A finite dimensional approximation for pricing American options on moving average

Peter Tankov

CMAP, Ecole Polytechnique

Joint with M. Bernhart and X. Warin (EDF R&D)

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Introduction

- Surge options: American-style options whose strike is adjusted daily to the moving average of the underlying price:

$$H_t = (S_t - X_t)^+, \quad X_t = \frac{1}{\delta}\int_{t-\delta}^{t} S_u du.$$

- The strike of indexed swing options (gas market) is linked to moving averages of different oil prices.
- Non-Markovian dynamics of the moving average leads to an infinite-dimensional optimal stopping problem:

$$dX_t = \frac{1}{\delta}(S_t - S_{t-\delta})\, dt.$$

- We propose a finite-dimensional approximation allowing to price moving average options in PDE or LS Monte Carlo framework.

The approximation problem

We consider general moving average processes of the form

$$M_t = \int_0^\infty S_{t-u}\mu(du)$$

where μ is a finite possibly signed measure on $[0, \infty)$ and we set $S_t = S_0$ for $t \le 0$.

We would like to find n processes $Y^1, \dots, Y^n$ such that $(S, Y^1, \dots, Y^n)$ are jointly Markov, and M_t is approximated by M_t^n which depends deterministically on $S_t, Y_t^1, \dots, Y_t^n$.

Assumptions on S

That the stock price S is a continuous Itô process:

$$S_t = S_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s$$

with

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |b_s|\right] < \infty \quad \text{and} \quad \mathbb{E}\left[\sup_{0\leq t\leq T} |\sigma_s|^{1+\gamma}\right] < \infty, \quad \gamma > 0$$

It can then be shown (Fischer and Nappo '10) that the modulus of continuity of S is integrable:

$$\mathbb{E}\left[\sup_{t,s\in[0,T]:|t-s|\leq h} |S_t - S_s|\right] \leq C\varepsilon(h), \quad \varepsilon(h) := \sqrt{h \ln\left(\frac{2T}{h}\right)}.$$

Comparing moving averages

Lemma

Let μ and ν be finite signed measures on $[0,\infty)$ such that $\mu^+(\mathbb{R}_+) > 0$, and let M and N be corresponding moving average processes. Then

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |M_t - N_t|\right] \leq C|\mu(\mathbb{R}_+) - \nu(\mathbb{R}_+)| + C\varepsilon\left(\frac{1}{\mu^+([0,T])}\int_0^T |F_\mu(t) - F_\nu(t)|dt\right)$$

where

$$F_\nu(t) := \nu([0,t]) \quad \text{and} \quad F_\mu(t) := \mu([0,t]).$$

Sketch of the proof

We first assume that μ and ν are probability measures. Let F_μ^{-1} and F_ν^{-1} be generalized inverses of μ and ν respectively. Then,

$$\mathbb{E}\left[\sup_{0\le t\le T}|M_t - N_t|\right] = \mathbb{E}\left[\sup_{0\le t\le T}\int_0^1 |S_{t-F_\mu^{-1}(u)} - S_{t-F_\nu^{-1}(u)}|du\right]$$
$$\le C\varepsilon\left(\int_0^1 |F_\mu^{-1}(u)\wedge T - F_\nu^{-1}(u)\wedge T|du\right).$$

The expression inside the brackets is the Wasserstein distance between μ and ν truncated at T. Therefore,

$$\mathbb{E}\left[\sup_{0\le t\le T}|M_t - N_t|\right] \le C\varepsilon\left(\int_0^T |F_\mu(t) - F_\nu(t)|dt\right).$$

Laguerre approximation: idea

- We assume

$$dY_t = -AYdt + \mathbf{1}(\alpha S_t dt + \beta dS_t), \quad M^n = B^\perp Y$$

- The solution can be written as

$$M_t^n = \int_{-\infty}^t B^\perp e^{-A(t-s)}\mathbf{1}(\alpha S_s ds + \beta dS_s)$$
$$= K_n S_t + \int_{-\infty}^t h_n(t-u) S_u du,$$

where h_n is of the form (Hankel approximation)

$$h_n(t) = \sum_{k=1}^K e^{-p_k t} \sum_{i=0}^{n_k} c_i^k t^i, \quad n_1 + \ldots + n_K + K = n$$

In this work we focus on a subclass for which h_n is of the form

$$h_n(t) = e^{-pt} \sum_{i=0}^{n-1} c_i t^i \quad \text{(Laguerre approximation)}$$

Laguerre polynomials and functions

Fix a scale parameter $p > 0$. The scaled Laguerre functions $(L_k^p)_{k \geq 0}$ are defined on $[0, +\infty)$ by

$$L_k^p(t) = \sqrt{2p}\; P_k(2pt) e^{-pt}, \quad \forall k \geq 0$$

where $(P_k)_{k \geq 0}$ are Laguerre polynomials:

$$P_k(t) = \sum_{i=0}^{k} \binom{k}{k-i} \frac{(-t)^i}{i!}, \quad \forall k \geq 0$$

The Laguerre functions $(L_k^p)_{k \geq 0}$ form an orthonormal basis of $L^2([0, \infty))$.

