A second order discretization and efficient simulation for Backward SDEs

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- Forward Backward SDEs and related PDEs.
- Second order discretization.
- Simulation with the cubature method
- Numerical examples.

$(W_t, \mathcal{F}_t)_{0\le t\le T}$ a d-dimensional BM on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(X, Y, Z) = \{(X_t, Y_t, Z_t)_{0\le t\le T}\} \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ be the solution of the (decoupled) system:

$$\begin{aligned} X_t &= x + \int_0^t V_0(X_s)ds + \sum_{i=1}^{d} \int_0^t V_i(X_s) \circ dW_s^i, \\ Y_t &= \Phi(X_T) + \int_t^T f(X_s, Y_s, Z_s)ds - \int_t^T Z_s \cdot dW_s, \end{aligned} \tag{1}$$

- $V_i : \mathbb{R}^d \to \mathbb{R}^d$ smooth vector fields.
- $\Phi(X_T)$ called *the final condition*
- $f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ Lipschitz, called "the driver".

Let $u \in C^{1,2}([0,T) \times \mathbb{R}^d)$ be the solution of the final value Cauchy problem

$$\begin{cases} \left(L^0 + L\right) u = -f\left(t, x, u, (\nabla u V)(x)\right), & t \in [0,T), x \in \mathbb{R}^d \\ u(T,x) = \Phi(x), \quad x \in \mathbb{R}^d \end{cases}, \qquad (2)$$

where

- L is the second order differential operator $L = \frac{1}{2}\sum_{i=1}^{d}\left(L^i\right)^2$.
- L^i, $i = 0, 1..., d$, are the first order differential operators associated to $V_i = (V_i^j)_{j=1}^d$.

$$L^i = \sum_{j=1}^{d} V_i^j \partial_{x_i}, \quad L^0 = \partial_t + \sum_{j=1}^{d} V_0^j \partial_{x_j}$$

- V is the matrix $V = (V_1, ..., V_d)$.

Theorem (Peng 1991,1992, Pardoux & Peng 1992)

The (viscosity or classical) solution of (2) *admits the following* **Feynman-Kac representation**

$$u(t,x) = Y_t^{t,x} = \mathbb{E}\left[\Phi(X_T^{t,x}) + \int_t^T f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds\right], \qquad (3)$$

where $(X^{t,x}, Y^{t,x}, Z^{t,x})$ *is the 'stochastic flow' associated to* (1)

$$\begin{aligned} dX_s^{t,x} &= V_0(X_s^{t,x})ds + \sum_{i=1}^d V_i(X_s^{t,x}) \circ dW_s^i, \quad s \in [t,T], \\ dY_s^{t,x} &= -f(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds + Z_s^{t,x} \cdot dW_s. \\ X_t^{t,x} &= x, \qquad Y_T^{t,x} = \Phi(X_T^{t,x}) \end{aligned} \qquad (4)$$

If in addition $u \in C_b^1(\mathbb{R}^d)$ *then* $Z_s^{t,x} = \nabla u(s, X_s^{t,x}) V(X_s^{t,x})$.

Notation:

We fix a a partition $\pi := \{0 = t_0 < t_1 < \ldots < t_n\}$.
$\mathcal{A}$ is the set of multi indices $\mathcal{A} := \cup_j \{0, 1, \ldots, d\}^j$.
Norm on multi indices

$$|\alpha| = \text{ length of } \alpha, \quad \|\alpha\| := |\alpha| + card\{j : \alpha_j = 0,\ 1 \leq j \leq |\alpha|\}.$$

For $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathcal{A}$, we denote by

$$L^\alpha u := L^{\alpha_1} \ldots L^{\alpha_k} u$$

and the iterated Stratonovich integral for appropriate $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}$:

$$J_\beta[g(\cdot, X_\cdot)]_{t,s} := \begin{cases} g(s, X_s) & |\beta| = 0 \\ \int_t^s J_{\beta-}[g(\cdot, X_\cdot)]_{t,u} du & l \geq 1,\ j_l = 0 \\ \int_t^s J_{\beta-}[g(\cdot, X_\cdot)]_{t,u} \circ dW^{j_l}(u) & l \geq 1, j_l \neq 0. \end{cases}$$

