Cross hedging, utility maximization and systems of FBSDE

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1 Cross hedging, optimal investment, exponential utility

for convex constraints: (N. El Karoui, R. Rouge '00; J. Sekine '02; J. Cvitanic, J. Karatzas '92, Kramkov, Schachermayer '99, Mania, Schweizer '05, Pham '07, Zariphopoulou '01,...)

maximal expected exponential utility from terminal wealth

\[ V(x) = \sup_{\pi \in A} EU(x + X^\pi_T + H) = \sup_{\pi \in A} E(-\exp(-\alpha(x + \int_0^T \pi_s[dW_s + \theta_s ds] + H))) \]

wealth on \([0, T]\) by investment strategy \(\pi\):

\[ \int_0^T \langle \pi_u, \frac{dS_u}{S_u} \rangle = \int_0^T \pi_u[dW_u + \theta_u du] = X^\pi_T, \]

\(H\) liability or derivative, correlated to financial market \(S\)

\(\pi \in A\) subject to \(\pi\) taking values in \(C\) closed

aim: use BSDE to represent optimal strategy \(\pi^*\)

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2 Martingale optimality

Idea: Construct family of processes $Q^{(\pi)}$ such that

\[
\begin{align*}
Q_0^{(\pi)} &= \text{constant}, \\
Q_T^{(\pi)} &= -\exp(-\alpha(x + X_T^{\pi} + H)), \\
Q_T^{(\pi)} &= \text{supermartingale, } \pi \in \mathcal{A}, \\
Q_{\pi^*} &= \text{martingale, for (exactly) one } \pi^* \in \mathcal{A}.
\end{align*}
\]

Then

\[
E(-\exp(-\alpha[x + X_T^{\pi} + H])) = E(Q_T^{(\pi)}) \\
\leq E(Q_0^{(\pi)}) \\
= E(Q_0^{(\pi^*)}) \\
= E(-\exp(-\alpha[x + X_T^{(\pi^*)} + H])).
\]

Hence $\pi^*$ optimal strategy.

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3 Solution method based on BSDE

Introduction of BSDE into problem

Find generator $f$ of BSDE

$$Y_t = H - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds, \quad Y_T = F,$$

such that with

$$Q^{(\pi)}_t = - \exp(-\alpha [x + X^\pi_t + Y_t]), \quad t \in [0, T],$$

we have

$$(\text{form 2})$$

$$Q^{(\pi)}_0 = - \exp(-\alpha (x + Y_0)) = \text{constant}, \quad (\text{fulfilled})$$

$$Q^{(\pi)}_T = - \exp(-\alpha (x + X^\pi_T + H)) \quad (\text{fulfilled})$$

$$Q^{(\pi)} \text{ supermartingale, } \pi \in \mathcal{A},$$

$$Q^{(\pi^*)} \text{ martingale, for (exactly) one } \pi^* \in \mathcal{A}.$$

This gives solution of valuation problem.

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4 Construction of generator of BSDE

How to determine $f$:

Suppose $f$ generator of BSDE. Then by Ito’s formula

$$Q_t^{(\pi)} = -\exp(-\alpha [x + X_t^{\pi} + Y_t])$$

$$= Q_0^{(\pi)} + M_t^{(\pi)} + \int_0^t \alpha Q_s^{(\pi)} [-\pi_s \theta_s - f(s, Z_s) + \frac{\alpha}{2} (\pi_s - Z_s)^2] ds,$$

with a local martingale $M^{(\pi)}$.

$Q^{(\pi)}$ satisfies (form 2) iff for

$$q(\cdot, \pi, z) = -f(\cdot, z) - \pi \theta + \frac{\alpha}{2} (\pi - z)^2, \quad \pi \in \mathcal{A}, z \in \mathbb{R},$$

we have

(form 3) $q(\cdot, \pi, z) \geq 0$, $\pi \in \mathcal{A}$ (supermartingale)

$q(\cdot, \pi^*, z) = 0$, for (exactly) one $\pi^* \in \mathcal{A}$ (martingale).

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4 Construction of generator of BSDE

Now

\[ q(\cdot, \pi, z) = -f(\cdot, z) - \pi \theta + \frac{\alpha}{2} (\pi - z)^2 \]

\[ = -f(\cdot, z) + \frac{\alpha}{2} (\pi - z)^2 - (\pi - z) \cdot \theta + \frac{1}{2\alpha} \theta^2 - z\theta - \frac{1}{2\alpha} \theta^2 \]

\[ = -f(\cdot, z) + \frac{\alpha}{2} [\pi - (z + \frac{1}{\alpha} \theta)]^2 - z\theta - \frac{1}{2\alpha} \theta^2. \]

