Generalized fractional smoothness and $\mathbb{L}_p$-variation of BSDEs with non-Lipschitz terminal condition

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Joint work with C. Geiss (Innsbruck University) and S. Geiss (Innsbruck University).

Framework: usual Markovian BSDE

$$\mathbf{Y_t} = \xi + \int_{\mathbf{t}}^{\mathbf{T}} \mathbf{f}(\mathbf{s}, \mathbf{X_s}, \mathbf{Y_s}, \mathbf{Z_s})\mathbf{ds} - \int_{\mathbf{t}}^{\mathbf{T}} \mathbf{Z_s dW_s}$$

with a **Lipschitz generator** f, where

- $X = (X_t)_{t\in[0,T]}$ is a $\mathbb{R}^d$-valued forward diffusion $dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$,
- $\xi = \mathbf{g}(\mathbf{X_{r_1}}, \ldots, \mathbf{X_{r_L}})$ for a given number L of times $0 = r_0 < r_1 < \cdots < r_L = T$ and a measurable function $g : (\mathbb{R}^d)^L \to \mathbb{R}$.

  We assume that $\xi \in \mathbb{L}_\mathbf{p}$ for a $\mathbf{p} \geq \mathbf{2}$.

Main concern: which connection between regularities of ξ and Z?

Our purpose

Provide necessary and sufficient conditions of the **$\mathbb{L}_p$-variations of the Y and Z** processes, in terms of the **fractional regularity of the terminal condition ξ**.

► See also El Karoui's talk about quadratic BSDE, providing estimates on $\mathbb{E}(\frac{1}{2}\int_S^T |Z_r|^2 dr|\mathcal{F}_S)$ in terms of $\Phi_{S,T}$.

Potential applications

- Error analysis of the convergence of the time discretization of BSDEs: it is known (in the $\mathbb{L}_2$-case) that one main contribution is due to

$$\mathcal{E}\big(Z,(t_i)_i\big) = \left(\mathbb{E}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} \|Z_t - \overline{Z}_{t_{i-1}}\|_2^2 dt\right)^{1/2}$$

where

$$\overline{Z}_{t_{i-1}} := \frac{1}{t_i - t_{i-1}}\mathbb{E}\left[\int_{t_{i-1}}^{t_i} Z_s ds|\mathcal{F}_{t_{i-1}}\right]$$

is an appropriate projection of Z.

$\rightsquigarrow$ Optimal choice of deterministic time grids.

- Tight estimates on the gradient of semi-linear PDEs.

Background results

- $\mathbb{L}_\infty$-functionals **[Zhang '04]**: if $\xi = \phi(X_t : t \leq T)$ with

$$|\phi(\mathbf{x}) - \phi(\mathbf{x}')| \leq \mathbf{C}_\phi \sup_{\mathbf{t}\leq\mathbf{T}} |\mathbf{x}(\mathbf{t}) - \mathbf{x}'(\mathbf{t})|,$$

then $$\mathcal{E}\big(\mathbf{Z}, (\mathbf{t_i})_\mathbf{i}\big) \leq \mathbf{C}|(\mathbf{t_i})_\mathbf{i}|^{\mathbf{1/2}}.$$

$\rightsquigarrow$ Uniform grids $t_i = i\frac{T}{N}$ yield the optimal rate of convergence $N^{1/2}$.

- Possible extensions to **jumps** **[Bouchard, Elie '08]**, to **RBSDE** **[Bouchard, Chassagneux '06]**, to **BSDE with random terminal time** **[Bouchard, Menozzi '09]**.

- Fractional regularity **[G', Makhlouf '10]** (see Makhlouf's talk): if $\xi = g(X_T)$ with

$$\|\mathbf{g}(\mathbf{X_T}) - \mathbb{E}(\mathbf{g}(\mathbf{X_T})|\mathcal{F}_\mathbf{t})\|_\mathbf{2} \leq \mathbf{c}(\mathbf{T}-\mathbf{s})^{\frac{\theta}{\mathbf{2}}}$$

then $\mathcal{E}\big(\mathbf{Z}, (\mathbf{t_i})_\mathbf{i}\big) \leq \frac{\mathbf{C}}{\mathbf{N}^{\theta/\mathbf{2}}}$ for uniform grids. The rate $N^{1/2}$ is achieved using non-uniform grids.

