An Exact Connection between two Solvable SDEs and a Non Linear Utility Stochastic PDEs

Mohamed MRAD
Joint work with Nicole El Karoui

Université Paris VI, ÉcolePolytechnique,
elkaroui@cmap.polytechnique.fr, mrad@cmap.polytechnique.fr
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Some remarks on martingale theory and utility functions in Investment Banking from M. Musiela, T. Zariphopoulo, C. Rogers +alii (2002-2009)

- No clear idea how to specify the utility function.
- Classical or recursive utilities are defined in isolation to the investment opportunities given to an agent.
- Explicit solutions to optimal investment problems can only be derived under very restrictive model and utility assumptions, as Markovian assumption which yields to HJB PDEs.
- In non-Markovian framework, theory is concentrated on the problem of existence and uniqueness of an optimal solution, often via the dual representation of utility.
- The investor may want to use intertemporal diversification, i.e., implement short, medium and long term strategies
- Can the same utility function be used for all time horizons?

Consistent Dynamic Utility

Let $\mathscr{X}$ be a convex family of positive portfolios, called Test porfolios
Definition : An $\mathscr{X}$-Consistent progressive utility $U(t,x)$ process is a positive adapted random field s.t.

- $\ast$ **Concavity assumption :** for $t \geq 0$, $x > 0 \mapsto U(t,x)$ is an increasing concave function, (in short utility function) .
- $\star$ **Consistency with the class of test portfolios** For any admissible wealth process $X \in \mathscr{X}$, $\mathbb{E}(U(t,X_t)) < +\infty$ and
$$\mathbb{E}(U(t,X_t)/\mathcal{F}_s) \leq U(s,X_s), \ \forall s \leq t.$$
- **Existence of optimal** For any initial wealth $x > 0$, there exists an optimal wealth process (benchmark) $X^* \in \mathscr{X}(X_0^* = x)$,
$$U(s,X_s^*) = \mathbb{E}(U(t,X_t^*)/\mathcal{F}_s) \ \forall s \leq t.$$
- $\odot$ **In short** for any admissible wealth $X \in \mathscr{X}$, $U(t,X_t)$ is a supermartingale, and a martingale for the optimal-benchmark wealth X^*.

The General Market Model

- The security market consists of one riskless asset S^0, $dS^0_t = S^0_t r_t dt$, and d continuous risky assets S^i, $\quad i = 1..d$ defined on a filtred Brownian space $(\Omega, \mathcal{F}_{t\geq 0}, \mathbb{P})$
$$\frac{dS^i_t}{S^i_t} = b^i_t dt + \sigma^i_t . dW_t, \qquad 1 \leq i \leq d$$
- Risk premium vector, η_t with $\quad b(t) - r(t)\mathbf{1} = \sigma_t \eta_t$

Def A positive wealth process is defined as a pair (x, π), $x > 0$ is the initial value of the portfolio and $\pi = (\pi^i)_{1\leq i\leq d}$ is the (predictable) proportion of each asset held in the portfolio, assumed to be S-integrable process.

- Thanks to AOA in the market, wealth process with π-strategy is driven by
$$\frac{dX^\pi_t}{X^\pi_t} = r_t dt + \sigma_t \pi_t . (dW_t + \eta_t dt),$$

For simplicity we denote by $\mathcal{R}^\sigma$ the range of the matrix $\sigma := (\sigma^i)_{i=1\ldots d}$, $\kappa := \sigma\pi, \ \pi \in \mathbb{R}^d$. The class of Test portfolio in what follows is
$$\mathscr{X} := \{(X^\kappa) : \tfrac{dX^\kappa_t}{X^\kappa_t} = r_t dt + \kappa_t . (dW_t + \eta^\sigma_t dt), \quad \kappa_t \in \mathcal{R}^\sigma_t\}$$

Consistent Utility of Itô's Type

Let U be a dynamic utility (concave, increasing) ,

$$dU(t,x) = \beta(t,x)dt + \gamma(t,x).dW_t$$

such that $U(t, X_t^\pi)$ is a supermartingale for $X^\pi \in \mathscr{X}(\mathcal{K})$ and a martingale for the optimal one.

Open questions

- What about the drift β of the utility?
- What about the volatility γ of the utility?
- Under which assumptions on (β, γ) can one be sure that solutions are concave, increasing and consistent?

Main difficulties come from the forward definition.

