L_2-time regularity of BSDEs with irregular terminal functions

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$$\begin{cases} -dY_t &= f(\omega, t, Y_t, Z_t)dt - Z_t dW_t, \\ Y_T &= \xi. \end{cases} \tag{1}$$

- T : terminal time
- ξ : terminal condition
- f : driver/generator
- Z : control variable

Theorem (existence and uniqueness)

Assume that

- $|\xi| + \int_0^T |f(t, 0, 0|dt \in \mathbf{L}_2$,
- f *is a.s. continuous w.r.t.* t *and* ***Lipschitz*** *w.r.t.* (y, z).

Then, BSDE (1) has a unique solution (Y, Z) *s.t.*

$$\mathbb{E}[\sup_{t\in[0,T]} |Y_t|^2] + \mathbb{E}[\int_0^T |Z_s|^2 ds] < +\infty.$$

Markovian BSDE

Forward component

$$\begin{cases} X_0 &= x_0, \\ dX_t &= b(t, X_t)dt + \sigma(t, X_t)dW_t, \end{cases}$$

$X_t \in \mathbb{R}^d, \quad W_t \in \mathbb{R}^q.$

Backward component

$$\begin{cases} -dY_t &= f(t, X_t, Y_t, Z_t)dt - Z_t dW_t, \\ Y_T &= g(X_T). \end{cases}$$

$Y_t \in \mathbb{R}$, $Z_t \in \mathbb{R}^{1\times q}$.

Link with PDEs

$\mathcal{L}_X$:= infinitesimal generator of X. If u is the smooth solution to

$\partial_t u(t,x) + \mathcal{L}_X u(t,x) + f(t,x,u(t,x),\nabla_x u(t,x)\sigma(t,x)) = 0, t < T,$
$u(T,x) = g(x),$

then

$$Y_t = u(t, X_t),$$
$$Z_t = \nabla_x u(t, X_t)\sigma(t, X_t).$$

For a fixed time mesh $\pi = \{0 = t_0 < \cdots < t_i < \cdots < t_N = T\}$,

L_2-time regularity modulus of Z

$$\mathcal{E}(Z,\pi) := \sum_{i=0}^{N-1} \mathbb{E} \int_{t_i}^{t_{i+1}} |Z_t - \bar{Z}_{t_i}|^2 dt$$

where $\bar{Z}_{t_i} := \frac{1}{t_{i+1}-t_i} \mathbb{E}^{\mathcal{F}_{t_i}} \int_{t_i}^{t_{i+1}} Z_s ds$.

- **Purpose** : estimate $\mathcal{E}(Z,\pi)$ according to
  - the regularity of g
  - $|\pi| = \sup_{0 \le i < N}(t_{i+1} - t_i)$.

Motivation

A usual numerical scheme for BSDEs is

$$\begin{cases} Y^{\pi}_{t_i} & = \mathbb{E}^{\mathcal{F}_{t_i}}(Y^{\pi}_{t_{i+1}} + (t_{i+1} - t_i) f(t_i, X_{t_i}, Y^{\pi}_{t_{i+1}}, Z^{\pi}_{t_i})), \\ Z^{\pi}_{t_i} & = \frac{1}{(t_{i+1} - t_i)} \mathbb{E}^{\mathcal{F}_{t_i}}(Y^{\pi}_{t_{i+1}} (W_{t_{i+1}} - W_{t_i})^*). \end{cases} \quad (2)$$

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It is known (Zhang'04) that

$$e(Y^\pi - Y, Z^\pi - Z, \pi) := \sup_{0 \le i \le N} \mathbb{E}(Y^\pi_{t_i} - Y_{t_i})^2 + \sum_{i=0}^{N-1} \mathbb{E} \int_{t_i}^{t_{i+1}} |Z^\pi_{t_i} - Z_t|^2 dt$$

$$\le C(|\pi| + \mathcal{E}(Z, \pi)).$$

- $\mathcal{E}(Z, \pi)$ plays a key role in the convergence of (2).

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- If $f = 0$ and $g(x) = (x - K)_+^a$ with $a \in (0, \frac{1}{2})$, then with uniform time net, $\mathcal{E}(Z,\pi) \sim N^{-(1/2+a)}$ (Gobet,Temam'01).

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- If $f = 0$ and $g = \mathbb{1}_{(0,\infty)}$, then with uniform time net, $\mathcal{E}(Z,\pi) \sim N^{-1/2}$ (Gobet, Temam'01).

