,T],\mathbb{R}^d) := \mathcal{C}([0,T],\mathbb{R}^d) \oplus \mathcal{M}^2([0,T],\mathbb{R}^d)$
- $\mathcal{H}^2([0,T],\mathbb{R}^p) := \{Z : \Omega \times [0,T] \to \mathbb{R}^p | Z$ predictable, $\|Z\|^2_{\mathcal{H}^2[0,T]} := E[\int_0^T |Z_t|^2 dt] < \infty\}$

Fully coupled forward–backward stochastic dynamics

- General filtration $\Rightarrow$ No martingale representation!
  $\Rightarrow$ Substitute Z by $L(M)$, where L nonlinear functional mapping $\mathcal{M}^2([0,T],\mathbb{R}^d)$ into p–dim. adapted processes
- This leads us to the following generalization of (1.2):

$$\begin{cases} dY_t = -f(t, X_t, Y_t, L(M)_t)dt + dM_t, \quad Y_T = \Phi(X_T), \\ dX_t = \mu(t, X_t, Y_t, L(M)_t)dt + \sigma(t, X_t, Y_t)dW_t, \quad X_0 = x. \end{cases} \tag{2.1}$$

A solution to (2.1) is then a triplet of adapted processes (X, Y, M) satisfying the integral formulation of (2.1) and such that M is a square-integrable martingale.

Fully coupled forward–backward stochastic dynamics

- General filtration $\Rightarrow$ No martingale representation!
  $\Rightarrow$ Substitute Z by $L(M)$, where L nonlinear functional mapping $\mathcal{M}^2([0,T],\mathbb{R}^d)$ into p–dim. adapted processes
- This leads us to the following generalization of (1.2):

$$\begin{cases} dY_t = -f(t, X_t, Y_t, L(M)_t)dt + dM_t, \quad Y_T = \Phi(X_T), \\ dX_t = \mu(t, X_t, Y_t, L(M)_t)dt + \sigma(t, X_t, Y_t)dW_t, \quad X_0 = x. \end{cases} \tag{2.1}$$

A solution to (2.1) is then a triplet of adapted processes (X, Y, M) satisfying the integral formulation of (2.1) and such that M is a square-integrable martingale.

Fully coupled forward–backward stochastic dynamics

- General filtration $\Rightarrow$ No martingale representation!
  $\Rightarrow$ Substitute Z by $L(M)$, where L nonlinear functional mapping $\mathcal{M}^2([0,T],\mathbb{R}^d)$ into p–dim. adapted processes
- This leads us to the following generalization of (1.2):

$$\begin{cases} dY_t = -f(t, X_t, Y_t, L(M)_t)dt + dM_t, \quad Y_T = \Phi(X_T), \\ dX_t = \mu(t, X_t, Y_t, L(M)_t)dt + \sigma(t, X_t, Y_t)dW_t, \quad X_0 = x. \end{cases} \tag{2.1}$$

A solution to (2.1) is then a triplet of adapted processes (X, Y, M) satisfying the integral formulation of (2.1) and such that M is a square-integrable martingale.

Fully coupled forward–backward stochastic dynamics

Reduce the problem (2.1) to a system of functional differential equations:

$$\begin{cases} dV_t = f(t, X_t, Y(V,X)_t, L(M(V,X))_t)dt, \\ dX_t = \mu(t, X_t, Y(V,X)_t, L(M(V,X))_t)dt + \sigma(t, X_t, Y(V,X)_t)dW_t, \end{cases} \tag{2.2}$$

with initial conditions $V_0 = 0$, $X_0 = x$.

Then: if (V, X) solves (2.2), $(X, Y(V,X), M(V,X))$ solves (2.1).

