# Cross hedging, utility maximization and systems of FBSDE

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# 1 Cross hedging, optimal investment, exponential utility

for convex constraints: (N. El Karoui, R. Rouge ’00; J. Sekine ’02; J. Cvitanic, J. Karatzas ’92, Kramkov, Schachermayer ’99, Mania, Schweizer ’05, Pham ’07, Zariphopoulou ’01,...)

maximal expected exponential utility from terminal wealth

$$V(x) = \sup_{\pi \in \mathcal{A}} EU(x + X_T^\pi + H) = \sup_{\pi \in \mathcal{A}} E(-\exp(-\alpha(x + \int_0^T \pi_s[dW_s + \theta_s ds] + H)))$$

wealth on $[0,T]$ by investment strategy $\pi$ :

$$\int_0^T \langle \pi_u, \frac{dS_u}{S_u} \rangle = \int_0^T \pi_u [dW_u + \theta_u du] = X_T^\pi,$$

$H$ liability or derivative, correlated to financial market $S$

$\pi \in \mathcal{A}$ subject to $\pi$ taking values in $C$ **closed**

aim: use BSDE to represent optimal strategy $\pi^*$

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# 2 Martingale optimality

**Idea:** Construct family of processes $Q^{(\pi)}$ such that

**(form 1)**

$$\begin{array}{rcl} Q_0^{(\pi)} & = & \text{constant}, \\ Q_T^{(\pi)} & = & -\exp(-\alpha(x + X_T^{\pi} + H)), \\ Q^{(\pi)} & & \text{supermartingale}, \quad \pi \in \mathcal{A}, \\ Q^{(\pi^*)} & & \text{martingale, for (exactly) one} \quad \pi^* \in \mathcal{A}. \end{array}$$

Then

$$\begin{aligned} E(-\exp(-\alpha[x + X_T^{\pi} + H])) &= E(Q_T^{(\pi)}) \\ &\leq E(Q_0^{\pi}) \\ &= E(Q_0^{(\pi^*)}) \\ &= E(-\exp(-\alpha[x + X_T^{(\pi^*)} + H])). \end{aligned}$$

Hence $\pi^*$ optimal strategy.

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# 3 Solution method based on BSDE

## Introduction of BSDE into problem

Find generator $f$ of BSDE

$$Y_t = H - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds, \quad Y_T = F,$$

such that with

$$Q_t^{(\pi)} = -\exp(-\alpha[x + X_t^\pi + Y_t]), \quad t \in [0, T],$$

we have

**(form 2)**

$$\begin{aligned} Q_0^{(\pi)} &= -\exp(-\alpha(x + Y_0)) = \text{constant}, && \text{(fulfilled)} \\ Q_T^{(\pi)} &= -\exp(-\alpha(x + X_T^\pi + H)) && \text{(fulfilled)} \\ Q^{(\pi)} & \quad \text{supermartingale}, \quad \pi \in \mathcal{A}, \\ Q^{(\pi^*)} & \quad \text{martingale, for (exactly) one} \quad \pi^* \in \mathcal{A}. \end{aligned}$$

This gives solution of valuation problem.

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# 4 Construction of generator of BSDE

**How to determine $f$:**

Suppose $f$ generator of BSDE. Then by Ito's formula

$$\begin{aligned} Q_t^{(\pi)} &= -\exp(-\alpha[x + X_t^\pi + Y_t]) \\ &= Q_0^{(\pi)} + M_t^{(\pi)} + \int_0^t \alpha Q_s^{(\pi)}[-\pi_s\theta_s - f(s, Z_s) + \frac{\alpha}{2}(\pi_s - Z_s)^2]ds, \end{aligned}$$

with a local martingale $M^{(\pi)}$.

