From bounds on optimal growth towards a theory of good-deal hedging

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Problem

- **Complete Market** (e.g. Black-Scholes)
  - unique martingale measure $Q$ for asset prices $S$
  - any claim $X \geq 0$ is priced by replication

\[
X = E_t^Q[X] + \int_t^\bar{T} \vartheta \, dS, \quad t \leq \bar{T}
\]

- **Incomplete Market**
  - infinitely many martingale measures $Q \in \mathcal{M}(S)$
  - No-arbitrage valuations bounds

\[
\inf_{Q \in \mathcal{M}} E_t^Q[X] \quad \text{and} \quad \sup_{Q \in \mathcal{M}} E_t^Q[X]
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- are the super-replication costs $\rightsquigarrow$ notion of hedging
- Problem: The bounds are typically too wide!
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“Solution”

- **Ad-hoc Solution**
  Get tighter bounds by using smaller subset $Q^{\text{ngd}} \subset \mathcal{M}$

  $$\inf_{Q \in Q^{\text{ngd}}} E_t^Q[X] \quad \text{and} \quad \sup_{Q \in Q^{\text{ngd}}} E_t^Q[X]$$

- **Questions**
  - Which subset $Q^{\text{ngd}}$ to choose?
  - ... for good mathematical dynamical valuation properties?
  - ... for financial meaning of such valuation bounds?
  - Can one associate to such bounds any notion of hedging?
“... we note that the good-deal bound theory is a pure pricing theory... one would expect that it should be possible to develop a dual 'good-deal hedging theory'. In our view, the task of developing such a theory constitutes a highly challenging open problem.”

(Björk/Slinko 2006, Towards a general theory of good-deal bounds)

Refs: Cochrane/Saa Reqquejo 2000 and Hodges/Cerny 2000
Outline

1. Bounds for Optimal Growth for Semimartingales by Duality
2. An Itô process model
3. Good-deal valuation and hedging via BSDE
Bounds on Optimal Growth

- discounted asset prices processes: Semimartingales $S \geq 0$
- **positive** (normalized) **wealth processes** = tradable numeraires

$$N_t = 1 + \int_0^t \theta dS > 0, \quad t \leq \bar{T}$$

- **cond. expected growth** over any period $\left\langle T, \tau \right\rangle$ is

$$E_T \left[ \log \frac{N_\tau}{N_T} \right]$$  \hspace{1cm} (1)

**Question**: Can we choose the set $Q^{\text{ngd}}$ such that a pre-specified **bound for growth** (1) is ensured for any **market extension** $\tilde{S} = (S, S')$ by derivative price processes $S'_t = E^Q_t [X]$ for $X \geq 0$ computed by $Q \in Q^{\text{ngd}}$?
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Ensuring Bounds for Optimal Growth

by defining a suitable set $\mathcal{Q}^{ngd}$ of pricing measures

- **Def**: Measures with finite (reverse) relative entropy

$$\mathcal{Q} := \{ Q \in \mathcal{M}^e(S) \mid E[ - \log Z_T ] < \infty \}$$

- Fix some predictable and bounded process $h = (h_t) > 0$, and
- **Def**: let $\mathcal{Q}^{ngd}$ contain $Q \in \mathcal{Q}$ iff density process $Z$ satisfies

$$E_T \left[ - \log \frac{Z_T}{Z_T} \right] \leq \frac{1}{2} E_T \left[ \int_T^\tau h_u^2 du \right] \quad \text{for all } T \leq \tau \leq \bar{T},$$

- ... equivalently with only deterministic times

$$E_s \left[ - \log \frac{Z_t}{Z_s} \right] \leq \frac{1}{2} E_s \left[ \int_s^t h_u^2 du \right] \quad \text{for all } s \leq t \leq \bar{T}$$
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- **Example**: For $h = \text{const}$ e.g.

  $$E_s \left[ - \log \frac{Z_t}{Z_s} \right] \leq \text{const}(t - s), \quad s \leq t \leq \bar{T}$$
Ensuring Bounds for Optimal Growth

- **Convex duality** yields: When pricing with $Q \in Q^{\text{ngd}}$, any extended market
  \[ \tilde{S}_t = (S_t, E^Q_t[X]) \]
  satisfies the bounds for expected growth of wealth
  \[ E_T \left[ \log \frac{\tilde{N}_T}{\tilde{N}_T} \right] \leq E_T \left[ - \log \frac{Z_T}{Z_T} \right] \]
  for all stopping times $T \leq \tau \leq \bar{T}$.