We can develop the Stratonovich-Taylor expansion for appropriate $g : [0,T] \times \mathbb{R}^d \to \mathbb{R}$: :

$$\begin{aligned} g(t, X_t^{0,x}) &= \sum_{\|\alpha\| \leq m} L^\alpha g(0,x) J^\alpha[1]_{0,t} + \sum_{\|\alpha\|=m+1,m+2} J^\alpha[L^\alpha g(\cdot, X_\cdot)]_{0,t} \\ &= Tayl(g,t) + R_m(t,g) \end{aligned} \tag{5}$$

$R_m(t,g)$ is called the remainder.

$$\mathbb{E}\left[\,|\,R_m(t,g)\,|\,\right] = \mathcal{O}(t^{(m+1)/2}), \quad t < 1$$

First revisit the Bouchard-Touzi-Zhang discretization (Euler style).
Assume that we “know” $\{X\}_{i=0}^{n}$:

$$
\begin{aligned}
Y_{t_n}^{\pi,1} &:= \Phi(X_n), \quad Z_{t_n}^{\pi,1} = 0 \\
Z_{t_i}^{\pi,1} &:= \frac{1}{\delta_{i+1}} \mathbb{E}_i \left[Y_{t_{i+1}}^{\pi,1} \Delta W_{i+1} \right] \\
Y_{t_i}^{\pi,1} &:= \mathbb{E}_i \left[Y_{t_{i+1}}^{\pi,1} \right] + \delta_{i+1} f \left(X_i, Y_{t_i}^{\pi,1}, Z_{t_i}^{\pi,1} \right), \quad i = n-1, \dots, 0.
\end{aligned}
$$

where $\delta_{i+1} = t_{i+1} - t_i$, $\Delta W_{i+1} = W_{t_{i+1}} - W_{t_i}$.

Theorem (Bouchard and Touzi(2004), Zhang(2004), Gobet and Labart(2007))

When coefficients of the FBSDE are smooth

$$\max_{0\leq i\leq n-1}\mathbb{E}\left[\left|Y_{t_i}-Y^{\pi,1}_{t_i}\right|^2+\delta_{i+1}\left|Z_{t_i}-Z^{\pi,1}_{t_i}\right|^2\right]=\mathcal{O}(\|\pi\|^2)$$

whereas, if these are Lipschitz continuous

$$\max_{0\leq i\leq n-1}\mathbb{E}\left[\left|Y_{t_i}-Y^{\pi,1}_{t_i}\right|^2+\delta_{i+1}\left|Z_{t_i}-Z^{\pi,1}_{t_i}\right|^2\right]=\mathcal{O}(\|\pi\|)$$

Towards the second order scheme:

For the driver, we use the Trapezoid rule rather than Euler. Assume that we also "know" $Z_i = \nabla u(t_i, X_{t_i}) V(X_{t_i})$:

$$Y_{t_i}^{\pi,2} := \mathbb{E}_i \left[Y_{t_{i+1}}^{\pi,2} \right] + \frac{\delta_{i+1}}{2} \left(f(X_{t_i}, Y_{t_i}^{\pi,2}, Z_{t_i}) + \mathbb{E}_i \left[f(X_{t_{i+1}}, Y_{t_{i+1}}^{\pi,2}, Z_{t_{i+1}}) \right] \right)$$

With standard arguments using the Stratonovich Taylor expansions, we can show :

$$\begin{aligned}
&\left| Y_{t_i} - Y_{t_i}^{\pi,2} \right| \\
&= \left| \mathbb{E}_i \left[Y_{t_{i+1}} - Y_{t_{i+1}}^{\pi,2} + \int_{t_i}^{t_{i+1}} f(\Theta_s) ds - \frac{\delta_{i+1}}{2} \left(f(\Theta_{t_i}^{\pi,2}) + \mathbb{E}_i \left[f(\Theta_{t_{i+1}}^{\pi,2}) \right] \right) \right] \right| \\
&\leq (1 + C\delta_{i+1}) \left| \mathbb{E}_i \left[Y_{t_{i+1}} - Y_{t_{i+1}}^{\pi,2} \right] \right| + \delta_{i+1}^3 \max_{\|\alpha\| \leq 4} \| L^\alpha u(t_{i+1}, \cdot) \|_\infty
\end{aligned}$$

where $\Theta_t = (X_t, Y_t, Z_t)$, $\Theta_{t_i}^{\pi,2} := (X_{t_i}, Y_{t_i}^{\pi,2}, Z_{t_i}^{\pi,2})$.