$Q^{(\pi)}$ satisfies **(form 2)** iff for

$$q(\cdot, \pi, z) = -f(\cdot, z) - \pi\theta + \frac{\alpha}{2}(\pi - z)^2, \quad \pi \in \mathcal{A}, z \in \mathbb{R},$$

we have

**(form 3)**
$$\begin{aligned} q(\cdot, \pi, z) &\geq 0, \quad \pi \in \mathcal{A} \quad \text{(supermartingale)} \\ q(\cdot, \pi^*, z) &= 0, \quad \text{for (exactly) one} \quad \pi^* \in \mathcal{A} \quad \text{(martingale)}. \end{aligned}$$

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# 4 Construction of generator of BSDE

Now

$$
\begin{aligned}
q(\cdot,\pi,z) &= -f(\cdot,z)-\pi\theta+\frac{\alpha}{2}(\pi-z)^2 \\
&= -f(\cdot,z)+\frac{\alpha}{2}(\pi-z)^2-(\pi-z)\cdot\theta+\frac{1}{2\alpha}\theta^2 \quad -z\theta-\frac{1}{2\alpha}\theta^2 \\
&= -f(\cdot,z)+\frac{\alpha}{2}[\pi-(z+\frac{1}{\alpha}\theta)]^2 \quad -z\theta-\frac{1}{2\alpha}\theta^2.
\end{aligned}
$$

Under **non-convex constraint** $p \in C$:

$$[\pi-(z+\frac{1}{\alpha}\theta)]^2 \geq \text{dist}^2(C, z+\frac{1}{\alpha}\theta).$$

with **equality** for at least one possible choice of $\pi^*$ due to **closedness** of $C$.
Hence **(form 3)** is solved by the choice (predictable selection)

**(form 4)**

$$
\begin{aligned}
f(\cdot,z) &= \frac{\alpha}{2}\text{dist}^2(C,z+\tfrac{1}{\alpha}\theta)-z\cdot\theta-\tfrac{1}{2\alpha}\theta^2 \quad \text{(supermartingale)} \\
\pi^* &: \text{dist}(C,z+\tfrac{1}{\alpha}\theta)=\text{dist}(\pi^*,z+\tfrac{1}{\alpha}\theta) \quad \text{(martingale)}.
\end{aligned}
$$

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## 5 Summary of results, exponential utility

Solve utility optimization problem

$$\sup_{\pi \in \mathcal{A}} EU(x + X_T^\pi + H)$$

by considering FBSDE

$$\begin{aligned} dX_t^\pi &= \pi_t[dW_t + \theta_t dt], \quad X_0^\pi = x, \\ dY_t &= Z_t dW_t + f(t, Z_t)dt, \quad Y_T = H \end{aligned}$$

with generator as described before; determine $\pi^*$ by previsible selection; coupling through requirement of martingale optimality

$$\begin{aligned} \sup_{\pi \in \mathcal{A}} EU(x + X_T^\pi + H) &= EU(x + X_T^{\pi^*} + H), \\ U'(x + X_t^{\pi^*} + Y_t) & \quad \text{martingale}. \end{aligned}$$

for general $U$: forward part depends on $\pi^*$, get fully coupled FBSDE

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# 6 Cross hedging, optimal investment, utility on $\mathbb{R}$

Lit: Mania, Tevzadze (2003)

$U : \mathbb{R} \to \mathbb{R}$ strictly increasing and concave; maximal expected utility from terminal wealth

$$(1)\ V(x) = \sup_{\pi \in \mathcal{A}} EU(x + X_T^\pi + H)$$

wealth on $[0, T]$ by investment strategy $\pi$ :

$$\int_0^T \langle \pi_u, \frac{dS_u}{S_u} \rangle = \int_0^T \pi_u [dW_u + \theta_u du] = X_T^\pi,$$

$H$ liability or derivative, correlated to financial market $S$, $W$ $d-$dimensional Wiener process, $W^1$ first $d_1$ components of $W$