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- If $f = 0$, $X = W$ (or 1-dim SDE), and $g \in \mathbf{B}^{\alpha}_{2,2}$, then (Geiss *et al.* '04,'07)
  - $\mathcal{E}(Z,\pi) \sim N^{-\alpha}$ with uniform time net,
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▶ **Our goal** : generalize w.r.t g, X and f.

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When $\mathbb{E}|g(X_T)|^2 < +\infty$, we define

$$V_{t,T}(g) := \mathbb{E}\left|g(X_T) - \mathbb{E}^{\mathcal{F}_t}(g(X_T))\right|^2.$$

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Definition

For a fixed $\alpha \in (0,1]$,

$$\mathbf{L}_{2,\alpha} = \left\{ g \text{ s.t. } \mathbb{E}(g(X_T)^2) + \sup_{0 \le t < T} \frac{V_{t,T}(g)}{(T-t)^\alpha} < +\infty \right\}.$$

Examples

- If g is Lipschitz, then $g \in \mathbf{L}_{2,1}$.
- Si g is Hölderian with exponent a, then $g \in \mathbf{L}_{2,(a+\frac{1}{2})\wedge 1}$!
- If $g(x) = \mathbb{1}_D(x)$, then $g \in \mathbf{L}_{2,\frac{1}{2}}$!

The cœfficients

Assumption $(\mathbf{A}_{b,\sigma})$

- b and σ are $\mathcal{C}_{\mathrm{b}}^{2+\gamma}$ $(\gamma \in (0,1])$ w.r.t. x.
- b and σ are Hölderian with exponent $\frac{1}{2}$ w.r.t. t.
- σ is uniformly elliptic : $\exists \delta > 0$ s.t.,
  $\forall (t,x) \in [0,T] \times \mathbb{R}^d,\ [\sigma\sigma^*](t,x) \geq \delta I_d$.

The time nets

For $\beta \in (0,1]$,

$$\pi^{(\beta)} := \left\{ t_k^{(N,\beta)} := T - T\left(1 - \frac{k}{N}\right)^{\frac{1}{\beta}}, 0 \leq k \leq N \right\}.$$

NB.

- $\pi^{(1)}$ = uniform grid.
- For $\beta < 1$, the nodes of $\pi^{(\beta)}$ are more concentrated near T.

Regularity of the linear BSDE ($f = 0$)

$(y, z) :=$ solution of the linear BSDE ($f = 0$).

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Uniform grid

Theorem

Assume $(\boldsymbol{A}_{b,\sigma})$, and $g \in \mathbf{L}_{2,\alpha}$ ($\alpha \in (0,1]$). Then, with uniform grid,

$$\mathcal{E}(z, \pi^{(1)}) = \sum_{k=0}^{N-1} \mathbb{E} \int_{t_k^{(N,1)}}^{t_{k+1}^{(N,1)}} \left| z_s - \bar{z}_{t_k^{(N,1)}} \right|^2 ds \leq \frac{C}{N^\alpha}$$

(C does not depend on N).

Non uniform grid

Theorem

Assume $(\boldsymbol{A}_{b,\sigma})$*, and* $g \in \mathbf{L}_{2,\alpha}$ $(\alpha \in (0,1])$*. Let* β *s.t.* $\beta = 1$*, if* $\alpha = 1$*, and* $\beta < \alpha$ *otherwise. Then, with the grid* $\pi^{(\beta)}$*,*

$$\mathcal{E}(z, \pi^{(\beta)}) = \sum_{k=0}^{N-1} \mathbb{E} \int_{t_k^{(N,\beta)}}^{t_{k+1}^{(N,\beta)}} \left| z_s - \bar{z}_{t_k^{(N,\beta)}} \right|^2 ds \leq \frac{C}{N}$$

(C does not depend on N).

Lemma

Assume $(\boldsymbol{A}_{b,\sigma})$ and g bounded. Then, $\exists C > 0$, s.t., $\forall t \in [0, T)$,

$$\mathbb{E}|u(t, X_t)|^2 \leq \mathbb{E}|g(X_T)|^2,$$
$$\mathbb{E}|\nabla_x u(t, X_t)|^2 \leq C\frac{V_{t,T}(g)}{T-t},$$
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Idea of the proof :

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Estimates for the derivatives

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- $\mathbb{E}^{\mathcal{F}_t}(H^{(1)}_{t,T}) = 0$
- $\mathbb{E}^{\mathcal{F}_t}|H^{(1)}_{t,T}|^2 \leq \frac{C}{T-t}$.