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$$= \frac{1}{\delta_{i+1}}\mathbb{E}_i\left[\Delta W^l_{i+1}\sum_{\|\alpha\|\leq 4} L^\alpha u(t_i, X_{t_i})J^\alpha[1]_{t_i,t_{i+1}}\right] + \frac{1}{\delta_{i+1}}\mathbb{E}_i\left[R_4(u,\delta_{i+1})\Delta W^l_{i+1}\right]$$

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$$= Z^l_{t_i} + \frac{1}{\delta_{i+1}}\sum_{\|\alpha\|=3} L^{\alpha}u(t_i, X_{t_i})\mathbb{E}_i\left[\Delta W^l_{i+1}J^{\alpha}[1]_{t_i,t_{i+1}}\right] + \delta^2_{i+1}\max_{\|\alpha\|=5,6}\|L^{\alpha}u(t_{i+1}, \cdot)\|_{\infty} \tag{6}$$

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Since

$$J^\alpha[1]_{t_i,t_{i+1}}J^{(l)}[1]_{t_i,t_{i+1}} = \sum_{j=1}^{k} J^{(\alpha_1,\ldots,\alpha_{j-1},l,\alpha_j,\ldots,\alpha_k)}[1]_{t_i,t_{i+1}}$$

$$\mathbb{E}_i\left[J^\alpha[1]_{t_i,t_{i+1}}\right] = 0, \quad \text{if } \|\alpha\| = 1 + 2\mathbb{N}_+$$

Which multi indices α, with $\|\alpha\| = 3$, satisfy

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Answer: $\alpha = (0, l),\ (l, 0), (l, k, k),\ (k, k, l), \quad k = 1, \ldots, d.$

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However, from the PDE

$$\left(L^0 + \frac{1}{2} \sum_{k=1}^{d} L^{(k,k)} \right) u(t_i, X_{t_i}) = -f(X_{t_i}, u(t_i, X_{t_i}), \nabla u(t_i, X_{t_i}))$$

We get

$$\left(L^{(l,0)} + \frac{1}{2} \sum_{k=1}^{d} L^{(l,k,k)} \right) u(t_i, X_{t_i}) = -L^l f(X_{t_i}, u(t_i, X_{t_i}), \nabla u(t_i, X_{t_i}))$$

On the other hand, with the same reasoning

$$\mathbb{E}_i \left[f(X_{t_{i+1}}, Y_{t_{i+1}}, Z_{t_{i+1}}) \Delta W_{i+1}^l \right] = \delta_{i+1} L^l f(X_{t_i}, Y_{t_i}, Z_{t_i}) + \delta_{i+1}^2 \max_{\|\alpha\|=3,4} \|L^\alpha u(t_{i+1}, \cdot)\|$$

Putting everything we have so far, together:

$$\frac{1}{\delta_{i+1}}\mathbb{E}_i\left[Y_{t_{i+1}}\Delta W^l_{i+1}\right]+\mathbb{E}_i\left[f(X_{t_{i+1}},Y_{t_{i+1}},Z_{t_{i+1}})\Delta W^l_{i+1}\right]$$

$$=Z^l_{t_i}+\frac{1}{\delta_{i+1}}\mathbb{E}_i\left[\Delta W^l_{i+1}\sum_{\alpha=(0,l),(k,k,l)}L^\alpha u(t_i,X_{t_i})J^\alpha[1]_{t_i,t_{i+1}}\right]+\mathcal{O}(\delta^2_{i+1})$$

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Final ingredient:

$$\mathbb{E}_i\left[Z^l_{i+1}\right]=\mathbb{E}_i\left[L^l u(t_{i+1},X_{i+1})\right]$$

$$=Z^l_i+\frac{1}{\delta_{i+1}}\mathbb{E}_i\left[\Delta W^l_{i+1}\sum_{\alpha=(0,l),(k,k,l)}L^\alpha u(t_i,X_{t_i})J^\alpha[1]_{t_i,t_{i+1}}\right]+\mathcal{O}(\delta^2_{i+1})$$

All these intuitive arguments tell us that

$$Z_i = \frac{2}{\delta_{i+1}}\mathbb{E}_i\left[Y_{t_{i+1}}\Delta W^l_{i+1}\right] - \mathbb{E}_i\left[Z_{i+1}\right] + \mathbb{E}_i\left[f(X_{t_{i+1}}, Y_{t_{i+1}}, Z_{t_{i+1}})\Delta W^l_{i+1}\right] + \mathcal{O}(\delta^2_{i+1})$$

