Discussion on “Generalized fractional smoothness and $L^p$-variation of BSDEs with non-Lipschitz terminal condition”

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Tamerza 2010
Motivations

- Want to approximate the solution of:

\[
\begin{align*}
X_t &= X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \\
Y_t &= g(X_1) + \int_t^1 f(X_s, Y_s, Z_s)ds - \int_t^1 Z_s dW_s
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- Formal approximation: \( \pi := (0 = t_0 < \cdots < t_i < \cdots < t_n = 1) \)

\[
Y_{t_i}^{\pi} \sim Y_{t_{i+1}}^{\pi} + (t_{i+1} - t_i)f(X_{t_i}^{\pi}, Y_{t_i}^{\pi}, Z_{t_i}^{\pi}) - Z_{t_i}^{\pi}(W_{t_{i+1}} - W_{t_i})
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- Euler scheme type approximation

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\begin{align*}
Y_{t_i}^\pi &= \mathbb{E} \left[ Y_{t_{i+1}}^\pi \mid \mathcal{F}_{t_i} \right] + (t_{i+1} - t_i)f (X_{t_i}^\pi, Y_{t_i}^\pi, Z_{t_i}^\pi) \\
Z_{t_i}^\pi &= (t_{i+1} - t_i)^{-1} \mathbb{E} \left[ Y_{t_{i+1}}^\pi (W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i} \right]
\end{align*}
\]
Error control

- Error term

\[
\text{Err}^2 := \max_i \mathbb{E} \left[ \sup_{t_i \leq t \leq t_{i+1}} |Y_{t_i}^{\pi} - Y_t|^2 \right] + \sum_i \int_{t_i}^{t_{i+1}} \mathbb{E} \left[ |Z_t - Z_{t_i}^{\pi}|^2 \right] dt
\]
Error control

- **Error term**

\[
\text{Err}^2 := \max_i \mathbb{E} \left[ \sup_{t_i \leq t \leq t_{i+1}} |Y^\pi_{t_i} - Y_t|^2 \right] + \sum_i \int_{t_i}^{t_{i+1}} \mathbb{E} \left[ |Z_t - Z^\pi_{t_i}|^2 \right] dt
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- **Important quantities**
Error control

- Error term

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- Important quantities
  - \( \mathbb{E} \left[ \sup_{t_i \leq t \leq t_{i+1}} |Y_{t_i} - Y_t|^2 \right] \)
Error control

- Error term

$$\text{Err}^2 := \max_i \mathbb{E} \left[ \sup_{t_i \leq t \leq t_{i+1}} |Y_t^\pi - Y_t|^2 \right] + \sum_i \int_{t_i}^{t_{i+1}} \mathbb{E} \left[ |Z_t - Z_t^\pi|^2 \right] dt$$

- Important quantities
  - $$\mathbb{E} \left[ \sup_{t_i \leq t \leq t_{i+1}} |Y_t - Y_t|^2 \right]$$
  - $$\int_0^1 \mathbb{E} \left[ |Z_t - \bar{Z}_t^\pi|^2 \right] dt$$

where

$$\bar{Z}_t^\pi := \frac{t_{i+1} - t_i}{t_{i+1}} \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} Z_s ds \mid \mathcal{F}_{t_i} \right] \text{ on } [t_i, t_{i+1})$$
For Lipschitz coefficients
Lipschitz case

- For Lipschitz coefficients
  - $Y = v(\cdot, X)$ with $v$ 1/2-Hölder in $t$ and Lipschitz in $x$
    - $\mathbb{E} \left[ \sup_{t_i \leq t \leq t_{i+1}} |Y_{t_i} - Y_t|^2 \right] = O(|\pi|)$
For Lipschitz coefficients

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- Ma and Zhang: \[ \int_0^1 \mathbb{E} \left[ |Z_t - \bar{Z}_t|^2 \right] dt = O(|\pi|) \text{ too!} \]
For Lipschitz coefficients

- \( Y = v(\cdot, X) \) with \( v \) 1/2-Hölder in \( t \) and Lipschitz in \( x \)
  \[ \Rightarrow \mathbb{E} \left[ \sup_{t_i \leq t \leq t_{i+1}} |Y_t - Y_{t_i}|^2 \right] = O(|\pi|) \]

- Ma and Zhang: \( \int_0^1 \mathbb{E} \left[ |Z_t - \bar{Z}_t^\pi|^2 \right] dt = O(|\pi|) \) too!

Estimates depend crucially on the Lipschitz continuity of \( g \) (and \( f \)).
Non-Lipschitz case but $f = 0$ (or linear in $(Y, Z)$)

- S. Geiss (with Ch. Geiss and A. Toivola) for $X = W$ or $X = e^W$
Non-Lipschitz case but $f = 0$ (or linear in $(Y, Z)$)

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  - Need to consider the *fractional smoothness of $g$* : $g \in B_{2,2}^\beta$, $\beta \in (0, 1]$, i.e
    \[
    \sum_{k \geq 1} (k + 1)^\beta \alpha_k^2 < \infty
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    with $\alpha_k$ the coefficients of $g$ in the Hermite polynomial basis.
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  - Leads to $\int_0^1 \mathbb{E} \left[ |Z_t - \tilde{Z}_t^\pi|^2 \right] dt = O(|\pi|^\beta)$
  - Need to adapt the time net accordingly: $t_i := 1 - (1 - i/n)^{1/\beta}$
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SUMER Geiss (with Ch. Geiss and A. Toivola) for $X = W$ or $X = e^W$

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- In this case can approximate with an error $1/\sqrt{n}$.
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  - Need to adapt the time net accordingly: $t_i := 1 - (1 - i/n)^{1/\beta}$
  - In this case can approximate with an error $1/\sqrt{n}$.
  - Results are sharp.
Non-Lipschitz case - General case

- Gobet and Makhlouf

- Need to revisit the notion of fractional smoothness of $g$:

  $g \in L_2$, $\beta \in (0, 1]$, i.e.

  $\sup_{t<1} \left( 1 - t \right)^{\beta} E \left[ \left| g(X_T) - E g(X_T) \right|^2 | F_t \right] < \infty$.

- From this (plus Malliavin calculus) obtain estimates on $\nabla v$ and $\nabla^2 v$.

- Also leads to $\int_0^1 E \left[ |Z_t - \bar{Z}_\pi_t|^2 \right] dt = O \left( |\pi|^{\beta} \right)$.

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  - From this (plus Malliavin calculus) obtain estimates on $\nabla v$ and $\nabla^2 v$.
  - Also leads to $\int_0^1 \mathbb{E} \left[ |Z_t - \bar{Z}_t^\pi|^2 \right] dt = O(|\pi|^\beta)$
  - Need to adapt the time net accordingly: $t_i := 1 - (1 - i/n)^{1/\beta}$
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Present paper

- \( g \) discretely path depend.
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- $g$ discretely path depend.
- $L^p$ estimates.
Questions

☐ $f$ irregular as in A. Richou?
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☐ Is the uniform ellipticity condition so crucial? (no need in the Lipschitz setting, even for Reflected BSDE - B. and Chassagneux - or Dirichlet type conditions - B. and Mennozì)

☐ Weak error as in Gobet and Labart?

☐ Pure numerical schemes?
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