Generalized fractional smoothness
and $L_p$-variation of BSDEs
with non-Lipschitz terminal condition

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Joint work with C. Geiss (Innsbruck University) and S. Geiss (Innsbruck University).
Framework: usual Markovian BSDE

\[ Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s \]

with a Lipschitz generator \( f \), where

- \( X = (X_t)_{t \in [0,T]} \) is a \( \mathbb{R}^d \)-valued forward diffusion \( dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \),
- \( \xi = g(X_{r_1}, \ldots, X_{r_L}) \) for a given number \( L \) of times \( 0 = r_0 < r_1 < \cdots < r_L = T \) and a measurable function \( g : (\mathbb{R}^d)^L \to \mathbb{R} \).

We assume that \( \xi \in L_p \) for a \( p \geq 2 \).

Main concern: which connection between regularities of \( \xi \) and \( Z \)?

Our purpose

Provide necessary and sufficient conditions of the \( L_p \)-variations of the \( Y \) and \( Z \) processes, in terms of the fractional regularity of the terminal condition \( \xi \).

See also El Karoui’s talk about quadratic BSDE, providing estimates on \( \mathbb{E}(\frac{1}{2} \int_S^T |Z_r|^2 dr |\mathcal{F}_S) \) in terms of \( \Phi_{S,T} \).
Potential applications

- Error analysis of the convergence of the time discretization of BSDEs: it is known (in the $L_2$-case) that one main contribution is due to

$$
\mathcal{E}(Z, (t_i)_i) = \left( \mathbb{E} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|Z_t - Z_{t_{i-1}}\|_2^2 dt \right)^{1/2}
$$

where

$$
\overline{Z}_{t_{i-1}} := \frac{1}{t_i - t_{i-1}} \mathbb{E} \left[ \int_{t_{i-1}}^{t_i} Z_s ds \mid \mathcal{F}_{t_{i-1}} \right]
$$

is an appropriate projection of $Z$.

$\leadsto$ Optimal choice of deterministic time grids.

- Tight estimates on the gradient of semi-linear PDEs.
Background results

- $L_\infty$-functionals [Zhang '04]: if $\xi = \phi(X_t : t \leq T)$ with

$$|\phi(x) - \phi(x')| \leq C_\phi \sup_{t \leq T} |x(t) - x'(t)|,$$

then

$$\mathcal{E}(Z, (t_i)_i) \leq C|t_i|^{1/2}.$$  

$\Rightarrow$ Uniform grids $t_i = i\frac{T}{N}$ yield the optimal rate of convergence $N^{1/2}$.

- Possible extensions to jumps [Bouchard, Elie '08], to RBSDE [Bouchard, Chassagneux '06], to BSDE with random terminal time [Bouchard, Menozzi '09].

- Fractional regularity [G’, Makhlouf ’10] (see Makhlouf’s talk): if $\xi = g(X_T)$ with

$$\|g(X_T) - \mathbb{E}(g(X_T)|\mathcal{F}_t)\|_2 \leq c(T - s)^{\frac{\theta}{2}}$$

then

$$\mathcal{E}(Z, (t_i)_i) \leq \frac{C}{N^{\theta/2}}$$

for uniform grids. The rate $N^{1/2}$ is achieved using non-uniform grids.

Our work: extension of G’-Makhlouf results for path-dependent $\xi$. 
Agenda of the talk

1. Definitions and assumptions

2. A general equivalence result: necessary and sufficient conditions

3. Simple sufficient conditions

4. Sketch of proofs
Definition of fractional regularity

Definition. Let $\Theta = (\theta_1, ..., \theta_L) \in (0, 1]^L$ and $2 \leq p < \infty$. Then we let $(\xi, f) \in B_{p, \infty}^\Theta(X)$ provided that there is some $c > 0$ such that

$$\|Y_{r_l} - \mathbb{E}(Y_{r_l} | \mathcal{F}_s)\|_p \leq c(r_l - s)^{\theta_l/2}$$

for all $l = 1, ..., L$ and $r_{l-1} \leq s < r_l$.

