WELLPOSEDNESS OF SECOND ORDER BSDEs

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BSDEs and Applications in Finance

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Outline

Standard BSDEs

$(\omega, \mathcal{F}, \mathbb{P})$, W Brownian motion, $\{\mathcal{F}_t, t \geq 0\}$ corresponding filtration. Pardoux and Peng introduced BSDE :

$$Y_t = \xi - \int_t^T F_t(Y_t, Z_t)dt + \int_t^T Z_t dW_t$$

and proved that for

$$\xi \in \mathbb{L}^2(\mathbb{P}), \quad H \text{ unif. Lipschitz in } (y,z) \quad \text{and } H_.(0,0) \in \mathbb{H}^2$$

there is a unique solution $(Y, Z) \in {}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P})$:

$$\|Y\|_2 := \mathbb{E}\Big[\sup_{t\in[t,T]} |Y_t|^2\Big] \quad \text{and} \quad \|Z\|_{\mathbb{H}^2} := \mathbb{E}\Big[\int_0^T |Z_t|^2 dt\Big]$$

Notice that $Z_t = D_t Y_t$. Then a BSDE can be viewed as a

first order equation in the Wiener space

(Mateo Casserini's talk, yesterday)

BSDEs and semilinear PDES

The Markov case corresponds to

$$F_t(\omega, y, z) = f(t, X_t(\omega), y, z) \quad \text{and} \quad \xi(\omega) = g\big(X_T(\omega)\big)$$

where $X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$

In this context, under the same conditions as before, we have

$$Y_t \quad = \quad V(t, X_t)$$

Moreover, if $V \in C^{1,2}$, then V is a classical solution of the semilinear PDE

$$\partial_t V + b \cdot DV + \frac{1}{2}\text{Tr}\big[\sigma\sigma^{\mathrm{T}} D^2 V\big] \quad = \quad f\big(., V, \sigma^{\mathrm{T}} DV\big)$$

Our objective

Extend the notion of BSDEs so as to correspond to fully nonlinear PDEs in the Markov case

First formulation of Second order BSDE

Cheridito, Soner, NT and Victoir 07 : Find an $(\mathcal{F}_t)_t$−prog. meas (Y, Z, Γ) satisfying :

$$dY_t = -H_t(Y_t, Z_t, \Gamma_t)dt + Z_t \circ dW_t, \;\; Y_1 = \xi, \quad \mathbb{P} - \text{a.s}$$

where

$$\Gamma_t dt := d\langle Z, W\rangle_t$$

and $\circ$ is the Fisk-Stratonovich stochastic integration

$$Z_t \circ dW_t = Z_t dW_t + \frac{1}{2} d\langle Z, W\rangle_t = Z_t dW_t + \frac{1}{2}\Gamma_t dt$$

Formally, $Z_t = D_t Y_t$ and $\Gamma_t = D_t Z_t = D_t^2 Y_t$, so

Second order differential equation in the Wiener space

The uniqueness result of CSTV (only Markov case)

Theorem If the comparison principle for viscosity solutions of PDE holds, then 2BSDE has a unique solution in class $\mathcal{Z}$

The admissibility set $\mathcal{Z}$ can not be relaxed as in standard BSDEs :

Counter-example For $c \neq \frac{1}{2}$, the linear 2BSDE with constant coefficients

$$dY_t = -c\Gamma_t dt + Z_t \circ dW_t, \qquad Y_1 = 0,$$

has a nonzero solution with $(Y, Z) \in \mathbb{S}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P})$

The admissibility set $\mathcal{Z}$ in CSTV

Definition $Z \in \mathcal{Z}$ if it is of the form

$$Z_t = \sum_{n=0}^{N-1} z_n \mathbb{1}_{\{t<\tau_{n+1}\}} + \int_0^t \alpha_s ds + \int_0^t \Gamma_s dW_s$$

