Introduction to dual quantization and first applications

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LPMA-Université Pierre et Marie Curie

New advances in BSDE's for financial engineering and applications

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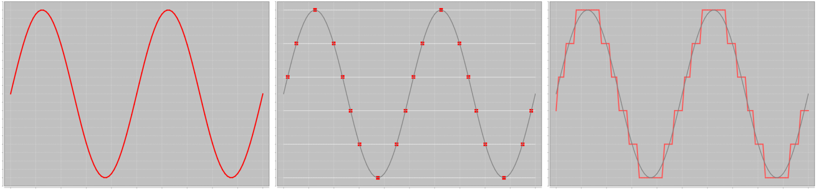
What is Quantization?

- Has its origin in the fields of signal processing in the late 1940's
- Describes the discretization of a random signal and analyses the recovery of the original signal from the discrete one



- Examples: Pulse-Code-Modulation(PCM), JPEG-Compression
- Extensive Survey about the IEEE-History: [Gray/Neuhoff '98]
- Mathematical Foundation of Quantization Theory: [Graf/Luschgy '00]

Definition of Optimal Quantization

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Let $X : (\Omega, \mathcal{S}, \mathbb{P}) \to (\mathbb{R}^d, \mathcal{B}^d)$ be a random vector with values in a normed space $(\mathbb{R}^d, \|\cdot\|)$ and

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The p-th *quantization error* for a grid $\Gamma \subset \mathbb{R}^d$ with size $|\Gamma| \leq n$, $n \in \mathbb{N}$ is given by

$$e^p(X; \Gamma) = \mathbb{E}\,\mathrm{dist}(X, \Gamma)^p = \mathbb{E}\min_{x \in \Gamma}\|X - x\|^p. \tag{1}$$

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The optimal quantization problem consists in minimizing (1) over all grids of size $|\Gamma| \leq n$.

We define the *optimal quantization error* of level n as

$$e_n^p(X) := \inf\Big\{\mathbb{E}\min_{x\in\Gamma}\|X - x\|^p : \Gamma \subset \mathbb{R}^d,\ |\Gamma| \leq n\Big\}.$$

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Theorem (Zador, Kiefer, Bucklew & Wise, Graf & Luschgy, cf. [Graf/Luschgy '00])

Let $X \in L^r(\mathbb{R}^d)$, $r > p$ and denote by φ the λ^d-density of the absolutely continuous part of $\mathbb{P}_X$.
Then

$$\lim_{n\to\infty} n^{p/d} \cdot e_n^p(X) = Q_{p,\|\cdot\|} \cdot \left(\int_{\mathbb{R}^d} \|\varphi\|^{d/(d+p)} \, d\lambda^d \right)^{(d+p)/d}$$

where $Q_{p,\|\cdot\|} = \lim_{n\to\infty} n^{p/d} \cdot e_n^p\big(U([0,1]^d)\big)$.

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- Let $\big(C_i(\Gamma)\big)_{1\le i\le n}$ be a *Voronoi partition* of $\mathbb{R}^d$ generated by Γ, i.e. $\big(C_i(\Gamma)\big)$ is a Borel partition of $\mathbb{R}^d$ satisfying

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- Let $\pi_\Gamma : \mathbb{R}^d \to \Gamma$ the Nearest Neighbor projection,

$$z \mapsto \sum_{i=1}^{n} x_i \mathbf{1}_{C_i(\Gamma)}(z).$$

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$\Rightarrow$ We define the *Voronoi Quantization* as

$$\widehat{X}^\Gamma = \pi_\Gamma(X) = \sum_{i=1}^n x_i \mathbf{1}_{C_i(\Gamma)}(X).$$

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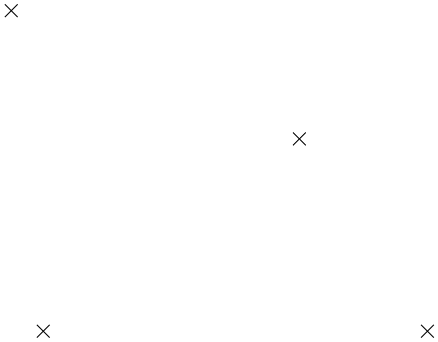

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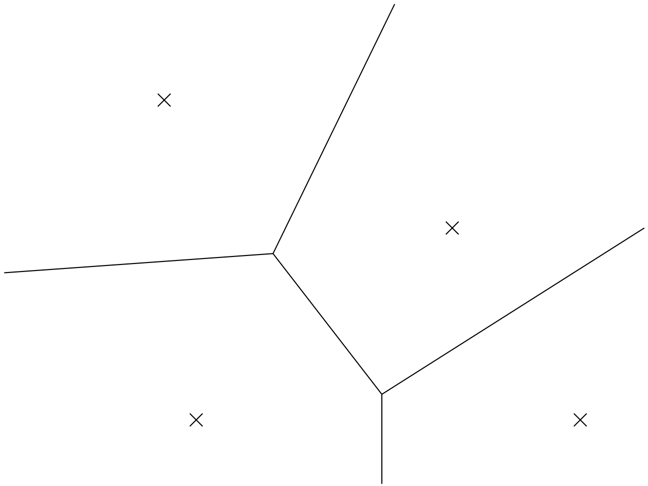

