Convex risk measures under model uncertainty

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INTRODUCTION

DYNAMIC RISK MEASURES ON A FILTERED PROBABILITY SPACE

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in R^+}, P)$ filtered probability space with a right continuous filtration.

- Coherent Dynamic Risk Measures: Delbaen (2002) and Artzner, Delbaen, Eber, Heat, Ku (2007)
- Convex dynamic risk measures considered in many papers, among them: Frittelli and Rosaza Gianin (2002), Klöppel, Schweizer (2007), Cheredito, Delbaen, Kupper (2006), Bion-Nadal (2008 and 2009), Föllmer and Penner (2006)
- g expectations or Backward Stochastic Differential Equations : Peng (2004), Rosazza Gianin (2004) and Barrieu El Karoui (2009)

DYNAMIC RISK MEASURES

DYNAMIC RISK MEASURES:
on $L^\infty(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in R^+}, P)$ (or $L^p(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in R^+}, P)$, $1 < p < \infty$).
$\rho_{\sigma,\tau} : L^\infty(\Omega, \mathcal{F}_\tau, P) \to L^\infty(\Omega, \mathcal{F}_\sigma, P)$, satisfying monotonicity, convexity, translation invariance and continuity from above.

- Time Consistency :

$$\forall \nu \leq \sigma \leq \tau, \quad \rho_{\nu,\tau}(X) = \rho_{\nu,\sigma}(-\rho_{\sigma,\tau}(X))$$

- Normalized ($\rho_{\sigma,\tau}(0) = 0$) time consistent dynamic risk measures have càdlàg paths (J B N 2009). Regularity is satisfied without the normalization assumption under some continuity assumption on the penalty,
- Families of dynamic risk measures constructed from right continuous BMO martingales generalizing B.S. D. E. and allowing for jumps. (J B N 2008 and 2009).

TIME CONSITENT DYNAMIC RISK MEASURES FROM BMO MARTINGALES

THEOREM

Let $(M_i)_{i\leq j}$ be strongly orthogonal right continuous BMO martingales. Let $\mathcal{M} = \{\sum H_i.M_i, H_i$ predictable$\}$. Let $\mathcal{S}$ be a stable subset of $\mathcal{Q}(\mathcal{M}) = \{Q_M \mid \frac{dQ_M}{dP} = \mathcal{E}(M)\ M \in \mathcal{M}\}$. Let b_i be measurable on $\boldsymbol{R}^+ \times \Omega \times \boldsymbol{R}^j$ admitting a quadratic bound from below. For $M = \sum H_i.M_i$,

$$\alpha_{\sigma,\tau}(Q_M) = E_{Q_M}(\int_\sigma^\tau b_i(s,\omega,H_1(\omega),..H_j(\omega))d[M_i,M_i]_s(\omega))|\mathcal{F}_\sigma)$$

- *If M_i are continuous*
- *or if the BMO norms of elements of $\mathcal{S}$ are bounded by $m < \frac{1}{16}$,*

$$\rho_{\sigma,\tau}(X) = \mathrm{esssup}_{\mathrm{Q_M}\in\mathcal{S},\mathrm{Q_M}_{|\mathcal{F}_\sigma}=\mathrm{P}}(\mathrm{E_{Q_M}}(-\mathrm{X}|\mathcal{F}_\sigma) - \alpha_{\sigma,\tau}(\mathrm{Q_M}))$$

defines a time consistent dynamic risk measure.

BROWNIAN FILTRATION

These exemples generalize the B.S.D.E. (which are convex and translation invariant)

QUADRATIC BACKWARDS Every solution of a BSDE with a convex driver independent of y and quadratic in z admits a dual representation of the preceding form (Barrieu and El Karoui 2009).