Outline

1. Brownian FBSDEs as functional differential equations
2. Fully coupled forward–backward stochastic dynamics
3. Existence and uniqueness of solutions
4. Related discretization algorithms for Brownian FBSDEs

Local existence and uniqueness

- Derive sufficient conditions on the coefficients and on L to guarantee existence and uniqueness of solutions
- For short time intervals: existence and uniqueness under weak assumptions on L
  $\Rightarrow$ Possibility to treat other types of functionals L not fitting in the classical framework

Local existence and uniqueness

- Derive sufficient conditions on the coefficients and on L to guarantee existence and uniqueness of solutions

- For short time intervals: existence and uniqueness under weak assumptions on L
  $\Rightarrow$ Possibility to treat other types of functionals L not fitting in the classical framework

Assumptions

Assumption (A1)

The coefficients f, μ, σ and Φ satisfy Assumption **(A1)** if there exists a constant $K > 0$ such that:

(A1.1) For any $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^p$, $f(\cdot, x, y, z)$, $\mu(\cdot, x, y, z)$ and $\sigma(\cdot, x, y)$ are $\mathbb{F}$-adapted and $\Phi(\cdot, x)$ is $\mathbb{F}_T$-measurable.

(A1.2) For every $t \in [0, T]$, $(x, y, z), (x', y', z') \in \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^p$,

$$|f(t, x, y, z) - f(t, x', y', z')| \leq K(|x - x'| + |y - y'| + |z - z'|),$$
$$|\Phi(x) - \Phi(x')| \leq K|x - x'|,$$
$$|\mu(t, x, y, z) - \mu(t, x, y', z')| \leq K(|y - y'| + |z - z'|),$$
$$|\sigma(t, x, y) - \sigma(t, x', y')|^2 \leq K(|x - x'|^2 + |y - y'|^2).$$

Assumptions

Assumption (A1)

(A1.3) For every $t \in [0, T]$, $(y, z) \in \mathbb{R}^d \times \mathbb{R}^p$, $x, x' \in \mathbb{R}^n$,

$$(x - x')^{\mathrm{T}}(\mu(t, x, y, z) - \mu(t, x', y, z)) \leq K|x - x'|^2.$$

(A1.4) For every $t \in [0, T]$, $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^p$,

$$\begin{aligned} |f(t, x, y, z)| &\leq K(1 + |x| + |y| + |z|), \\ |\Phi(x)| &\leq K(1 + |x|), \\ |\mu(t, x, y, z)| &\leq K(1 + |x| + |y| + |z|), \\ |\sigma(t, x, y)| &\leq K(1 + |x| + |y|). \end{aligned}$$

(A1.5) The functions $u \mapsto \mu(t, u, y, z)$ is continuous for all $t \in [0, T]$, $(y, z) \in \mathbb{R}^d \times \mathbb{R}^p$.

Assumptions

Assumption (A2)

The functional L satisfies Assumption **(A2)** if there exists a constant $K > 0$ such that:

(A2.1) L maps $\mathcal{M}^2([0,T],\mathbb{R}^d)$ into $\mathcal{O}([0,T],\mathbb{R}^p)$, where $\mathcal{O}([0,T],\mathbb{R}^p) \in \{\mathcal{H}^2([0,T],\mathbb{R}^p), \mathcal{C}([0,T],\mathbb{R}^p)\}$.

(A2.2) L is bounded and Lipschitz continuous, i.e.

$$\|L(M)\|_{\mathcal{O}[0,T]} \leq K\|M\|_{\mathcal{C}[0,T]},$$
$$\|L(M) - L(M')\|_{\mathcal{O}[0,T]} \leq K\|M - M'\|_{\mathcal{C}[0,T]} \quad \forall\ M, M' \in \mathcal{M}^2.$$

Examples for L

Example (1)

- $(\mathcal{F}_t)_{0\le t\le T}$ *augmented filtration generated by* W
- *Choose* $\mathcal{O}([0,T],\mathbb{R}^p) = \mathcal{H}^2([0,T],\mathbb{R}^{d\times m})$
- $L : \mathcal{M}^2([0,T],\mathbb{R}^d) \to \mathcal{H}^2([0,T],\mathbb{R}^{d\times m})$ *defined via the Itô representation theorem, i.e.*

$$M_t^i = E[M_t^i] + \sum_{j=1}^{m} \int_0^t L(M)_s^{i,j} dW_s^j, \quad i = 1,\dots,d.$$

- *Classical fully coupled FBSDEs* $(L(M(X,V)) = Z(X,V))$

Examples for L

Example (2)