$\pi \in \mathcal{A}$ subject to convex constraint $\pi = (\pi^1, 0)$, $\pi^1$ $d_1-$dimensional, hence incomplete market

aim: use FBSDE system to describe optimal strategy $\pi^*$

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# 7 Verification theorems

**Thm 1**

Assume $U$ is three times differentiable, $U'$ regular enough. If there exists $\pi^*$ solving (1), and $Y$ is the predictable process for which $U'(X^{\pi^*}+Y)$ is square integrable martingale, then with $Z=\frac{d}{dt}\langle Y,W\rangle$

$$(\pi^*)^1 = -\theta^1 \frac{U'}{U''}(X^{\pi^*}+Y) - Z^1.$$

**Pf:**

$$\alpha \;=\; \mathbb{E}(U'(X_T^{\pi^*}+H)|\mathcal{F}_\cdot),\; Y=(U')^{-1}(\alpha)-X^{\pi^*}.$$

Use Itô's formula and martingale property. Find

$$Y = H - \int_\cdot^T Z_s dW_s - \int_\cdot^T f(s, X_s^{\pi^*}, Y_s, Z_s)ds,$$

with

$$f(s, X_s^{\pi^*}, Y_s, Z_s) = -\frac{1}{2}\frac{U^{(3)}}{U''}(X^{\pi^*}+Y)|\pi_s^*+Z_s|^2 - \pi_s^*\theta_s.$$

Use variational maximum principle to derive formula for $\pi^*$.

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# 7 Verification theorems

From preceding theorem derive the FBSDE system

**Thm 2**

Assumptions of Thm 1; then optimal wealth process $X^{\pi^*}$ given as component $X$ of solution $(X, Y, Z)$ of fully coupled FBSDE system

$$
\begin{aligned}
X &= x - \int_0^{\cdot} (\theta_s^1 \frac{U'}{U''}(X_s + Y_s) + Z_s^1) dW_s^1 - \int_0^{\cdot} (\theta_s^1 \frac{U'}{U''}(X_s + Y_s) + Z_s^1)\theta_s^1 ds, \\
Y &= H - \int_{\cdot}^{T} Z_s dW_s \\
& \quad - \int_{\cdot}^{T} [|\theta_s^1|^2((-\frac{1}{2}\frac{U^{(3)}U'^2}{(U'')^3} + \frac{U'}{U''})(X_s + Y_s) + Z_s^1 \cdot \theta_s^1) \\
& \qquad -\frac{1}{2}|Z_s^2|^2 \frac{U^{(3)}}{U''}(X_s + Y_s)] ds. \quad (2)
\end{aligned}
$$

**Pf:**

Use expression for $f$ and formula for $\pi^*$ from Thm 1.

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# 8 Representation of optimal strategy

Invert conclusion of Thm 2 to give representation of optimal strategy

**Thm 3**
Let $(X, Y, Z)$ be solution of (2), $U(X_T + H)$ integrable, $U'(X_T + H)$ square integrable. Then

$$(\pi^*)^1 = -\theta^1 \frac{U'}{U''}(X + Y) + Z^1$$

is optimal solution of (1).

**Pf:**
By concavity for any admissible $\pi$

$$U(X^\pi + Y) - U(X + Y) \leq U'(X + Y)(X^\pi - X).$$

Now prove that

$$U'(X + Y)(X^\pi - X) = U'(X^{\pi^*} + Y)(X^\pi - X^{\pi^*})$$ is a martingale!

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# 9 The complete case

Formula representing $\pi^*$ $\longrightarrow$ martingale representation

$$U'(X^{\pi^*} + Y) = U'(x + Y_0)\mathcal{E}(-\theta \cdot W).$$

Aim: show existence for fully coupled system of Thm 2.

