2BSDEs with Continuous Coefficients

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New advances in Backward SDEs for financial engineering applications
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Introduction

Motivated by applications in financial mathematics and probabilistic numerical schemes for PDEs, Soner, Touzi and Zhang introduced recently the notion of second order backward stochastic differential equations (2BSDEs for short) [10], which are connected to the larger class of fully non-linear PDEs. They provided a complete theory of existence and uniqueness for 2BSDEs under uniform Lipschitz conditions similar to those of Pardoux and Peng, so our aim here is twofold

- we want to relax the Lipschitz assumptions on the driver to a linear growth framework as in Lepeltier and San Martin [6] or Matoussi [7].
- we want to highlight the major difficulties and differences from the classical BSDE case.

Plan

The local martingale measures

Let $\Omega := \left\{\omega \in C([0,1],\mathbb{R}^d) : \omega_0 = 0\right\}$ be the canonical space equipped with the uniform norm $\|\omega\|_\infty := \sup_{0\leq t\leq T} |\omega_t|$, B the canonical process, $\mathbb{P}_0$ the Wiener measure, $\mathbb{F} := \{\mathcal{F}_t\}_{0\leq t\leq T}$ the filtration generated by B, and $\mathbb{F}^+ := \left\{\mathcal{F}_t^+\right\}_{0\leq t\leq T}$ the right limit of $\mathbb{F}$. We first recall the notations introduced Soner, Touzi and Zhang.

$\mathbb{P}$ is a local martingale measure if the canonical process B is a local martingale under $\mathbb{P}$. By Föllmer [5], there exists an $\mathbb{F}$-progressively measurable process, denoted as $\int_0^t B_s dB_s$, which coincides with the Itô's integral, $\mathbb{P} - a.s.$ for all local martingale measure $\mathbb{P}$. This provides a pathwise definition of

$$\langle B\rangle_t := B_t B_t^T - 2\int_0^t B_s dB_s^T \text{ and } \widehat{a}_t := \limsup_{\epsilon \searrow 0} \frac{1}{\epsilon}\left(\langle B\rangle_t - \langle B\rangle_{t-\epsilon}\right).$$

The local martingale measures

Let $\overline{\mathcal{P}}_W$ denote the set of all local martingale measures $\mathbb{P}$ such that

$\langle B \rangle_t$ is absolutely continuous in t and $\widehat{a}$ takes values in $\mathbb{S}_d^{>0}$, $\mathbb{P}-a.s.$

We concentrate on the subclass $\overline{\mathcal{P}}_s \subset \overline{\mathcal{P}}_W$ consisting of all probability measures

$$\mathbb{P}^\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1} \text{ where } X_t^\alpha := \int_0^t \alpha_s^{1/2} dB_s,\ t \in [0,1],\ \mathbb{P}_0 - a.s.$$

for some $\mathbb{F}$-progressively measurable process α taking values in $\mathbb{S}_d^{>0}$ with $\int_0^T |\alpha_t| dt < +\infty$, $\mathbb{P}_0 - a.s.$

The non-linear generator

We consider a map $H_t(\omega, y, z, \gamma) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \to \mathbb{R}$, where $D_H \subset \mathbb{R}^{d\times d}$ is a given subset containing 0.

Define the corresponding conjugate of H w.r.t.γ by

$$F_t(\omega, y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2}\mathrm{Tr}(a\gamma) - H_t(\omega, y, z, \gamma) \right\} \text{ for } a \in S_d^{>0},$$

$$\widehat{F}_t(y, z) := F_t(y, z, \widehat{a}_t) \text{ and } \widehat{F}_t^0 := \widehat{F}_t(0, 0).$$

We fix a constant $\kappa \in (1, 2]$ and restrict to $\mathcal{P}_H^\kappa \subset \overline{\mathcal{P}}_S$

$$\underline{a}_{\mathbb{P}} \leq \widehat{a} \leq \bar{a}_{\mathbb{P}},\ dt \times d\mathbb{P} - as \text{ for some } \underline{a}_{\mathbb{P}}, \bar{a}_{\mathbb{P}} \in S_d^{>0}$$

$$\sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^{\mathbb{P}} \left[\operatorname*{ess\,sup}^{\mathbb{P}}_{0\leq t\leq T} \left(\mathbb{E}_t^{H,\mathbb{P}} \left[\int_0^T \left| \widehat{F}_s^0 \right|^\kappa ds \right] \right)^{\frac{2}{\kappa}} \right] < +\infty$$

The non-linear generator

We assume

(i) The domain $D_{F_t(y,z)} = D_{F_t}$ is independent of (ω, y, z).

(ii) For fixed (y, z, γ), F is $\mathbb{F}$-progressively measurable in D_{F_t}.