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We suggest the following scheme:

$$Y^{\pi,2}_{t_n} = \Phi(X_n),$$

$$Z^{\pi,2}_{t_n} = \begin{cases} \nabla\Phi(X_n)V(X_n), & \text{if } \Phi \text{ smooth} \\ 0, & \text{else} \end{cases}$$

If Φ Lipschitz $Z^{\pi,2}_{t_{n-1}} := Z^{\pi,1}_{t_{n-1}}, \quad Y^{\pi,2}_{t_{n-1}} = Y^{\pi,1}_{t_{n-1}}$

For $i = n-2, \ldots, 0$

$$Z^{\pi,2}_{t_i} = 2\mathbb{E}_i\left[Y^{\pi,2}_{t_{i+1}}\frac{\Delta W_{i+1}}{\delta_{i+1}}\right] - \mathbb{E}_i\left[Z^{\pi,2}_{t_{i+1}}\right] + \mathbb{E}_i\left[f(X_{i+1}, Y^{\pi,2}_{t_{i+1}}, Z^{\pi,2}_{t_{i+1}})\Delta W_{i+1}\right],$$

$$Y^{\pi,2}_{t_i} = \mathbb{E}_i\left[Y^{\pi,2}_{t_{i+1}}\right] + \frac{\delta_{i+1}}{2}\left(f\left(X_i, Y^{\pi,2}_{t_i}, Z^{\pi,2}_{t_i}\right) + \mathbb{E}_i\left[f\left(X_{i+1}, Y^{\pi,2}_{t_{i+1}}, Z^{\pi,2}_{t_{i+1}}\right)\right]\right).$$

Theorem

Let $\{V_i\}_{i=0}^d$, f *be smooth.*

$$\max_{0\le i\le n-1} \mathbb{E}\left[\left| Y_i - Y_{t_i}^{\pi,2} \right|^2\right] \simeq C\|\nabla\Phi\|_\infty \sqrt{\delta_n} + C\sum_{i=1}^{n-1} \delta_{i+1}^5 \max_{\|\alpha\|\le 5} \|L^\alpha u(t_{i+1},\cdot)\|_\infty \tag{7}$$

Theorem

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Theorem (F. Delarue 2010)

In the context of the above theorem, assume that Φ is Lipschitz continuous. Then the solution of the PDE is smooth on $C_b^{\lceil m/2 \rceil, m}\left([0,T) \times \mathbb{R}^d\right)$ and

$$\|D^\alpha u(t,\cdot)\|_\infty \leq C \frac{\|\nabla \Phi\|_\infty}{(T-t)^{(\|\alpha\|-1)/2}}, \quad t \in [0,T),$$

Corollary

Within the same framework, consider the second order scheme along the partition

$$\pi := \{t_i := T\left(1-\left(1-\frac{i}{n}\right)^2\right), \quad i=0,\dots,n, \quad n \in \mathbb{N}_+\}.$$

Then, there exists a constant C independent of the partition such that

$$\max_{0\le i\le n-1} \mathbb{E}\left[\left|Y_{t_i} - Y_{t_i}^{\pi,2}\right|^2\right]^{1/2} \le \frac{C\|\nabla\Phi\|_\infty}{n^2}$$

We approximate the involved expectations with the cubature method. The details:
Given a finite measure μ on $\left(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\right)$, the points $\{x_i\}_{i=1}^N \subset \mathbb{R}^d$ and numbers $\lambda_i \in \mathbb{R}_+$, $i = 1, \ldots, N$ define a cubature formula of order m w.r.t μ if

$$\int_{\mathbb{R}^d} x^k \mu(dx) = \sum_{i=1}^N \lambda_i x_i^k \equiv \sum_{i=1}^N \lambda_i \delta_{x_i}(x^k), \quad \forall k \leq m$$

$\rightarrow \int_{\mathbb{R}^d} f(x)\mu(dx) \simeq \sum_{i=1}^N \lambda_i f(x_i)$, $\forall f$ smooth.

$$\begin{array}{ccc} (\mathbb{R}^d, \mu) & & (C_0[0,T], \mathbb{P}) \quad (\mathbb{P} \sim \text{ Wiener measure}) \\ x^k & \longleftrightarrow & \int \ldots \int \circ dW^{i_1} \ldots \circ dW^{i_k} \\ x \in \mathbb{R}^d & \longleftrightarrow & \omega \in C_0[0,T] \end{array}$$