- That is, derivatives price processes are taken such that there arise no dynamic trading opportunities which offer deals that are ‘too good’!
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(2)

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Multiplicative Stability

For any \( Q \in \mathcal{Q} \), we have a **Doob-Meyer decomposition**

\[
- \log Z_t = M_t + A_t
\]

with \( M = \text{UI-martingale}, A = \text{predictable, increasing, integrable} \)

Additive functional for \( T \leq \tau \):

\[
E_T \left[ - \log \frac{Z_\tau}{Z_T} \right] = E_T[A_\tau - A_T]
\]

\( \sim \mathcal{Q}_{\text{ngd}} \) is **multiplicative stable**

\( \sim \) Dynamic good-deal valuation bounds

\[
\pi^u_t(X) = \sup_{Q \in \mathcal{Q}_{\text{ngd}}} E^Q_t[X] \quad \text{and} \quad \pi^\ell_t(X) = \inf_{Q \in \mathcal{Q}_{\text{ngd}}} E^Q_t[X] = -\pi^u_t(-X)
\]

have **good dynamic behavior** over time....
**Good Dynamic Valuation Bound Properties**

**Thm:** Mappings $X \mapsto \pi^u_t(X)$ ($t \leq \bar{T}$) from $L^\infty \to L^\infty(\mathcal{F}_t)$ satisfies

- **(nice paths)** For any $X \in L^\infty$ there is an RCLL-version of $(\pi^u_t(X))_{t \leq \bar{T}}$

  $$\pi^u_T(X) = \text{ess sup}_{Q \in S} E^Q_T[X] \quad \text{for all stopping times } T \leq \bar{T}.$$ 

- **(recursiveness)** For any stopping times $T \leq \tau \leq \bar{T}$ holds that

  $$\pi^u_T(X) = \pi^u_T(\pi^u_\tau(X)).$$

- **(Stopping-time consistency)** For stopping times $T \leq \tau \leq \bar{T}$ the inequality $\pi^u_\tau(X^1) \geq \pi^u_\tau(X^2)$ implies $\pi^u_T(X^1) \geq \pi^u_T(X^2)$.
Thm (cont.)

- **(dynamic coherent risk measure)** For any stopping time \( T \leq \bar{T} \) and \( m_T, \alpha_T, \lambda_T \in L^\infty(\mathcal{F}_T) \) with \( 0 \leq \alpha_T \leq 1 \), \( \lambda_T \geq 0 \), the mapping \( X \mapsto \pi^u_T(X) \) satisfies the properties:
  - monotonicity: \( X^1 \geq X^2 \) implies \( \pi^u_T(X^1) \geq \pi^u_T(X^2) \)
  - translation invariance: \( \pi^u_T(X + m_T) = \pi^u_T(X) + m_T \)
  - convexity:
    \[
    \pi^u_T(\alpha_T X^1 + (1 - \alpha_T) X^2) \leq \alpha_T \pi^u_T(X^1) + (1 - \alpha_T) \pi^u_T(X^2)
    \]
  - positive homogeneity:
    \[
    \pi^u_T(\lambda_T X) = \lambda_T \pi^u_T(X)
    \]

- **No arbitrage consistency**: \( \pi^u_T(X) = x + \vartheta \cdot S_T \) for any \( X = x + \vartheta \cdot S_T \) with \((\vartheta \cdot S_t)_{t \leq \bar{T}}\) being uniformly bounded.
Itô price process model

- Take **more explicit** model for **more constructive** results:
- Filtration \((\mathcal{F}_t)_{t \leq \bar{T}}\) generated by \(n\)-dim Brownian motion \(W\)
- Market with \(d\) assets, \(d \leq n\).
- **Itô prices processes**

\[
dS_t = \text{diag}(S_t) \sigma_t (\xi_t \, dt + dW_t), \quad t \leq \bar{T},
\]

where \(\sigma, \xi\) are predictable, \(\sigma_t \in \mathbb{R}^{d \times n}\) has full rank \(d \leq n\).

- (minimal) **market price of risk** process \(\xi\) bounded,
  \(\xi_t \in \text{Im} \sigma_t^{\text{tr}} = (\text{Ker} \sigma_t)^\perp\)
Trading strategies

- **Trading strategy** $\varphi$ (wealth invested in assets) yields wealth process

  $$dV_t = \varphi_t^{\text{tr}} dR_t = \varphi_t^{\text{tr}} \sigma_t (\xi_t dt + dW_t)$$