Under non-convex constraint \( p \in C \):

\[ [\pi - (z + \frac{1}{\alpha} \theta)]^2 \geq \text{dist}^2(C, z + \frac{1}{\alpha} \theta). \]

with equality for at least one possible choice of \( \pi^* \) due to closedness of \( C \).
Hence (form 3) is solved by the choice (predictable selection)

(form 4) \[ \begin{align*}
\pi^* & : \text{dist}(C, z + \frac{1}{\alpha} \theta) = \text{dist}(\pi^*, z + \frac{1}{\alpha} \theta) \quad \text{(martingale).}
\end{align*} \]

\[ f(\cdot, z) = \frac{\alpha}{2} \text{dist}^2(C, z + \frac{1}{\alpha} \theta) - z \cdot \theta - \frac{1}{2\alpha} \theta^2 \quad \text{(supermartingale)} \]

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5 Summary of results, exponential utility

Solve utility optimization problem

$$\sup_{\pi \in \mathcal{A}} EU(x + X^{\pi}_T + H)$$

by considering FBSDE

\[
\begin{align*}
    dX^{\pi}_t &= \pi_t[dW_t + \theta_t dt], \quad X^{\pi}_0 = x, \\
    dY_t &= Z_t dW_t + f(t, Z_t) dt, \quad Y_T = H
\end{align*}
\]

with generator as described before; determine $\pi^*$ by previsible selection; coupling through requirement of martingale optimality

$$\sup_{\pi \in \mathcal{A}} EU(x + X^{\pi}_T + H) = EU(x + X^{\pi^*}_T + H),$$

$$U'(x + X^{\pi^*}_t + Y_t) \quad \text{martingale.}$$

for general $U$: forward part depends on $\pi^*$, get fully coupled FBSDE
6 Cross hedging, optimal investment, utility on $\mathbb{R}$

Lit: Mania, Tevzadze (2003)

$U : \mathbb{R} \to \mathbb{R}$ strictly increasing and concave; maximal expected utility from terminal wealth

$$V(x) = \sup_{\pi \in \mathcal{A}} EU(x + X_T^\pi + H)$$

wealth on $[0, T]$ by investment strategy $\pi$:

$$\int_0^T \langle \pi_u, \frac{dS_u}{S_u} \rangle = \int_0^T \pi_u [dW_u + \theta_u du] = X_T^\pi,$$

$H$ liability or derivative, correlated to financial market $S$, $W$ $d$–dimensional Wiener process, $W^1$ first $d_1$ components of $W$

$\pi \in \mathcal{A}$ subject to convex constraint $\pi = (\pi^1, 0)$, $\pi^1$ $d_1$–dimensional, hence incomplete market

aim: use FBSDE system to describe optimal strategy $\pi^*$

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7 Verification theorems

Thm 1
Assume $U$ is three times differentiable, $U'$ regular enough. If there exists $\pi^*$ solving (1), and $Y$ is the predictable process for which $U'(X^\pi + Y)$ is square integrable martingale, then with $Z = \frac{d}{dt} \langle Y, W \rangle$

$$(\pi^*)^1 = -\theta^1 \frac{U'}{U''}(X^\pi + Y) - Z^1.$$

Pf:

$$\alpha = \mathbb{E}(U'(X^\pi + H)|\mathcal{F}), \ Y = (U')^{-1}(\alpha) - X^\pi.$$

Use Itô’s formula and martingale property. Find

$$Y = H - \int_0^T Z_s dW_s - \int_0^T f(s, X^\pi_s, Y_s, Z_s) ds,$$

with

$$f(s, X^\pi_s, Y_s, Z_s) = -\frac{1}{2} \frac{U^{(3)}}{U''}(X^\pi + Y)|\pi^* + Z_s|^2 - \pi^* \theta_s.$$

Use variational maximum principle to derive formula for $\pi^*$.

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7 Verification theorems

From preceding theorem derive the FBSDE system

Thm 2
Assumptions of Thm 1; then optimal wealth process \( X^{\pi^*} \) given as component \( X \) of solution \((X, Y, Z)\) of fully coupled FBSDE system

\[
X = x - \int_0^T (\theta_s^1 \frac{U'}{U''} (X_s + Y_s) + Z_s^1) dW_s^1 - \int_0^T (\theta_s^1 \frac{U'}{U''} (X_s + Y_s) + Z_s^1) \theta_s^1 ds,
\]

\[
Y = H - \int_0^T Z_s dW_s
\]

\[
- \int_0^T |\theta_s^1|^2 ((-\frac{1}{2} \frac{U^{(3)} U'^2}{(U'')^3} + \frac{U'}{U''}) (X_s + Y_s) + Z_s^1 \cdot \theta_s^1)
\]

\[
- \frac{1}{2} |Z_s^2|^2 \frac{U^{(3)}}{U''} (X_s + Y_s)] ds. \quad (2)
\]

Pf:
Use expression for \( f \) and formula for \( \pi^* \) from Thm 1.
8 Representation of optimal strategy

Invert conclusion of Thm 2 to give representation of optimal strategy

Thm 3
Let \( (X, Y, Z) \) be solution of (2), \( U(X_T + H) \) integrable, \( U'(X_T + H) \) square integrable. Then

\[
(\pi^*)^1 = -\theta^1 \frac{U'}{U''}(X + Y) + Z^1
\]

is optimal solution of (1).

