

# BSDEs With Rough Drivers

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New advances in Backward SDEs  
Tamerza

# Outline

BSDEs With Rough Drivers

Markovian Setting - Connection To Rough PDEs

Connection To BDSDEs

Classical BSDE theory concerns the solution of

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r) dr - \int_t^T Z_r dW_r.$$

We are interested in giving meaning to

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r) dr + \sum_{k=1}^d \int_t^T H_k(Y_r) d\zeta_r^k - \int_t^T Z_r dW_r.$$

When  $\zeta$  is smooth, this is just a Lipschitz BSDE

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## Short reminder on rough path theory

The solution mapping  $\zeta \mapsto x$  to the (deterministic) ODE

$$x_t = x_0 + \int_0^t H(x_r) d\zeta_r = x_0 + \int_0^t H(x_r) \dot{\zeta}_r dr,$$

defined for *smooth paths*  $\zeta$  is in general *not* continuous in supremum norm.

For  $p \geq 1$ , rough path theory

- ▶ embeds smooth paths  
 $\zeta \in C^\infty([0, T], \mathbb{R}^d) \subset C([0, T], G^{[p]}(\mathbb{R}^d)),$
- ▶ defines a metric  $d_{p\text{-var}}$  on this space ( $p$ -variation),
- ▶ can then handle the closure of all smooth paths with respect to this metric (*geometric  $p$ -rough paths*).

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## Short reminder on rough path theory - Continued

We can then solve *rough differential equations*

$$x_t = x_0 + \int_0^t H(x_r) d\zeta_r.$$

The theory yields (for smooth enough  $H$ )

- ▶ flow of diffeomorphisms
- ▶ stability of the flow, its inverse and derivatives



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## Back to BSDEs

We factorize the randomness of the vector field  $H$  through an Ito process, i.e. we consider

$$X_t = x + \int_0^t b_r dr + \int_0^t \sigma_r dW_r,$$
$$Y_t = \xi + \int_t^T f(\omega; r, Y_r, Z_r) dr + \int_t^T H(X_r, Y_r) d\zeta_r - \int_t^T Z_r dW_r.$$

Where

- ▶  $\xi \in L^\infty(\mathcal{F}_T)$ ,
- ▶  $\zeta$  is a  $p$ -rough path for a  $p \geq 1$ ,
- ▶  $b, \sigma$  are predictable and bounded,
- ▶  $f$  is predictable and Lipschitz in  $(y, z)$ ,
- ▶  $H = (H_1, \dots, H_d)$  has bounded derivatives up to order  $2 + p$ .

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## Theorem (Definition/Existence/Uniqueness of solution)

Assume  $\zeta^n \rightarrow \zeta$  in  $p$ -variation. Let  $(Y^n, Z^n)$  solve

$$Y_t^n = \xi + \int_t^T f(r, Y_r^n, Z_r^n) dr + \int_t^T H(X_r, Y_r^n) d\zeta_r^n - \int_t^T Z_r^n dW_r.$$

Then there exists a process  $(Y, Z)$  such that

$$Y^n \rightarrow Y, \quad Z^n \rightarrow Z.$$

It only depends on  $\zeta$  and not on the approximating sequence.

We write (formally)

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r) dr + \int_t^T H(X_r, Y_r) d\zeta_r - \int_t^T Z_r dW_r.$$

## Proof idea

- ▶ Define the (deterministic) flow

$$\phi^n(t, x, y) = y + \int_t^T H(x, \phi^n(r, x, y)) d\zeta_r^n.$$

- ▶  $\bar{Y}_t^n := [\phi^n]^{-1}(t, X_t, Y_t^n)$ ,  $\bar{Z}_t^n := \dots$  then satisfies the BSDE

$$\bar{Y}_t^n = \xi + \int_t^T \bar{f}^n(r, X_r, \bar{Y}_r^n, \bar{Z}_r^n) dr - \int_t^T \bar{Z}_r^n dW_r.$$

where  $\bar{f}^n$  is quadratic in  $z$  (e.g. if  $f \equiv 0$  then

$$\bar{f}^n(t, x, y, z) = \frac{\partial_{yy}\phi^n(t, x, y)}{\partial_y\phi^n(t, x, y)} |z|^2).$$

- ▶ Use standard rough path results to get  $\bar{f}^n \rightarrow \bar{f}$  and use comparison for BSDEs (Kobylanski '00) to show that  $\bar{Y}^n$  and  $\bar{Z}^n$  converge.

