From bounds on optimal growth towards a theory of good-deal hedging

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Tamerza, Tunesia, Oct.2010

Problem

- **Complete Market** (e.g Black-Scholes)
  - unique martingale measure Q for asset prices S
  - any claim $X \geq 0$ is priced by replication

$$X = \underbrace{E_t^Q[X]}_{\text{replication cost}} + \underbrace{\int_t^{\bar{T}} \vartheta \, dS}_{hedging}, \quad t \leq \bar{T}$$

- **Incomplete Market**
  - infinitely many martingale measures $Q \in \mathcal{M}(S)$
  - No-arbitrage valuations bounds

$$\inf_{Q \in \mathcal{M}} E_t^Q[X] \quad \text{and} \quad \sup_{Q \in \mathcal{M}} E_t^Q[X]$$

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"Solution"

- **Ad-hoc Solution**
  Get tighter bounds by using smaller subset $\mathcal{Q}^{\mathrm{ngd}} \subset \mathcal{M}$

$$\inf_{Q \in \mathcal{Q}^{\mathrm{ngd}}} E_t^Q[X] \quad \text{and} \quad \sup_{Q \in \mathcal{Q}^{\mathrm{ngd}}} E_t^Q[X]$$

- **Questions**
  - Which subset $\mathcal{Q}^{\mathrm{ngd}}$ to choose ?
  - ... for good mathematical dynamical valuation properties ?
  - ... for financial meaning of such valuation bounds ?
  - Can one associate to such bounds any notion of hedging ?

"... we note that the good-deal bound theory is a pure pricing theory... one would expect that it should be possible to develop a dual 'good-deal hedging theory'. In our view, the task of developing such a theory constitutes a highly challenging open problem."

(Björk/Slinko 2006, Towards a general theory of good-deal bounds)

Refs: Cochrane/Saa Reqquejo 2000 and Hodges/Cerny 2000

Outline

1. Bounds for Optimal Growth for Semimartingales by Duality
2. An Itô process model
3. Good-deal valuation and hedging via BSDE

Bounds on Optimal Growth

- discounted asset prices processes: Semimartingales $S \geq 0$
- **positive** (normalized) **wealth processes** = tradable numeraires

$$N_t = 1 + \int_0 \vartheta dS > 0\,, \quad t \leq \bar{T}$$

- **cond. expected growth** over any period $]\!] T, \tau]\!]$ is

$$E_T\left[\log \frac{N_\tau}{N_T}\right] \tag{1}$$

- **Question**: Can we choose the set $\mathcal{Q}^{\mathrm{ngd}}$ such that a pre-specified **bound for growth** (1) is ensured for any **market extension** $\bar{S} = (S, S')$ by derivative price processes $S'_t = E^Q_t[X]$ for $X \geq 0$ computed by $Q \in \mathcal{Q}^{\mathrm{ngd}}$ **?**

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Ensuring Bounds for Optimal Growth

by defining a suitable set $\mathcal{Q}^{\mathrm{ngd}}$ of pricing measures

- **Def**: Measures with finite (reverse) relative entropy

$$\mathcal{Q} := \left\{ Q \in \mathcal{M}^e(S) \,\middle|\, E[-\log Z_{\bar{T}}] < \infty \right\}$$

- Fix some predictable and bounded process $h = (h_t) > 0$, and
- **Def**: let $\mathcal{Q}^{\mathrm{ngd}}$ contain $Q \in \mathcal{Q}$ iff density process Z satisfies

$$E_T \left[-\log \frac{Z_\tau}{Z_T} \right] \leq \frac{1}{2} E_T \left[\int_T^\tau h_u^2 \, du \right] \quad \text{for all } T \leq \tau \leq \bar{T} \, ,$$

- ... **equivalently** with only deterministic times

$$E_s \left[-\log \frac{Z_t}{Z_s} \right] \leq \frac{1}{2} E_s \left[\int_s^t h_u^2 du \right] \quad \text{for all } s \leq t \leq \bar{T}$$