Property. It measures the rate of best approximation in $\mathbb{L}_p$ of $Y_{r_l}$ by a $\mathcal{F}_s$-measurable random variable, as $s \to r_l$.

Proof: $\inf_{V \in \mathbb{L}_p(\mathcal{G})} \|U - V\|_p \leq \|U - \mathbb{E}(U|\mathcal{G})\|_p \leq 2 \inf_{V \in \mathbb{L}_p(\mathcal{G})} \|U - V\|_p$.

Remark. Specializing to the linear one-step Gaussian case ($X = W$, $T = L = 1$ and $f = 0$) we obtain that

$$g(W_1) \in B_{p, \infty}^{(\theta)}(W) \text{ if and only if } g \in B_{p, \infty}^\theta(\mathbb{R}^d, \gamma_d),$$

i.e. the usual interpolation space taking the standard Gaussian measure $\gamma_d$ on $\mathbb{R}^d$ (see [GH07, Toi09]).
Standing assumptions on $X$, $f$ and $g$

\((A_{\sigma,b})\) We have $1 \leq d < \infty$, $\sigma \sigma^* \geq \delta I_{\mathbb{R}^d}$ for some $\delta > 0$, and the functions $b, \sigma$ are bounded and $C^2$ with respect to the space variable, with uniformly bounded and $\gamma$-Hölder $C^0$ derivatives, for some $\gamma \in (0, 1]$. In addition, $b$ and $\sigma$ are $\frac{1}{2}$-Hölder $C^0$ in time uniformly in space.

\((A_f)\) The function $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is continuous in $(t, x, y, z)$ and continuously differentiable in $x, y$ and $z$ with uniformly bounded derivatives. In particular, there is some $L_f > 0$ such that

$$|f(s, x_1, y_1, z_1) - f(s, x_2, y_2, z_2)| \leq L_f [|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|].$$

\((A_g)\) The terminal condition $\xi := g(X_{r_1}, ..., X_{r_L})$ is such that $\xi \in \mathbb{L}_p$ ($p \geq 2$).

\((A_{g,pol})\) The terminal functional is at most of polynomial growth (stronger than $A_g$).
A preliminary result (extension of [Zha05])

**Proposition.** Under \((A_{\sigma,b}), (A_f)\) and \((A_{pol}^g)\), the solution \((Y, Z)\) is on \([r_{l-1}, r_l)\) by

\[
Y_t = u_l(\bar{X}_{l-1}; t, X_t) \quad \text{and} \quad Z_t = v_l(\bar{X}_{l-1}; t, X_t)\sigma(t, X_t),
\]

where we set \(\bar{X}_{l-1} := (X_1, \ldots, X_{l-1})\), for some measurable functions \(u_l\) and \(v_l\) that satisfy the following properties:

(i) \(u_l(\bar{x}_{l-1}; \cdot, \cdot) : [r_{l-1}, r_l) \times \mathbb{R}^d \to \mathbb{R}\) is continuous and continuously differentiable w.r.t. the space variable with

\[
\nabla_x u_l(\bar{x}_{l-1}; t, x) = v_l(\bar{x}_{l-1}; t, x),
\]

where \(\bar{x}_{l-1} = (x_1, \ldots, x_{l-1})\),

(ii) there are \(\alpha_l, q_{l,1}, \ldots, q_{l,l} \in [1, \infty)\) such that

\[
\sup_{t \in [r_{l-1}, r_l)} |u_l(\bar{x}_{l-1}; t, x)| + \sup_{t \in [r_{l-1}, r_l)} \sqrt{r_l - t}|v_l(\bar{x}_{l-1}; t, x)| \leq \alpha_l(1 + |x_1|^{q_{l,1}} + \cdots + |x_{l-1}|^{q_{l,l-1}} + |x|^{q_{l,l}}).
\]
**A general equivalence result**

**Theorem.** Assume that \((A_{\sigma,b}), (A_f)\) and \((A_{g}^{pol})\) are satisfied. For \(2 \leq p < \infty\) and \(\Theta \in (0, 1]^L\) consider the following conditions:

(C1) There is some \(c_1 > 0\) such that, for \(r_{l-1} \leq s < t < r_l\),

\[
\|Z_t - Z_s\|_p \leq c_1 \left( \int_s^t (r_l - r)^{\theta_l - 2} \, dr \right)^{\frac{1}{2}}.
\]

(C2) There is some \(c_2 > 0\) with \(\|Z_t\|_p \leq c_2 (r_l - t)^{\frac{\theta_l - 1}{2}}\) for \(r_{l-1} \leq t < r_l\).