- (τ_n) is an $\nearrow$ seq. of stop. times, z_n are $\mathcal{F}_{\tau_n}-$measurable, $\|N\|_\infty < \infty$
- Z_t and Γ_t are $\mathbb{L}^\infty-$bounded up to some polynomial of X_t
- $\Gamma_t = \Gamma_0 + \int_0^t a_s ds + \int_0^t \xi_s dW_s$, $0 \leq t \leq T$, and

$$\|\alpha\|_{B,b} + \|a\|_{B,b} + \|\xi\|_{B,2} < \infty, \qquad \|\phi\|_{B,b} := \left\| \sup_{0\leq t\leq T} \frac{|\phi_r|}{1+X_t^B} \right\|_{\mathbb{L}^b}$$

Hedging under Gamma constraints

Given a derivative security ξ, solve the **super-hedging problem**

$$V \;:=\; \inf\left\{y \;:\; Y_T^{y,Z} \geq \xi \text{ for some "admissible" } Z,\; \Gamma_t \in [\underline{\Gamma}, \overline{\Gamma}]\right\}$$

where, for continuous Z :

$$Y_T^{y,z} \;=\; Y_0 + \int_0^T Z_t dS_t$$

- The backward SDE formulation is :

$$dY_t = -\chi_{[\underline{\Gamma},\overline{\Gamma}]}(\Gamma_t)dt + Z_t \sigma dB_t \quad \text{and} \quad Y_T = \xi$$

Hedging under illiquidity cost, Cetin Jarrow Protter

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where, for continuous Z :

$$Y_T^{y,z} \quad = \quad Y_0 + \int_0^T Z_t dS_t - \int_0^T \frac{\partial S}{\partial \nu}(S_t, 0)\, d\langle Z \rangle_t$$

- The backward SDE formulation is :

$$dY_t = Z_t \sigma dB_t - \frac{\partial S}{\partial \nu}(S_t, 0)\, d\langle Z \rangle_t \quad \text{and} \quad Y_T = \xi$$

Hedging under volatility uncertainty

Avellaneda Levy Paras, Lyons, Denis Martin :

$$V_0 \;\; := \;\; \inf\left\{ Y_0 : Y_0 + \int_0^T Z_t dW_t \geq \xi \ \mathbb{P}-\text{a.s. for all } \mathbb{P}\in\mathcal{P} \right\}$$

where $\mathcal{P}$ is the collection of all probability measures $\mathbb{Q}$ such that W is a $\mathbb{Q}-$martingale with quadratic variation

$$\underline{a}\ t \;\leq\; d\langle W\rangle_t \;\leq\; \overline{a}\ t \quad \mathbb{Q}-\text{a.s. for all} \quad \mathbb{Q}\in\mathcal{P}$$

with corresponding BSDE formulation if existence...

Corresponds to the fully nonlinear PDE

$$-\partial_t V - G(D^2V) = 0 \quad \text{where} \quad G(\gamma) := \frac{1}{2}\left(\overline{a}\gamma^+ - \underline{a}\gamma^-\right)$$

No-arbitrage bounds

Galichon Henry-Labordère NT :

$$V_0 \quad := \quad \inf \left\{ Y_0 : Y_0 + \int_0^T Z_t dW_t \geq \xi \ \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}(\mu) \right\}$$

where $\mathcal{P}(\mu)$ is the collection of all probability measures $\mathbb{Q}$ such that W is a $\mathbb{Q}-$martingale with quadratic variation a.c. with respect to Lebesgue measure, and

$$W_T \quad \sim \quad \mu$$

with corresponding BSDE formulation if existence...

Outline

Main references

L. Denis and C. Martini 2006 : Quasi-sure analysis

Peng 2007 : $G-$Brownian motion

M. Soner, NT and J. Zhang (2010a,2010b,2010c,2010d)

Nutz 2010, Kervarec Bion-Nadal 2010, ...

Intuition from PDEs

Let V be a solution of

$$-\partial_t V - h(., V, DV, D^2 V) = 0 \quad \text{and} \quad V(T,.) = g$$

and suppose

$$h(x, r, p, \gamma) \;=\; \sup_{a \geq 0} \left\{ \frac{1}{2} a\gamma - f(x, r, p, a) \right\}$$

Then $V = \sup_a V^a$ where V^a is a solution of

$$-\partial_t V^a - \frac{1}{2} a\gamma + f(., V^a, DV^a, a) = 0 \quad \text{and} \quad V^a(T,.) = g$$

a semilinear PDE which corresponds to a BSDE...