- Convenient: **Re-parameterize** strategy set by $\phi \in \Phi$

  $$\phi_t = \sigma_t^{\text{tr}} \varphi_t \in \text{Im} \sigma_t^{\text{tr}} \quad \text{and} \quad \varphi = (\sigma \sigma^{\text{tr}})^{-1} \sigma \phi$$
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- Later useful: **orthogonal projections**

$$\Pi_t : \mathbb{R}^n \to \text{Im} \sigma^{\text{tr}}_t \quad \text{and} \quad \Pi^\perp_t : \mathbb{R}^n \to (\text{Im} \sigma^{\text{tr}}_t)^\perp = \text{Ker} \sigma_t$$
Equivalent martingale measures

Convenient parameterization of $Q^{\text{ngd}}$ by Girsanov kernels

- Any $Q \in \mathcal{M}$ has a density process of the form

$$Z_t := \left. \frac{dQ}{dP} \right|_t = \mathcal{E} \left( \int \lambda dW \right)_t = \mathcal{E} \left( - \int \xi dW \right)_t \mathcal{E} \left( \int \eta dW \right)_t$$

with (possible) market price of risk $\lambda = -\xi + \eta$ predictable s.t. $\Pi_t(\lambda_t) = -\xi_t$ and $\Pi_t^\perp(\lambda_t) = \eta_t$.

- For $Q \in Q^{\text{ngd}} \subset \mathcal{M}$ holds $|\lambda|^2 = |\xi|^2 + |\eta|^2 \leq h^2 \ (P \times dt$-a.e.)

- **Vice versa** any predictable $\lambda$ with $|\lambda|^2 \leq h^2$ and $\Pi_t(\lambda_t) = -\xi_t \ (P \times dt$-a.e.) defines a density process $Z$ for some $Q \in Q^{\text{ngd}}$ with $\eta = \Pi^\perp(\lambda)$.
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BSDE description of good-deal valuation bounds

- Upper good-deal bound \( \pi_t^u(X) = \text{ess sup}_{Q \in Q_{\text{ngd}}} E_t^Q [X], X \in L^2 \)
- maximizing over linear BSDE generators
  \( (-\xi_t^{\text{tr}} \Pi_t(Z_t) + \eta_t^{\text{tr}} \Pi_{\perp t}(Z_t)) \) yields upper good-deal valuation process

  \[ \pi_t^u(X) = \text{ess sup}_{Q \in Q_{\text{ngd}}} E_t^Q [X] = E_t^{\tilde{Q}} [X] = Y_t, \quad t \leq \tilde{T} \]

- ...where \((Y, Z)\) is solution to the BSDE with \(Y_{\tilde{T}} = X\) and

  \[ -dY_t = \left( -\xi_t^{\text{tr}} \Pi_t(Z_t) + \sqrt{h_t^2 - |\xi_t|^2} \left| \Pi_{\perp t}(Z_t) \right| \right) dt - Z_t \, dW_t \]

- Density of ‘worst case’ scenario measure \(\tilde{Q}\) is described too.
BSDE description of good-deal valuation bounds

- Upper good-deal bound $\pi^u_t(X) = \operatorname{ess sup}_{Q \in Q^{ngd}} E^Q_t[X]$, $X \in L^2$

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Good-deal hedging
Motivation

General

Itô processes

Good deals by BSDEs

Valuation bounds

Hedging

Ambiguity

Ambiguity

Fine

Illustration

Spanned by returns

Minimal Market Price of Risk

... and Orthogonal Component

Radius h

A-priori Pricing Measures

Good-Deal Pricing measures

Orthogonal subspace

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Good-deal hedging
What hedging notion can we associate to good-deal valuation bounds?

- Define dynamic ‘a-priori’ coherent risk measure

\[ \rho_t(X) := \text{ess sup}_{Q \in \mathcal{P}^{ngd}} E_q^Q[X], \quad t \leq \bar{T}, \]

for \( \mathcal{P}^{ngd} := \left\{ Q \sim P \ \left| \ \frac{dQ}{dP} \bigg|_{\mathcal{F}} = \mathcal{E} \left( \int \lambda dW \right) \right. \right\} \) with \(|\lambda| \leq h\)

- Note 1) \( \mathcal{P}^{ngd} \supset Q^{ngd} \)
- 2) analogous ‘no-good-deal type’ structure as \( Q^{ngd} \)
- As before, get BSDE description for \( \rho_t(X) = Y_t \):

\[ -dY_t = h_t|Z_t| \ dt - Z_t \ dW_t, \quad t \leq \bar{T} \quad \text{with} \quad Y_{\bar{T}} = X \]
Motivation  General  Itô processes  Good deals by BSDEs  Valuation bounds  Hedging  Ambiguity  Ambiguity  Fine

BSDE description for good-deal hedging

- Applying again the optimality methods for BSDEs...
- ... yields

\[
\pi^u_t(X) = Y_t = \operatorname{ess} \inf_{\phi \in \Phi} \rho_t \left( X - \int_t^\tilde{T} \phi \, d\tilde{W} \right) = \rho_t \left( X - \int_t^\tilde{T} \phi^* \, d\tilde{W} \right)
\]

- ... where the **hedging strategy** \( \phi^* \) is explicitly given in terms of the \( \pi^u \)-BSDE solution \((Y, Z)\) as

\[
\phi^* = \frac{|\Pi(Z)|}{\sqrt{h^2 - |\xi|^2}} \xi + \Pi(Z)
\]
BSDE description for good-deal hedging

Tracking error (cost process) of **hedging** strategy?