From Kunita Book, Carmona-Nualart

- Let ϕ be a semimartingale random field satisfying

$$d\phi(t,x) = \mu(t,x)dt + \gamma(t,x).dW_t, \tag{1}$$

- The pair (μ, γ) is called the local characteristic of ϕ, and γ is referred as the volatility random field.
- A semimartingale random field ϕ is said to be Itô-Ventzel regular if
  - ϕ is a continuous $\mathcal{C}^{2+\cdots}$-process in x
  - local characteristic (μ, γ) are $\mathcal{C}^1$ in x
  - additional assumptions as more regularity, uniform integrability are need to guarantee smoothness of ϕ and its derivatives, and the existence of regular version of these random fields

Itô-Ventzel's Formula (Kunita)

- Let ϕ and ψ be Itô-Ventzel's regular one-dimensional stochastic flows

$$d\phi(t,x) = \mu(t,x)dt + \gamma(t,x).dW_t, \quad d\psi(t,x) = \alpha(t,x)dt + \nu(t,x).dW_t.$$

- The compound random field $\phi o \psi(t,x) = \phi(t,\psi(t,x))$ is a regular semimartingale

Itô-Ventzel's Formula

$$\begin{aligned} d(\phi o \psi)(t,x) &= \mu(t,\psi(t,x))dt + \gamma(t,\psi(t,x)).dW_t \\ &+ \phi_x(t,\psi(t,x))d\psi(t,x) + \frac{1}{2}\phi_{xx}(t,x)(t,\psi(t,x))||\nu(t,x)||^2 dt \\ &+ \langle \gamma_x(t,\psi(t,x)), \nu(t,x)\rangle dt. \end{aligned}$$

The volatility of $\phi o \psi$ is given by $\nu^{\phi o \psi}(t,x) = \gamma(t,\psi(t,x)) + \phi_x(t,\psi(t,x))\nu(t,x)$.

Drift Constraint

Let U be a progressive utility of class $\mathcal{C}^{(2)}$ in the sense of Kunita with local characteristics (β, γ) and risk tolerance coefficient $\alpha_t^U(t,x) = -\frac{U_x(t,x)}{U_{xx}(t,x)}$. We introduce the utility risk premium $\eta^U(t,x) = \frac{\gamma_x(t,x)}{U_x(t,x)}$. Then, for any admissible portfolio X^κ,

$$\begin{aligned} dU(t, X_t^\kappa) &= \Big(U_x(t, X_t^\kappa) X_t^\kappa\, \kappa_t + \gamma(t, X_t^\kappa)\Big).dW_t \\ &+ \Big(\beta(t, X_t^\kappa) + U_x(t, X_t^\kappa) r_t X_t^\kappa + \frac{1}{2} U_{xx}(t, X_t^\kappa) \mathcal{Q}(t, X_t^\kappa, \kappa_t)\Big) dt, \\ \text{where } x^2 \mathcal{Q}(t, x, \kappa) &:= \|x\kappa_t\|^2 - 2\alpha^U(t,x)(x\kappa_t).\big(\eta_t^\sigma + \eta^{U,\sigma}(t,x)\big). \end{aligned}$$

Let γ_x^σ be the orthogonal projection of γ_x on $\mathcal{R}^\sigma$. Let $\mathcal{Q}^*(t,x) = \inf_{\kappa \in \mathcal{R}^\sigma} \mathcal{Q}(t,x,\kappa)$; the minimum of this quadratic form is achieved at the optimal policy κ^* given by

$$\begin{cases} x\kappa_t^*(x) &= -\frac{1}{U_{xx}(t,x)}(U_x(t,x)\eta_t^\sigma + \gamma_x^\sigma(t,x)) = \alpha^U(t,x)\big(\eta_t^\sigma + \eta^{U,\sigma}(t,x)\big) \\ x^2 \mathcal{Q}^*(t,x) &= -\frac{1}{U_{xx}(t,x)^2}||U_x(t,x)\eta_t^\sigma + \gamma_x^\sigma(t,x))||^2 = -||x\kappa_t^*(x)||^2. \end{cases}$$

Verification Theorem: I

Let U be a progressive utility of class $\mathcal{C}^{(2)}$ in the sense of Kunita with local characteristics (β, γ).