NORMALIZED TIME CONSISTENT DYNAMIC RISK MEASURES IN A BROWNIAN FILTRATION For every normalized time consistent dynamic risk measure on the Brownian filtration the penalty term associated to $(\frac{dQ}{dP}) = \mathcal{E}(q.B)$ can be written:

$$c_{\sigma,\tau}(Q) = E_Q\left(\int_\sigma^\tau f(u, q_u)du | \mathcal{F}_\sigma\right) \ \ \forall 0 \leq \sigma \leq \tau$$

Delbaen Peng and Rosazza Gianin (2009)

Model Uncertainty

Financial framework: No reference probability measure is given. Instead a weakly relatively compact set of probability measures is given.
Motivations:
Example of Uncertain Volatility

$$dX_t^\sigma = b_t dt + \sigma_t dW_t \quad \sigma_t \in [\underline{\sigma}, \overline{\sigma}]$$

The set of the laws of X_t^σ: weakly relatively compact set $\mathcal{P}$ of probability measures not all absolutely continuous with respect to some probability measure.
Denis Martini (2006)
$\Omega = \mathcal{C}_0(\boldsymbol{R}^+, \boldsymbol{R}^d)$, B_t coordinate process.
$\mathcal{P}$: weakly relatively compact set of orthogonal martingale measures for B_t
Pricing function $\Lambda(f) = \sup_{P \in \mathcal{P}} E_P(f)$.

Introduction

S. Peng: G-Expectations (2007), (2008)
G-expectation $\mathbb{E}$ is defined on *Lip* , subset of $\mathcal{C}_b(\Omega)$ using PDE.
Denis Hu Peng (2010) Every G expectation admits the representation

$$\forall f \in Lip \quad \mathbb{E}(f) = \sup_{P \in \mathcal{P}^1} E_P(f)$$

$\mathcal{P}^1$ is weakly relatively compact.

In both cases, $\Pi(f) = \sup_{P \in \mathcal{P}} E_P(f)$ $\mathcal{P}$ weakly relatively compact.
Π is sublinear monotone translation invariant and regular $\Pi(X_n) \to 0$ when $X_n \downarrow 0$.
Soner, Touzi, Zhang (2010), Nutz (2010)
Same framework $\Omega = \mathcal{C}_0(\boldsymbol{R}^+, \boldsymbol{R}^d)$, $\mathcal{P}$ is a set of probability measures.
$\Pi(f) = \sup_{P \in \mathcal{P}} E_P(f)$

- Either $f \in \mathcal{C}_b(\Omega)$ and $\mathcal{P}$ is weakly relatively compact.
- Or $f \in \mathcal{UC}_b(\Omega)$ and no restriction on $\mathcal{P}$.

INTRODUCTION

REGULAR CONVEX RISK MEASURES ON $\mathcal{C}_b(\Omega)$

Ω is a Polish space. For example $\Omega = \mathcal{C}(\boldsymbol{R}^+, \boldsymbol{R}^d)$ or $\Omega = D([0,\infty[, \boldsymbol{R}^d)$ the space of càdlàg functions, endowed with the Skorokhod topology.

Regularity (for sublinear risk measures): $\rho(-X_n) \to 0$ when $X_n \downarrow 0$.
Regularity $\iff$ continuity with respect to a certain capacity c.
If ρ is sublinear, $c(X) = \rho(-|X|)$
$c(f) = \sup_{P\in\mathcal{P}} E_P(|f|))$ $\mathcal{P}$ weakly relatively compact.

$L^1(c)$ Banach space obtained by completion and separation of $\mathcal{C}_b(\Omega)$ for the semi-norm c.
$L^1(c)$: introduced by Feyel and de la Pradelle (1989).
Thus we study $L^1(c)$ and convex risk measures on $L^1(c)$.
We prove that there is an equivalence class of probability measures canonically associated to ρ, characterizing the riskless elements.

OUTLINE

1. $L^1(c)$
    - Topological properties of the dual space of $L^1(c)$
    - Convex risk measures on $L^1(c)$
2. EQUIVALENCE CLASS OF PROBABILITY MEASURES ASSOCIATED TO A NON DOMINATED SET OF PROBABILITY MEASURES
3. REGULAR CONVEX RISK MEASURES ON $\mathcal{C}_b(\Omega)$
    - Examples

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CAPACITY

Ω: Polish space (metrizable and separable space and complete for some metric defining the topology)
$\mathcal{L}$: linear vector subspace of $\mathcal{C}_b(\Omega)$ containing the constants, generating the topology of Ω and which is a vector lattice.

