A finite dimensional approximation for pricing American options on moving average

Peter Tankov

CMAP, Ecole Polytechnique

Joint with M. Bernhart and X. Warin (EDF R&D)

New advances in Backward SDEs for financial engineering applications
Tamerza, Tunisia, October 25–28, 2010
Surge options: American-style options whose strike is adjusted daily to the moving average of the underlying price:

\[ H_t = (S_t - X_t)^+, \quad X_t = \frac{1}{\delta} \int_{t-\delta}^{t} S_u du. \]

The strike of indexed swing options (gas market) is linked to moving averages of different oil prices.

Non-Markovian dynamics of the moving average leads to an infinite-dimensional optimal stopping problem:

\[ dX_t = \frac{1}{\delta} (S_t - S_{t-\delta}) \, dt. \]

We propose a finite-dimensional approximation allowing to price moving average options in PDE or LS Monte Carlo framework.
The approximation problem

We consider general moving average processes of the form

$$M_t = \int_0^\infty S_{t-u} \mu(du)$$

where $\mu$ is a finite possibly signed measure on $[0, \infty)$ and we set $S_t = S_0$ for $t \leq 0$.

We would like to find $n$ processes $Y^1, \ldots, Y^n$ such that $(S, Y^1, \ldots, Y^n)$ are jointly Markov, and $M_t$ is approximated by $M^n_t$ which depends deterministically on $S_t, Y^1_t, \ldots, Y^n_t$. 
Assumptions on $S$

That the stock price $S$ is a continuous Itô process:

$$S_t = S_0 + \int_0^t b_s \, ds + \int_0^t \sigma_s \, dW_s$$

with

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |b_s| \right] < \infty \quad \text{and} \quad \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\sigma_s|^{1+\gamma} \right] < \infty, \quad \gamma > 0$$

It can then be shown (Fischer and Nappo ’10) that the modulus of continuity of $S$ is integrable:

$$\mathbb{E} \left[ \sup_{t,s \in [0,T]:|t-s| \leq h} |S_t - S_s| \right] \leq C \varepsilon(h), \quad \varepsilon(h) := \sqrt{h \ln \left( \frac{2T}{h} \right)}.$$
Lemma

Let $\mu$ and $\nu$ be finite signed measures on $[0, \infty)$ such that $\mu^+(\mathbb{R}_+) > 0$, and let $M$ and $N$ be corresponding moving average processes. Then

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |M_t - N_t| \right] \leq C |\mu(\mathbb{R}_+) - \nu(\mathbb{R}_+)|
$$

$$
+ C\varepsilon \left( \frac{1}{\mu^+(\mathbb{R}])} \int_0^T |F_\mu(t) - F_\nu(t)| dt \right)
$$

where

$$
F_\nu(t) := \nu([0, t]) \quad \text{and} \quad F_\mu(t) := \mu([0, t]).
$$
We first assume that $\mu$ and $\nu$ are probability measures. Let $F_{\mu}^{-1}$ and $F_{\nu}^{-1}$ be generalized inverses of $\mu$ and $\nu$ respectively. Then,

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |M_t - N_t| \right] = \mathbb{E} \left[ \sup_{0 \leq t \leq T} \int_0^1 |S_{t - F_{\mu}^{-1}(u)} - S_{t - F_{\nu}^{-1}(u)}| du \right]
$$

$$
\leq C\varepsilon \left( \int_0^1 |F_{\mu}^{-1}(u) \wedge T - F_{\nu}^{-1}(u) \wedge T| du \right).
$$

The expression inside the brackets is the Wasserstein distance between $\mu$ and $\nu$ truncated at $T$. Therefore,

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |M_t - N_t| \right] \leq C\varepsilon \left( \int_0^T |F_{\mu}(t) - F_{\nu}(t)| dt \right).
$$
Laguerre approximation: idea

- We assume
  \[ dY_t = -AY dt + 1(\alpha S_t dt + \beta dS_t), \quad M^n = B^\perp Y \]
- The solution can be written as
  \[ M^n_t = \int_{-\infty}^t B^\perp e^{-A(t-s)} 1(\alpha S_s ds + \beta dS_s) \]
  \[ = K_n S_t + \int_{-\infty}^t h_n(t-u) S_u du, \]
where \( h_n \) is of the form (Hankel approximation)
  \[ h_n(t) = \sum_{k=1}^{K} e^{-p_k t} \sum_{i=0}^{n_k} c_i^k t^i, \quad n_1 + \ldots + n_K + K = n \]
In this work we focus on a subclass for which \( h_n \) is of the form
  \[ h_n(t) = e^{-pt} \sum_{i=0}^{n-1} c_i t^i \] (Laguerre approximation)
Fix a scale parameter $p > 0$. The scaled Laguerre functions $(L^p_k)_{k \geq 0}$ are defined on $[0, +\infty)$ by

$$L^p_k(t) = \sqrt{2p} \, P_k(2pt) e^{-pt}, \quad \forall k \geq 0$$

where $(P_k)_{k \geq 0}$ are Laguerre polynomials:

$$P_k(t) = \sum_{i=0}^{k} \binom{k}{k-i} \left( -t \right)^i \frac{1}{i!}, \quad \forall k \geq 0$$

The Laguerre functions $(L^p_k)_{k \geq 0}$ form an orthonormal basis of $L^2([0, \infty))$. 
Laguerre approximations for moving averages

- Let $H(x) = \mu([x, +\infty))$.

- Compute the Laguerre coefficients of the function $H$:
  
  $$A_p^k = \langle H, L^p_k \rangle.$$ 

  Set $H_p^p(t) = \sum_{k=0}^{n-1} A_p^k L^p_k(t)$ and $h_p^p(t) = -\frac{d}{dt} H_p^p(t)$.