Given a nice function f, the Sratonovich-Taylor expansion tells us

$$f(X_t^{0,x}) = \sum_{\|\alpha\| \leq m} L^\alpha f(0,x) \int_{0<t_1<\ldots<t_k<t} \circ dW_{t_1}^{i_1} \ldots \circ dW_{t_k}^{i_k} + R_m(t,x,f)$$

$$\|R_m(t,x,f)\|_p = O(t^{\frac{m+1}{2}})$$

Let $\mathbb{Q}$ be another measure on $(C([0,t]), \mathcal{B}(C([0,t])))$ with

$$(\mathbb{E}^{\mathbb{Q}} - \mathbb{E}) \left[\int_{0<t_1<\ldots<t_k<t} \circ dW_{t_1}^{i_1} \ldots \circ dW_{t_k}^{i_k} \right] = 0, \quad (i_1, \ldots, i_k) \in \mathcal{A}_m$$

$\mathbb{E}^{\mathbb{Q}}[|R_m(t,x,f)|^p]$ small for small t

$$\rightarrow \mathbb{E}^{\mathbb{Q}} \left[f(X_t^{0,x}) \right] \simeq \mathbb{E} \left[f(X_t^{0,x}) \right]$$

Theorem (Lyons & Victoir (2004), Litterer& Lyons (2008))

Cubature paths exist. Moreover there exists explicit construction for $m = 3, 5$ any dimension and $m = 7$ and dimension $1, 2$.

This new measure, called cubature measure, will be denoted by $\mathbb{Q}_t^m$, i.e $\mathbb{Q}_t^m := \sum_{i=1}^N \lambda_i \delta_{\omega_i}$
Let $\omega \in C_{0,bv}([0,T];\mathbb{R}^d)$. What is $\mathbb{E}^{\delta_\omega}[f(X_t^{0,x})]$?

$$\mathbb{E}^{\delta_\omega}[f(X_t^{0,x})] = f\left(x + \int_0^t V_0(X_s(\omega))ds + \sum_{i=1}^d \int_0^t V_i(X_s(\omega))d\omega^i(s)\right)$$

Put differently, if $\Xi_{T,x}(\omega)$ the solution at time T of the ODE $dy_{t,x} = \sum_{i=0}^{d} V_i(y_{t,x})d\omega^i(t),\ y_0 = x$, then

$$\mathbb{E}_{\mathbb{Q}^m_{h_{i+1}}}[f(X^{t_i,x}_{t_{i+1}})] = \sum_{j=1}^{N} \lambda_j f\left(\Xi_{h_{i+1},x}(\omega_j)\right).$$

Over multiple steps, the above measure compounds as

$$\mathbb{E}^{\mathbb{Q}^m_{t_k}}\left[f(X_{t_k})|X_0 = x\right] = \sum_{i_1,\ldots i_k=1}^{N} \lambda_{i_1}\ldots\lambda_{i_k}\delta_{\Xi_{t_k,x}\left(\omega_{i_1}\otimes\ldots\otimes\omega_{i_k}\right)}(f) \equiv \mathbb{E}^{cub_m}[f]$$

Along the partition π we build the cubature tree $\bigcup_{k=0}^{n} \mathcal{N}_k$:

$$\mathcal{N}_0 := \{x = X_0\}, \quad \mathcal{N}_k^x := \{\Xi_{t_k,x}(\omega_j) \,|\, j = 1, \ldots, N\}, \quad x \in \mathcal{N}_{k-1}$$

$$\mathcal{N}_k := \bigcup_{x \in \mathcal{N}_{k-1}} \mathcal{N}_k^x$$

We define recursively the family of vectors $\{\psi_k(x), \zeta_k(x), x \in \mathcal{N}_k\}_{k=0}^{n}$ (each of length N^k), as

$$\psi_n(x) = \Phi(x), \; \zeta_n(x) = 0, \quad j = 1, \ldots, N, \quad x \in \mathcal{N}_n$$

$$\zeta_i(x) = \sum_{\bar{x} \in \mathcal{N}_{i+1}^x} \lambda_{\bar{x}} \frac{2}{\delta_{i+1}^{1/2}} \left(\psi_{i+1}(\bar{x}) \right) - \zeta_{i+1}(\bar{x}) + f(\bar{x}, \psi_{i+1}(\bar{x}), \zeta_{i+1}(\bar{x}))\delta_{i+1}^{1/2}$$

$$\psi_i(x) = \sum_{\bar{x} \in \mathcal{N}_{i+1}^x} \lambda_{\bar{x}} \psi_{i+1}(\bar{x})$$