(iii) We have the following uniform Lipschitz-type property

$$\forall (y, z, z', t),\ \left|\widehat{F}_t(y,z) - \widehat{F}_t(y,z')\right| \le C\left|\widehat{a}_t^{1/2}(z - z')\right|,\ \mathcal{P}_H^\kappa - q.s.$$

(iv) F is uniformly continuous in ω for the $||\cdot||_\infty$ norm.

(v) F is continuous in y and has the following growth property

$$\exists C > 0 \text{ s.t. } \quad |F_t(\omega, y, 0, a)| \le |F_t(\omega, 0, 0, a)| + C(1 + |y|),\ \mathcal{P}_H^\kappa - q.s.$$

(vi) We have the following monotonicity condition

$$\exists \mu > 0 \text{ s.t. } \quad (y_1 - y_2)(F_t(\omega, y_1, z, \gamma) - F_t(\omega, y_2, z, \gamma)) \le \mu \left|y_1 - y_2\right|^2.$$

The non-linear generator

Let us comment on these assumptions

- Assumptions (i) and (iv) are taken from [10] and are needed to deal with the technicalities induced by the quasi-sure framework.
- Assumptions (ii) and (iii) are quite standard in the classical BSDE litterature.
- Assumptions (v) and (vi) where introduced by Pardoux in [8] in a more general setting (namely with a general growth condition in y) and are also quite commonplace in the litterature (see e.g. Briand et al. [1], [2]).

The spaces and norms

For $p \geq 1$, $L_H^{p,\kappa}$ denotes the space of all $\mathcal{F}_T$-measurable scalar r.v. ξ with

$$\|\xi\|_{L_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P}\left[|\xi|^p\right] < +\infty.$$

$\mathbb{H}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}^d$-valued processes Z with

$$\|Z\|_{\mathbb{H}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P}\left[\left(\int_0^T |\widehat{a}_t^{1/2} Z_t|^2 dt\right)^{\frac{p}{2}}\right] < +\infty.$$

$\mathbb{D}_H^{p,\kappa}$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}$-valued processes Y with

$\mathcal{P}_H^\kappa - q.s.$ càdlàg paths, and $\|Y\|_{\mathbb{H}_H^{p,\kappa}}^p := \sup_{\mathbb{P}\in\mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P}\left[\sup_{0\leq t\leq T} |Y_t|^p\right] < +\infty$

The spaces and norms

For each $\xi \in L^{1,\kappa}_H$, $\mathbb{P} \in \mathcal{P}^\kappa_H$ and $t \in [0, T]$ denote

$$\mathbb{E}^{H,\mathbb{P}}_t[\xi] := \operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}^\kappa_H(t^+,\mathbb{P})} {}^{\mathbb{P}}\mathbb{E}^{\mathbb{P}'}_t[\xi],$$

where $\mathcal{P}^\kappa_H(t^+, \mathbb{P}) := \left\{ \mathbb{P}' \in \mathcal{P}^\kappa_H : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}^+_t \right\}$.

Then we define for each $p \geq \kappa$,

$$\mathbb{L}^{p,\kappa}_H := \left\{ \xi \in L^{p,\kappa}_H : \|\xi\|_{\mathbb{L}^{p,\kappa}_H} < +\infty \right\},$$

where $\|\xi\|^p_{\mathbb{L}^{p,\kappa}_H} := \sup_{\mathbb{P} \in \mathcal{P}^\kappa_H} \mathbb{E}^{\mathbb{P}} \left[\operatorname*{ess\,sup}^{\mathbb{P}}_{0 \leq t \leq T} \left(\mathbb{E}^{H,\mathbb{P}}_t[|\xi|^\kappa] \right)^{\frac{p}{\kappa}} \right]$.

The spaces and norms

Finally, we denote by $\mathrm{UC}_b(\Omega)$ the collection of all bounded and uniformly continuous maps $\xi : \Omega \to \mathbb{R}$ with respect to the $\|\cdot\|_\infty$-norm, and we let

$$\mathcal{L}_H^{p,\kappa} := \text{the closure of } \mathrm{UC}_b(\Omega) \text{ under the norm } \|\cdot\|_{\mathbb{L}_H^{p,\kappa}}.$$

Formulation

Definition

For $\xi \in \mathrm{L}_{\mathrm{H}}^{2,\kappa}$, we say $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ is a solution to the 2BSDE if :

- $Y_T = \xi \ \mathcal{P}_H^\kappa - qs.$
- $\forall \mathbb{P} \in \mathcal{P}_H^\kappa$, the process $K^\mathbb{P}$ has non-decreasing paths $\mathbb{P} - as$

$$K_t^\mathbb{P} := Y_0 - Y_t - \int_0^t \widehat{F}_s(Y_s, Z_s)ds + \int_0^t Z_s dB_s, \ 0 \leq t \leq T.$$