(C3) There is some \(c_3 > 0\) such that, for \(r_{l-1} \leq s < t \leq r_l\),

\[
\|Y_t - Y_s\|_p \leq c_3 \left( \int_s^t (r_l - r)^{\theta_l - 1} \, dr \right)^{\frac{1}{2}}.
\]

(C4) \((\xi, f) \in B_{p,\infty}^\Theta(X)\).

Then one has that \(\text{(C1)} \overset{\Theta \in (0, 1]^L}{\iff} \text{(C2)} \iff \text{(C3)} \iff \text{(C4)} \implies \text{(C1)}\).

\(\text{⚠️ (C1)} \implies \text{(C2)}\) is false in general (explicit counter-example with \(X = W, f \equiv 0\)).
Another equivalence result - In terms of the second derivatives of the linear PDE

We consider the piece-wise linearization of the backward equation by letting

$$g_l(x_1, \ldots, x_l) := u_l(x_1, \ldots, x_{l-1}; r_l, x_l)$$

and $F_l(x_{l-1}; \cdot, \cdot) : [r_{l-1}, r_l] \times \mathbb{R}^d \to \mathbb{R}$ by

$$F_l(x_1, \ldots, x_{l-1}; t, x) = F_l(x_{l-1}; t, x) := \mathbb{E}g_l(x_1, \ldots, x_{r_l-1}, X_{r_l}^{t,x}),$$

which solves a linear PDE on the interval $[r_{l-1}, r_l)$ for fixed $x_1, \ldots, x_{l-1} \in \mathbb{R}^d$.

**Theorem.** Under $(A_{\sigma,b})$, $(A_f)$ and $(A_{g^{pol}})$, we have $(C4) \iff (C5)$ where

(C4) $(\xi, f) \in B^\Theta_{p,\infty}(X)$.

(C5) There is some $c_5 > 0$ such that, for any $l$ and for $r_{l-1} \leq t < r_l$,

$$\left\| \left( \int_{r_{l-1}}^t |(D^2 F_l)(X_{l-1}; r, X_r)|^2 dr \right)^{\frac{1}{2}} \right\|_p \leq c_5 (r_l - t)^{\frac{\theta_l-1}{2}}.$$
Another equivalence result - In terms of adapted splines

Non regular grids

A common idea when we deal with singularities located at the times
$0 < r_1 < \cdots < r_L = T$ is to compensate them with adapted time-nets, that are
more concentrated near these points.

**Definition.** For $\Theta \in (0, 1]^L$ we let $\tau^{n, \Theta} = (t^{n, \Theta}_k)_{k=1}^{nL}$ be given by $t^{n, \Theta}_0 := 0$ and

$$
t^{n, \Theta}_k := r_{l-1} + (r_l - r_{l-1}) \left( 1 - \left( 1 - \frac{k - (l - 1)n}{n} \right) \frac{1}{\theta_l} \right)
$$

for $(l - 1)n < k \leq ln$.

**Remark.** In [GM10], taking this type of grid with $\theta_l$ strictly smaller that the
regularity index of $(\xi, f)$ yields an optimal rate of $\mathbb{L}_2$-convergence w.r.t. $n$ (see also
[GG04] for $f \equiv 0$).
**Definition.** **Adapted splines.** Given a time-net $\tau = (t_k)_{k=0}^n$ with $0 = t_0 < \cdots < t_n = T$ we say that the process $S = (S_t)_{t \in [0, T]}$ is an **adapted spline based on** $\tau$ provided that $S_{t_k}$ is $\mathcal{F}_{t_k}$-measurable for all $k = 0, \ldots, n$ and

$$S_t := \frac{t_k - t}{t_k - t_{k-1}} S_{t_{k-1}} + \frac{t - t_{k-1}}{t_k - t_{k-1}} S_{t_k} \quad \text{for} \quad t_{k-1} \leq t \leq t_k.$$ 

This is a useful concept to find efficient approximation schemes for stochastic processes where the whole path needs to be approximated but the adaptedness of the approximation is not fully needed, see [CMGR07].