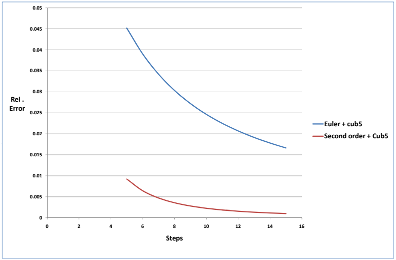
$$+ \frac{\delta_{i+1}}{2} f\left(x, \psi_i(x), \zeta_i(x) \right) + \frac{\delta_{i+1}}{2} f \sum_{\bar{x} \in \mathcal{N}_{i+1}^x} f\left(\bar{x}, \psi_{i+1}(\bar{x}), \zeta_{i+1}(\bar{x}) \right)$$

Conside the (F)BSDE:

$$dX_t = \mu X_t dt + \sqrt{1 + X_t^2}\, dW_t, \quad X_0 = 1$$

$$Y_t = \arctan(X_T) + \int_t^T e^{r(T-s)}(1-\mu) X_s Z_s^2 - rY_s ds - \int_t^T Z_s dW_s,$$

$$\text{Solution } :(Y_t, Z_t) = \left(e^{-r(T-t)} \arctan(X_t),\ \frac{e^{-r(T-t)}}{\sqrt{1+X_t^2}} \right).$$



The cubature method produces a finitely supported measure on $\mathbb{R}^d$, at every time t_k: $\mu_k^{cub_m} = \sum \lambda_{i_1} \dots \lambda_{i_k} \delta_{x_{(i_1,\dots,i_k)}}$ where the point $x_{(i_1,\dots,i_k)}$ is obtained by solving ODEs along $\omega_{i_1}, \dots, \omega_{i_k}$.

The support of the measure explodes exponentially.

At time t_k we have $(N^{k+1} - 1)/(N - 1)$ points , N the number of cubature paths.

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The support of the measure explodes exponentially.

At time t_k we have $(N^{k+1}-1)/(N-1)$ points , N the number of cubature paths. But we do not need this much !.

Let $g : \mathbb{R}^d \to \mathbb{R}$ be smooth with compact support. Then

$$\begin{aligned} g(x) &= \sum_{|\alpha|\leq m} \frac{D^\alpha g(x_0)}{\alpha!}(x-x_0)^\alpha + R_m(g,x_0) \\ &= Tay_m(g,x_0) + R_m(g,x_0), \quad x \in B(x_0,\delta) \end{aligned}$$

Any measure $\tilde{\mu}$ such that $\tilde{\mu}\,[\,(\,x - x_0\,)^{\alpha}\,] = \mu\,[\,(\,x - x_0\,)^{\alpha}\,]\,,\ \forall|\alpha| \leq m$ will give us

$$\tilde{\mu}(g|_{B(x_0,\delta)}) \simeq \mu(g|_{B(x_0,\delta)})$$

There are $\binom{d+m}{d}$ monomials x^{α}, $|\alpha| \leq m$ in d variables.

We can reduce the $supp(\mu_k^{cub_m}) \cap B(x_0, \delta)$, to $\binom{d+m}{d} + 1$ points
Namely, find at most $\binom{d+m}{d} + 1$ points among all points in $supp(\mu_k^{cub_m}) \cap B(x_0, \delta)$ with new weights, that integrate polynomials same as $\mu_k^{cub_m}$. Call this measure $\mu^{cub_m,red}$.

- $\left|\,\mu^{cub_m,red}(g) - \mathbb{E}\,[\,g(X_{t_k})\,]\,\right| = \mathcal{O}\left(\,\left|\,\mu^{cub_m}(g) - \mathbb{E}\,[\,g(X_{t_k})\,]\,\right|\,\right)$
- The number of knods grows only polynomially.

All the above carry over in the BSDE framework with two differentiations:

- The condition for the reduce measure is

$$\mu^{cub_m,red}\left(\,(x^{\alpha})^{p}\,\right) = \mu^{cub_m}\left(\,(x^{\alpha})^{p}\,\right), \quad |\alpha| \leq m,\ p > 1.$$

- Children of the same parent knod, all die or all survive

. The overall rate of convergence stays the same!

$$dX_t^i = \mu_i X_t dt + \sigma_i X_t^i dW_t^i, \quad i = 1, 2, 3$$
$$dY_t = rY_t dt + \theta \cdot Z_t dt + Z_t \cdot dW_t,$$
$$Y_t = \left(K - \prod_{i=1}^{3} X_t^i \right)_+$$