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- **Example**: For $h = const$ e.g.

$$E_s\left[-\log \frac{Z_t}{Z_s}\right] \le const(t-s)\,, \quad s \le t \le \bar{T}$$

Ensuring Bounds for Optimal Growth

- **Convex duality** yields: When pricing with $Q \in \mathcal{Q}^{\mathrm{ngd}}$, any extended market

$$\bar{S}_t = (S_t, E^Q_t[X])$$

satisfies the bounds for expected growth of wealth

$$E_T\left[\log \frac{\bar{N}_\tau}{\bar{N}_T}\right] \le E_T\left[-\log \frac{Z_\tau}{Z_T}\right] \tag{2}$$

for all stopping times $T \le \tau \le \bar{T}$.

- That is, derivatives price processes are taken such that there arise no dynamic trading opportunities which offer deals that are 'too good'!

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Multiplicative Stability

- For any $Q \in \mathcal{Q}$, we have a **Doob-Meyer decomposition**

$$-\log Z_t = M_t + A_t$$

with $M=$ UI-martingale, $A=$ predictable, increasing, integrable

- Additive functional for $T \leq \tau$: $E_T\left[-\log \frac{Z_\tau}{Z_T}\right] = E_T[A_\tau - A_T]$
- $\rightsquigarrow$ $\mathcal{Q}^{\mathrm{ngd}}$ is **multiplicative stable**
- $\rightsquigarrow$ Dynamic good-deal valuation bounds

$$\pi_t^u(X) = \sup_{Q \in \mathcal{Q}^{\mathrm{ngd}}} E_t^Q[X] \quad \text{and} \quad \pi_t^\ell(X) = \inf_{Q \in \mathcal{Q}^{\mathrm{ngd}}} E_t^Q[X] = -\pi_t^u(-X)$$

have **good dynamic behavior** over time....

Good Dynamic Valuation Bound Properties

Thm: Mappings $X \mapsto \pi_t^u(X)$ $(t \le \bar{T})$ from $L^\infty \to L^\infty(\mathcal{F}_t)$ satisfies

- (**nice paths**) For any $X \in L^\infty$ there is an RCLL-version of $(\pi_t^u(X))_{t \le \bar{T}}$

$$\pi_T^u(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{S}} E_T^Q[X] \quad \text{for all stopping times } T \le \bar{T}.$$

- (**recursiveness**) For any stopping times $T \le \tau \le \bar{T}$ holds that

$$\pi_T^u(X) = \pi_T^u(\pi_\tau^u(X)).$$

- (Stopping-time **consistency**) For stopping times $T \le \tau \le \bar{T}$ the inequality $\pi_\tau^u(X^1) \ge \pi_\tau^u(X^2)$ implies $\pi_T^u(X^1) \ge \pi_T^u(X^2)$.

Good Valuation Bound properties (cont.)

Thm (cont.)

- (**dynamic coherent risk measure**) For any stopping time $T \le \bar{T}$ and $m_T, \alpha_T, \lambda_T \in L^\infty(\mathcal{F}_T)$ with $0 \le \alpha_T \le 1$, $\lambda_T \ge 0$, the mapping $X \mapsto \pi^u_T(X)$ satisfies the properties:
  - monotonicity: $X^1 \ge X^2$ implies $\pi^u_T(X^1) \ge \pi^u_T(X^2)$
  - translation invariance: $\pi^u_T(X + m_T) = \pi^u_T(X) + m_T$
  - convexity:
    $\pi^u_T(\alpha_T X^1 + (1-\alpha_T)X^2) \le \alpha_T \pi^u_T(X^1) + (1-\alpha_T)\pi^u_T(X^2)$
  - positive homogeneity: $\pi^u_T(\lambda_T X) = \lambda_T \pi^u_T(X)$
- **No arbitrage consistency**: $\pi^u_T(X) = x + \vartheta \cdot S_T$ for any $X = x + \vartheta \cdot S_{\bar{T}}$ with $((\vartheta \cdot S_t)_{t \le \bar{T}})$ being uniformly bounded.