Hyp Assume the drift constraint to be Hamilton-Jacobi-Bellman nonlinear type

$$\beta(t,x) = -U_x(t,x)r_t x + \frac{1}{2}U_{xx}(t,x)\|x\kappa_t^*(t,x)\|^2 \tag{2}$$

where κ^* is the optimal policy given by

$$x\kappa_t^*(x) = -\frac{1}{U_{xx}(t,x)}(U_x(t,x)\eta_t^\sigma + \gamma_x^\sigma(t,x))$$

Then the progressive utility is solution of the following forward HJB-SPDE

$$dU(t,x) = \big(- U_x(t,x)r_t x + \tfrac{1}{2}\tfrac{(U_x(t,x))^2}{U_{xx}(t,x)}||\eta_t^\sigma + \tfrac{\gamma_x^\sigma(t,x)}{U_x(t,x)}||^2\big)dt + \gamma(t,x).dW_t,$$

and for any admissible wealth X_t^κ, the process $U(t, X_t^\kappa)$ is a supermartingale.

Verification Theorem: II

Theorem

Under previous hypothesis,

- Assume that $\kappa^*(t,x)$ is sufficiently smooth so that the equation

$$dX_t^* = X_t^*(r_t dt + \kappa^*(t, X_t^*).(dW_t + \eta_t^\sigma dt)$$

has a (unique? strong ?) positive solution for any initial wealth $x > 0$.

$\Rightarrow$ Then, the progressive increasing utility U is a consistent utility, with optimal wealth X^*.

Inverse flows

Let ϕ be a strictly monotone Itô-Ventzel regular flow with inverse process $\xi(t,y) = \phi(t,.)^{-1}(y)$. Assume $d\phi(t,x) = \mu(t,x)dt + \gamma(t,x).dW_t$,

i) The inverse flow $\xi(t,y)$ has as dynamics in old variables

$$d\xi(t,y) = -\xi_y(t,y)(\mu(t,\xi)dt + \gamma(t,\xi).dW_t) + \frac{1}{2}\partial_y \frac{\|\gamma(t,\xi)\|^2}{\phi_x(t,\xi)} dt$$

ii) In terms of new variable, with $\nu^\xi(t,y) = -\xi_y \gamma(t,\xi)$

$$d\xi(t,y) = \nu^\xi(t,y).dW_t + \left(\frac{1}{2}\partial_y\Big(\frac{\|\nu^\xi(t,y)\|^2}{\xi_y}\Big) - \mu(t,\xi)\xi_y(t,y)\right)dt$$

iii) If $\phi = \Phi_x(t,x)$ with $d\Phi(t,x) = M(t,x)dt + C(t,x).dW_t$, then $\xi = \Xi_y(t,y)$

$$d\Xi(t,y) = -C(t,\xi).dW_t - M(t,\xi)dt + \frac{1}{2}\frac{\|C_x(t,\xi)\|^2}{\Phi_{xx}(t,\xi)} dt$$

Duality: Convex conjugate SPDE I

Let U be a consistent progressive utility of class $\mathcal{C}^{(3)}$, in the sense of Kunita, satisfying the β constraint (2), then the convex conjugate $\tilde{U}(t,y) \stackrel{def}{=} \inf_{x\in Q_+^*}\big(U(t,x) - x\,y\big)$ satisfies

$$\begin{aligned} d\tilde{U}(t,y) &= \Big[\frac{1}{2\tilde{U}_{yy}(t,y)}\big(\|\tilde{\gamma}_y(t,y)\|^2 - \|\tilde{\gamma}_y^\sigma(t,y) + y\tilde{U}_{yy}(t,y)\eta_t^\sigma\|^2\big) + y\tilde{U}_y(t,y)r_t\Big]dt \\ &+ \tilde{\gamma}(t,y).dW_t \quad \text{with } \tilde{\gamma}(t,y) = \gamma(t,-\tilde{U}_y(t,y)). \end{aligned}$$

- The drift $\tilde{\beta}(t,y)$ is the value of an optimization program achieved on the optimal policy $\nu^*(t,y) = \theta^*(t,-\tilde{U}(t,y)) = -\tilde{\gamma}_y^\perp(t,y)/y\tilde{U}_{yy}(t,y)$.
- $\tilde{\beta}$ can be written us the solution of the following optimization program

$$\tilde{\beta}(t,y) = y\tilde{U}_y(t,y)r_t - \frac{1}{2}y^2\tilde{U}_{yy}(t,y)\inf_{\nu_t\in\mathcal{R}^{\sigma,\perp}}\{\|\nu_t-\eta_t^\sigma\|^2 + 2(\nu_t-\eta_t^\sigma).(\frac{\tilde{\gamma}_y(t,y)}{y\tilde{U}_{yy}(t,y)})\}$$

with $-\tilde{\gamma}_y(t,y)/y\tilde{U}_{yy}(t,y) = \eta^U(t,-\tilde{U}(t,y)) = \gamma_x(t,-\tilde{U}(t,y))/y$.