CAPACITY

DEFINITION

a capacity on $\mathcal{L}$ is a semi norm c defined on $\mathcal{L}$ satisfying the following properties:

1. monotonicity: $\forall f, g \in \mathcal{L}$ such that $|f| \leq |g|, \quad c(f) \leq c(g)$
2. regularity along sequences: for every sequence $f_n \in \mathcal{L}$ decreasing to 0, $\lim c(f_n) = 0$

THE BANACH SPACE $L^1(c)$

EXTENSION OF THE CAPACITY Feyel and de la Predelle (1989)
The semi-norm c is extended to all real functions on Ω:

$$\forall f \; l.s.c., \; f \geq 0, \quad c(f) = \sup\{c(\phi) | 0 \leq \phi \leq f, \; \phi \in \mathcal{L}\} \tag{1}$$

$$\forall g, \quad c(g) = \inf\{c(f) | \, f \geq |g|, \, f \; l.s.c.\} \tag{2}$$

THE BANACH SPACE $L^1(c)$
$\mathcal{L}^1(c)$: closure of $\mathcal{L}$ in the set $\{g| \; c(g) < \infty\}$.
$\mathcal{L}^1(c)$ contains $\mathcal{C}_b(\Omega)$. (Feyel and de la Pradelle)

Let $L^1(c)$ be the quotient of $\mathcal{L}^1(c)$ by the c null elements.
$L^1(c)$ is a Banach space.

THE DUAL SPACE OF $L^1(c)$

PROPOSITION

Let c be a capacity on a Polish space Ω. Every continuous linear form L on $L^1(c)$ admits a representation:

$$L(f) = \int f d\mu \ \ \forall f \in L^1(c) \tag{3}$$

where μ is a regular bounded signed measure defined on a σ-algebra containing the Borel σ-algebra of Ω denoted $\mathcal{B}(\Omega)$.
If L is a non negative linear form , the measure μ is non negative finite.

Topological Properties of the dual space of $L^1(c)$

WEAK AND WEAK* TOPOLOGIES

- Weak topology on $\mathcal{M}_+(\Omega)$, the set of non negative finite measures on $(\Omega, \mathcal{B}(\Omega))$:
  coarsest topology for which the mappings
  $$\mu \in \mathcal{M}_+(\Omega) \to \int f d\mu$$
  are continuous for every given f in $\mathcal{C}_b(\Omega)$.
- Weak* topology on $L^1(c)^*$: $\sigma(L^1(c)^*, L^1(c))$ topology
  i.e. coarsest topology for which the mappings
  $$L \in L^1(c)^* \to L(X)$$
  are continuous for every given X in $L^1(c)$.

Topological Properties of the dual space of $L^1(c)$

Proposition

Let c be a capacity on a Polish space Ω. On the non negative part K_+ of the unit ball of $L^1(c)^*$, the weak topology coincides with the weak topology.*

Proposition

The set K_+ is compact metrizable for the weak topology, as well as for the weak topology.*

Convex risk measures

Definition

Let $\rho : L^1(c) \to \boldsymbol{R}$.

- ρ is monotonic if $\rho(X) \geq \rho(Y)$ for every $X, Y \in L^1(c)$, such that $X \leq Y$.
- ρ is convex if for every $X, Y \in L^1(c)$, for every $0 \leq \lambda \leq 1$, $\rho(\lambda X + (1-\lambda)Y \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$
- ρ is translation invariant if $\rho(X + a) = \rho(X) - a$ for every $X \in L^1(c)$ and $a \in \boldsymbol{R}$.

ρ is a convex risk measure if it satisfies all these conditions.

Definition

A convex risk measure ρ on $L^1(c)$ is normalized if $\rho(0) = 0$.

REPRESENTATION

THEOREM

Assume that c is a capacity on a Polish space Ω. Let ρ be a convex risk measure on $L^1(c)$. Then, ρ is continuous and admits a representation of the form:

$$\forall X \in L^1(c),\ \rho(X) = \sup_{Q \in \mathcal{P}'} \left(E_Q[-X] - \alpha(Q)\right) \tag{4}$$

where

$$\alpha(Q) = \sup_{X \in L^1(c)} \left(E_Q[-X] - \rho(X)\right) \tag{5}$$

$\mathcal{P}'$ is the set of probability measures on $(\Omega, \mathcal{B}(\Omega))$ belonging to $L^1(c)^*$.