Nondominated family of measures on canonical space

$\Omega := C([0,1], \mathbb{R}^d)$, B : coordinate process, $\mathbb{P}_0$: Wiener measure
$\mathbb{F} := \{\mathcal{F}_t\}_{0\le t\le 1}$: filtration generated by B

$\mathbb{P}$ is a local martingale measure if B local martingale under $\mathbb{P}$

Föllmer 73 : $\int_0^t B_s dB_s$, defined $\omega-$wise, coincides with Itô integral, $\mathbb{P}-$a.s. for all loc mart measure $\mathbb{P}$. Then

$$\langle B\rangle_t := B_t B_t^{\mathrm{T}} - 2\int_0^t B_s dB_s^{\mathrm{T}} \quad \text{and} \quad \hat{a}_t := \limsup_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\Big(\langle B\rangle_t - \langle B\rangle_{t-\varepsilon}\Big),$$

defined $\omega-$wise

$\overline{\mathcal{P}}_W$:set of all local martingale measures $\mathbb{P}$ such that

$$\langle B\rangle_t \quad \text{is a. c. in } t \text{ and} \quad \hat{a} \text{ takes values in } \mathbb{S}_d^{>0}(\mathbb{R}),\ \mathbb{P}-\text{a.s.}$$

General framework, continued

For every $\mathbb{F}-$prog. meas. α valued in $\mathbb{S}_d^{>0}(\mathbb{R})$ with $\int_0^1 |\alpha_t|dt < \infty$, $\mathbb{P}_0-$a.s. Define

$$\mathbb{P}^\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1} \quad \text{where} \quad X_t^\alpha := \int_0^t \alpha_s^{1/2} dB_s, t \in [0,1], \mathbb{P}_0 - \text{a.s.}$$

$\overline{\mathcal{P}}_S \subset \overline{\mathcal{P}}_W$: collection of all such $\mathbb{P}^\alpha$

Then every $\mathbb{P} \in \overline{\mathcal{P}}_S$

- satisfies the Blumenthal zero-one law
- and the martingale representation property

Generator $H_t(\omega, y, z, \gamma) : [0,1] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \to \mathbb{R}$

- Convex conjugate :

$$F_t(\omega, y, z, a) := \sup_{\gamma \in D_H} \left\{ \tfrac{1}{2}\mathrm{Tr}[a\gamma] - H_t(\omega, y, z, \gamma) \right\}, \quad a \in \mathbb{S}_d^{>0}(\mathbb{R});$$

$$\hat{F}_t(y, z) := F_t(y, z, \hat{a}_t) \quad \text{and} \quad \hat{F}_t^0 := \hat{F}_t(0, 0)$$

$$\mathcal{P}_H^\kappa = \left\{ \mathbb{P} \in \overline{\mathcal{P}}_S : \hat{a}, \hat{a}^{-1} \text{ bdd}, \ \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^1 |\hat{F}_t^0|^\kappa dt\right)^{2/\kappa}\right] < \infty \right\}, \ \kappa \in (1, 2]$$

Def (Denis-Martini 06) $\mathcal{P}_H^\kappa$–q.s. means $\mathbb{P}$–a.s. for all $\mathbb{P} \in \mathcal{P}_H^\kappa$

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Assumptions

- Domain of $a \longmapsto F_t(y, z, a)$ independent of (ω, y, z)
- F is $\mathbb{F}^+-$progressively measurable, uniformly continuous in ω
- $|\hat{F}_t(y, z) - \hat{F}_t(y', z')| \leq C\left(|y - y'| + |\hat{a}^{1/2}(z - z')|\right)$, $\mathcal{P}_H^\kappa-$q.s.