Our work: extension of G'-Makhlouf results for path-dependent ξ.

Agenda of the talk

1. Definitions and assumptions
2. A general equivalence result: necessary and sufficient conditions
3. Simple sufficient conditions
4. Sketch of proofs

Definition of fractional regularity

Definition. Let $\Theta = (\theta_1, ..., \theta_L) \in (0,1]^L$ and $2 \leq p < \infty$. Then we let $(\xi, f) \in B^{\Theta}_{p,\infty}(X)$ provided that there is some $c > 0$ such that

$$\|\mathbf{Y}_{\mathbf{r_l}} - \mathbb{E}(\mathbf{Y}_{\mathbf{r_l}}|\mathcal{F}_{\mathbf{s}})\|_{\mathbf{p}} \leq \mathbf{c}(\mathbf{r_l} - \mathbf{s})^{\frac{\theta_l}{2}}$$

for all $l = 1, ..., L$ and $r_{l-1} \leq s < r_l$.

Property. It measures the rate of best approximation in $\mathbb{L}_p$ of Y_{r_l} by a $\mathcal{F}_s$-measurable random variable, as $s \to r_l$.

Proof: $$\inf_{\mathbf{V} \in \mathbb{L}_{\mathbf{p}}(\mathcal{G})} \|\mathbf{U} - \mathbf{V}\|_{\mathbf{p}} \leq \|\mathbf{U} - \mathbb{E}(\mathbf{U}|\mathcal{G})\|_{\mathbf{p}} \leq \mathbf{2} \inf_{\mathbf{V} \in \mathbb{L}_{\mathbf{p}}(\mathcal{G})} \|\mathbf{U} - \mathbf{V}\|_{\mathbf{p}}.$$

Remark. Specializing to the linear one-step Gaussian case ($X = W$, $T = L = 1$ and $f = 0$) we obtain that

$$g(W_1) \in B^{(\theta)}_{p,\infty}(W) \text{ if and only if } g \in B^{\theta}_{p,\infty}(\mathbb{R}^d, \gamma_d),$$

i.e. the usual interpolation space taking the standard Gaussian measure γ_d on $\mathbb{R}^d$ (see [GH07, Toi09]).

Standing assumptions on X, f and g

($A_{\sigma,b}$) We have $1 \leq d < \infty$, $\sigma\sigma* \geq \delta I_{\mathbb{R}^d}$ for some $\delta > 0$, and the functions b, σ are bounded and C^2 with respect to the space variable, with uniformly bounded and γ-Hölder C^0 derivatives, for some $\gamma \in (0,1]$. In addition, b and σ are $\frac{1}{2}$-Hölder C^0 in time uniformly in space.

(A_f) The function $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is continuous in (t, x, y, z) and continuously differentiable in x, y and z with uniformly bounded derivatives. In particular, there is some $L_f > 0$ such that

$$|f(s, x_1, y_1, z_1) - f(s, x_2, y_2, z_2)| \leq L_f[|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|].$$

(A_g) The terminal condition $\xi := g(X_{r_1}, ..., X_{r_L})$ is such that $\xi \in \mathbb{L}_p$ $(p \geq 2)$.

(A_g^{pol}) The terminal functional is at most of polynomial growth (stronger than A_g).

A preliminary result (extension of [Zha05])

Proposition. Under $(A_{\sigma,b})$, (A_f) and (A_g^{pol}), the solution (Y, Z) is on $[r_{l-1}, r_l)$ by

$$Y_t = u_l(\overline{X}_{l-1}; t, X_t) \text{ and } Z_t = v_l(\overline{X}_{l-1}; t, X_t)\sigma(t, X_t),$$

where we set $\overline{X}_{l-1} := (X_1, \ldots, X_{l-1})$, for some measurable functions u_l and v_l that satisfy the following properties:

(i) $u_l(\overline{x}_{l-1}; \cdot, \cdot) : [r_{l-1}, r_l) \times \mathbb{R}^d \to \mathbb{R}$ is continuous and continuously differentiable w.r.t. the space variable with $\nabla_x u_l(\overline{x}_{l-1}; t, x) = v_l(\overline{x}_{l-1}; t, x)$, where $\overline{x}_{l-1} = (x_1, \ldots, x_{l-1})$,