Risk measures represented by a weakly relatively compact set of probability measures

Proposition

Let $\rho : L^1(c) \to \mathbf{R}$ be a normalized convex risk measure. The following conditions are equivalent:

1. *ρ is majorized by a sublinear risk measure*
2. $\forall X \in L^1(c)$, $\sup_{\lambda>0} \frac{\rho(\lambda X)}{\lambda} < \infty$
3. *there exits $K > 0$ such that $\forall X \in L^1(c)$, $|\rho(X)| \leq Kc(X)$*
4. *ρ is represented by a set $\mathcal{Q}$ of probability measures in $L^1(c)^*$ relatively compact for the weak* topology, i.e.*

$$\forall X \in L^1(c),\ \rho(X) = \sup_{Q \in \mathcal{Q}} \left(E_Q[-X] - \alpha(Q)\right) \tag{6}$$

Risk measures represented by a weakly relatively compact set of probability measures

Theorem

Let ρ be a convex risk measure on $L^1(c)$. Assume that ρ is represented by

$$\rho(X) = \sup_{Q \in \mathcal{Q}} (E_Q(-X) - \alpha(Q))$$

where $\mathcal{Q}$ is a set of probability measures in $L^1(c)^*$ relatively compact for the weak topology. Let $\overline{\mathcal{Q}}$ be the closure of $\mathcal{Q}$ for the weak* topology. Then for every $X \in L^1(c)$, there is a probability measure $Q_X \in \overline{\mathcal{Q}}$ such that*

$$\rho(X) = E_{Q_X}(-X) - \alpha(Q_X) \tag{7}$$

REPRESENTATION WITH A COUNTABLE SET OF PROBABILITY MEASURES

THEOREM

Assume that c is a capacity on a Polish space Ω. Every convex risk measure on $L^1(c)$ can be represented by a countable set of probability measures $\{R_n,\ n \in N\}$ belonging to $L^1(c)^*$.

$$\forall X \in L^1(c),\ \rho(X) = \sup_{n \in \boldsymbol{N}} (E_{R_n}(-X) - \alpha(R_n)) \tag{8}$$

where

$$\alpha(R) = \sup_{X \in L^1(c)} (E_R[-X] - \rho(X)) \tag{9}$$

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Capacity defined from a weakly relatively compact set of probability measures

$\mathcal{P}$ weakly relatively compact set of probability measures. Capacity $c_{p,\mathcal{P}}$ defined on $\mathcal{C}_b(\Omega)$ by $c_{p,\mathcal{P}}(f) = \sup_{Q\in\mathcal{P}} E_Q(|f|^p)^{\frac{1}{p}}$

Lemma

For all X in $L^1(c_{p,\mathcal{P}})$, $c_{p,\mathcal{P}}(X) = \sup_{Q\in\mathcal{P}} E_Q(|X|^p)^{\frac{1}{p}}$.
There is $(Q_n)_{n\in N}$ in $\mathcal{P}$ such that $c_{p,\mathcal{P}}(X) = \sup_{n\in N} E_{Qn}(|X|^p)^{\frac{1}{p}}$.

Remark

It can happen that for a certain Borelian set A, the above equation is not satisfied for $X = 1_A$, i.e. $c_{p,\mathcal{P}}(1_A) \neq \sup_{Q\in\mathcal{P}} Q(A)^{\frac{1}{p}}$. ($\geq$ is always satified)
Example: $\Omega = [0,1]$. Let $x_n \in]0,1[$ be a sequence converging to 0. Let $A = [0,1] - \{x_n, n\in N\}$. Let $Q_n = \delta_{x_n}$. Let $\mathcal{P} = \{Q_n,\ n \in N\}$. Then $c_{p,\mathcal{P}}(1_A) = 1$ and $\sup_{Q\in\mathcal{P}} Q(A)^{\frac{1}{p}} = 0$

CANONICAL CLASS OF PROBABILITY MEASURE ASSOCIATED TO $L^1(c_{p,\mathcal{P}}$

USUAL EQUIVALENCE CLASS OF MEASURES

A non negative measure ν on $(\Omega, \mathcal{B}(\Omega)$ belongs to the (usual) equivalence class of the probability measure P if and only if

$$\forall A \in \mathcal{B}(\Omega), \quad P(A) = 0 \iff \nu(A) = 0$$

Or equivalently if $\nu \in (L^1(\Omega, \mathcal{B}(\Omega), P))^*$,

$$P \sim \nu \iff [\forall X \in L^1(\Omega, \mathcal{B}(\Omega), P)_+, \quad X = 0 \iff \nu(X) = \int X d\nu = 0]$$