Itô price process model

- Take **more explicit** model for **more constructive** results:
- Filtration $(\mathcal{F}_t)_{t\le\bar{T}}$ generated by n-dim Brownian motion W
- Market with d assets, $d \le n$.
- **Itô prices processes**

$$dS_t = \mathrm{diag}(S_t)\,\sigma_t\,(\xi_t\,dt + dW_t)\;, \quad t \le \bar{T},$$

  where σ, ξ are predictable, $\sigma_t \in \mathbb{R}^{d\times n}$ has full rank $d \le n$.
- (minimal) **market price of risk** process ξ bounded, $\xi_t \in \mathrm{Im}\,\sigma_t^{\mathrm{tr}} = (\mathrm{Ker}\,\sigma_t)^{\perp}$

Trading strategies

- Trading **strategy** φ (wealth invested in assets) yields wealth process

$$\begin{aligned} dV_t &= \varphi_t^{\mathrm{tr}} dR_t = \varphi_t^{\mathrm{tr}} \sigma_t (\xi_t dt + dW_t) \\ &= \end{aligned}$$

- Convenient: **Re-parameterize** strategy set by $\phi \in \Phi$

$$\phi_t = \sigma_t^{\mathrm{tr}} \varphi_t \in \mathrm{Im}\, \sigma_t^{\mathrm{tr}} \quad \text{and} \quad \varphi = (\sigma \sigma^{\mathrm{tr}})^{-1} \sigma \phi$$

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- Later useful: **orthogonal projections**

$$\Pi_t : \mathbb{R}^n \to \mathrm{Im}\, \sigma_t^{\mathrm{tr}} \quad \text{and} \quad \Pi_t^{\perp} : \mathbb{R}^n \to (\mathrm{Im}\, \sigma_t^{\mathrm{tr}})^{\perp} = \mathrm{Ker}\, \sigma_t$$

Equivalent martingale measures

Convenient **parameterization** of $\mathcal{Q}^{\mathrm{ngd}}$ **by Girsanov kernels**

- Any $Q \in \mathcal{M}$ has a density process of the form

$$Z_t := \frac{dQ}{dP}\bigg|_t = \mathcal{E}\left(\int \lambda dW\right)_t = \mathcal{E}\left(-\int \xi\, dW\right)_t \mathcal{E}\left(\int \eta\, dW\right)_t$$

  with (possible) **market price of risk** $\lambda = -\xi + \eta$ predictable s.t. $\Pi_t(\lambda_t) = -\xi_t$ and $\Pi_t^{\perp}(\lambda_t) = \eta_t$.
- For $Q \in \mathcal{Q}^{\mathrm{ngd}} \subset \mathcal{M}$ holds $|\lambda|^2 = |\xi|^2 + |\eta|^2 \leq h^2$ ($P \times dt$-a.e.)
- **Vice versa** any predictable λ with $|\lambda|^2 \leq h^2$ and $\Pi_t(\lambda_t) = -\xi_t$ ($P \times dt$-a.e.) defines a density process Z for some $Q \in \mathcal{Q}^{\mathrm{ngd}}$ with $\eta = \Pi^{\perp}(\lambda)$.