(ii) there are $\alpha_l, q_{l,1}, ..., q_{l,l} \in [1, \infty)$ such that

$$\sup_{t\in[r_{l-1},r_l)} |u_l(\overline{x}_{l-1}; t, x)| + \sup_{t\in[r_{l-1},r_l)} \sqrt{r_l - t}|v_l(\overline{x}_{l-1}; t, x)| \le \alpha_l(1 + |x_1|^{q_{l,1}} + \cdots + |x_{l-1}|^{q_{l,l-1}} + |x|^{q_{l,l}}).$$

A general equivalence result

Theorem. Assume that $(A_{\sigma,b})$, (A_f) and (A_g^{pol}) are satisfied. For $2 \leq p < \infty$ and $\Theta \in (0,1]^L$ consider the following conditions:

(C1) There is some $c_1 > 0$ such that, for $r_{l-1} \leq s < t < r_l$,

$$\|Z_t - Z_s\|_p \leq c_1 \Big(\int_s^t (r_l - r)^{\theta_l - 2} dr \Big)^{\frac{1}{2}}.$$

(C2) There is some $c_2 > 0$ with $\|Z_t\|_p \leq c_2 (r_l - t)^{\frac{\theta_l - 1}{2}}$ for $r_{l-1} \leq t < r_l$.

(C3) There is some $c_3 > 0$ such that, for $r_{l-1} \leq s < t \leq r_l$,

$$\|Y_t - Y_s\|_p \leq c_3 \Big(\int_s^t (r_l - r)^{\theta_l - 1} dr \Big)^{\frac{1}{2}}.$$

(C4) $(\xi, f) \in B_{p,\infty}^{\Theta}(X)$.

Then one has that $\mathbf{(C1)} \overset{\Theta \in (0,1)^L}{\Longrightarrow} \mathbf{(C2)} \Longleftrightarrow \mathbf{(C3)} \Longleftrightarrow \mathbf{(C4)} \Longrightarrow \mathbf{(C1)}$.

$(C1) \Longrightarrow (C2)$ is false in general (explicit counter-example with $X = W$, $f \equiv 0$).

Another equivalence result - In terms of the second derivatives of the linear PDE

We consider the piece-wise linearization of the backward equation by letting

$$g_l(x_1, ..., x_l) := u_l(x_1, ..., x_{l-1}; r_l, x_l)$$

and $F_l(\overline{x}_{l-1}; \cdot, \cdot) : [r_{l-1}, r_l] \times \mathbb{R}^d \to \mathbb{R}$ by

$$F_l(x_1, ..., x_{l-1}; t, x) = F_l(\overline{x}_{l-1}; t, x) := \mathbb{E} g_l(x_1, ..., x_{r_{l-1}}, X_{r_l}^{t,x}),$$

which solves a linear PDE on the interval $[r_{l-1}, r_l)$ for fixed $x_1, ..., x_{l-1} \in \mathbb{R}^d$.

Theorem. Under $(A_{\sigma,b})$, (A_f) and (A_g^{pol}), we have $(C4) \iff (C5)$ where

(C4) $(\xi, f) \in B_{p,\infty}^{\Theta}(X)$.

(C5) There is some $c_5 > 0$ such that, for any l and for $r_{l-1} \le t < r_l$,

$$\left\| \left(\int_{r_{l-1}}^{t} |(D^2 F_l)(\overline{X}_{l-1}; r, X_r)|^2 dr \right)^{\frac{1}{2}} \right\|_p \le c_5 (r_l - t)^{\frac{\theta_l - 1}{2}}.$$

Another equivalence result - In terms of adapted splines

Non regular grids

A common idea when we deal with singularities located at the times $0 < r_1 < \cdots < r_L = T$ is to compensate them with adapted time-nets, that are more concentrated near these points.