**Theorem.** Under $(A_{\sigma,b})$, $(A_f)$ and $(A_{pol}^g)$, we have $(C4) \iff (C6)$ where

$(C4)$ $(\xi, f) \in B_{p,\infty}^\Theta(X)$.

$(C6)$ There is some $c_6 > 0$ such that for all $n = 1, 2, \ldots$ there is an adapted spline $S^n = (S^n_t)_{t \in [0,T]}$ based on $\tau^n, \Theta$ such that

$$\|Y_{r_l} - Y_{r_{l-1}}\|_p + \sqrt{n} \sup_{t \in [0, T]} \|Y_t - S^n_t\|_p \leq c_6.$$
**Sufficient conditions (independent from the generator $f$)**

**First approach.** We use the Hirsch fractional smoothness approach [Hir99]. Consider $B$ an independent copy of $W$, supported on the same $(\Omega, \mathcal{F}, \mathbb{P})$.

For a given measurable function $\eta : [0, T] \mapsto [-1, 1]$, we define $W^\eta$ a new BM:

$$W^\eta_t := \int_0^t \sqrt{1 - \eta(s)^2} dW_s + \int_0^t \eta(s) dB_s.$$  

We denote by $(\mathcal{F}^\eta_t)_{0 \leq t \leq T}$ the $\mathbb{P}$-augmented natural filtration of $W^\eta$.

Then, define the strong solution to

$$X^\eta_t = x_0 + \int_0^t b(s, X^\eta_s) ds + \int_0^t \sigma(s, X^\eta_s) dW^\eta_s$$

and for $\xi^\eta \in \mathbb{L}_p(\mathcal{F}^\eta_T)$, define the $\mathbb{L}_p$-solution (in the filtration $(\mathcal{F}^\eta_t)_{0 \leq t \leq T}$) to

$$Y^\eta_t = \xi^\eta + \int_t^T f(s, X^\eta_s, Y^\eta_s, Z^\eta_s) ds - \int_t^T Z^\eta_s dW^\eta_s.$$  

$\Rightarrow \mathbb{L}_p$-distance between $(X, Y, Z)$ and $(X^\eta, Y^\eta, Z^\eta)$?
Main tool

Theorem. Assume that $(A_{\sigma,b})$ and $(A_f)$ are satisfied. Then for $2 \leq p < \infty$ and $\xi, \xi^\eta \in \mathbb{L}_p$ we have that

$$\| \sup_{0 \leq t \leq T} |X_t^\eta - X_t| \|_p + \| \sup_{0 \leq t \leq T} |Y_t^\eta - Y_t| \|_p + \left\| \left( \int_0^T |Z_t^\eta - Z_t|^2 dt \right)^{1/2} \right\|_p \leq c \left[ \| \xi^\eta - \xi \|_p + \sqrt{\int_0^T \eta(t)^2 dt} \right]$$

where $c > 0$ depends at most on $p, T, f, b$ and $\sigma$.

Proof. Based on a clever application of a priori estimates for BSDE in $\mathbb{L}_p$. 
Applications (relaxing $A_{ag}^{pol}$ to $A_g$)

We choose specific perturbations $\eta$ defined as follows: for $0 \leq t < r \leq T$ we let

$$\eta_{t,r}(s) := \chi(t,r)(s).$$

**Corollary.** Assume $(A_{b,\sigma})$, $(A_f)$ and $(A_g)$ for some $p \geq 2$. Let

$$\xi^{t,r} := g(X_{r_1}^{\eta_{t,r}}, \ldots, X_{r_L}^{\eta_{t,r}})$$

for $0 \leq t < r \leq T$. Setting $\Theta = (\theta_1, \ldots, \theta_L) \in (0,1]^L$, if there is a constant $c > 0$ such that one has that

$$\|\xi - \xi^{t,r_1}\|_p \leq c(r_1 - t)^{\theta_1}$$

for all $l = 1, \ldots, L$ and $r_{l-1} \leq t < r_l$, then $(\xi, f) \in B_{p,\infty}^\Theta$.