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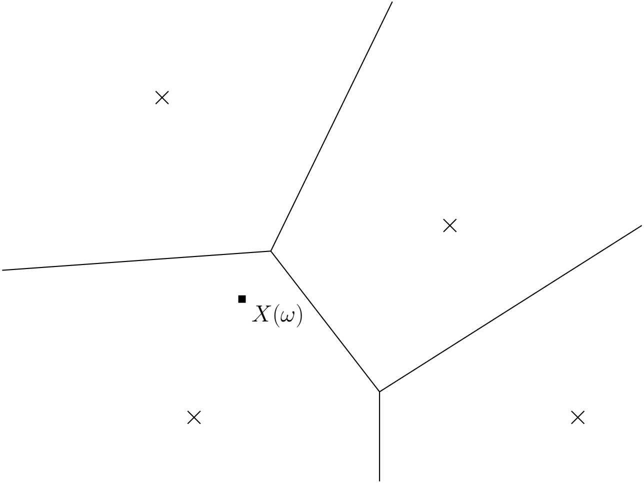

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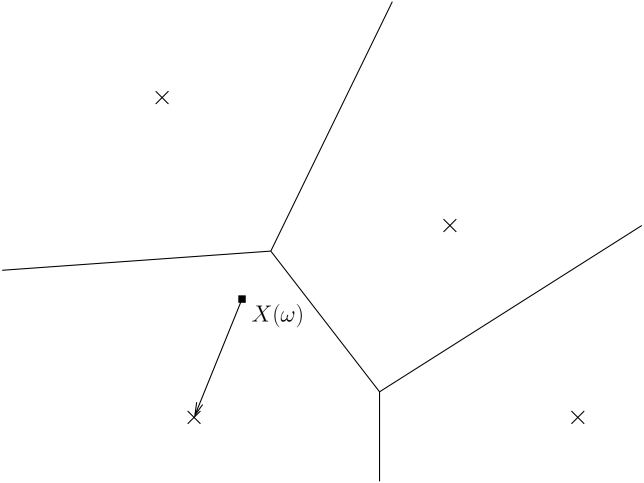

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A further characterization for the optimal quantization error is given by

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⇒ The Nearest Neighbor projection is the coding rule, which yields the smallest L^p-mean approximation error for X.

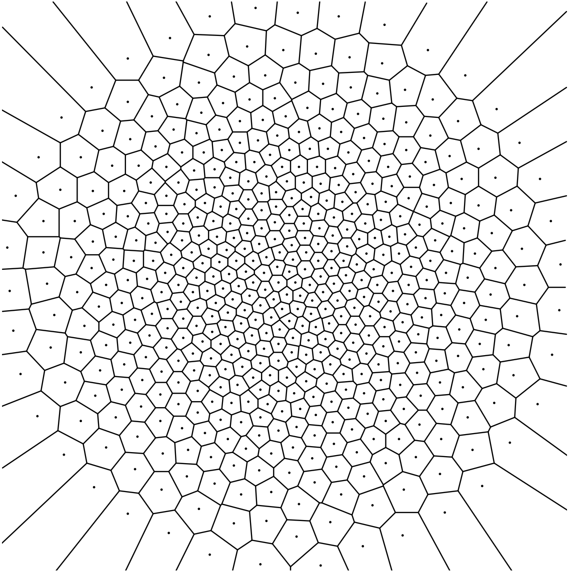

Figure: A Quantizer for $\mathcal{N}(0, I_2)$ of size 500 in $(R^2, \|\cdot\|_2)$.

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Application as Cubature formula

Assume that we have access to the Voronoi-Cell weights

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⇒ The computation of $\mathbb{E}F(\widehat{X}^\Gamma)$ for some Lipschitz continuous $F : \mathbb{R}^d \to \mathbb{R}$ becomes straightforward:

$$\mathbb{E}F(\widehat{X}^\Gamma) = \mathbb{E}F\Big(\sum_{i=1}^{n} x_i \mathbf{1}_{C_i(\Gamma)}(X)\Big) = \sum_{i=1} w_i(\Gamma) F(x_i).$$

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As a first error estimate, we clearly have

$$|\mathbb{E}F(X) - \mathbb{E}F(\widehat{X}^\Gamma)| \leq [F]_{\text{Lip}}\, \mathbb{E}\|X - \widehat{X}^\Gamma\|.$$

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Second order rate

If $F \in C^1_{\mathrm{Lip}}$ and the grid Γ is a *stationary*, i.e.

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$$|\mathbb{E}F(X) - \mathbb{E}F(\widehat{X}^\Gamma)| \leq [F']_{\mathrm{Lip}} \cdot \mathbb{E}\|X - \widehat{X}^\Gamma\|^2.$$

Furthermore, if F is convex, then Jensen's inequality implies for stationary Γ

$$\mathbb{E}F(\widehat{X}^\Gamma) \leq \mathbb{E}F(X).$$

Further Applications

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Approximation of conditional expectations in non-linear problems by means of the Backward Dynamic Programming Principle

- Obstacle Problems: Valuation of Bermudan and American options ([Bally/Pagès '03])
- δ-Hedging for American options ([Bally/Pagès/Printems '05])
- Optimal Stochastic Control problems, e.g. Pricing of Swing options ([Bronstein/Pagès/W. '09] and [Bardou/Bouthemy/Pagès '09])

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Do not map $X(\omega)$ to its nearest neighbor, but split up the projection randomly between the "surrounding" neighbors of $X(\omega)$.