Equivalence class of probability measures associated to a non dominated set of probability measures

Equivalence relation on $\mathcal{M}^+(c_p)$

When $\mathcal{P}$ is fixed, write c_p instead of $c_{p,\mathcal{P}}$ $\mathcal{M}^+(c_p)$: set of non negative finite measures on $(\Omega, \mathcal{B}(\Omega))$ defining an element of $L^1(c_p)^*$. Define on $\mathcal{M}^+(c_p)$ the relation $\mathcal{R}_{c_p}$ by

$$\mu \mathcal{R}_{c_p} \nu \iff \tag{10}$$

$$\forall X \in L^1(c_p), X \geq 0, \quad \{\mu(X) = 0 \iff \nu(X) = 0\}$$

Lemma

$\mathcal{R}_{c_p}$ defines an equivalence relation on $\mathcal{M}^+(c_p)$.

Canonical equivalence class of probability measures associated to $c_{p,\mathcal{P}}$

Theorem

Ω Polish space. $\mathcal{P}$ weakly relatively compact.
There is a unique $\mathcal{R}_{c_p}$ equivalence class in $\mathcal{M}^+(c_p)$ such $\mu \in \mathcal{M}^+(c_p)$ belongs to this class if and only if

$$\forall X \in L^1(c_p), X \geq 0, \qquad \{\mu(X) = 0\} \iff \{X = 0 \text{ in } L^1(c_p)\}$$

This class is referred as the canonical c_p-class.
For every countable weakly relatively compact set $\{Q_n,\ n \in \mathbf{N}\}$ such that for every $X \in L^1(c_p)$ $c_p(X) = \sup_{n\in\mathbf{N}}(E_{Q_n}(|X|^p))^{\frac{1}{p}}$,
for $\alpha_n > 0$ such that $\sum_{n\in\mathbf{N}} \alpha_n = 1$ the probability measure $\sum_{n\in\mathbf{N}} \alpha_n Q_n$ belongs to the canonical c_p-class.

THE CANONICAL c_p-CLASS

PROPERTY

Let P be a probability measure belonging to the canonical c_p-class. Let X be an element of $L^1(c_p)$. Then $X \geq 0$ (for the order in $L^1(c_p)$) if and only $X \geq 0$ P $a.s.$

REMARK

When $\mathcal{P} = \{P\}$ the canonical c_p-class is the restriction to $\mathcal{M}^+(c_p)$ of the usual equivalence class of the probability measure P.

When $\mathcal{P}$ is a finite set, $\mathcal{P} = \{P_1, ...P_n\}$ the canonical c_p-class is the restriction to $\mathcal{M}^+(c_p)$ of the equivalence class (in the usual sense) of the probability measure $P = \frac{\sum_{1 \leq i \leq n} P_i}{n}$.

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REGULARITY

$\mathcal{L}$: linear vector subspace of $\mathcal{C}_b(\Omega)$ containing the constants, generating the topology of Ω and which is a vector lattice.

DEFINITION

- A sublinear risk measure ρ on $\mathcal{L}$ is regular if for every decreasing sequence X_n of elements of $\mathcal{L}$ with limit 0, $\rho(-X_n)$ tends to 0.
- A normalized convex risk measure is uniformly regular if for all X $\sup_{\lambda>0} \frac{\rho(\lambda X)}{\lambda} < \infty$, and for every decreasing sequence X_n of elements of $\mathcal{L}$ with limit 0, $\frac{\rho(-\lambda X_n)}{\lambda}$ converges to 0 uniformly in $\lambda > 0$.

LEMMA

Let ρ be a normalized convex risk measure uniformly regular. $\rho_{min}(X) = \sup_{\lambda>0} \frac{\rho(\lambda X)}{\lambda}$ *defines a regular sublinear risk measure. It is the minimal sublinear risk measure on $\mathcal{L}$ majorizing ρ.*

Extension of a risk measure

Lemma

ρ: normalized convex risk measure uniformly regular on $\mathcal{L}$.
$c_\rho(X) = \rho_{min}(-|X|)$ defines a capacity on $\mathcal{L}$.
ρ (resp ρ_{min})has a unique continuous extension into a normalized convex risk measure $\overline{\rho}$ (resp a sublinear risk measure $\overline{\rho}_{min}$)on $L^1(c_\rho)$. $\overline{\rho}$ is majorized by $\overline{\rho}_{min}$.