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BSDE description of good-deal valuation bounds

- Upper good-deal bound $\pi^u_t(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}^{\mathrm{ngd}}} E^Q_t[X]$, $X \in L^2$
- **maximizing** over linear BSDE generators $(-\xi^{\mathrm{tr}}_t \Pi_t(Z_t) + \eta^{\mathrm{tr}}_t \Pi^{\perp}_t(Z_t))$ **yields upper good-deal valuation** process

$$\pi^u_t(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}^{\mathrm{ngd}}} E^Q_t[X] = E^{\bar{Q}}_t[X] = Y_t\,, \quad t \le \bar{T}$$

- ...where (Y, Z) is **solution to the BSDE** with $Y_{\bar{T}} = X$ and

$$-dY_t = \left(-\xi^{\mathrm{tr}}_t \Pi_t(Z_t) + \sqrt{h^2_t - |\xi_t|^2}\,\left|\Pi^{\perp}_t(Z_t)\right|\right) dt - Z_t\, dW_t$$

- Density of **'worst case' scenario measure** $\bar{Q}$ is described too.

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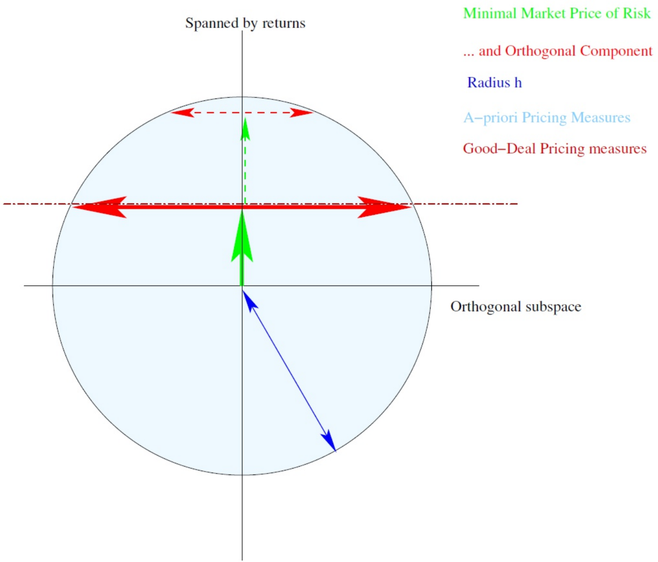
Illustration



BSDE description for good-deal hedging

What **hedging** notion can we associate to good-deal valuation bounds **?**

- Define dynamic **'a-priori' coherent risk measure**

$$\rho_t(X) := \operatorname*{ess\,sup}_{Q \in \mathcal{P}^{\mathrm{ngd}}} E_t^Q[X]\,, \quad t \le \bar{T}\,,$$

for $\mathcal{P}^{\mathrm{ngd}} := \left\{ Q \sim P \;\middle|\; \frac{dQ}{dP}\Big|_{\mathbb{F}} = \mathcal{E}\left(\int \lambda dW\right) \;\text{ with } |\lambda| \le h \right\}$

- Note 1) $\mathcal{P}^{\mathrm{ngd}} \supset \mathcal{Q}^{\mathrm{ngd}}$
  2) analogous 'no-good-deal type' structure as $\mathcal{Q}^{\mathrm{ngd}}$
- As before, get **BSDE description for** $\rho_t(X) = Y_t$:

$$-dY_t \;=\; h_t|Z_t|\,dt - Z_t\,dW_t\,, \quad t \le \bar{T} \quad \text{with } \; Y_{\bar{T}} = X$$

BSDE description for good-deal hedging

- Applying again the optimality methods for BSDEs...
- ... yields

$$\pi_t^u(X) = Y_t = \operatorname*{ess\,inf}_{\phi\in\Phi} \rho_t\left(X - \int_t^{\bar{T}} \phi\, d\widehat{W}\right) = \rho_t\left(X - \int_t^{\bar{T}} \phi^*\, d\widehat{W}\right)$$

- ... where the **hedging strategy** ϕ^* is explicitly given in terms of the π^u-BSDE solution (Y, Z) as

$$\phi^* = \frac{|\Pi^\perp(Z)|}{\sqrt{h^2 - |\xi|^2}}\xi + \Pi(Z)$$

BSDE description for good-deal hedging

Tracking error (cost process) **of hedging** strategy ?