Definition. For $\Theta \in (0,1]^L$ we let $\tau^{n,\Theta} = (t_k^{n,\Theta})_{k=1}^{nL}$ be given by $t_0^{n,\Theta} := 0$ and

$$t_k^{n,\Theta} := r_{l-1} + (r_l - r_{l-1}) \left(1 - \left(1 - \frac{k-(l-1)n}{n}\right)^{\frac{1}{\theta_l}}\right) \text{ for } (l-1)n < k \leq ln.$$

Remark. In [GM10], taking this type of grid with θ_l strictly smaller that the regularity index of (ξ, f) yields an optimal rate of $\mathbb{L}_2$-convergence w.r.t. n (see also [GG04] for $f \equiv 0$).

Definition. **Adapted splines.** Given a time-net $\tau = (t_k)_{k=0}^n$ with $0 = t_0 < \cdots < t_n = T$ we say that the process $S = (S_t)_{t\in[0,T]}$ **is an adapted spline based on τ** provided that S_{t_k} is $\mathcal{F}_{t_k}$-measurable for all $k = 0, ..., n$ and

$$S_t := \frac{t_k - t}{t_k - t_{k-1}} S_{t_{k-1}} + \frac{t - t_{t_{k-1}}}{t_k - t_{k-1}} S_{t_k} \text{ for } t_{k-1} \le t \le t_k.$$

This is a useful concept to find efficient approximation schemes for stochastic processes where the whole path needs to be approximated but the adaptedness of the approximation is not fully needed, see [CMGR07].

Theorem. Under $(A_{\sigma,b})$, (A_f) and (A_g^{pol}), we have $(C4) \Longleftrightarrow (C6)$ where

(C4) $(\xi, f) \in B^{\Theta}_{p,\infty}(X)$.

(C6) There is some $c_6 > 0$ such that for all $n = 1, 2, ...$ there is an adapted spline $S^n = (S^n_t)_{t\in[0,T]}$ based on $\tau^{n,\Theta}$ such that

$$\|Y_{r_l} - Y_{r_{l-1}}\|_p + \sqrt{n} \sup_{t\in[0,T]} \|Y_t - S^n_t\|_p \le c_6.$$

Sufficient conditions (independent from the generator f)

First approach. We use the Hirsch fractional smoothness approach [Hir99]. Consider B an independent copy of W, supported on the same $(\Omega, \mathcal{F}, \mathbb{P})$.

For a given measurable function $\eta : [0, T] \mapsto [-1, 1]$, we define W^η a new BM:

$$W_t^\eta := \int_0^t \sqrt{1 - \eta(s)^2} dW_s + \int_0^t \eta(s) dB_s.$$

We denote by $(\mathcal{F}_t^\eta)_{0 \le t \le T}$ the $\mathbb{P}$-augmented natural filtration of W^η.

Then, define the strong solution to

$$X_t^\eta = x_0 + \int_0^t b(s, X_s^\eta) ds + \int_0^t \sigma(s, X_s^\eta) dW_s^\eta$$

and for $\xi^\eta \in \mathbb{L}_p(\mathcal{F}_T^\eta)$, define the $\mathbb{L}_p$-solution (in the filtration $(\mathcal{F}_t^\eta)_{0 \le t \le T}$) to

$$Y_t^\eta = \xi^\eta + \int_t^T f(s, X_s^\eta, Y_s^\eta, Z_s^\eta) ds - \int_t^T Z_s^\eta dW_s^\eta.$$

$\rightsquigarrow$ **$\mathbb{L}_p$-distance between (X, Y, Z) and (X^η, Y^η, Z^η)?**

Main tool

Theorem. Assume that $(A_{\sigma,b})$ and (A_f) are satisfied. Then for $2 \leq p < \infty$ and $\xi, \xi^\eta \in \mathbb{L}_p$ we have that

$$\left\| \sup_{0\leq t\leq T} |X_t^\eta - X_t| \right\|_p + \left\| \sup_{0\leq t\leq T} |Y_t^\eta - Y_t| \right\|_p + \left\| \left(\int_0^T |Z_t^\eta - Z_t|^2 dt \right)^{1/2} \right\|_p$$

$$\leq c \left[\|\xi^\eta - \xi\|_p + \sqrt{\int_0^T \eta(t)^2 dt} \right]$$

where $c > 0$ depends at most on p, T, f, b and σ.