Crucial observation: $P = X + Y$ solves forward SDE

$$P = x + Y_0 - \int_0^\cdot \theta_s \frac{U'}{U''}(P_s)dW_s - \int_0^\cdot \frac{1}{2}\frac{U^{(3)}U'^2}{(U'')^3}(P_s)ds.$$

Idea: forward SDE

$$P^m = x + m - \int_0^\cdot \theta_s \frac{U'}{U''}(P_s^m)dW_s - \int_0^\cdot \frac{1}{2}\frac{U^{(3)}U'^2}{(U'')^3}(P_s^m)ds$$

has solution; now decouple again, by considering BSDE

$$Y^m = H - \int_\cdot^T Z_s^m dW_s - \int_\cdot^T (|\theta_s|^2[-\frac{1}{2}\frac{U^{(3)}U'^2}{(U'')^3} + \frac{U'}{U''}](P_s^m) + Z_s^m\theta_s)ds.$$

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## 9 The complete case

Solve for $(Y^m, Z^m)$, use continuity of $m \mapsto Y_0^m$ to find $m$ such that $Y_0^m = m$.

This gives

**Thm 4**
Assume $\frac{U^{(3)}U'^2}{(U'')^3}$ and $\frac{U'}{U''}$ are Lipschitz and bounded. Then the system of FBSDE

$$\begin{aligned} X &= x - \int_0^\cdot (\theta_s \frac{U'}{U''}(X_s + Y_s) + Z_s) dW_s^1 - \int_0^\cdot (\theta_s \frac{U'}{U''}(X_s + Y_s) + Z_s)\theta_s ds, \\ Y &= H - \int_\cdot^T Z_s dW_s - \int_\cdot^T [|\theta_s|^2((-\frac{1}{2}\frac{U^{(3)}U'^2}{(U'')^3} + \frac{U'}{U''})(X_s + Y_s) + Z_s \cdot \theta_s)] ds \quad (3) \end{aligned}$$

has solution $(X, Y, Z)$ such that $U(X_T + H)$ is integrable, $U'(X_T + H)$ square integrable.

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# 10 Utility function on $\mathbb{R}_+$

Replace $U'(X^{\pi^*}+Y)$ with $U'(X^{\pi^*})\exp(\tilde{Y})$. Then $(X,Y,Z)$ satisfies (3) if and only if $(X,\tilde{Y},\tilde{Z})$ satisfies (4) ($\tilde{Z}=\frac{d}{dt}\langle W,\tilde{Y}\rangle$):

**Thm 5**
Let $(X,\tilde{Y},\tilde{Z})$ be solution of the fully coupled FBSDE

$$\begin{aligned} X &= x-\int_0^{\cdot}(\frac{U'}{U''}(X_s)(\theta_s^1+Z_s^1)dW_s^1-\int_0^{\cdot}(\frac{U'}{U''}(X_s)(\theta_s^1+Z_s^1)\theta_s^1 ds, \\ Y &= \ln(\frac{U'(X_T+H)}{U'(X_T)})-\int_{\cdot}^T Z_s dW_s \\ &\quad -\int_{\cdot}^T [|Z_s^1+\theta_s^1|^2((1-\frac{1}{2}\frac{U^{(3)}U'}{(U'')^2})(X_s)-\frac{1}{2}|Z_s|^2]ds. \ (4) \end{aligned}$$

such that $U(X_T^{\pi^*}+H)$ is integrable and $U'(X_T^{\pi^*}+H)$ is square integrable. Then

$$(\pi^*)^1=-\frac{U'}{U''}(X)(Z^1+\theta^1)$$

solves (1).

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## 11 The complete case

Using forward equation for $P = U'(X)\exp(Y)$ as above we obtain

**Thm 6**
Assume $\frac{U^{(3)}U'}{(U'')^2}$ and $\frac{U'}{U''}$ are Lipschitz and bounded. Then system of FBSDE

$$\begin{aligned} X &= x - \int_0^{\cdot} (\frac{U'}{U''}(X_s)(\theta_s + Z_s) dW_s - \int_0^{\cdot} \frac{U'}{U''}(X_s)(\theta_s + Z_s)\theta_s ds, \\ Y &= \ln(\frac{U'(X_T + H)}{U'(X_T)}) - \int_{\cdot}^T Z_s dW_s \\ &\quad - \int_{\cdot}^T [|Z_s + \theta_s|^2((1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2})(X_s) - \frac{1}{2}|Z_s|^2] ds. \end{aligned}$$

has solution $(X, Y, Z)$ such that $U(X_T)$ is integrable, $U'(X_T)$ square integrable.