- $(\mathcal{F}_t)_{0\leq t\leq T}$ *with usual assumptions*
- *Choose* $\mathcal{O}([0,T],\mathbb{R}^p)=\mathcal{H}^2([0,T],\mathbb{R}^{d\times m})$
- $L:\mathcal{M}^2([0,T],\mathbb{R}^d)\to\mathcal{H}^2([0,T],\mathbb{R}^{d\times m})$ *given by the integrand process in the orthogonal decomposition w.r.t.* W, *i.e.*

$$M_t^i=E[M_t^i]+\sum_{j=1}^m\int_0^t L(M)_s^{i,j}dW_s^j+(M')_t^i,\quad i=1,\dots,d.$$

Examples for L

Example (3)

- $(\mathcal{F}_t)_{0\le t\le T}$ *quasi-left continuous*
- *For* $M \in \mathcal{M}^2([0,T],\mathbb{R})$ *consider the decomposition*

$$M = M^c + M^d$$

M^c *continuous martingale null at 0,* M^d *purely discontinuous martingale*

- *Choose* $\mathcal{O}([0,T],\mathbb{R}^p) = \mathcal{C}([0,T],\mathbb{R}^d)$.
$L : \mathcal{M}^2([0,T],\mathbb{R}^d) \to \mathcal{C}([0,T],\mathbb{R}^d)$ *defined by*

$$L(M)^i_t := \sqrt{\langle (M^c)^i, (M^c)^i \rangle_t}, \quad i = 1, \ldots, d.$$

Existence of local solutions

Theorem

Under the assumptions **(A1)** *and* **(A2)** *there is a constant* τ_K *so that, for* $T \leq \tau_K$, (2.2) *admits a unique solution* (X, V) *satisfying*

$$\|X\|_{\mathcal{C}[0,T]} + \|V\|_{\mathcal{C}[0,T]} < \infty.$$

Moreover, the solution processes V *and* X *are continuous.*

Sketch of proof

Sketch of proof

Define the mapping $\mathbb{L} : \mathcal{C}([0,T],\mathbb{R}^n) \times \mathcal{C}([0,T],\mathbb{R}^d) \to \mathcal{C}([0,T],\mathbb{R}^n) \times \mathcal{C}([0,T],\mathbb{R}^d)$ by $\mathbb{L}(X,V) := (\widetilde{X},\widetilde{V})$, where $\widetilde{X}$ solution of the forward SDE

$$\begin{cases} \widetilde{X}_0 = x, \\ d\widetilde{X}_t = \mu(t,\widetilde{X}_t, Y(V,X)_t, L(M(V,X))_t)dt + \sigma(t,\widetilde{X}_t, Y(V,X)_t)dW_t, \end{cases}$$

whereas $\widetilde{V}$ is explicitly given by

$$\widetilde{V}_t = \int_0^t f(s,\widetilde{X}_s, Y(V,X)_s, L(M(V,X))_s)ds.$$

(X,V) solves (2.2) if and only if it is a fixed point of $\mathbb{L}$.

Global solution

- Extension of the local solutions to global ones: still work in progress
- The study of the simple decoupled case suggests that additional assumptions on L are needed!
- For $[T_2, T_1] \subset [0, T]$, define the restriction $L_{[T_2,T_1]}$ from $\mathcal{M}^2([T_2, T_1], \mathbb{R}^d)$ to $\mathcal{O}([T_2, T_1], \mathbb{R}^m)$ by

$$L_{[T_2,T_1]}(N)_t := L(\widetilde{N})_t, \quad N \in \mathcal{M}^2([T_2, T_1], \mathbb{R}^d),$$

where $\widetilde{N}_t := E[N_{T_1}|\mathcal{F}_t]$, $t \in [0, T]$, is the extension of N to $\mathcal{M}^2([0, T], \mathbb{R}^d)$.