Proof. Based on a clever application of a priori estimates for BSDE in $\mathbb{L}_p$.

Applications (relaxing A_g^{pol} to A_g)

We choose specific pertubations η defined as follows: for $0 \le t < r \le T$ we let

$$\eta_{\mathbf{t},\mathbf{r}}(\mathbf{s}) := \chi_{(\mathbf{t},\mathbf{r}]}(\mathbf{s}).$$

Corollary. Assume $(A_{b,\sigma})$, (A_f) and (A_g) for some $p \ge 2$. Let

$$\xi^{\mathbf{t},\mathbf{r}} := \mathbf{g}(\mathbf{X}_{\mathbf{r_1}}^{\eta_{\mathbf{t},\mathbf{r}}}, ..., \mathbf{X}_{\mathbf{r_L}}^{\eta_{\mathbf{t},\mathbf{r}}})$$

for $0 \le t < r \le T$. Setting $\Theta = (\theta_1, ..., \theta_L) \in (0,1]^L$, if there is a constant $c > 0$ such that one has that

$$\|\xi - \xi^{\mathbf{t},\mathbf{r_l}}\|_{\mathbf{p}} \le \mathbf{c}(\mathbf{r_l} - \mathbf{t})^{\frac{\theta_l}{2}}$$

for all $l = 1, ..., L$ and $r_{l-1} \le t < r_l$, then $(\xi, \mathbf{f}) \in \mathbf{B}^{\mathbf{\Theta}}_{\mathbf{p},\infty}$.

Consequently, for some $c > 0$ we have

(C'1) $\|\mathbf{Z_t} - \mathbf{Z_s}\|_{\mathbf{p}} \le \mathbf{c_1}\left(\int_{\mathbf{s}}^{\mathbf{t}} (\mathbf{r_l} - \mathbf{r})^{\theta_l - 2} \mathbf{dr}\right)^{\frac{1}{2}}$, for a.e. (s,t) s.t. $r_{l-1} \le s < t < r_l$;

(C'2) $\|\mathbf{Z_t}\|_{\mathbf{p}} \le \mathbf{c_2}(\mathbf{r_l} - \mathbf{t})^{\frac{\theta_l - 1}{2}}$, for a.e. t $r_{l-1} \le t < r_l$;

(C'3) $\|\mathbf{Y_t} - \mathbf{Y_s}\|_{\mathbf{p}} \le \mathbf{c_3}\left(\int_{\mathbf{s}}^{\mathbf{t}} (\mathbf{r_l} - \mathbf{r})^{\theta_l - 1} \mathbf{dr}\right)^{\frac{1}{2}}$, for every (s,t) s.t. $r_{l-1} \le s < t \le r_l$.

Proof of $\|\xi - \xi^{t,r_l}\|_p \le c(r_l - t)^{\frac{\theta_l}{2}} \Longrightarrow (\xi, f) \in B^{\Theta}_{p,\infty}$.

For $r_{l-1} \le t < r_l$ we get that

$$\begin{aligned}
\|Y_{r_l} - \mathbb{E}(Y_{r_l}|\mathcal{F}_t)\|_p &= \|Y_{r_l} - \mathbb{E}^W(Y_{r_l}|\mathcal{F}_t)\|_p \\
&= \|Y_{r_l} - \mathbb{E}^B(Y_{r_l}^{\eta_{t,r_l}}|\mathcal{F}_t)\|_p \\
&\le \|Y_{r_l} - Y_{r_l}^{\eta_{t,r_l}}\|_p \\
&\le c\left[\|\xi - \xi^{t,r_l}\|_p + \sqrt{\int_0^T \eta_{t,r_l}(r)^2 dr}\right] \\
&\le c\left[c(r_l - t)^{\frac{\theta_l}{2}} + \sqrt{r_l - t}\right].
\end{aligned}$$

$\square$

Applications (Cont'd)

Proposition. Let $g_1, ..., g_L$ be of bounded variation, i.e.