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# 12 Link to stochastic maximum principle, complete case

$H = 0$, $\tilde{X}^\pi = U(X^\pi)$; value function

$$J(\pi) = \mathbb{E}(U(X_T^\pi)) = \mathbb{E}(\tilde{X}_T^\pi)$$

Using Peng (1993) obtain system of FBSDE

$$\begin{aligned}
d\tilde{X}_t^\pi &= U'(U^{-1}(\tilde{X}_t^\pi))\pi_t dW_t + [U'(U^{-1}(\tilde{X}_t^\pi))\pi_t\theta_t + \frac{1}{2}U''(U^{-1}(\tilde{X}_t^\pi))|\pi_t|^2]dt, \\
& \tilde{X}_0^\pi = U(x), \\
-dp_t &= \frac{U''}{U'}(\tilde{X}_t^\pi)k_t\pi_t dW_t + [\frac{U''}{U'}(U^{-1}(\tilde{X}_t^\pi))\pi_t\theta_t + \frac{1}{2}\frac{U^{(3)}}{U'}(U^{-1}(\tilde{X}_t^\pi))|\pi_t|^2]dt, \\
& p_T = 1 \ (5).
\end{aligned}$$

maximization of Hamiltonian

$$H(x,\pi,p,k) = p[U'(U^{-1}(x))\pi\theta + \frac{1}{2}U''(U^{-1}(x)|\pi|^2] + kU'(U^{-1}(x))\pi$$

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# 12 Link to stochastic maximum principle, complete case

gives

$$\pi^* = -\frac{U'}{U''}(U^{-1}(\tilde{X}))[\frac{k}{p} + \theta].$$

Use this in (4), apply Cole-Hopf transformation

$$Y = \ln(p), \quad Z = \frac{d}{dt}\langle Y, W\rangle = \frac{k}{p}$$

to get

$$\begin{aligned} d\tilde{X}_t^{\pi^*} &= -\frac{U'}{U''}(\tilde{X}_t^{\pi^*})(Z_t + \theta_t)(dW_t + \theta_t dt), \ \tilde{X}_0^{\pi^*} = U(x), \\ dY_t &= [(Z_t + \theta_t)^2(1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2}(\tilde{X}_t^{\pi^*}) - \frac{1}{2}|Z_t|^2]dt + Z_t dW_t, \ Y_T = 0 \ (6). \end{aligned}$$

FBSDE system identical to the one obtained above, decouples if $\frac{U^{(3)}U'}{(U'')^2}$ is constant, i.e. for exponential, power and logarithmic utility

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# 13 Example: power utility, general liability, incomplete case

Lit: Nutz (2010) for $H = 0$

$U(x) = \frac{1}{\gamma} x^\gamma$ for some $\gamma < 1$, $W = (W^1, W^2)$ two-dimensional Wiener process, $dS_t^i = dW_t^i + \theta_t^i dt, i = 1, 2,$ investment $dX_t^\pi = \pi_t dS_t^1$, liability $H = \phi(S_T^2),$ with $\phi$ positive, bounded

(4) transforms into

$$
\begin{aligned}
dX_t &= \frac{1}{1-\gamma} X_s (Z_t^1 + \theta_t^1)(dW_t + \theta_t^1 dt), \;\; X_0 = x, \\
dY_t &= -[\frac{\gamma}{2(\gamma - 1)}(Z_t^1 + \theta_t^1)^2 - \frac{1}{2}|Z_t|^2] dt + Z_t dW_t,
\end{aligned}
$$

$$
Y_T = (\gamma - 1) \ln(1 + \frac{H}{X_T}) \;(7).
$$

optimal solution

$$
(\pi^*)^1 = \frac{1}{1-\gamma}(Z^1 + \theta^1)
$$

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