Consequently, for some $c > 0$ we have

(C’1) $\|Z_t - Z_s\|_p \leq c_1\left(\int_s^t (r_1 - r)^{\theta_1 - 2}dr\right)^{\frac{1}{2}}$, for a.e. $(s,t)$ s.t. $r_{l-1} \leq s < t < r_l$;

(C’2) $\|Z_t\|_p \leq c_2(r_1 - t)^{\theta_1 - \frac{1}{2}}$, for a.e. $t \ r_{l-1} \leq t < r_l$;

(C’3) $\|Y_t - Y_s\|_p \leq c_3\left(\int_s^t (r_1 - r)^{\theta_1 - 1}dr\right)^{\frac{1}{2}}$, for every $(s,t)$ s.t. $r_{l-1} \leq s < t \leq r_l$.
Proof of \( \|\xi - \xi_{t,r_l}\|_p \leq c(r_l - t)^{\frac{\theta_l}{2}} \Rightarrow (\xi, f) \in B_{p,\infty}^{\Theta} \).

For \( r_{l-1} \leq t < r_l \) we get that

\[
\|Y_{r_l} - \mathbb{E}(Y_{r_l}|\mathcal{F}_t)\|_p = \|Y_{r_l} - \mathbb{E}^W(Y_{r_l}|\mathcal{F}_t)\|_p \\
= \|Y_{r_l} - \mathbb{E}^B(Y_{r_l}^{\eta_t,r_l}|\mathcal{F}_t)\|_p \\
\leq \|Y_{r_l} - Y_{r_l}^{\eta_t,r_l}\|_p \\
\leq c \left[ \|\xi - \xi_{t,r_l}\|_p + \sqrt{\int_0^T \eta_{t,r_l}(r)^2 dr} \right] \\
\leq c \left[ c(r_l - t)^{\frac{\theta_l}{2}} + \sqrt{r_l - t} \right].
\]

\( \Box \)
Applications (Cont’d)

**Proposition.** Let $g_1, \ldots, g_L$ be of bounded variation, i.e.

$$
\sup_N \sup_{-\infty < x_0 < \cdots < x_N < \infty} \sum_{k=1}^N |g_l(x_k) - g(x_{k-1})| < \infty
$$

for any $l$. Consider

$$
\xi = \Phi(g_1(X_{r_1}), \ldots, g_L(X_{r_L}))
$$

such that

$$
|\Phi(x_1, \ldots, x_L) - \Phi(y_1, \ldots, y_L)| \leq \kappa \left( |x_1 - y_1|^\alpha + \cdots + |x_L - y_L|^\alpha \right).
$$

Then one has that $(\xi, f) \in \bigcap_{0 < \theta < \frac{\alpha}{2p}} B_{p,\infty}^{(\theta, \ldots, \theta)}(X)$. 
Proof

According to [Avikainen ’09] one has

\[ \mathbb{E}|g(X) - g(Y)|^p \leq c(p, q, g, X)\|X - Y\|^\frac{q}{q+1} \]

whenever \( g \in BV, 1 \leq p, q < \infty \), where \( X \) has a bounded density. Hence,

\[ \|\xi - \xi^{t, r_l}\|_p \leq \kappa \sum_{j=1}^{L} \left\| g_j(X_{r_j}) - g_j(X_{\eta t, r_l}^{r_j}) \right\|_p^\alpha \]

\[ \leq \kappa \sum_{j=l}^{L} \left\| g_j(X_{r_j}) - g_j(X_{\eta t, r_l}^{r_j}) \right\|_p^\alpha \]

\[ \leq c' \sum_{j=l}^{L} \left( X_{r_j} - X_{\eta t, r_l}^{r_j} \right) \|_q^{\frac{\alpha q}{(q+1)p}} \]

\[ \leq c'' (r_l - t) \frac{\alpha q}{(q+1)^2p} . \]

Now we can take a large \( q \). \qed
Sufficient conditions (Cont’d)

Second approach. Relies on a simple iteration procedure.