Global solution

- Extension of the local solutions to global ones: still work in progress
- The study of the simple decoupled case suggests that additional assumptions on L are needed!
- For $[T_2, T_1] \subset [0, T]$, define the restriction $L_{[T_2,T_1]}$ from $\mathcal{M}^2([T_2, T_1], \mathbb{R}^d)$ to $\mathcal{O}([T_2, T_1], \mathbb{R}^m)$ by

$$L_{[T_2,T_1]}(N)_t := L(\widetilde{N})_t, \quad N \in \mathcal{M}^2([T_2, T_1], \mathbb{R}^d),$$

where $\widetilde{N}_t := E[N_{T_1}|\mathcal{F}_t]$, $t \in [0, T]$, is the extension of N to $\mathcal{M}^2([0, T], \mathbb{R}^d)$.

Global solution

- Extension of the local solutions to global ones: still work in progress
- The study of the simple decoupled case suggests that additional assumptions on L are needed!
- For $[T_2, T_1] \subset [0, T]$, define the restriction $L_{[T_2,T_1]}$ from $\mathcal{M}^2([T_2, T_1], \mathbb{R}^d)$ to $\mathcal{O}([T_2, T_1], \mathbb{R}^m)$ by

$$L_{[T_2,T_1]}(N)_t := L(\widetilde{N})_t, \quad N \in \mathcal{M}^2([T_2, T_1], \mathbb{R}^d),$$

where $\widetilde{N}_t := E[N_{T_1}|\mathcal{F}_t]$, $t \in [0, T]$, is the extension of N to $\mathcal{M}^2([0, T], \mathbb{R}^d)$.

Global solution

Assumption (A2')

We say that L satisfies **(A2')** if it satisfies **(A2)** as well as

(A2.3) (Local-in-time property) For $0 \leq T_2 < T_1 \leq T$ and $M \in \mathcal{M}^2([0, T], \mathbb{R}^d)$,

$$L(M) = L_{[T_2, T_1]}(\widehat{M}) \text{ on } (T_2, T_1), \quad \text{where } \widehat{M} = M\big|_{[T_2, T_1]}.$$

(A2.4) (Differential property) For $0 \leq T_2 < T_1 \leq T$ and $N \in \mathcal{M}^2([T_2, T_1], \mathbb{R}^d)$,

$$L_{[T_2, T_1]}(N - N_{T_2}) = L_{[T_2, T_1]}(N) \text{ on } (T_2, T_1).$$

Global solution

- **Main idea:** Derive some uniform estimates for the solution over short time intervals, extend the solution to any time interval while still keeping that estimate.
- Well known from classical theory ([Delarue], [Zhang]): additional assumptions on the coefficients are needed.
- It can be proven that, under the assumption (A2') on L and under the same assumptions on the coefficients as in [Delarue] or [Zhang], the system (2.2) has a unique solution on any time interval

Global solution

- **Main idea:** Derive some uniform estimates for the solution over short time intervals, extend the solution to any time interval while still keeping that estimate.
- Well known from classical theory ([Delarue], [Zhang]): additional assumptions on the coefficients are needed.
- It can be proven that, under the assumption (A2') on L and under the same assumptions on the coefficients as in [Delarue] or [Zhang], the system (2.2) has a unique solution on any time interval

Global solution

- **Main idea:** Derive some uniform estimates for the solution over short time intervals, extend the solution to any time interval while still keeping that estimate.
- Well known from classical theory ([Delarue], [Zhang]): additional assumptions on the coefficients are needed.
- It can be proven that, under the assumption (A2') on L and under the same assumptions on the coefficients as in [Delarue] or [Zhang], the system (2.2) has a unique solution on any time interval

Outline

Brownian FBSDEs

The functional differential equation approach and the related contraction mapping opens the door to a new class of discretization algorithms.
Assume we have a classical FBSDE in a Brownian filtration:

$$\begin{cases} dY_t & = -f(t, X_t, Y_t, Z_t)dt + Z_t dW_t, \quad Y_T = \Phi(X_T), \\ dX_t & = \mu(t, X_t, Y_t, Z_t)dt + \sigma(t, X_t, Y_t)dW_t, \quad X_0 = x. \end{cases}$$

$$\Leftrightarrow \begin{cases} dV_t = f(t, X_t, Y(V, X)_t, Z(V, X)_t)dt, \\ dX_t = \mu(t, X_t, Y(V, X)_t, Z(V, X)_t)dt + \sigma(t, X_t, Y(V, X)_t)dW_t. \end{cases}$$