$$\sup_{\mathbf{N}} \sup_{-\infty<\mathbf{x_0}<\cdots<\mathbf{x_N}<\infty} \sum_{\mathbf{k=1}}^{\mathbf{N}} |\mathbf{g_l}(\mathbf{x_k}) - \mathbf{g}(\mathbf{x_{k-1}})| < \infty$$

for any l. Consider

$$\xi = \Phi(g_1(X_{r_1}), ..., g_L(X_{r_L}))$$

such that

$$|\mathbf{\Phi}(\mathbf{x_1}, ..., \mathbf{x_L}) - \mathbf{\Phi}(\mathbf{y_1}, ..., \mathbf{y_L})| \leq \kappa \left(|\mathbf{x_1} - \mathbf{y_1}|^\alpha + \cdots + |\mathbf{x_L} - \mathbf{y_L}|^\alpha\right).$$

Then one has that $(\xi, \mathbf{f}) \in \bigcap_{\mathbf{0}<\theta<\frac{\alpha}{\mathbf{2p}}} \mathbf{B}_{\mathbf{p},\infty}^{(\theta,\ldots,\theta)}(\mathbf{X})$.

Proof

According to **[Avikainen '09]** one has

$$\mathbb{E}|g(X)-g(Y)|^p \le c(p,q,g,X)\|X-Y\|_q^{\frac{q}{q+1}}$$

whenever $g \in BV$, $1 \le p, q < \infty$, where X has a bounded density. Hence,

$$\begin{aligned}
\|\xi - \xi^{t,r_l}\|_p &\le \kappa \sum_{j=1}^{L} \left\| |g_j(X_{r_j}) - g_j(X_{r_j}^{\eta_{t,r_l}})|^\alpha \right\|_p \\
&\le \kappa \sum_{j=l}^{L} \|g_j(X_{r_j}) - g_j(X_{r_j}^{\eta_{t,r_l}})\|_p^\alpha \\
&\le c' \sum_{j=l}^{L} \|X_{r_j} - X_{r_j}^{\eta_{t,r_l}}\|_q^{\frac{\alpha q}{(q+1)p}} \\
&\le c''(r_l - t)^{\frac{\alpha q}{(q+1)2p}}.
\end{aligned}$$

Now we can take a large q. $\square$

Sufficient conditions (Cont'd)

Second approach. Relies on a simple iteration procedure.

Theorem. Assume that

$$|\mathbf{\Phi}(\mathbf{x_1},\dots,\mathbf{x_L})-\mathbf{\Phi}(\mathbf{x'_1},\dots,\mathbf{x'_L})| \le \sum_{\mathbf{l=1}}^{\mathbf{L}} [|\mathbf{g_l}(\mathbf{x_l})-\mathbf{g_l}(\mathbf{x'_l})| + \psi_{\mathbf{l}}(\mathbf{x_1},\dots,\mathbf{x_l};\mathbf{x'_1},\dots,\mathbf{x'_l})|\mathbf{x_l}-\mathbf{x'_l}|]$$

where the functions Φ, g_l and ψ_l are polynomially bounded Borel functions such that

$$\|\mathbf{g_l}(\mathbf{X_{r_l}})-\mathbb{E}(\mathbf{g_l}(\mathbf{X_{r_l}})|\mathcal{F}_{\mathbf{t}})\|_{\mathbf{p}} \le \mathbf{c}(\mathbf{r_l}-\mathbf{t})^{\frac{\theta_{\mathbf{l}}}{\mathbf{2}}}$$

for $l=1,\dots,L$, $0<\theta_l\le 1$, and $r_{l-1}\le t\le r_l$.

Then,

$$(\xi,\mathbf{f})\in \mathbf{B}^{\mathbf{\Theta}}_{\mathbf{p},\infty}(\mathbf{X}).$$

Example: for $\mathbf{\Phi}(\mathbf{x}) = \mathbf{1}_{\underline{\mathbf{a}}_\mathbf{1}<\mathbf{x_1}<\bar{\mathbf{a}}_\mathbf{1}}\cdots\mathbf{1}_{\underline{\mathbf{a}}_\mathbf{L}<\mathbf{x_L}<\bar{\mathbf{a}}_\mathbf{L}}$, we have $\theta_{\mathbf{l}}=\frac{\mathbf{1}}{\mathbf{2p}}$.