Theorem. Assume that

\[ |\Phi(x_1, ..., x_L) - \Phi(x'_1, ..., x'_L)| \leq \sum_{l=1}^{L} [|g_l(x_l) - g_l(x'_l)| + \psi_l(x_1, ..., x_l; x'_1, ..., x'_l)|x_l - x'_l|] \]

where the functions \( \Phi, g_l \) and \( \psi_l \) are polynomially bounded Borel functions such that

\[ \|g_l(X_{r_l}) - \mathbb{E}(g_l(X_{r_l})|\mathcal{F}_t)\|_p \leq c(r_l - t)^{\theta_l/2} \]

for \( l = 1, ..., L, 0 < \theta_l \leq 1, \) and \( r_{l-1} \leq t \leq r_l. \)

Then,

\[ (\xi, f) \in B^{\Theta}_{\infty}(X). \]

Example: for \( \Phi(x) = 1_{a_1 < x_1 < a_1} \cdots 1_{a_L < x_L < a_L}, \) we have \( \theta_1 = \frac{1}{2p}. \)
Sketch of proofs of the main equivalence results

\((C1) \implies (C2)\) for \(0 < \theta_l < 1\)

Quite easy since

\[
\|Z_t\|_p \leq \|Z_{r_l-1}\|_p + c_1 \left( \int_{r_{l-1}}^{t} (r_l - r)^{\theta_l-2} \, dr \right)^{\frac{1}{2}}
\]

\[
= \|Z_{r_{l-1}}\|_p + c_1 \left( \frac{1}{1 - \theta_l} \left[ (r_l - t)^{\theta_l-1} - (r_l - r_{l-1})^{\theta_l-1} \right] \right)^{\frac{1}{2}}
\]

\[
\leq \|Z_{r_{l-1}}\|_p + c_1 (1 - \theta_l)^{-\frac{1}{2}} (r_l - t)^{\frac{\theta_l-1}{2}}.
\]
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$(C2) \implies (C3)$

Quite easy since

$$
\|Y_t - Y_s\|_p = \left\| \int_s^t f(r, X_r, Y_r, Z_r)dr - \int_s^t Z_r dW_r \right\|_p
$$

$$
\leq \int_s^t \|f(r, X_r, Y_r, Z_r)\|_p dr + a_p \left( \int_s^t \|Z_r\|_p^2 dr \right)^{\frac{1}{2}} \quad \text{(BDG + } p \geq 2) 
$$

$\leq \cdots$

$\square$
(C3) $\implies$ (C4)

Quite easy since

$$\|Y_{r_l} - \mathbb{E}(Y_{r_l} | \mathcal{F}_s)\|_p \leq \|Y_{r_l} - Y_s\|_p + \|Y_s - \mathbb{E}(Y_{r_l} | \mathcal{F}_s)\|_p$$

$$\leq 2\|Y_{r_l} - Y_s\|_p$$

$$\leq 2c_3\left(\int_s^{r_l} (r_l - r)^{\theta_l-1} dr\right)^{\frac{1}{2}}$$

$$= 2c_3\sqrt{\frac{1}{\theta_l}}(r_l - s)^{\frac{\theta_l}{2}}.$$
\[(C4) \implies (C5)\]

A bit less easy (inspired from [GM10]).

**Crucial Malliavin calculus \(L_p\)-estimates on PDE, under fractional smoothness assumptions**

Set

\[R_s := \|Y_{r_1} - \mathbb{E}(Y_{r_1}|\mathcal{F}_s)\|_p.\]

Then

\[\|\nabla_x F_1(\overline{X}_{1-1}; s, X_s)\|_p \leq \kappa_p' \frac{\|Y_{r_1} - \mathbb{E}(Y_{r_1}|\mathcal{F}_s)\|_p}{\sqrt{r_1 - s}}\]

and

\[\|D^2 F_1(\overline{X}_{1-1}; s, X_s)\|_p \leq \kappa_p' \frac{\|Y_{r_1} - \mathbb{E}(Y_{r_1}|\mathcal{F}_s)\|_p}{r_1 - s}.\]