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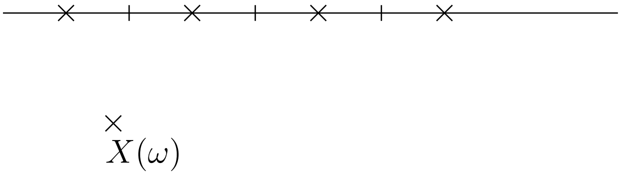

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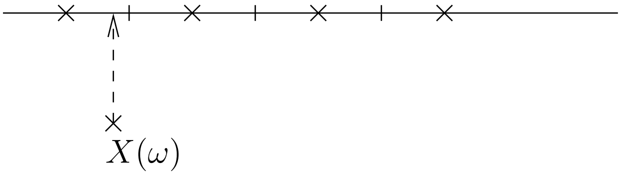

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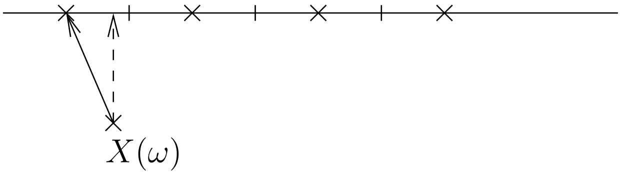

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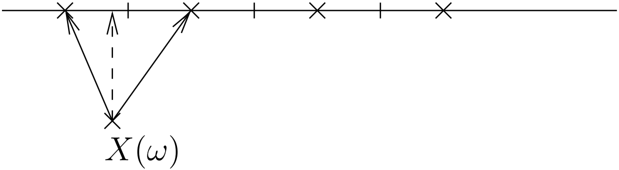

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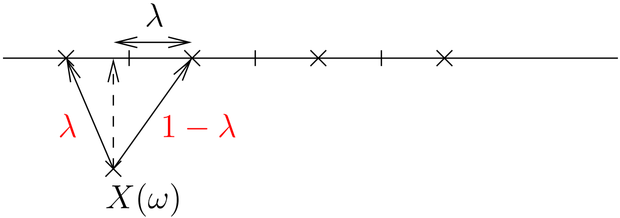

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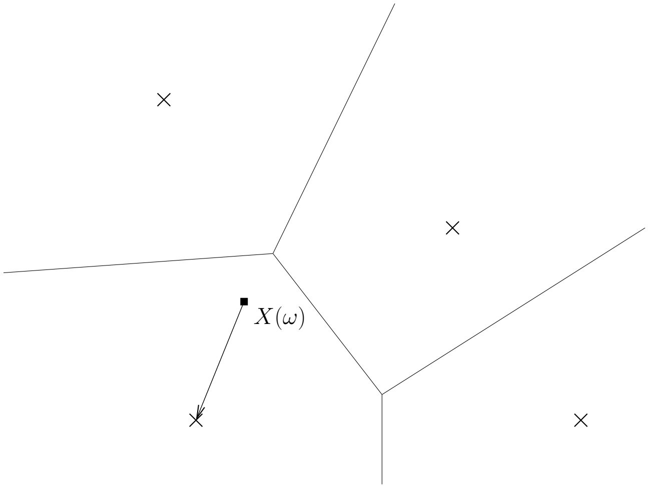

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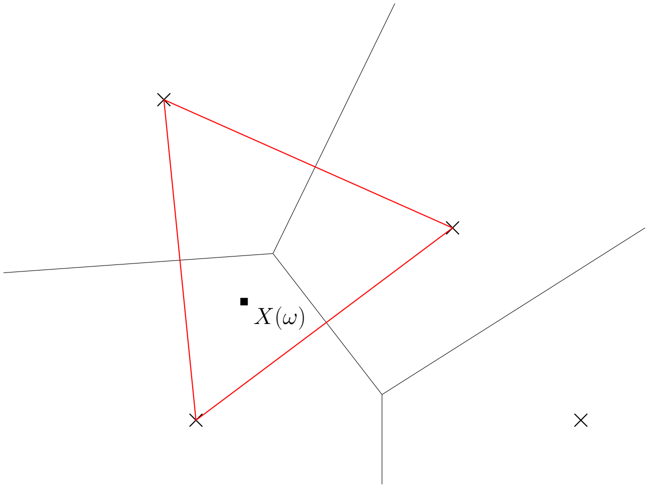

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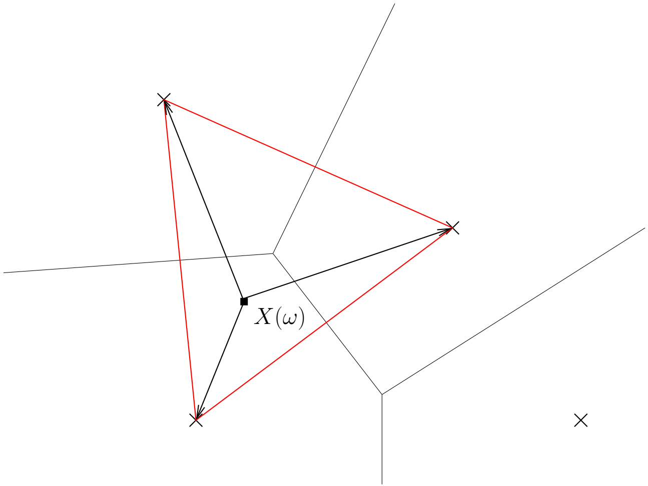

Ideas behind Dual Quantization

Suppose that $\tau = \{t_1, \ldots, t_{d+1}\} \subset \mathbb{R}^d$ spans a d-simplex in $\mathbb{R}^d$, i.e. $t_1, \ldots, t_{d+1}$ are affinely independent.

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Moreover, let $U \sim \mathcal{U}[0,1]$ be defined on some exogenous probability space $(\Omega_0, \mathcal{S}_0, \mathbb{P}_0)$.
Denoting by $\lambda(\xi)$ the barycentric coordinate of $\xi \in \mathrm{conv}\{\tau\}$, we define a dual quantization operator $\mathcal{J}_\tau^U : \mathrm{conv}\{\tau\} \to \tau$ as

$$\xi \mapsto \sum_{i=1}^{d+1} t_i \mathbf{1}_{\left\{\sum_{j=1}^{i-1} \lambda_j(\xi) \leq U < \sum_{j=1}^{i} \lambda_j(\xi)\right\}}.$$

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This operator satisfies a mean preserving property:

$$\mathbb{E}_0\big(\mathcal{J}_\tau^U(\xi)\big) = \sum_{i=1}^{d+1} \lambda_i(\xi) \cdot t_i = \xi, \qquad \forall \xi \in \mathrm{conv}\{\tau\}. \tag{2}$$

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Similarly, we can construct such an operator for any triangulation on a grid $\Gamma = \{x_1, \ldots, x_n\}$, so that (2) holds for any $\xi \in \operatorname{conv}\{\Gamma\}$.