Under previous assumption,

- The conjugate Utility $\tilde{U}(t,y)$ is a convex decreasing stochastic flows,
- consistent with the family $\mathcal{Y}$ of semimartingales Y^ν, defined from

$$\frac{dY_t}{Y_t} = -r_t dt + (\nu_t - \eta_t^\sigma).dW_t, \quad \nu_t \in \mathcal{K}_t^{\sigma,\perp}$$

- There exists a dual optimal choice $Y_t^* = Y_t^{\nu^*}$ satisfying the dual identity

$$Y^*(t,y) = U_x(t, X_t^*((U_x)^{-1}(0,y)), \quad \mathcal{Y}(t,x) := U_x(t, X_t^*(x))$$

Assume $X_t^*(x)$ is strictly monotone in x, by taking the inverse $\mathcal{X}(t,x)$,

$$\Rightarrow U_x(t,x) = Y_t^*\big(u_x(\mathcal{X}(t,x))\big)$$

$$\Rightarrow U(t,x) = \int_0^x Y_t^*(u_x(\mathcal{X}(t,z)))dz$$

Req: $x \mapsto X_t^*(x)$ is increasing $\Rightarrow$ $y \mapsto Y_t^*(y)$ is increasing.

Flows Assumption

Let $X^*(x)$ be any wealth process and $Y^*(y)$ be any state price density assumed to be continuous and increasing in x (resp. in y) from 0 to $+\infty$. Moreover, X^* and Y^* are Itô-Ventzel regular

$$dX_t^*(x) = X_t^*(x) r_t dt + X_t^*(x) \kappa^*(t, X^*).(dW_t + \eta_t^\sigma dt), \quad \kappa^*(t,x) \in \mathcal{R}_t^\sigma$$
$$dY_t^*(y) = -Y_t^*(y) r_t dt + (\nu^*(t, Y_t^*) - \eta_t^\sigma).dW_t, \quad \nu^*(t,y) \in \mathcal{R}_t^{\sigma,\perp}$$

Note that the Monotony Assumption is

- true in a lot examples,
- may be a consequence of no arbitrage opportunity.
- from flows point of view, it is implied by coefficient regularity.

Theorem: Utility Charracterization, Basic Example

Let $\mathcal{X}(t,z)$ be the inverse flow of $X^*(t,z)$, satisfying $X^* Y^\nu$ $(\nu \in \mathcal{R}^{\sigma,\perp})$ is a martingale. Then for any utility function u such that $u_x(\mathcal{X}(t,z))$ is locally integrable near $z=0$, the stochastic process U defined by

$$U(t,x) = Y_t^\nu(1) \int_0^x u_x(\mathcal{X}(t,z))dz, \quad U(t,0) = 0 \tag{3}$$

is a $\mathscr{X}$-Consistent utility. The associated optimal wealth process is X^* and the optimal dual choice $Y^*(y) = yY^\nu(1)$. Moreover

$$\gamma_x(t,x) = U_x(t,x)(\nu_t - \eta_t^\sigma) - U_{xx}(t,x)\kappa^*(t,x).$$

Furthermore, the conjugate process of U denoted by $\tilde{U}$, is given by

$$\tilde{U}(t,y) = \int_y^{+\infty} X^*(t, -\tilde{u}_y(z/Y_t^\nu(1))dz, \tag{4}$$

With this utility characterization, the study of the risk tolerance coefficient, taken along the optimal wealth, is greatly simplified. In particular, the nice martingale property established in He and Huang in 1992, in a complete market, may be generalized to consistent utilities.