- The family $\left\{K^\mathbb{P}, \mathbb{P} \in \mathcal{P}_H^\kappa\right\}$ satisfies the minimum condition

$$K_t^\mathbb{P} = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})}{\operatorname{ess\,inf}^\mathbb{P}} \mathbb{E}_t^{\mathbb{P}'}\left[K_T^{\mathbb{P}'}\right], \ 0 \leq t \leq T, \ \mathbb{P} - as, \ \forall \mathbb{P} \in \mathcal{P}_H^\kappa.$$

Plan

Representation Formula

For any $\mathbb{P} \in \mathcal{P}_H^\kappa$, $\mathbb{F}$-stopping time τ, and $\mathcal{F}_\tau$-measurable random variable $\xi \in \mathbb{L}^2(\mathbb{P})$, consider the BSDE

$$y_t^{\mathbb{P}} = \xi + \int_t^\tau \widehat{F}_s(y_s^{\mathbb{P}}, z_s^{\mathbb{P}})ds - \int_t^\tau z_s^{\mathbb{P}} dB_s,\ 0 \leq t \leq \tau,\ \mathbb{P} - a.s.$$

Theorem

Assume $\xi \in \mathbb{L}_H^{2,\kappa}$ and that $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ is a solution to the 2BSDE. Then, for any $\mathbb{P} \in \mathcal{P}_H^\kappa$ and $0 \leq t_1 < t_2 \leq T$,

$$Y_{t_1} = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1, \mathbb{P})} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}),\ \mathbb{P} - a.s.$$

Consequently, the 2BSDE has at most one solution in $\mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$.

Comments on the proof of uniqueness

- As in the Lipschitz case, uniqueness follows from a stochastic representation suggested by the optimal control interpretation, and because of the non-decreasing process $K^{\mathbb{P}}$, we were unable to use fixed-point arguments.
- For the proof to work, you need a comparison theorem for the underlying BSDE.
- With our assumptions the monotonicity condition is crucial to obtain uniqueness.

Plan

2. Continuous 2BSDE with monotonicity condition
   - Preliminaries
   - Uniqueness
   - Approximation and Existence of a solution
   - Limitations

Approximation by inf-convolution

Lemma

Define

$$\widehat{F}^n_t(y,z) := \inf_{(u,v)\in\mathbb{Q}^{d+1}} \left\{ \widehat{F}_t(u,v) + n\,|y-u| + n\left|\widehat{a}_t^{1/2}(z-v)\right|^2 \right\}.$$

(i) $\widehat{F}^n$ *is well defined for n large enough and we have*

$$\left|\widehat{F}^n_t(y,z)\right| \leq \left|\widehat{F}^0_t\right| + C(1+|y|+|\widehat{a}_t^{1/2}z|),\ \mathbb{P}-as,\ \forall \mathbb{P}\in\mathcal{P}^\kappa_H.$$

(ii) $|\widehat{F}^n_t(y,z_1) - \widehat{F}^n_t(y,z_2)| \leq C|\widehat{a}_t^{1/2}(z_1-z_2)|,\ \mathbb{P}-as,\ \forall \mathbb{P}\in\mathcal{P}^\kappa_H.$

(iii) $|\widehat{F}^n_t(y_1,z) - \widehat{F}^n_t(y_2,z)| \leq n\,|y_1-y_2|\,,\ \mathbb{P}-as,\ \forall \mathbb{P}\in\mathcal{P}^\kappa_H.$

(iv) $\widehat{F}^n_t(y,z) \nearrow.$

(v) *If* $\widehat{F}$ *is decreasing in* y, *then so is* $\widehat{F}^n$.

Approximation by inf-convolution

- As usual with monotonicity condition in dimension 1 we can assume without loss of generality that $\widehat{F}$ is decreasing in y.
- Our aim is to use monotonic approximation in order to obtain existence in our framework, by building on the results of Soner, Touzi and Zhang in the Lipschitz case.
- We do not use linear inf-convolution for our approximation, as in Lepeltier and San Martin [6] or Matoussi [7] but a mix of linear and quadratic inf-convolution. This is due to the fact that we absolutely need our approximation to remain uniformly Lipschitz in z with a constant which do not depend on n.
- The major difficulty here is that since we are working with a family of mutually singular probability measures, monotone and dominated convergence theorem may fail.

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Approximation by inf-convolution

Therefore we need to assume a strong type of convergence for our approximation. Namely, we assume that one of these assumptions hold true

(i) The sequence $\widehat{F}_n$ converges uniformly in z for all y, uniformly in y for all z.