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Connection To BDSDEs



If  $\zeta$  is a smooth path, it is well known that the FBSDE

$$\begin{aligned} X_t^{s,x} &= x + \int_s^t \sigma(r, X_r^{s,x}) dW_r + \int_s^t b(r, X_r^{s,x}) dr, \\ Y_t^{s,x} &= g(X_T^{s,x}) + \int_t^T f(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x}) dr \\ &\quad + \int_t^T H(X_r^{s,x}, Y_r^{s,x}) d\zeta_r - \int_t^T Z_r^{s,x} dW_r. \end{aligned}$$

is connected (via  $u(t, x) := Y_t^{t,x}$ ) to the PDE

$$\begin{aligned} \partial_t u + Lu + f(t, x, u, Du \cdot \sigma(t, x)) + H(x, u) \dot{\zeta}_t &= 0, \\ u(T, x) &= g(x). \end{aligned}$$

where

$$Lu(t, x) := \frac{1}{2} \text{Tr}[\sigma(t, x)\sigma(t, x)^T D^2 u(t, x)] + \langle b(t, x), Du(t, x) \rangle$$

If  $\zeta^n \rightarrow \zeta$ , in  $p$ -variation, then by the previous section

$$u^n(t, x) \rightarrow u(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^n.$$

- ▶ could hence define a solution to a *rough PDE*
- ▶ but: not straightforward, via this approach, to show uniform convergence on compacta and continuity of the solution map
- ▶ hence work directly on PDE (Caruana/Friz/Oberhauser '09)

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## Theorem

Assume  $\zeta^n \rightarrow \zeta$  in  $p$ -variation. Let  $u^n \in BUC([0, T] \times \mathbb{R}^n)$  be the viscosity solution to

$$\begin{aligned}\partial_t u^n + Lu^n + f(t, x, u^n, Du^n \cdot \sigma(t, x)) + H(x, u^n) \dot{\zeta}_t^n &= 0, \\ u^n(T, x) &= g(x).\end{aligned}$$

Then (under appropriate assumptions on  $g, f$  and  $H$ ) there exists a  $u \in BUC([0, T] \times \mathbb{R}^n)$ , only dependent on  $\zeta$  but not on the approximating sequence  $\zeta^n$ , such that

$$u^n \rightarrow u \quad \text{locally uniformly.}$$

We write (formally)

$$\begin{aligned}du + [Lu + f(t, x, u, Du \cdot \sigma(t, x))] dt + H(x, u) d\zeta_t &= 0, \\ u(T, x) &= g(x).\end{aligned}$$

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Consider two Wiener spaces,

$$\begin{aligned}(\Omega^B, \mathcal{F}^B, \mathbb{P}^B) &= (C([0, T], \mathbb{R}^d), \mathcal{F}^B, \mathbb{P}^B), \\(\Omega^W, \mathcal{F}^W, \mathbb{P}^W) &= (C([0, T], \mathbb{R}^m), \mathcal{F}^W, \mathbb{P}^W).\end{aligned}$$

And consider the product space

$$\Omega = \Omega^B \times \Omega^W, \mathbb{P} := \mathbb{P}^B \otimes \mathbb{P}^W.$$

Then

$$B(\omega^B, \omega^W) := \omega^B, \quad W(\omega^B, \omega^W) := \omega^W.$$

are independent Brownian motions.

In this setting Pardoux/Peng '92 introduced the notion of *BDSDEs*

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r) dr + \int_t^T H(X_r, Y_r) \circ dB_r - \int_t^T Z_r dW_r.$$

We note that  $Y$  is  $\mathcal{F}_t := \mathcal{F}_{t,T}^B \vee \mathcal{F}_{0,t}^W$  adapted.

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We note that  $Y$  is  $\mathcal{F}_t := \mathcal{F}_{t,T}^B \vee \mathcal{F}_{0,t}^W$  adapted.

Let  $p \in (2, 3)$ . Let  $\mathbf{B}_t = \exp(B_t + A_t)$  be the Enhanced Brownian motion (over  $B$ ), i.e.

$$\mathbf{B}_t = \left( B_t, \begin{pmatrix} \int_0^t B_r^1 \circ dB_r^1 & \cdots & \int_0^t B_r^1 \circ dB_r^d \\ \cdots & \cdots & \cdots \\ \int_0^t B_r^d \circ dB_r^1 & \cdots & \int_0^t B_r^d \circ dB_r^d \end{pmatrix} \right)$$

Especially  $\mathbf{B} \in C_0^{p\text{-var}}([0, T], G^2(\mathbb{R}^d))$ ,  $\mathbb{P}^B$ -a.s..

For almost every  $\omega^B \in \Omega^B$  we can construct the solution to the BSDE with rough driver

$$Y_t^{rp} = \xi + \int_t^T f(r, Y_r^{rp}, Z_r^{rp}) dr + \int_t^T H(X_r, Y^{rp}) d\mathbf{B}_r(\omega^B) - \int_t^T Z^{rp} dW_r, \quad t \in [0, T].$$



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## Theorem

Let  $(Y, Z)$  be the solution to the BDSDE

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Then  $\mathbb{P}$ -a.s.

$$Y_t = Y_t^{rp}, \quad t \leq T.$$

## Outlook

- ▶ numerical scheme / intrinsic definition
- ▶ can also define SDE with rough drift
- ▶ use for Feynman-Kac representation rough PDEs of different type
- ▶ application to pathwise stochastic control (Buckdahn/Ma '01)

Thank you!



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