Stationarity

Motivated by this observation, we call a random splitting operator $\mathcal{J}_\Gamma : \Omega_0 \times \mathbb{R}^d \to \Gamma$ for a grid $\Gamma \subset \mathbb{R}^d$ *intrinsic stationary*, if

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The deeper meaning of this definition is revealed by the following Proposition.

Proposition

$\mathcal{J}_\Gamma$ is intrinsic stationary, if and only if it satisfies the stationarity condition

$$\mathbb{E}_{\mathbb{P}\otimes\mathbb{P}_0}\big(\mathcal{J}_\Gamma(Y)|Y\big) = Y$$

for any r.v. $Y : (\Omega, \mathcal{S}, \mathbb{P}) \to (\mathbb{R}^d, \mathcal{B}^d)$ *with* $\mathrm{supp}(\mathbb{P}_Y) \subset \mathrm{conv}\{\Gamma\}$.

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Note that this kind of stationarity now is very robust, since it holds by construction for any r.v. Y with support in Γ.

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Proposition

(a) Let $F \in C^1_{Lip}$, $\Gamma \subset \mathbb{R}^d$ *and* $\mathcal{J}_\Gamma$ *be intrinsic stationary. Then it holds for any r.v.* $Y \in L^2(\mathbb{P})$ *with* $\mathrm{supp}(\mathbb{P}_Y) \subset \mathrm{conv}\{\Gamma\}$,

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(b) If F *is convex, then Jensen's inequality implies*

$$\mathbb{E}F(\mathcal{J}_\Gamma(X)) \geq \mathbb{E}F(X)$$

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▷ The optimal p-th dual quantization error is then defined as

$$d_n^p(X) = \inf\left\{\mathbb{E}\, F^p(X;\Gamma) :\ \Gamma \subset \mathbb{R}^d, |\Gamma| \leq n\right\}.$$

Optimality regions for $F^p(\xi;\Gamma)$

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$$D_I(\Gamma) = \big\{\xi \in \mathbb{R}^d : \exists I^*(\xi) \subset I\big\},$$

or equivalently in term of linear programming

$$D_I(\Gamma) = \Big\{\xi \in \mathbb{R}^d : \lambda^I = A_I^{-1}\left[\begin{smallmatrix}\xi\\1\end{smallmatrix}\right] \geq 0 \text{ and } \sum_{i\in I} \lambda_i^I \|\xi - x_i\|^p = F^p(\xi;\Gamma)\Big\},$$

where

$$I \in \mathcal{I}(\Gamma) = \Big\{J \subset \{1,\dots,n\} : |J| = d+1,\ \mathrm{rk}(A_J) = d+1\Big\}$$

and A_I denotes the submatrix of $\left[\begin{smallmatrix}x_1 & \dots & x_n\\1 & \dots & 1\end{smallmatrix}\right]$ whose columns are given by I.

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The following theorem is an extention of an important theorem by Rajan ([Rajan '91]).

Theorem

Let $\|\cdot\| = |\cdot|_2$, $p = 2$, *and* $\Gamma = \{x_1, \dots, x_n\} \subset \mathbb{R}^d$ *with* $\operatorname{aff.dim}\{\Gamma\} = d$.
(a) If $I \in \mathcal{I}(\Gamma)$ *defines a Delaunay triangle (or d-simplex), then*

$$\lambda^I = A_I^{-1} \begin{bmatrix} \xi \\ 1 \end{bmatrix}$$

provides a solution to $F^p(\xi; \Gamma)$ *for every* $\xi \in \operatorname{conv}\{x_j : j \in I\}$ *i.e.*

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The *optimal dual quantization operator* $\mathcal{J}^*_\Gamma$ is defined as

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One easily checks that this operator is intrinsic stationary.

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Extension to unbounded support

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Since it is not possible to obtain intrinsic stationarity for $\xi \notin \operatorname{conv}\{\Gamma\}$, we have to limit the claim for stationarity to a subset of $\operatorname{supp}(\mathbb{P}_X)$ in order to extend the dual quantization problem to distributions with unbounded support.

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This means that we use a Nearest Neighbor projection beyond $\mathrm{conv}\{\Gamma\}$ while preserving stationarity in the interior of $\mathrm{conv}\{\Gamma\}$.

Existence of optimal dual quantizers

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Theorem ([Pagès/W. '10a])

(a) Let $p > 1$ and assume that $\mathbb{P}_X$ has a compact support. Then for every $n \geq d+1$ optimal dual quantizers actually exist, i.e. the dual quantization problem $d_n^p(X)$ attains its infimum. Moreover, $d_n^p(X)$ is (strictly) decreasing to 0 as $n \to \infty$, if it does not vanish.

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(b) Let $p > 1$ and assume that the distribution $\mathbb{P}_X$ is strongly continuous. Then also optimal quantizers for $\bar{d}_n^p(X)$ exists and $\bar{d}_n^p(X)$ is (strictly) decreasing to 0 as $n \to \infty$, if it does not vanish.

Asymptotic behavior

Theorem ([Pagès/W. '10b])

(a) *Let* $X \in L^{p+}(\mathbb{R}^d)$ *and denote by* φ *the* λ^d*-density of the absolutely continuous part of* $\mathbb{P}_X$.