(ii) The sequence $\widehat{F}_n$ converges uniformly globally in (y,z).

These are implicit assumptions on the driver $\widehat{F}$, which are satisfied if for example $\widehat{F}$ is uniformly continuous in y, uniformly in z, t and ω or if $\widehat{F}$ takes the special form $\widehat{F}_t(y,z) := \phi_t(z) + \psi_t(y)$ and the first derivative of $\widehat{F}$ with respect to y (which exists *a.e.* since $\widehat{F}$ is decreasing in y) is bounded near $\pm\infty$, uniformly in z.

Sketch of the proof of existence

For a fixed n, consider the following 2BSDE

$$Y_t^n = \xi + \int_t^T \widehat{F}_s^n(Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dB_s + K_T^n - K_t^n,\ 0 \leq t \leq T,\ \mathcal{P}_H^\kappa - qs.$$

and the corresponding BSDE

$$y_t^{\mathbb{P},n} = \xi + \int_t^T \widehat{F}_s^n(y_s^{\mathbb{P},n}, z_s^{\mathbb{P},n})ds - \int_t^T z_s^{\mathbb{P},n} dB_s,\ 0 \leq t \leq T,\ \mathbb{P} - as,$$

Then you start by proving a priori estimates uniform in n by using standard arguments for the classical BSDE and then the representation formula for the 2BSDE.

Sketch of the proof of existence

- Then, using the fact that the approximation is monotone and comparison theorems, you obtain that Y^n converges quasi-surely to some processus Y, and a similar result for $y^{\mathbb{P},n}$ for all $\mathbb{P}$.
- However, this is not sufficient to obtain convergence in an $\mathbb{L}^2$ sense, since we cannot use monotone convergence theorem $\Rightarrow$ we use our uniform convergence assumption to conclude.
- Use representation in order to control the $\mathbb{D}_H^{2,\kappa}$ norm of $(Y^n - Y)$ by the supremum over $\mathbb{P}$ of the norms of $(y^{\mathbb{P},n} - y^{\mathbb{P}})$. You then get convergence of Y^n.
- Use classical estimates to get the convergence of Z^n and then our uniform convergence assumption to get the convergence of $K^{\mathbb{P},n}$.

Plan

2. Continuous 2BSDE with monotonicity condition
   - Preliminaries
   - Uniqueness
   - Approximation and Existence of a solution
   - Limitations

Limitations

- The proof relies heavily on the approximation by inf-convolution which is completely explicit and has very nice properties $\Rightarrow$ we probably won't be able to prove uniform convergence of the approximation for more general growth conditions in y.
- Can we relax the Lipschitz assumption on z ?

Plan

3. Continuous 2BSDEs with linear growth
   - Weak Compactness
   - New Hypotheses

Weak Compactness

- Our problem earlier was that the monotone convergence theorem did not hold. However, if we assume that the family $\mathcal{P}_H^\kappa$ is weakly relatively compact, then it will still hold.
- As proven by Denis, Hu and Peng[4] or Denis and Martini [3], it will be the case if for instance we assume uniform bounds in $\mathbb{P}$ for the density of the quadratic variation of the canonical process.

Plan

3. Continuous 2BSDEs with linear growth
   - Weak Compactness
   - New Hypotheses

New Hypotheses

We can now consider the weaker hypotheses

(i) The domain $D_{F_t(y,z)} = D_{F_t}$ is independent of (ω, y, z).

(ii) For fixed (y, z, a), F is $\mathbb{F}$-progressively measurable in D_{F_t}.

(iii) F is uniformly continuous in ω for the $||\cdot||_\infty$ norm.

(iv) F is continuous in y and z and has the following growth property

$$|F_t(\omega, y, z, a)| \leq |F_t(\omega, 0, 0, a)| + C(1 + |y| + \left|\widehat{a}_t^{1/2} z\right|), \ \mathcal{P}_H^\kappa - q.s.$$

and the following approximation

$$\widehat{F}_t^n(y, z) := \inf_{(u,v)\in\mathbb{Q}^{d+1}} \left\{ \widehat{F}_t(u, v) + n\,|y - u| + n|\widehat{a}_t^{1/2}(z - v)| \right\}.$$

A simple modification of the previous proofs gives us existence of a minimal and a maximal solution.

Conclusion

- In the most general case, we cannot expect the monotonic approximation to work.
- You could try instead to use the regular conditional probability distribution to prove existence and uniqueness for $\xi \in UC_b(\Omega)$ and pass to the limit in its closure $\mathcal{L}_H^{2,\kappa}$, but it will ignore our second case where there is no uniqueness .
- Current work on quadratic 2BSDEs with possible applications to utility maximization and superhedging.

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Thank you for your attention !

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