Sketch of proofs of the main equivalence results

$(C1) \Longrightarrow (C2)$ for $0 < \theta_l < 1$

Quite easy since

$$\begin{aligned}
\|Z_t\|_p &\leq \|Z_{r_{l-1}}\|_p + c_1\Big(\int_{r_{l-1}}^{t} (r_l - r)^{\theta_l - 2} dr\Big)^{\frac{1}{2}} \\
&= \|Z_{r_{l-1}}\|_p + c_1\Big(\frac{1}{1-\theta_l}\big[(r_l - t)^{\theta_l - 1} - (r_l - r_{l-1})^{\theta_l - 1}\big]\Big)^{\frac{1}{2}} \\
&\leq \|Z_{r_{l-1}}\|_p + c_1(1-\theta_l)^{-\frac{1}{2}}(r_l - t)^{\frac{\theta_l - 1}{2}}.
\end{aligned}$$

$\square$

$$(C2) \Longrightarrow (C3)$$

Quite easy since

$$\begin{aligned}
\|Y_t - Y_s\|_p &= \left\| \int_s^t f(r, X_r, Y_r, Z_r) dr - \int_s^t Z_r dW_r \right\|_p \\
&\le \int_s^t \|f(r, X_r, Y_r, Z_r)\|_p dr + a_p \left(\int_s^t \|Z_r\|_p^2 dr \right)^{\frac{1}{2}} \quad (\text{BDG} + p \ge 2) \\
&\le \cdots
\end{aligned}$$

$\square$

$(C3) \Longrightarrow (C4)$

Quite easy since

$$\begin{aligned}
\|Y_{r_l} - \mathbb{E}(Y_{r_l}|\mathcal{F}_s)\|_p &\leq \|Y_{r_l} - Y_s\|_p + \|Y_s - \mathbb{E}(Y_{r_l}|\mathcal{F}_s)\|_p \\
&\leq 2\|Y_{r_l} - Y_s\|_p \\
&\leq 2c_3\Big(\int_s^{r_l}(r_l - r)^{\theta_l - 1}dr\Big)^{\frac{1}{2}} \\
&= 2c_3\sqrt{\frac{1}{\theta_l}}(r_l - s)^{\frac{\theta_l}{2}}.
\end{aligned}$$

□

$$(C4) \Longrightarrow (C5)$$

A bit less easy (inspired from [GM10]).

Crucial Malliavin calculus $\mathbb{L}_p$-estimates on PDE, under fractional smoothness assumptions

Set

$$\mathbf{R_s} := \|\mathbf{Y_{r_l}} - \mathbb{E}(\mathbf{Y_{r_l}}|\mathcal{F}_\mathbf{s})\|_\mathbf{p}.$$

Then

$$\left\|\nabla_\mathbf{x}\mathbf{F_l}(\overline{\mathbf{X}}_{\mathbf{l-1}};\mathbf{s},\mathbf{X_s})\right\|_\mathbf{p} \leq \kappa_{\mathbf{p'}}\frac{\|\mathbf{Y_{r_l}} - \mathbb{E}(\mathbf{Y_{r_l}}|\mathcal{F}_\mathbf{s})\|_\mathbf{p}}{\sqrt{\mathbf{r_l}-\mathbf{s}}}$$

and

$$\left\|\mathbf{D^2F_l}(\overline{\mathbf{X}}_{\mathbf{l-1}};\mathbf{s},\mathbf{X_s})\right\|_\mathbf{p} \leq \kappa_{\mathbf{p'}}\frac{\|\mathbf{Y_{r_l}} - \mathbb{E}(\mathbf{Y_{r_l}}|\mathcal{F}_\mathbf{s})\|_\mathbf{p}}{\mathbf{r_l}-\mathbf{s}}.$$

(see Makhlouf's talk).