Definition

For $\mathcal{F}_1-$meas. ξ, consider the 2BSDE :

$$dY_t = \hat{F}_t(Y_t, Z_t)dt + Z_t dB_t - dK_t,\ 0 \le t \le 1,\ \ Y_T = \xi,\ \mathcal{P}_H^\kappa - \text{q.s.}$$

We say $(Y, Z) \in \mathbb{D}_H^2 \times \mathbb{H}_H^2$ is a solution to the 2BSDE if

- $Y_T = \xi$, $\mathcal{P}_H^\kappa-$q.s.
- For each $\mathbb{P} \in \mathcal{P}_H^\kappa$, $K^{\mathbb{P}}$ has nondecreasing paths, $\mathbb{P}-$a.s. :

$$K_t^{\mathbb{P}} := Y_0 - Y_t + \int_0^t \hat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s,\ \ t \in [0,1],\ \ \mathbb{P} - \text{a.s.}$$

- The family of processes $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H^\kappa\}$ satisfies :

$$K_t^{\mathbb{P}} = \operatorname*{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t,\mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[K_1^{\mathbb{P}'}],\ \ \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}_H^\kappa, t \le T$$

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$$dY_t = \hat{F}_t(Y_t, Z_t)dt + Z_t dB_t - dK_t,\ 0 \le t \le 1,\ Y_T = \xi,\ \mathcal{P}^\kappa_H - \text{q.s.}$$

We say $(Y, Z) \in \mathbb{D}^2_H \times \mathbb{H}^2_H$ is a solution to the 2BSDE if

- $Y_T = \xi$, $\mathcal{P}^\kappa_H-$q.s.
- For each $\mathbb{P} \in \mathcal{P}^\kappa_H$, $K^\mathbb{P}$ has nondecreasing paths, $\mathbb{P}-$a.s. :

$$K^\mathbb{P}_t := Y_0 - Y_t + \int_0^t \hat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s,\ t \in [0,1],\ \mathbb{P} - \text{a.s.}$$

- The family of processes $\{K^\mathbb{P}, \mathbb{P} \in \mathcal{P}^\kappa_H\}$ satisfies :

$$K^\mathbb{P}_t = \operatorname*{ess\,inf}_{\mathbb{P}' \in \mathcal{P}^\kappa_H(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}_t[K^{\mathbb{P}'}_1],\ \ \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}^\kappa_H, t \le T$$

Back to standard BSDEs

For standard BSDEs

$$dY_t := -H_t^0(Y_t, Z_t)dt + Z_t dB_t, \;\; Y_T = \xi$$

the nonlinearity and the corresponding conjugate :

$$H_t(., \gamma) := -\tfrac{1}{2}\mathrm{Tr}[\gamma] + H_t^0(.), \quad F_t(., a) = \begin{cases} -H_t^0(.) & \text{for } a = I_d \\ \infty & \text{otherwise} \end{cases}$$

Then $\mathcal{P}_H^\kappa = \{\mathbb{P}_0\}$, $K^{\mathbb{P}_0} \equiv 0$, and the previous definition reduces to the standard definition

$$dY_t := -H_t^0(Y_t, Z_t)dt + Z_t dB_t, \;\; Y_T = \xi, \;\; \mathbb{P}_0 - \text{a.s.}$$

Benchmark example : uncertain volatility model, $G-$expectation (Peng)

Let $d = 1$, and $H_t(y, z, \gamma) := G(\gamma) = \overline{a}\gamma^+ - \underline{a}\gamma^-$, and suppose that the PDE

$$\frac{\partial u}{\partial t} + G(u_{xx}) = 0, \quad \text{and} \quad u(T, .) = g$$

has a smooth solution. Then

$$Y_t := u(t, B_t), \quad Z_t := Du(t, B_t),$$

is a solution of the 2BSDE with

$$K_t := \int_0^t \left(G(u_{xx}) - \frac{1}{2}\hat{a}_s u_{xx} \right)(s, B_s)ds$$

