# Discussion on "Generalized fractional smoothness and $L^p$-variation of BSDEs with non-Lipschitz terminal condition"

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# Motivations

☐ Want to approximate the solution of :

$$\begin{cases} X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \ , \\ Y_t = g(X_1) + \int_t^1 f(X_s, Y_s, Z_s)ds - \int_t^1 Z_s dW_s \end{cases}$$

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- Formal approximation : $\pi := (0 = t_0 < \cdots < t_i < \cdots < t_n = 1)$

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- Euler scheme type approximation

$$\begin{aligned} Y^\pi_{t_i} &= \mathbb{E}\left[Y^\pi_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + (t_{i+1} - t_i) f\left(X^\pi_{t_i}, Y^\pi_{t_i}, Z^\pi_{t_i}\right) \\ Z^\pi_{t_i} &= (t_{i+1} - t_i)^{-1} \mathbb{E}\left[Y^\pi_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right] \end{aligned}$$

# Error control

- Error term

$$\mathrm{Err}^2 := \max_i \mathbb{E}\left[\sup_{t_i \le t \le t_{i+1}} |Y^\pi_{t_i} - Y_t|^2\right] + \sum_i \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|Z_t - Z^\pi_{t_i}|^2\right] dt$$

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- Important quantities
  - $\mathbb{E}\left[\sup_{t_i \le t \le t_{i+1}} |Y_{t_i} - Y_t|^2\right]$
  - $\int_0^1 \mathbb{E}\left[|Z_t - \bar{Z}^\pi_t|^2\right] dt$ where

$$\bar{Z}^\pi_t := (t_{i+1} - t_i)^{-1} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} Z_s ds \mid \mathcal{F}_{t_i}\right] \text{ on } [t_i, t_{i+1})$$

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☐ Estimates depend crucially on the Lipschitz continuity of $g$ (and $f$).

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$$\sum_{k \geq 1} (k+1)^{\beta} \alpha_k^2 < \infty$$

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- Results are sharp.

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□ Weak error as in Gobet and Labart ?

□ Pure numerical schemes ?