Then, using ellipticity and BDG inequalities, one has

$$
\begin{aligned}
& \left\| \left(\int_{r_{l-1}}^{t} |(D^2 F_l)(\overline{X}_{l-1}; s, X_s)|^2 ds \right)^{\frac{1}{2}} \right\|_p \\
\leq \quad & c \sum_{k=1}^{d} \left\| \int_{r_{l-1}}^{t} (\nabla_x (\partial_{x_k} F_l) \sigma)(\overline{X}_{l-1}; s, X_s) dW_s \right\|_p .
\end{aligned}
$$

Thanks to the PDE solved by F_l, we have

$$
\begin{aligned}
& \partial_{x_k} F_l(\overline{X}_{l-1}; t, X_t) - \partial_{x_k} F_l(\overline{X}_{l-1}; r_{l-1}, X_{r_{l-1}}) \\
= \quad & - \int_{r_{l-1}}^{t} \left\{ \langle \partial_{x_k} b, \nabla_x F_l \rangle + \frac{1}{2} \langle \partial_{x_k} A, D^2 F_l \rangle \right\} (\overline{X}_{l-1}; s, X_s) ds \\
& + \int_{r_{l-1}}^{t} \left\{ \nabla_x (\partial_{x_k} F_l) \sigma \right\} (\overline{X}_{l-1}; s, X_s) dW_s .
\end{aligned}
$$

Then, this implies that

$$
\begin{aligned}
&\left\| \int_{r_{l-1}}^{t} (\nabla_x(\partial_{x_k} F_l)\sigma)(\overline{X}_{l-1}; s, X_s) dW_s \right\|_p \\
\leq\ & \|\nabla_x F_l(\overline{X}_{l-1}; t, X_t)\|_p + \|\nabla_x F_l(\overline{X}_{l-1}; r_{l-1}, X_{r_{l-1}})\|_p \\
&+ \left\| \int_{r_{l-1}}^{t} \big\{\langle \partial_{x_k} b, \nabla_x F_l\rangle + \frac{1}{2}\langle \partial_{x_k} A, D^2 F_l\rangle\big\}(\overline{X}_{l-1}; s, X_s) ds \right\|_p \\
\leq\ & \kappa_p\sqrt{d}\frac{R_t}{\sqrt{r_l - t}} + \kappa_p\sqrt{d}\frac{R_{r_{l-1}}}{\sqrt{r_l - r_{l-1}}} + \kappa_p\sqrt{d}\|\partial_{x_k} b\|_\infty \int_{r_{l-1}}^{r_l} \frac{R_s}{\sqrt{r_l - s}} ds \\
&+ \kappa_p d \frac{\|\partial_{x_k} A\|_\infty}{2} \int_{r_{l-1}}^{r_l} \frac{R_s}{r_l - s} ds \\
&\ldots
\end{aligned}
$$

$\square$

$(C5) \Longrightarrow (C2)$

Decomposition of the Z process: $\delta \mathbf{Z_r} := \mathbf{Z_r} - \nabla_{\mathbf{x}} \mathbf{F_l}(\overline{\mathbf{X}}_{\mathbf{l-1}}; \mathbf{r}, \mathbf{X_r})\sigma(\mathbf{r}, \mathbf{X_r})$.

Two steps:

- prove that $\sup_{r_{l-1} \leq r < r_l} \|\delta Z_r\|_p < \infty$ (crucially linked to the fact that **this difference of BSDEs has zero terminal condition**).
- estimate $\nabla_x F_l(\overline{X}_{l-1}; r, X_r)$ in $\mathbb{L}_p$. It relies on the

Lemma. There exists a constant $c > 0$ such that, for all $r_{l-1} \leq s < t < r_l$,

$$\begin{aligned}
&\|\nabla_x F_l(\overline{X}_{l-1}; t, X_t) - \nabla_x F_l(\overline{X}_{l-1}; s, X_s)\|_p \\
&\qquad \leq c(t-s)\|\nabla_x F_l(\overline{X}_{l-1}; r_{l-1}, X_{r_{l-1}})\|_p \\
&\qquad\quad + c(t-s)\left\|\Big(\int_{r_{l-1}}^s |D^2 F_l(\overline{X}_{l-1}; v, X_v)|^2 dv\Big)^{\frac{1}{2}}\right\|_p \\
&\qquad\quad + c\left\|\Big(\int_s^t |D^2 F_l(\overline{X}_{l-1}; v, X_v)|^2 dv\Big)^{\frac{1}{2}}\right\|_p .
\end{aligned}$$

$$(C4) \Longrightarrow (C1)$$

We linearize the BSDE, following the approach of [GM10].

Long and technical…

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