Optimality

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Lemma

*Let $\alpha \in (0,1]$, and assume **($A_{b,\sigma}$)** and g bounded. Then, the following assertions are equivalent :*

(i) $g \in \mathbf{L}_{2,\alpha}$.

(ii)

$$\int_0^t \mathbb{E}\left|D^2 u(s, X_s)\right|^2 ds \le \frac{C}{(T-t)^{1-\alpha}}.$$

(iii)

$$\mathbb{E}\left|\nabla_x u(t, X_t)\right|^2 \le \frac{C}{(T-t)^{1-\alpha}}.$$

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$$\mathbb{E}\left|\nabla_x u(t, X_t)\right|^2 \le \frac{C}{(T-t)^{1-\alpha}}.$$

- **NB.** Geiss *et al.* have proved that (iii) is equivalent to $\mathcal{E}(z,\pi) \le \frac{C}{N^\alpha}$, for every grid π, in the case X = BM/GBM/1-dim SDE.

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Theorem

Assume $(\boldsymbol{A}_{b,\sigma})$, g bounded and $g \in \mathbf{L}_{2,\alpha}$ ($\alpha \in (0,1]$). Then, $\forall t \in [0,T)$,

$$|Z_t - z_t| \leq C \int_t^T \frac{\sqrt{\mathbb{E}^{\mathcal{F}_t}\left[\left(g(X_T) - \mathbb{E}^{\mathcal{F}_s} g(X_T)\right)^2\right]}}{T-s} ds + C(T-t).$$

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- **Ex**. If g is bounded and Hölderian with exponent α, then

$$|Z_t - z_t| \leq C(T-t)^{\frac{\alpha}{2}}.$$

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  $$f^0(t,x,y,z) := f\left(t, x, y + u(t,x), z + \nabla_x u(t,x)\sigma(t,x)\right).$$

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- If $\mathbb{E}\int_0^T \int_0^T |D_\theta f^0(s, X_s, y, z)|^2 ds d\theta < \infty$, then $Z_t^0 = D_t Y_t^0 =$ Linear BSDE (El Karoui *et al.* '97).

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  But : this condition is not satisfied ($\mathbb{E}|D^2 u(t, X_t)|^2$ may not be integrable).

- Localization : for $\varepsilon > 0$, $f^{\varepsilon}(t,x,y,z) := f^{0}(t,x,y,z)\mathbb{1}_{t\leq T-\varepsilon}$,

$$Y_t^{\varepsilon} = \int_t^T f^{\varepsilon}(s, X_s, Y_s^{\varepsilon}, Z_s^{\varepsilon})ds - \int_t^T Z_s^{\varepsilon} dW_s.$$

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- When $\varepsilon \to 0$,

$$\begin{aligned} Z_t^\varepsilon &\to Z_t^0, \\ Z_t^\varepsilon &= D_t Y_t^\varepsilon, \\ D_t Y_t^\varepsilon &\to \mathcal{D}_t Y_t^0. \end{aligned}$$

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$\Longrightarrow Z_t^0 = \mathcal{D}_t Y_t^0 = \text{Linear BSDE}$

Theorem

Assume $(\boldsymbol{A}_{b,\sigma})$ *and* $g \in \mathbf{L}_{2,\alpha}$ $(\alpha \in (0,1])$. *Then,* $\exists C$ *s.t. , for every grid* $\pi = \{t_k : k = 0...N\}$

$$\mathcal{E}(Z,\pi) = \sum_{k=0}^{N-1} \mathbb{E} \int_{t_k}^{t_{k+1}} \left| Z_s - \bar{Z}_{t_k} \right|^2 ds$$
$$\leq C\mathcal{E}(z,\pi) + C|\pi|.$$

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Assume $(\boldsymbol{A}_{b,\sigma})$ *and* $g \in \mathbf{L}_{2,\alpha}$ $(\alpha \in (0,1])$. *Then,*

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$$\mathcal{E}(Z, \pi^{(1)}) \leq \frac{C}{N^{\alpha}};$$

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a) *with the uniform grid* $\pi^{(1)}$,

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b) *with the non uniform grid* $\pi^{(\beta)}$, *where* $\beta = 1$ *if* $\alpha = 1$, *and* $\beta < \alpha$ *if* $\alpha < 1$,

$$\mathcal{E}(Z, \pi^{(\beta)}) \leq \frac{C}{N}.$$