- **Tracking error** :=

\[
\pi_0^u(X) - \pi_t^u(X) + \int_0^t \phi_s^* \, d\hat{W}_s, \quad t \leq \bar{T}
\]

of the good-deal hedging strategy \( \phi^* \) is submartingale under any \( Q \in \mathcal{P}^{\text{ngd}} \) and a martingale under a worst-case measure \( Q^\lambda \in \mathcal{P}^{\text{ngd}} \), whose density is explicitly known in terms of the \( \pi^u \)-BSDE solution \((Y, Z)\).

- Hedging strategy is “**super-mean-self-financing**” under all generalized scenarios \( Q \in \mathcal{P}^{\text{ngd}} \).
BSDE description for good-deal hedging

Tracking error (cost process) of hedging strategy?

- Tracking error :=
  \[ \underbrace{\pi^u_0(X) - \pi^u_t(X)}_{\text{regul. capital reqmmt}} + \underbrace{\int_0^t \phi^*_s \, d\hat{W}_s}_{\text{P+L from trading}}, \quad t \leq \bar{T} \]

of the good-deal hedging strategy \( \phi^* \) is submartingale under any \( Q \in \mathcal{P}^{\text{ngd}} \) and a martingale unter a worst-case measure \( Q^\lambda \in \mathcal{P}^{\text{ngd}} \), whose density is explicitly known in terms of the \( \pi^u \)-BSDE solution \( (Y, Z) \).

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- Hedging strategy is “super-mean-self-financing” under all generalized scenarios \( Q \in \mathcal{P}^{ngd} \).
Problem: We do not really know market prices for risk

\[ d\widehat{W} = \xi' dt + dW \]
Ambiguity

- **Aim:** Robustness wrt uncertainty of market prices for risk:

\[ d\widehat{W} = \xi^\nu dt + dW^\nu := (\hat{\xi} + \nu)dt + dW^\nu \]

with \( \nu \in \{ \nu \in \text{Ker } \sigma_t : |\nu| \leq \delta \} \). (="Confidence region")

- Instead of single reference probability \( P = P^0 \) consider set

\[ \{ P^\nu \mid dP^\nu = \mathcal{E}(\nu \cdot W^0) dP^0 \} \]

- \( \leadsto \) A-priori dynamic risk measure to be minimized becomes

\[ \rho_t(X) = \text{ess sup}_\nu E^\nu_t[X] = \text{ess sup}_Q E^Q_t[X] \]

with \( \bar{\mathcal{P}} := \bigcup_{\nu} \mathcal{P}^{\text{ngd}}(P^\nu) \) being m-stable.
Robust Hedging

- **Note:** There is a ‘worst case’ measure $P^{ν^*}$ yielding the widest (highest) good-deal bounds $π^{u,ν}(X)$.

- **But:** Good-deal hedging strategy wrt to ‘worst case’ measure $P^{ν^*}$ does not ensure submartingale property for tracking errors of the hedge uniformly for all $P^{ν} \in \overline{P}$!
Robust Hedging

- BSDE solution

\[-dY_t = f(t, Z_t) \, dt - Z_t \, dW_t, \quad t \leq \bar{T}, \quad \text{with } Y_{\bar{T}} = X\]

for \( f(t, Z_t) = \min_{\phi \in \Phi} \left( -\xi_t^{tr} \phi_t + \delta \left| \phi_t - \Pi_t(Z) \right| + h \left| \phi_t - Z_t \right| \right) \)

for robust Valuation:

\[\bar{\pi}_t^u(X) = \operatorname{ess} \inf_{\phi} \operatorname{ess} \sup_{\nu} E_\nu^t \left[ X - \int_t^{\bar{T}} \phi \, d\hat{W} \right] = Y_t\]

- and for robust Hedging:

\[\bar{\phi}^* = \operatorname{argmin}_{\phi \in \Phi} \left( -\xi_t^{tr} \phi_t + \delta \left| \phi_t - \Pi_t(Z) \right| + h \left| \phi_t - Z_t \right| \right)\]
Thank you!