Asymptotic behavior

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(a) Let $X \in L^{p+}(\mathbb{R}^d)$ and denote by φ the λ^d-density of the absolutely continuous part of $\mathbb{P}_X$. Then

$$\lim_{n\to\infty} n^{p/d} \cdot \bar{d}_n^p(X) = Q_{d,p,\|\cdot\|} \cdot \left(\int_{\mathbb{R}^d} \varphi^{d/(d+p)} \, d\lambda^d \right)^{\frac{d+p}{d}}$$

where $Q_{d,p,\|\cdot\|} = \lim_{n\to\infty} n^{p/d} \cdot d_n^p\big(U([0,1]^d)\big)$.

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(b) If $d = 1$, $Q_{d,p,\|\cdot\|} = \frac{2^{p+1}}{p+2} \lim_{n\to\infty} n^{p/d} \cdot e_n^p\big(U([0,1])\big)$. *If* $d \geq 2$, *???*

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- Use Differentiation of measure to cover the general case (still compact support)
- Random dual quantization argument (so-called extended Pierce Lemma) to get the unbounded case.

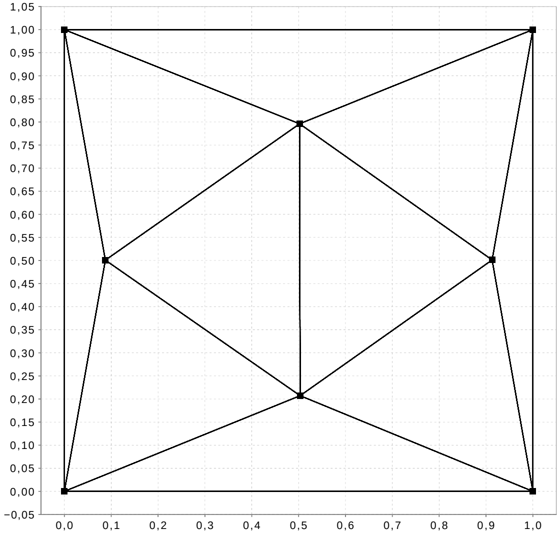

Figure: Dual Quantization for $\mathcal{U}([0,1]^2)$ and $n = 8$

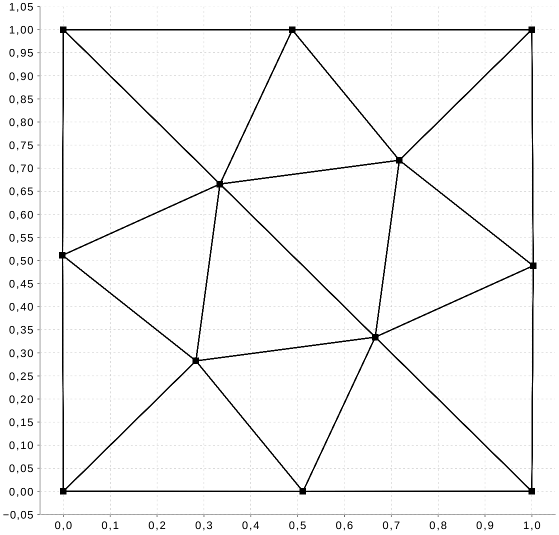

Figure: Dual Quantization for $\mathcal{U}([0,1]^2)$ and $n = 12$

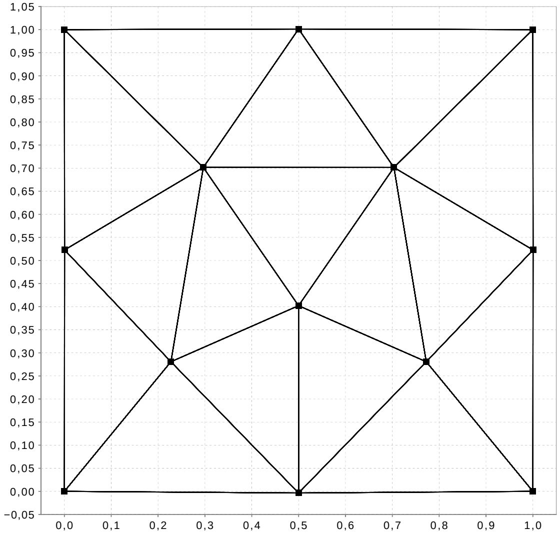

Figure: Dual Quantization for $\mathcal{U}([0,1]^2)$ and $n = 13$

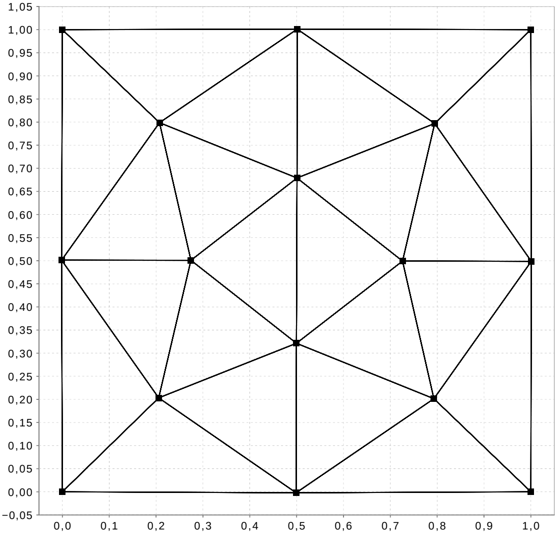

Figure: Dual Quantization for $\mathcal{U}([0,1]^2)$ and $n = 16$

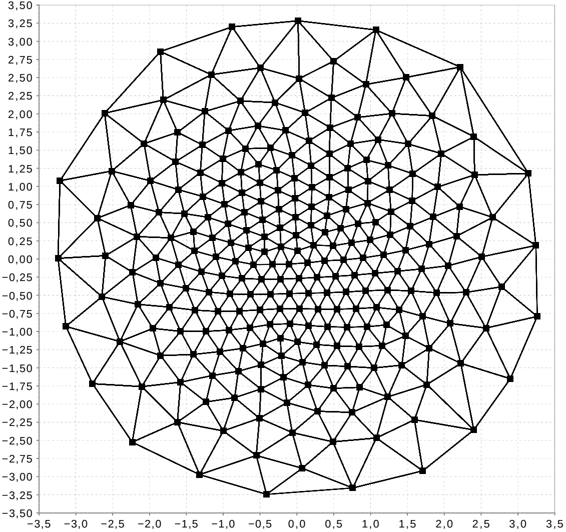

Figure: Dual Quantization for $\mathcal{N}(0, I_2)$ and $N = 250$

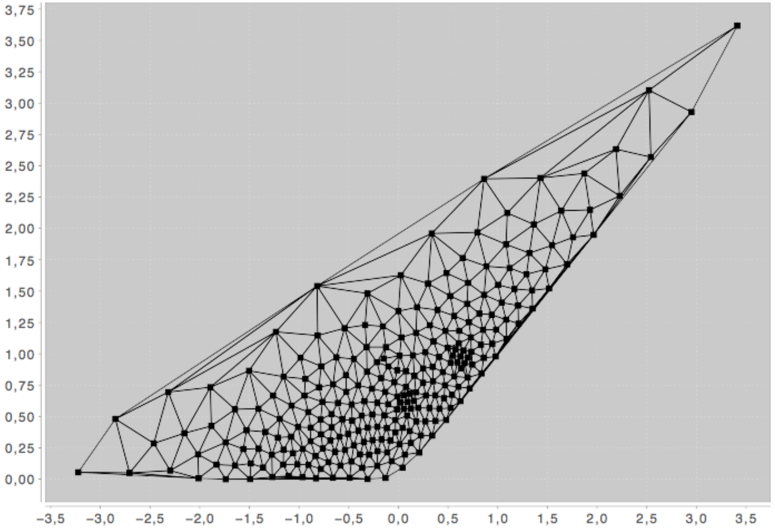

Figure: Joint Dual Quantization of the BM and its supremum, $N = 250$

Numerical Applications

Pricing of Early Exercise Options:

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Using a Backward-Dynamic-Programming principle for the valuation of early exercise options with underlying Markov dynamics $(X_k)_{1\leq k\leq N}$ the numerical challenge in this approach consists in the approximation of conditional expectations

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As in the case of Quantization for numerical cubature, we may replace the Markov chain (X_k) by a Quantization $(\widehat{X}_k)$, so that the the computation of $\mathbb{E}\big(f(X_{k+1})|X_k\big)$ becomes straightforward as

$$\mathbb{E}(f(\hat{X}_{k+1})|\hat{X}_k = x_i^k) = \sum_{j=1}^{n_{k+1}} f(x_j^{k+1})\pi_{ij}^k,$$

with transition probabilities

$$\pi_{ij}^k = \mathbb{P}(\hat{X}_{k+1} = x_j^{k+1}|\hat{X}_k = x_i^k).$$

Numerical Applications