Outline

Spaces and norms

$$L_H^{2,\kappa} := \{\xi \ \mathcal{F}_1 - \text{meas.} : \ \|\xi\|_{L_H^{2,\kappa}} < \infty\}, \ \|\xi\|^2_{L_H^{2,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}[|\xi|^2]$$

$$\mathbb{H}_H^{2,\kappa} := \left\{ Z \ \mathbb{F}^+ - \text{prog. meas. in } \mathbb{R}^d : \ \|Z\|^p_{\mathbb{H}_H^{2,\kappa}} < \infty \right\}$$

$$\|Z\|^2_{\mathbb{H}_H^{2,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\int_0^1 |\hat{a}_t^{1/2} Z_t|^2 dt \right]$$

$$\mathbb{D}_H^{2,\kappa} := \left\{ Y \ \mathbb{F}^+ - \text{prog. in } \mathbb{R} \text{ càdlàg } \mathcal{P}_H^\kappa - \text{q.s. } \|Y\|_{\mathbb{D}_H^{2,\kappa}} < \infty \right\}$$

$$\|Y\|^2_{\mathbb{D}_H^{2,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\le t\le 1} |Y_t|^2 \right]$$

$$\mathbb{L}_H^{2,\kappa} := \{\xi \in L_H^{2,\kappa} : \|\xi\|_{\mathbb{L}_H^{2,\kappa}} < \infty\}, \ \mathbb{E}_t^{H,\mathbb{P}}[\xi] := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}[\xi|\mathcal{F}_t]$$

$$\|\xi\|_{\mathbb{L}_H^{2,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\le t\le 1} \left(\mathbb{E}_t^{H,\mathbb{P}}[|\xi|^\kappa]\right)^{2/\kappa} \right]$$

Spaces and norms

$$L^{2,\kappa}_H := \left\{\xi\ \mathcal{F}_1 - \text{meas.}:\ \|\xi\|_{L^{2,\kappa}_H} < \infty\right\},\quad \|\xi\|^2_{L^{2,\kappa}_H} := \sup_{\mathbb{P}\in\mathcal{P}^\kappa_H} \mathbb{E}^{\mathbb{P}}[|\xi|^2]$$

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Spaces and norms

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$$\|\xi\|_{\mathbb{L}^{2,\kappa}_H} := \sup_{\mathbb{P}\in\mathcal{P}^\kappa_H} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\le t\le 1} \left(\mathbb{E}^{H,\mathbb{P}}_t[|\xi|^\kappa]\right)^{2/\kappa}\right]$$

Spaces and norms

$$L_H^{2,\kappa} := \left\{\xi \ \mathcal{F}_1 - \text{meas.} : \ \|\xi\|_{L_H^{2,\kappa}} < \infty\right\}, \ \|\xi\|^2_{L_H^{2,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}[|\xi|^2]$$

$$\mathbb{H}_H^{2,\kappa} := \left\{Z \ \mathbb{F}^+ - \text{prog. meas. in } \mathbb{R}^d : \ \|Z\|^p_{\mathbb{H}_H^{2,\kappa}} < \infty\right\}$$

$$\|Z\|^2_{\mathbb{H}_H^{2,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\int_0^1 |\hat{a}_t^{1/2} Z_t|^2 dt\right]$$

$$\mathbb{D}_H^{2,\kappa} := \left\{Y \ \mathbb{F}^+ - \text{prog. in } \mathbb{R} \text{ càdlàg } \mathcal{P}_H^\kappa - \text{q.s. } \|Y\|_{\mathbb{D}_H^{2,\kappa}} < \infty\right\}$$

$$\|Y\|^2_{\mathbb{D}_H^{2,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\le t\le 1} |Y_t|^2\right]$$

$$\mathbb{L}_H^{2,\kappa} := \left\{\xi \in L_H^{2,\kappa} : \|\xi\|_{\mathbb{L}_H^{2,\kappa}} < \infty\right\}, \ \mathbb{E}_t^{H,\mathbb{P}}[\xi] := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}[\xi|\mathcal{F}_t]$$

$$\|\xi\|_{\mathbb{L}_H^{2,\kappa}} := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\le t\le 1} \left(\mathbb{E}_t^{H,\mathbb{P}}[|\xi|^\kappa]\right)^{2/\kappa}\right]$$