- **Tracking error** :=

$$\underbrace{\pi_0^u(X) - \pi_t^u(X)}_{\text{regul.capital reqrmnt}} + \underbrace{\int_0^t \phi_s^* \, d\widehat{W}_s}_{\text{P+L from trading}} \quad , \qquad t \le \bar{T}$$

  of the good-deal hedging strategy ϕ^* is submartingale under any $Q \in \mathcal{P}^{\mathrm{ngd}}$ and a martingale unter a worst-case measure $Q^\lambda \in \mathcal{P}^{\mathrm{ngd}}$, whose density is explicitly known in terms of the π^u-BSDE solution (Y, Z).

- Hedging strategy **is "super-mean-self-financing"** under all generalized scenarios $Q \in \mathcal{P}^{\mathrm{ngd}}$.

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Ambiguity

- Problem: We do not really know market prices for risk
- $\rightsquigarrow$ **Model uncertainty** ("Knightean uncertainty")
- Aim: Robustness wrt uncertainty of market prices for risk :

$$d\widehat{W} = \xi^{\nu}dt + dW$$

Ambiguity

- Aim: Robustness wrt uncertainty of market prices for risk :

$$d\widehat{W} = \xi^{\nu} dt + dW^{\nu} := (\widehat{\xi} + \nu)dt + dW^{\nu}$$

with $\nu \in \{\nu \in \mathrm{Ker}\, \sigma_t : |\nu| \leq \delta\}$. (= "Confidence region")

- Instead of single reference probability $P = P^0$ consider set

$$\left\{ P^{\nu} \,\middle|\, dP^{\nu} = \mathcal{E}\left(\nu \cdot W^0\right) \, dP^0 \right\}$$

- $\rightsquigarrow$ A-priori dynamic risk measure to be minimized becomes

$$\rho_t(X) = \operatorname*{ess\,sup}_{\nu} E_t^{\nu}[X] = \operatorname*{ess\,sup}_{Q \in \bar{\mathcal{P}}} E_t^{Q}[X]$$

with $\bar{\mathcal{P}} := \cup_{\nu} \mathcal{P}^{\mathrm{ngd}}(P^{\nu})$ being m-stable.

Robust Hedging

- Note: There is a 'worst case' measure $P^{\nu*}$ yielding the widest (highest) good-deal bounds $\pi^{u,\nu}(X)$.
- **But**: Good-deal hedging strategy wrt to 'worst case' measure $P^{\nu*}$ does **not** ensure submartingale property for tracking errors of the hedge uniformly for all $P^{\nu} \in \bar{\mathcal{P}}$!

Robust Hedging

- **BSDE solution**

$$
\begin{aligned}
-dY_t &= f(t,Z_t)\,dt - Z_t\,dW^0_t\,, \quad t \le \bar{T}, \quad \text{with } Y_{\bar{T}} = X \\
&\text{for} \quad f(t,Z_t) = \min_{\phi\in\Phi}\left(-\widehat{\xi}^{\mathrm{tr}}_t\phi_t + \delta\Big|\phi_t - \Pi_t(Z)\Big| + h\Big|\phi_t - Z_t\Big|\right)
\end{aligned}
$$

for robust Valuation:

$$
\bar{\pi}^u_t(X) = \operatorname*{ess\,inf}_{\phi}\ \operatorname*{ess\,sup}_{\nu} E^{\nu}_t\left[X - \int_t^{\bar{T}} \phi\, d\widehat{W}\right] = Y_t
$$

- and **for robust Hedging**:

$$
\bar{\phi}^* = \operatorname{argmin}_{\phi\in\Phi}\left(-\widehat{\xi}^{\mathrm{tr}}_t\phi_t + \delta\Big|\phi_t - \Pi_t(Z)\Big| + h\Big|\phi_t - Z_t\Big|\right)
$$

Thank you !