- Approximate the moving average $M$ with
  
  $$M^n_{t,p} = (H(0) - H^n(0)) S_t + \int_0^{+\infty} h^n_p(u) S_{t-u} du.$$
Laguerre approximations for moving averages

Introduce \( n \) random processes

\[
X_{t}^{p,k} = \int_{0}^{+\infty} L_{k}(v) S_{t-v} dv, \quad k = 0, \ldots, n-1.
\]

They are related to the moving average approximation by

\[
M_{t}^{n,p} = (H(0) - H_{n}(0)) S_{t} + \sum_{k=0}^{n-1} a_{k} X_{t}^{p,k}, \quad \forall t \geq 0.
\]

and have Markovian dynamics

\[
\begin{align*}
\left\{ 
\begin{array}{l}
\quad dX_{t}^{p,0} = \left( \sqrt{2p} S_{t} - p X_{t}^{p,0} \right) dt \\
\quad \ldots \\
\quad dX_{t}^{p,k} = \left( \sqrt{2p} S_{t} - 2p \sum_{i=0}^{k-1} X_{t}^{p,i} - p X_{t}^{p,k} \right) dt
\end{array}
\right.
\end{align*}
\]

with initial values

\[
X_{0}^{p,k} = S_{0}(-1)^{k} \frac{\sqrt{2p}}{p}, \quad \forall k \geq 0.
\]
Theorem

Suppose that the moving average process $M$ is of the form

$$M_t = K_0 S_t + \int_0^\infty S_{t-u} h(u) du$$

where $K_0$ is a constant and the function $h$ has compact support, finite variation on $\mathbb{R}$ and is constant in the neighborhood of zero. Then the approximation error admits the bound

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |M_t - M_{t,n,p}^n| \right] \leq C\varepsilon (n^{-\frac{3}{4}}).$$
Approximating option prices

One can approximate

$$\sup_{\tau \in T} \mathbb{E}[\phi(S_\tau, M_\tau)] \quad \text{by} \quad \sup_{\tau \in T} \mathbb{E}[\phi(S_\tau, M^{n,p}_\tau)].$$

Corollary

Let the payoff function $\phi$ be Lipschitz in the second variable, then the pricing error admits the bound

$$\left| \sup_{\tau \in T} \mathbb{E}[\phi(S_\tau, M_\tau)] - \sup_{\tau \in T} \mathbb{E}[\phi(S_\tau, M^{n,p}_\tau)] \right| \leq C\varepsilon(n^{-\frac{3}{4}}).$$

where $C > 0$ is a constant independent of $n$. 
Uniformly weighted moving averages

Let

$$\mu(dx) = h(x)dx = \frac{1}{\delta}1_{[0,\delta]}dx \quad \Rightarrow \quad H(x) = \frac{1}{\delta} (\delta - x)^+$$

The coefficients $A_{\delta,p}^k = \langle H, L_{\delta}^p \rangle$ can be computed explicitly. We determine the optimal scale parameter $p_{opt}(\delta, n)$ as

$$p_{opt}(\delta, n) = \arg \min_{p>0} \| H - H_n^p \|_2 = \arg \min_{p>0} \left\{ \frac{\delta}{3} - \sum_{k=0}^{n-1} |A_{\delta,p}^k|^2 \right\}.$$ 

It satisfies the scaling relation

$$p_{opt}(\delta, n) = \frac{p_{opt}(1, n)}{\delta},$$

and the values $p_{opt}(1, n)$ can be tabulated.
Simulated trajectory of the moving average process and its Laguerre approximations.
Replace the American option by a Bermudan one with a discrete grid $\pi = \{0 = t_0, t_1, \ldots, t_N = T\}$ of exercise dates.

Simulate $M$ paths of stock price and Laguerre processes.

Compute optimal exercise times by backward induction:

1. Initialization: $\tau_\pi^{\pi,m} = T$, $m = 1, \ldots, M$
2. Backward induction for $i = N - 1, \ldots, N_\delta$, $m = 1, \ldots, M$:

$$\begin{cases} 
\tau_i^{\pi,m} = t_i 1_{A_i^m} + \tau_{i+1}^{\pi,m} 1_{C A_i^m} \\
A_i^m = \left\{ \phi \left( S_{t_i}^{\pi,m}, M_{t_i}^{n,\pi,m} \right) \geq \mathbb{E}_{t_i}^M \left[ \phi \left( S_{\tau_{i+1}^{\pi,\pi}}, M_{\tau_{i+1}^{\pi,\pi}}^{n,\pi} \right) \right] \right\}
\end{cases}$$

3. Estimation of the option price at time 0:

$$V_0^{\pi} = \frac{1}{M} \sum_{m=1}^M \phi \left( S_{\tau_N^{\pi,\pi}}, M_{\tau_N^{\pi,\pi}}^{n,\pi,m} \right)$$

The conditional expectations are estimated by regression on the basis functions of state variables.
In numerical examples, we find that best results are obtained if the approximate moving average $M_{n,\pi}$ is replaced by the true moving average $M^{\pi}$ in the pay-off function, while estimating the conditional expectations by regressions on $S^{\pi}, X_{0,\pi}, \ldots, X_{n,\pi}$.

The suboptimal approach often used by practitioners consists in estimating the conditional expectations by regression on $(S^{\pi}, M^{\pi})$ only.
Numerical examples: convergence

Left: Laguerre approximation vs. the improved method.
Right: zoom for the improved method and the practitioner’ method.
For moving average options with time delay whose pay-off depends on $X_\tau = \frac{1}{\delta} \int_{\tau-l-\delta}^{\tau-l} S_u du$, the Laguerre approximation leads to a substantial improvement compared to the practitioner’s method.