Pf:
By concavity for any admissible \( \pi \)

\[
U(X^\pi + Y) - U(X + Y) \leq U'(X + Y)(X^\pi - X).
\]

Now prove that

\[
U'(X + Y)(X^\pi - X) = U'(X^{\pi*} + Y)(X^\pi - X^{\pi*}) \quad \text{is a martingale!}
\]
9 The complete case

Formula representing $\pi^* \longrightarrow$ martingale representation

$$U'(X^{\pi^*} + Y) = U'(x + Y_0)\mathcal{E}(-\theta \cdot W).$$

Aim: show existence for fully coupled system of Thm 2.

Crucial observation: $P = X + Y$ solves forward SDE

$$P = x + Y_0 - \int_0^\cdot \theta_s U' U''(P_s) dW_s - \int_0^\cdot \frac{1}{2} U^{(3)} U'^2 (P_s) ds.$$

Idea: forward SDE

$$P^m = x + m - \int_0^\cdot \theta_s U' U''(P^m_s) dW_s - \int_0^\cdot \frac{1}{2} U^{(3)} U'^2 (P^m_s) ds$$

has solution; now decouple again, by considering BSDE

$$Y^m = H - \int_0^T Z^m_s dW_s - \int_0^T (|\theta_s|^2 \left[ -\frac{1}{2} \frac{U^{(3)} U'^2}{U''^3} + \frac{U'}{U''} \right] (P^m_s) + Z^m_s \theta_s) ds.$$
9 The complete case

Solve for \((Y^m, Z^m)\), use continuity of \(m \mapsto Y_0^m\) to find \(m\) such that \(Y_0^m = m\).

This gives

**Thm 4**

Assume \(\frac{U^{(3)}U'}{(U'')^3}\) and \(\frac{U'}{U''}\) are Lipschitz and bounded. Then the system of FBSDE

\[
X = x - \int_0^\cdot (\theta_s \frac{U'}{U''}(X_s + Y_s) + Z_s) dW_s^1 - \int_0^\cdot (\theta_s \frac{U'}{U''}(X_s + Y_s) + Z_s) \theta_s ds,
\]

\[
Y = H - \int_\cdot^T Z_s dW_s - \int_\cdot^T [||\theta_s||^2((\frac{1}{2} \frac{U^{(3)}U'}{(U'')^3} + \frac{U'}{U''})(X_s + Y_s) + Z_s \cdot \theta_s)] ds \tag{3}
\]

has solution \((X, Y, Z)\) such that \(U(X_T + H)\) is integrable, \(U'(X_T + H)\) square integrable.
10 Utility function on $\mathbb{R}_+$

Replace $U'(X^{\pi^*} + Y)$ with $U'(X^{\pi^*}) \exp(\tilde{Y})$. Then $(X, Y, Z)$ satisfies (3) if and only if $(X, \tilde{Y}, \tilde{Z})$ satisfies (4) ($\tilde{Z} = \frac{d}{dt} \langle W, \tilde{Y} \rangle$):

**Thm 5**

Let $(X, \tilde{Y}, \tilde{Z})$ be solution of the fully coupled FBSDE

$$
X = x - \int_0^T \frac{U'}{U''}(X_s)(\theta^1_s + Z^1_s)dW^1_s - \int_0^T \frac{U'}{U''}(X_s)(\theta^1_s + Z^1_s){\theta^1_s}ds,
$$

$$
Y = \ln\left(\frac{U'(X_T + H)}{U'(X_T)}\right) - \int_0^T Z_s dW_s
$$

$$
- \int_0^T |Z^1_s + \theta^1_s|^2((1 - \frac{1}{2}U^{(3)}U')\frac{1}{(U'')^2})(X_s) - \frac{1}{2}|Z_s|^2]ds. (4)
$$

such that $U(X^{\pi^*}_T + H)$ is integrable and $U'(X^{\pi^*}_T + H)$ is square integrable. Then

$$(\pi^*)^1 = -\frac{U'}{U''}(X)(Z^1 + \theta^1)$$

solves (1).