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For the approximation error the following result can be derived.

Proposition

If the mappings $f : \mathbb{R}^d \to \mathbb{R}$ *and*

$$\Phi_{f,k} : \mathbb{R}^d \to \mathbb{R}, x \mapsto \mathbb{E}\big(f(X_{k+1})|X_k = x\big)$$

are Lipschitz, then it holds

$$\begin{aligned}\|\mathbb{E}(f(X_{k+1})|X_k) - \mathbb{E}(f(\hat{X}_{k+1})|\hat{X}_k)\|_p \leq [\Phi_{f,k}]_{Lip} \cdot \|X_k - \hat{X}_k\|_p \\ + [f]_{Lip} \cdot \|X_{k+1} - \hat{X}_{k+1}\|_p.\end{aligned}$$

Valuation of Swing options

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Swing options - A common contract in energy markets

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The right to buy every day a certain quantity of gas/electricity for a given price, where the bought quantity has to respect certain daily and global constraints.

The fair premium of such an contract leads to a stochastic control problem (SCP)

$$\operatorname{esssup}\left\{\mathbb{E}\left(\sum_{k=0}^{n-1} q_k v_k(X_k)|\mathcal{F}_0\right), q_k : (\Omega, \mathcal{F}_k) \to [0,1], \bar{q}_n \in [Q_{\min}, Q_{\max}]\right\}$$

for $\bar{q}_k := \sum_{l=0}^{k-1} q_l$.

Backward Dynamic Programming Principle

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It was shown in [Bardou/Bouthemy/Pagès '09] that (SCP) can be solved by the Backward Dynamic Programming Principle with bang-bang control, i.e we set

$$P_n^n \equiv 0$$
$$P_k^n(Q^k) = \max\left\{ x v_k(X_k) + \mathbb{E}(P_{k+1}^n(\chi^{n-k-1}(Q^k, x))|X_k), x \in \{0,1\} \cap I_{Q^k}^{n-k-1} \right\}$$

with admissible set $I_{Q^k}^M := [(Q_{\min}^k - M)^+ \wedge 1, Q_{\max}^k \wedge 1]$ and $\chi^M(Q^k, x) := \big((Q_{\min}^k - x)^+, (Q_{\max}^k - x) \wedge M\big)$.

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Then $P_0^n(Q_{\min}, Q_{\max})$ is a solution to (SCP).

Backward Dynamic Programming Principle

Using the Quantization $(\widehat{X}_k)$ we define an approximation of (P_k) as

$$\hat{P}_n^n \equiv 0$$

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Under the same assumptions on (X_k) and $f = v_k$ as in the above Proposition about the approximation power of $\mathbb{E}(f(\hat{X}_{k+1})|\hat{X}_k)$ one gets

$$|P_0^n(Q) - \hat{P}_0^n(Q)| \leq C \sum_{k=0}^{n-1} \mathbb{E}\|X_k - \hat{X}_k\|$$

for any reasonable initial global constraints $Q = (Q_{\min}, Q_{\max})$ (see [Bardou/Bouthemy/Pagès '10]).