Representation and uniqueness

Assumption $\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\le t\le 1} \mathbb{E}_t^{H,\mathbb{P}}\left[\int_t^1 |\hat{F}_s^0|^2 ds\right]\right] < \infty$

Theorem For $\xi \in \mathbb{L}_H^{2,\kappa}$ the 2BSDE has at most one solution $(Y,Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ satisfying

$$Y_t = \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t_1,\mathbb{P})}{}^{\mathbb{P}} \mathcal{Y}_t^{\mathbb{P}'}(1,\xi), \quad \mathbb{P}-\text{a.s. for all } \mathbb{P}\in\mathcal{P}_H^\kappa$$

where $\mathcal{Y}_t^{\mathbb{P}}(1,\xi) = y_t$, where $(y_s, z_s)_{t\le s\le 1}$ is the solution of

$$y_t = \xi - \int_t^1 \hat{F}_s(y_s,z_s)ds + \int_t^1 z_s dB_s, \ \mathbb{P}-\text{a.s.}$$

Moreover, comparison holds true

Existence

Theorem For any $\xi \in \mathcal{L}_H^{2,\kappa}$, the 2BSDE admits a unique solution $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$

where $\mathcal{L}_H^{2,\kappa}$:= closure of $\mathrm{UC}_b(\Omega)$ under the norm $\mathbb{L}_H^{2,\kappa}$:

$$\|\xi\|_{\mathbb{L}_H^{2,\kappa}} \quad := \quad \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\le t\le 1}\big(\mathbb{E}_t^{H,\mathbb{P}}\big[|\xi|^\kappa\big]\big)^{2/\kappa}\Big]$$

and

$$\mathbb{E}_t^{H,\mathbb{P}}[\xi] := \operatorname*{ess\,sup}_{\mathbb{P}'\in\mathcal{P}_H^\kappa(t,\mathbb{P})}{}^{\mathbb{P}}\ \mathbb{E}^{\mathbb{P}'}[\xi|\mathcal{F}_t]$$

Connection with PDEs

Let

$$H_t(\omega, y, z, \gamma) =: h(t, \omega_t, y, z, \gamma) \quad \text{and} \quad \xi(\omega) =: g(\omega_1)$$

In particular $F_t(\omega, y, z, a) =: f(t, \omega_t, y, z, a)$. Define

$$\hat{h}(t, x, y, z, \gamma) := \sup_{a \in \mathbb{S}_d^+(\mathbb{R})} \left\{ \frac{1}{2} \mathrm{Tr}[a\gamma] - f(t, x, y, z, a) \right\}, \ \gamma \in \mathbb{R}^{d \times d}$$

convex nondecreasing in γ

Theorem Under "natural conditions", the solution of the 2BSDE satisfies $Y_t = u(t, B_t)$, $t \in [0, T]$, $\mathcal{P}_H^\kappa$−q.s. and u is a viscosity solution of

$$-\partial_t u(t, x) - \hat{h}\Big(t, x, u(t, x), Du(t, x), D^2 u(t, x)\Big) = 0, \quad t < 1$$

$$u(1, x) = g(x)$$

Feynman-Kac formula for fully nonlinear Cauchy problem

Assumption D_f is independent of t and bounded both from above and away from 0

Theorem Let $g(B_1) \in \mathbb{L}^{2,\kappa}_H$ and $v \in C^{1,2}([0,1), \mathbb{R}^d)$ a classical solution of the PDE. Then, under regularity conditions on h :

$$Y_t := v(t, B_t), \;\; Z_t := Dv(t, B_t), \;\; K_t := \int_0^t k_s ds$$

with

$$k_t := \hat{h}(t, B_t, Y_t, Z_t, \Gamma_t) - \frac{1}{2}\mathrm{Tr}[\hat{a}_t \Gamma_t] + f(t, B_t, Y_t, Z_t, \hat{a}_t)$$
$$\Gamma_t := D^2 v(t, B_t)$$

is the unique solution of the 2BSDE