(see Makhlouf’s talk).
Then, using ellipticity and BDG inequalities, one has
\[
\left\| \left( \int_{r_{l-1}}^{t} \left| (D^2 F_l)(X_{l-1}; s, X_s) \right|^2 ds \right) \right\|_p^{\frac{1}{2}} \leq c \sum_{k=1}^{d} \left\| \int_{r_{l-1}}^{t} (\nabla_x (\partial_{x_k} F_l) \sigma)(X_{l-1}; s, X_s) dW_s \right\|_p.
\]

Thanks to the PDE solved by $F_l$, we have
\[
\partial_{x_k} F_l(X_{l-1}; t, X_t) - \partial_{x_k} F_l(X_{l-1}; r_{l-1}, X_{r_{l-1}}) = - \int_{r_{l-1}}^{t} \left\{ \langle \partial_{x_k} b, \nabla_x F_l \rangle + \frac{1}{2} \langle \partial_{x_k} A, D^2 F_l \rangle \right\} (X_{l-1}; s, X_s) ds + \int_{r_{l-1}}^{t} \left\{ \nabla_x (\partial_{x_k} F_l) \sigma \right\} (X_{l-1}; s, X_s) dW_s.
\]
Then, this implies that

\[
\left\| \int_{r_{l-1}}^{t} \left( \nabla_x (\partial_{x_k} F_l) \sigma(\overline{X}_{l-1}; s, X_s) dW_s \right) \right\|_p \\
\leq \left\| \nabla_x F_l(\overline{X}_{l-1}; t, X_t) \right\|_p + \left\| \nabla_x F_l(\overline{X}_{l-1}; r_{l-1}, X_{r_{l-1}}) \right\|_p \\
+ \left\| \int_{r_{l-1}}^{t} \left\{ \langle \partial_{x_k} b, \nabla_x F_l \rangle + \frac{1}{2} \langle \partial_{x_k} A, D^2 F_l \rangle \right\} (\overline{X}_{l-1}; s, X_s) ds \right\|_p \\
\leq \kappa_p \sqrt{d} \frac{R_t}{\sqrt{r_{l-t}}} + \kappa_p \sqrt{d} \frac{R_{r_{l-1}}}{\sqrt{r_{l-1}-r_{l-1}}} + \kappa_p \sqrt{d} \| \partial_{x_k} b \|_\infty \int_{r_{l-1}}^{r_l} \frac{R_s}{\sqrt{r_l-s}} ds \\
+ \kappa_p d \| \partial_{x_k} A \|_\infty \int_{r_{l-1}}^{r_l} \frac{R_s}{r_l-s} ds \\
\ldots
\]
(C5) $\Rightarrow$ (C2)

Decomposition of the $Z$ process: $\delta Z_r := Z_r - \nabla_x F_l(X_{l-1}; r, X_r)\sigma(r, X_r)$.

Two steps:

- prove that $\sup_{r_{l-1} \leq r < r_l} \|\delta Z_r\|_p < \infty$ (crucially linked to the fact that this difference of BSDEs has zero terminal condition).

- estimate $\nabla_x F_l(X_{l-1}; r, X_r)$ in $\mathbb{L}_p$. It relies on the

**Lemma.** There exists a constant $c > 0$ such that, for all $r_{l-1} \leq s < t < r_l$,

$$
\|\nabla_x F_l(X_{l-1}; t, X_t) - \nabla_x F_l(X_{l-1}; s, X_s)\|_p
\leq c(t - s)\|\nabla_x F_l(X_{l-1}; r_{l-1}, X_{r_{l-1}})\|_p
+ c(t - s) \left\| \left( \int_{r_{l-1}}^{s} |D^2 F_l(X_{l-1}; v, X_v)|^2 dv \right)^{\frac{1}{2}} \right\|_p
+ c \left\| \left( \int_{s}^{t} |D^2 F_l(X_{l-1}; v, X_v)|^2 dv \right)^{\frac{1}{2}} \right\|_p.
$$
\[(C4) \implies (C1)\]

We linearize the BSDE, following the approach of [GM10].

Long and technical...
References