Example: Gaussian 2-factor model

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In this model, the dynamics of the underlying are given as

$$S_t = s_0 \exp\Big(\sigma_1 \int_0^t e^{-\alpha_1(t-s)} dW_s^1 + \sigma_2 \int_0^t e^{-\alpha_2(t-s)} dW_s^2 - \frac{1}{2}\mu_t\Big)$$

for Brownian Motions W^1 and W^2 with some correlation parameter ρ.

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for Brownian Motions W^1 and W^2 with some correlation parameter ρ. For a time-step parameter Δt we consider the 2-dimensional Markov process

$$X_k = \Big(\int_0^{k\Delta t} e^{-\alpha_1(k\Delta t - s)} dW_s^1, \int_0^{k\Delta t} e^{-\alpha_2(k\Delta t - s)} dW_s^2\Big).$$

Numerical Results

Example

Gaussian 2-factor with parameters

$$s_0 = 20,\ \alpha_1 = 1.11,\ \alpha_2 = 5.4,\ \sigma_1 = 0.36,\ \sigma_2 = 0.21,\ \rho = -0.11$$

and $n = 30$ exercise days for the swing contract.
Results in the Benchmark case of a Call-Strip, i.e. the global consumption constraints are

$$(Q_{\min}, Q_{\max}) = (0, n).$$

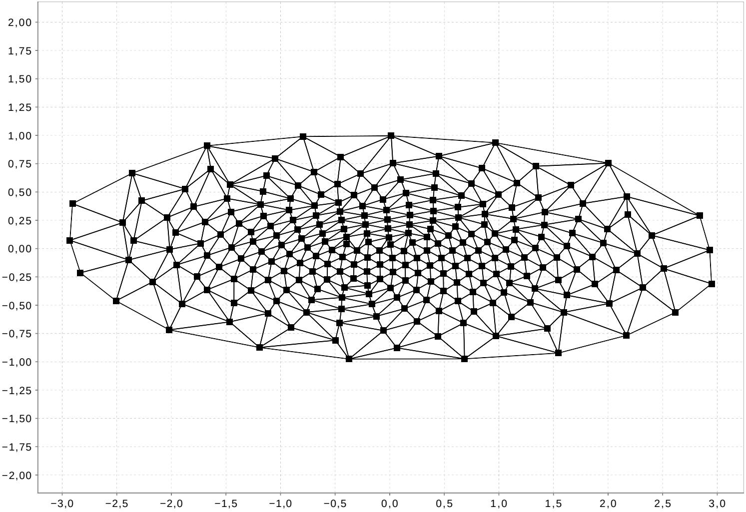
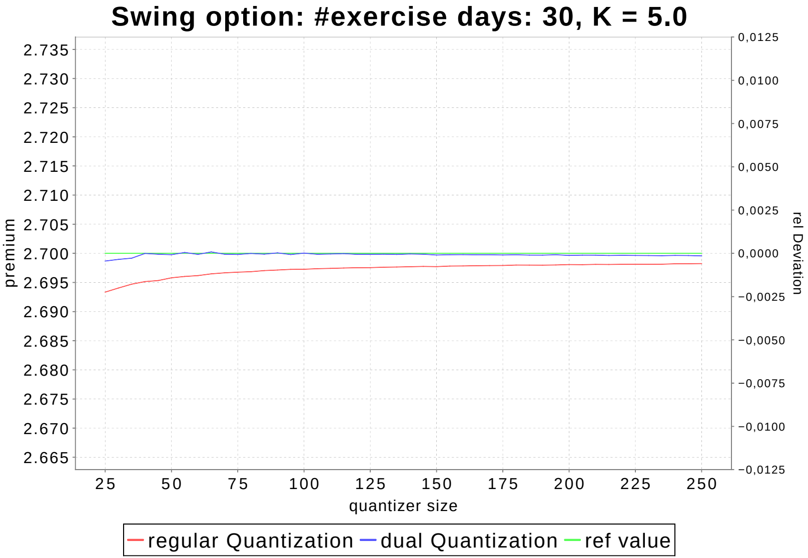
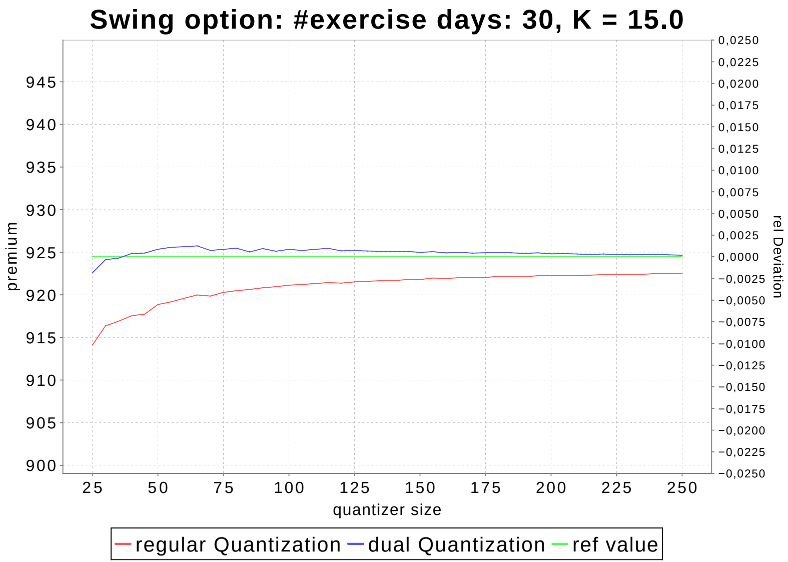