Numerical approximation

$\pi = (t_0, \cdots, t_N)$ partition of $[0, T]$. For $p \in \mathbb{N}$, define $V^{\pi,p}$ and $X^{\pi,p}$ recursively on π by $V^{\pi,0} \equiv 0$, $X^{\pi,0} \equiv x$ and

$$
\begin{aligned}
X_0^{\pi,p+1} &= x, \quad V_0^{\pi,p+1} = 0,\\
X_{t_{i+1}}^{\pi,p+1} &= X_{t_i}^{\pi,p+1} + \mu(t_i, X_{t_i}^{\pi,p+1}, Y(V^{\pi,p}, X^{\pi,p})_{t_i}, Z(V^{\pi,p}, X^{\pi,p})_{t_i})\Delta t_i\\
&\qquad + \sigma(t_i, X_{t_i}^{\pi,p+1}, Y^{\pi,p}(V, X)_{t_i})(\Delta W_{t_i})^T,\\
V_{t_{i+1}}^{\pi,p+1} &= V_{t_i}^{\pi,p+1} + f(t_i, X_{t_i}^{\pi,p+1}, Y(V^{\pi,p}, X^{\pi,p})_{t_i}, Z(V^{\pi,p}, X^{\pi,p})_{t_i})\Delta t_i
\end{aligned}
$$

for $i = 0, \cdots, N-1$ and $p \geq 1$, where

$$
\begin{aligned}
Y(V^{\pi,p}, X^{\pi,p})_{t_i} &= E[\Phi(X_T^{\pi,p}) + V_T^{\pi,p} | \mathcal{F}_{t_i}] - V_{t_i}^{\pi,p},\\
Z(V^{\pi,p}, X^{\pi,p})_{t_i} &= \frac{1}{\Delta t_i} E\left[Y(V^{\pi,p}, X^{\pi,p})_{t_{i+1}} (\Delta W_{t_i})^T | \mathcal{F}_{t_i} \right].
\end{aligned}
$$

Numerical approximation

- Motivated by the continuous time results
- **Advantage:** Avoid the nesting of conditional expectations (arising in most numerical approaches to BSDEs), thus reducing the amplification of the error.
- Conjecture: the algorithm converges to the true solution of the FBSDE

Numerical approximation

- Motivated by the continuous time results
- **Advantage:** Avoid the nesting of conditional expectations (arising in most numerical approaches to BSDEs), thus reducing the amplification of the error.
- Conjecture: the algorithm converges to the true solution of the FBSDE

Numerical approximation

- Motivated by the continuous time results
- **Advantage:** Avoid the nesting of conditional expectations (arising in most numerical approaches to BSDEs), thus reducing the amplification of the error.
- Conjecture: the algorithm converges to the true solution of the FBSDE

The decoupled case

The convergence can easily be proved in the decoupled case:

Theorem

Assume that f, μ, σ and Φ are Lipschitz in the space variables and $1/2$-Hölder in the time variable. Then there is a constant C, depending only on the Lipschitz constants involved and the dimension of the problem, such that

$$\sup_{0\le t\le T} E[|V_t - V_t^{p,\pi}|^2] + \sup_{0\le t\le T} E[|X_t - X_t^{p,\pi}|^2] \le C\left(|\pi| + \left(\frac{1}{2} + C|\pi|\right)^p\right).$$

Some references

Christian Bender and Jianfeng Zhang.
Time discretization and Markovian iteration for coupled FBSDEs.
Ann. Appl. Probab., 18(1):143–177, 2008.

M. Casserini and G. Liang.
A functional differential equation approach to the numerical solution of BSDEs.
Working paper, 2010.

G. Liang, T. Lyons, and Z. Qian.
Backward stochastic dynamics on a filtered probability space.
Working paper, 2009.

Jianfeng Zhang.
The wellposedness of FBSDEs.
Discrete Contin. Dyn. Syst. Ser. B, 6(4):927–940 (electronic), 2006.

Thank you for your attention!