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11 The complete case

Using forward equation for $P = U'(X) \exp(Y)$ as above we obtain

**Thm 6**
Assume $\frac{U(3)U'}{(U'')^2}$ and $\frac{U'}{U''}$ are Lipschitz and bounded. Then system of FBSDE

\[
X = x - \int_0^\cdot \left( \frac{U'}{U''}(X_s)(\theta_s + Z_s) \right) dW_s - \int_0^\cdot \frac{U'}{U''}(X_s)(\theta_s + Z_s) \theta_s ds,
\]
\[
Y = \ln\left( \frac{U'(X_T + H)}{U'(X_T)} \right) - \int_0^T Z_s dW_s - \int_0^T [\|Z_s + \theta_s\|^2((1 - \frac{1}{2}(U(3)U')^2)(X_s) - \frac{1}{2}\|Z_s\|^2] ds.
\]

has solution $(X, Y, Z)$ such that $U(X_T)$ is integrable, $U'(X_T)$ square integrable.

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12 Link to stochastic maximum principle, complete case

\( H = 0, \; \tilde{X}^\pi = U(X^\pi); \) value function

\[ J(\pi) = \mathbb{E}(U(X^\pi_T)) = \mathbb{E}(\tilde{X}^T) \]

Using Peng (1993) obtain system of FBSDE

\[
\begin{align*}
  d\tilde{X}^\pi_t &= U'(U^{-1}(\tilde{X}^\pi_t))\pi_t dW_t + [U'(U^{-1}(\tilde{X}^\pi_t))\pi_t \theta_t + \frac{1}{2}U''(U^{-1}(\tilde{X}^\pi_t))|\pi_t|^2] dt, \\
  \tilde{X}^\pi_0 &= U(x), \\
  -dp_t &= \frac{U''}{U'}(\tilde{X}^\pi_t)k_t \pi_t dW_t + [\frac{U''}{U'}(U^{-1}(\tilde{X}^\pi_t))\pi_t \theta_t + \frac{1}{2}\frac{U(3)}{U'}(U^{-1}(\tilde{X}^\pi_t))|\pi_t|^2] dt, \\
  p_T &= 1 \quad (5). 
\end{align*}
\]

maximization of Hamiltonian

\[ H(x, \pi, p, k) = p[U'(U^{-1}(x))\pi \theta + \frac{1}{2}U''(U^{-1}(x)|\pi|^2] + kU'(U^{-1}(x))\pi \]

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12 Link to stochastic maximum principle, complete case 
gives 
\[ \pi^* = -\frac{U'}{U''} (U^{-1}(\tilde{X})) \left[ \frac{k}{p} + \theta \right]. \]

Use this in (4), apply Cole-Hopf transformation 
\[ Y = \ln(p), \quad Z = \frac{d}{dt} \langle Y, W \rangle = \frac{k}{p} \]
to get 
\[ d\tilde{X}_{t}^{\pi^*} = -\frac{U'}{U''} (\tilde{X}_{t}^{\pi^*}) (Z_t + \theta_t) (dW_t + \theta_t dt), \quad \tilde{X}_{0}^{\pi^*} = U(x), \]
\[ dY_t = \left[(Z_t + \theta_t)^2 \left(1 - \frac{1}{2} \frac{U^{(3)} U'}{(U'')}^2 (\tilde{X}_{t}^{\pi^*}) \right) - \frac{1}{2} |Z_t|^2 \right] dt + Z_t dW_t, \quad Y_T = 0 \] (6).

FBSDE system identical to the one obtained above, decouples if \( \frac{U^{(3)} U'}{(U'')}^2 \) is constant, i.e. for exponential, power and logarithmic utility.

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13 Example: power utility, general liability, incomplete case

Lit: Nutz (2010) for $H = 0$

$U(x) = \frac{1}{\gamma}x^{\gamma}$ for some $\gamma < 1$, $W = (W^1, W^2)$ two-dimensional Wiener process, $dS_t^i = dW_t^i + \theta_t^i dt$, $i = 1, 2$, investment $dX^\pi_t = \pi_t dS_t^1$, liability $H = \phi(S_T^2)$, with $\phi$ positive, bounded

(4) transforms into

\[
\begin{align*}
    dX_t &= \frac{1}{1 - \gamma} X_s(Z_t^1 + \theta^1_t)(dW_t + \theta^1_t dt), \quad X_0 = x, \\
    dY_t &= -\left[\frac{\gamma}{2(\gamma - 1)}(Z_t^1 + \theta^1_t)^2 - \frac{1}{2}|Z_t|^2\right]dt + Z_t dW_t, \\
    Y_T &= (\gamma - 1) \ln(1 + \frac{H}{X_T}) \quad (7).
\end{align*}
\]

optimal solution

\[
    (\pi^*)^1 = \frac{1}{1 - \gamma}(Z^1 + \theta^1) \quad (\pi^*)^2 = \frac{1}{1 - \gamma}(Z^2 + \theta^2)
\]

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