Figure: Triangulation for X_n and $N = 250$ in Gaussian 2-factor model


Swing option: #exercise days: 30, K = 5.0
premium
2.735
2.730
2.725
2.720
2.715
2.710
2.705
2.700
2.695
2.690
2.685
2.680
2.675
2.670
2.665
rel Deviation
0,0125
0,0100
0,0075
0,0050
0,0025
0,0000
−0,0025
−0,0050
−0,0075
−0,0100
−0,0125
25 50 75 100 125 150 175 200 225 250
quantizer size
regular Quantization
dual Quantization
ref value


Swing option: #exercise days: 30, K = 15.0
premium
rel Deviation
quantizer size
945
940
935
930
925
920
915
910
905
900
25 50 75 100 125 150 175 200 225 250
0,0250
0,0225
0,0200
0,0175
0,0150
0,0125
0,0100
0,0075
0,0050
0,0025
0,0000
−0,0025
−0,0050
−0,0075
−0,0100
−0,0125
−0,0150
−0,0175
−0,0200
−0,0225
−0,0250
regular Quantization
dual Quantization
ref value

Bermudan options

In the same way we use the BDP-Principle for the valuation of Bermudan options:

BDP for Bermudan options

$$
\begin{aligned}
\widehat{V}_n &= \varphi_{t_n}(\widehat{X}_n) \\
\widehat{V}_k &= \max\Big\{\varphi_{t_k}(\widehat{X}_k);\, \mathbb{E}\Big(\widehat{V}_{k+1}\Big|\widehat{X}_k\Big)\Big\},\ 0 \le k \le n-1,
\end{aligned}
$$

so that $\widehat{V}_0$ yields an approximation to the Bermudan option premium

$$V_0 = \operatorname{esssup}\{\mathbb{E}\varphi(X_\tau) : \tau \text{ is } \{t_0, \dots, t_n\}\text{-valued stopping time}\}.$$

Numerical Results

Example

2-asset Black-Scholes model with

$$s_0^1 = s_0^2 = 40,\ r = 0.05,\ \sigma_1 = 0.2,\ \sigma_2 = 0.3,\ \rho = 0.5,\ K = 40,$$

for a put on the min, i.e. payoff

$$\varphi(S_t^1, S_t^2) = (K - \min\{S_t^1, S_t^2\})^+.$$

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As underlying Markov process we have chosen a 2-dimensional Brownian Motion with correlation ρ.

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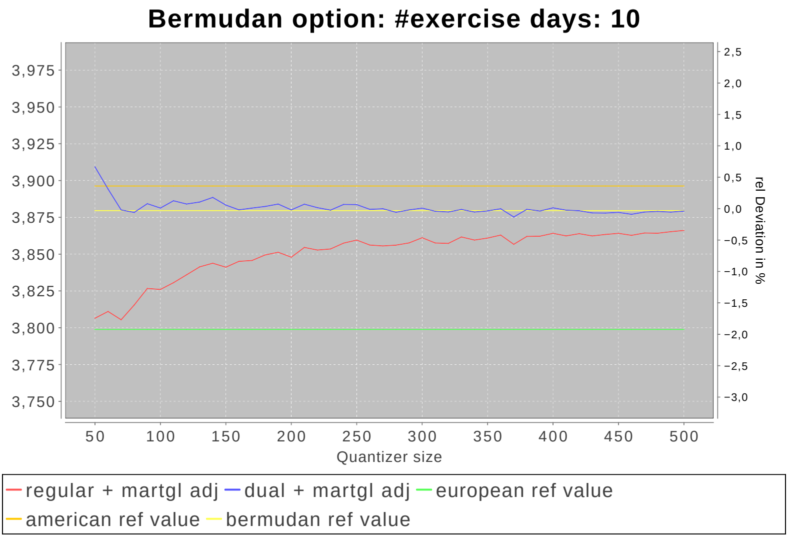
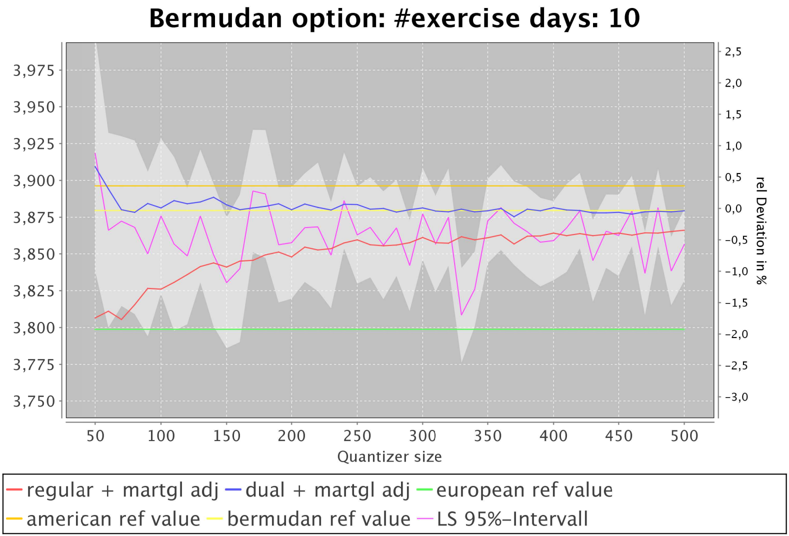
As underlying Markov process we have chosen a 2-dimensional Brownian Motion with correlation ρ.

Reference values were computed using a Boyle-Evnine-Gibbs tree with 10.000 timesteps.

Martingale Adjustment



Martingale Adjustment



Conclusion / Outlook

Conclusion / Summary

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- Application to 3-factor models, etc.

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