Proposition

Let $\alpha^U(t,x) = -\frac{U_x(t,x)}{U_{xx}(t,x)}$ be the risk tolerance coefficient of U.
Then $\alpha^U(t, X^*(t,x)) = \alpha^u(x) X^*_x(t,x)$, where $X^*_x(t,x)$ is the derivative (assumed to exist) of $X^*(t,x)$ with respect to x. Moreover, denoting Y^*_y the partial derivative of Y^* with respect to its initial condition, the process $Y^0_t \alpha^U(t, X^*(t,x)) \equiv Y^*_y(t,y)\alpha^U(t, X^*(t,x))$ is a local martingale, since $X^*_x(t,x)$ is also an admissible portfolio with initial wealth 1.

Theorem

Let $(X_t^*(x))$, and $Y^*(t,y)$ two regular stochastic flows as above and u an utility function. Denote by $\mathcal{X}$ and $\mathcal{Y}$ the inverse flows and assume that $x \mapsto Y_t^*(u_x(\mathcal{X}(t,y)))$ is locally integrable near $z=0$. Define the processes U and $\tilde{U}$ by

$$U(t,x) = \int_0^x Y_t^*(u_x(\mathcal{X}(t,z)))dz, \quad \tilde{U}(t,y) = \int_y^{+\infty} X_t^*(-\tilde{u}_y(\mathcal{Y}(t,z)))dz.$$

Then U is a consistent utility, whose the convex conjugate is $\tilde{U}$, and the dynamics

$$dU(t,x) = \Big(- xU_x(t,x)r_t + \frac{1}{2U_{xx}(t,x)}||\gamma_x^\sigma(t,x) + U_x(t,x)\eta_t^\sigma||^2\Big)dt + \gamma(t,x).dW_t,$$

with volatility vector γ given by

$$\gamma(t,x) = -U(t,x)\eta_t^\sigma - \int_0^x \Big(zU_{xx}(t,z)\kappa^*(t,z) - \nu_t^*(U_x(t,z))\Big)dz.$$

The associated optimal portfolio and the optimal dual process are X^* and Y^*.

Proposition

Under the same assumptions as in the previous theorem, the risk tolerance coefficient α^U of U is given by

$$\alpha^U(t,x) = \frac{\mathcal{Y} \circ \mathcal{X}(t,x)}{\mathcal{Y}_x \circ \mathcal{X}(t,x)} X_x^* \circ \mathcal{X}(t,x).$$

Where , $\mathcal{Y}(t,x) := Y_t^*(u_x(x))$. Moreover, $\alpha^U(t, X_t^*(x)) = \frac{Y_t^*(u_x(x))}{Y_y^*(t,u_x(x))u_{xx}(t,x)} X_x^*(t,x)$ and satisfies: $Y_y^*(t,y)\alpha^U(t,X_t^*(x))$ is a local martingale.

Converse point of view

Consider a utility stochastic PDE with initial condition $u(.)$,

$$dU(t,x) = \Big(-xU_x(t,x)r_t + \frac{1}{2U_{xx}(t,x)}||\gamma_x^\sigma(t,x) + U_x(t,x)\eta_t^\sigma||^2\Big)dt + \gamma(t,x).dW_t. \quad (5)$$

Where the derivative γ_x of γ is the operator given by

$$\gamma_x(t,x) = -U_x(t,x)\eta_t^\sigma - xU_{xx}(t,x)\kappa^*(t,x) + \nu_t^*(U_x(t,x)),\ \kappa_t^* \in \mathcal{R}_t^\sigma,\ \nu_t^* \in \mathcal{R}_t^{\sigma,\perp},\ t \geq 0.$$

Assume that the both equations

$$\frac{dX_t^*(x)}{X_t^*(x)} = r_t dt + \kappa^*(t, X_t^*(x)).(dW_t + \eta_t^\sigma dt), \quad \frac{dY_t^*(y)}{Y_t^*(y)} = -r_t dt + (\nu_t^*(Y_t^*(y)) - \eta_t^\sigma).dW_t$$

admit solutions and that X^* is monotonous regular flow in the sense of Kunita $\Rightarrow$ there exists a solution U of the SPDE (5) given by

$$U(t,x) = \int_0^x Y_t^*(u_x(\mathcal{X}(t,z)))dz$$

- If X^* and Y^* are increasing regular flows $\Rightarrow$ U is an increasing and concave solution of the SPDE (5).
- If X^* and Y^* are unique $\Rightarrow$ U is the unique solution of (5).

The main assumption is that the optimal portfolio is increasing in x, because we have the same characterization in more abstract form (minimal regularities assumption), based on the properties of the optimum.

Thank you for your attention