Definition

Let $X \in \mathcal{C}_b(\Omega)$, X is riskless if for all $\lambda > 0$, $\rho(\lambda X) \leq 0$.
For $X \leq 0$ this is equivalent to $\rho(\lambda X) = 0$ for every $\lambda > 0$

Representation of uniformly regular convex risk measures

Theorem

Let ρ be a normalized uniformly regular convex risk measure on $\mathcal{L}$.
Then ρ extends uniquely to $\mathcal{C}_b(\Omega)$. There is a countable weakly relatively compact set $\{Q_n,\ n \in N\}$ such that

$$\forall X \in \mathcal{C}_b(\Omega) \;\; \rho(X) = \sup_{n \in N}(E_{Q_n}(-X) - \alpha(Q_n)) \tag{11}$$

Furthermore for $\alpha_n > 0$ such that $\sum_{n \in N} \alpha_n = 1$ the probability measure $P = \sum_{n \in N} \alpha_n Q_n$ characterizes the riskless non positive elements of $\mathcal{C}_b(\Omega)$, that is $X \leq 0$ is riskless iff $X = 0$ P a.s.
For every $X \in \mathcal{C}_b(\Omega)$, there is a probability measure Q_X in the weak closure of $\{Q_n,\ n \in N\}$, such that

$$\rho(X) = E_{Q_X}(-X) - \alpha(Q_X) \tag{12}$$

G-EXPECTATIONS

$\Omega = \mathcal{C}_0([0,T], \boldsymbol{R}^d)$, G-expectations where introduced by S. Peng (2007).
From Denis Hu and Peng (2009) $\mathbb{E}(f) = \sup_{P \in \mathcal{P}} E_P(f)$ $\mathcal{P}$ is weakly relatively compact
$\rho(f) = \mathbb{E}(-f)$ is a sublinear regular risk measure on $\mathcal{C}_b(\Omega)$.

PROPOSITION

There is a countable weakly relatively compact set $\{Q_n, n \in N\}$ of probability measures, $Q_n \in \mathcal{P}$ such that

$$\forall X \in \mathcal{C}_b(\Omega) \;\; \mathbb{E}(X) = \sup_{n \in \boldsymbol{N}} E_{Q_n}(X) \tag{13}$$

Let $P = \sum_{n \in \boldsymbol{N}^*} \alpha_n Q_n$ ($\alpha_n > 0$ and $\sum \alpha_n = 1$). For all $f \geq 0$ in $\mathcal{C}_b(\Omega)$, $\mathbb{E}(f) = 0$ iff $f = 0$ P a.s.
For every $X \in \mathcal{C}_b(\Omega)$, there is a probability measure Q_X in the weak closure of $\{Q_n, n \in N^*\}$, such that $\mathbb{E}(X) = E_{Q_X}(X)$.

$\Omega = \mathcal{C}_0([0,T],\boldsymbol{R}^d)$

$\mathcal{C}_b(\Omega)$, $\mathcal{P}$ is weakly relatively compact

$\Pi(f) = \sup_{P\in\mathcal{P}} E_P(f)$ or $\rho(f) = \sup_{P\in\mathcal{P}} E_P(-f)$ All our previous results apply. Framework considered in Denis and Martini, also in Soner Touzi Zhang.

$\mathcal{UC}_b(\Omega)$, $\mathcal{P}$ is not necessarily weakly relatively compact

Framework considered by Soner Touzi Zhang, and Nutz.

$n(f) = \sup_{P\in\mathcal{P}} E_P(|f|)$ is a semi-norm.

The closure of $\mathcal{UC}_b(\Omega)$ for the semi-norm n leads to a separable Banach space $L^1(n)$.

Thus the unit ball of the dual space is metrizable compact for the weak* topology. Notice that in this case the unit ball itself and not only its non negative part is metrizable compact. Therefore we get similar results: The norm on $L^1(n)$ can be defined using a numerable subset in $\mathcal{P}$, there is a canonical class of probability measures...