Laguerre approximations for moving averages

- Let $H(x) = \mu([x, +\infty))$.
- Compute the Laguerre coefficients of the function H:

$$A_k^p = \langle H, L_k^p \rangle.$$

Set $H_n^p(t) = \sum_{k=0}^{n-1} A_k^p L_k^p(t)$ and $h_n^p(t) = -\frac{d}{dt} H_n^p(t)$.

- Approximate the moving average M with

$$M_t^{n,p} = (H(0) - H_n^p(0))S_t + \int_0^{+\infty} h_n^p(u) S_{t-u} du.$$

Laguerre approximations for moving averages

Introduce n random processes

$$X_t^{p,k} = \int_0^{+\infty} L_k^p(v) S_{t-v} dv, \quad k = 0, \dots, n-1.$$

They are related to the moving average approximation by

$$M_t^{n,p} = (H(0) - H_n^p(0)) S_t + \sum_{k=0}^{n-1} a_k^p X_t^{p,k}, \quad \forall t \geq 0.$$

and have Markovian dynamics

$$\begin{cases} dX_t^{p,0} = \left(\sqrt{2p} S_t - p X_t^{p,0}\right) dt \\ \dots \\ dX_t^{p,k} = \left(\sqrt{2p} S_t - 2p \sum_{i=0}^{k-1} X_t^{p,i} - p X_t^{p,k}\right) dt \end{cases}$$

with initial values

$$X_0^{p,k} = S_0 (-1)^k \frac{\sqrt{2p}}{p}, \forall k \geq 0.$$

Convergence rate

Theorem

Suppose that the moving average process M is of the form

$$M_t = K_0 S_t + \int_0^\infty S_{t-u} h(u) du$$

where K_0 is a constant and the function h has compact support, finite variation on $\mathbb{R}$ and is constant in the neighborhood of zero. Then the approximation error admits the bound

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |M_t - M_t^{n,p}|\right] \leq C\varepsilon(n^{-\frac{3}{4}}).$$

Approximating option prices

One can approximate

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}\left[\phi\left(S_\tau, M_\tau\right)\right] \quad \text{by} \quad \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[\phi\left(S_\tau, M_\tau^{n,p}\right)\right].$$

Corollary

Let the payoff function ϕ be Lipschitz in the second variable, then the pricing error admits the bound

$$\left| \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[\phi\left(S_\tau, M_\tau\right)\right] - \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[\phi\left(S_\tau, M_\tau^{n,p}\right)\right] \right| \leq C\varepsilon(n^{-\frac{3}{4}}).$$

where $C > 0$ is a constant independent of n.

Uniformly weighted moving averages

Let

$$\mu(dx) = h(x)dx = \frac{1}{\delta}1_{[0,\delta]}dx \quad \Rightarrow \quad H(x) = \frac{1}{\delta}(\delta - x)^+$$

The coefficients $A_k^{\delta,p} = \langle H, L_k^p \rangle$ can be computed explicitly. We determine the optimal scale parameter $p_{opt}(\delta, n)$ as

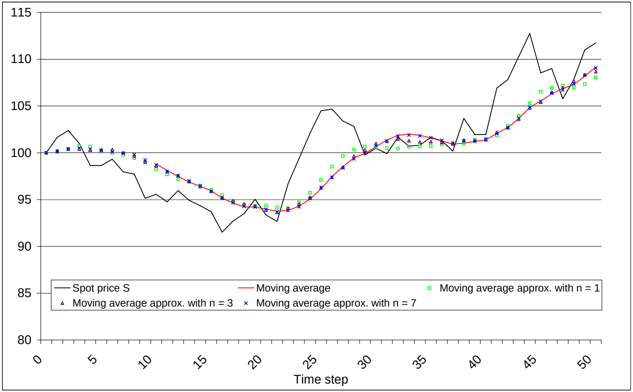
$$p_{opt}(\delta, n) = \arg\min_{p>0} \|H - H_n^p\|_2 = \arg\min_{p>0} \left\{ \frac{\delta}{3} - \sum_{k=0}^{n-1} \left| A_k^{\delta,p} \right|^2 \right\}.$$

It satisfies the scaling relation

$$p_{opt}(\delta, n) = \frac{p_{opt}(1, n)}{\delta},$$

and the values $p_{opt}(1, n)$ can be tabulated.

Uniformly weighted moving average: illustration



Simulated trajectory of the moving average process and its Laguerre approximations.

Least squares Monte Carlo

- Replace the American option by a Bermudan one with a discrete grid $\pi = \{0 = t_0, t_1, \dots, t_N = T\}$ of exercise dates.
- Simulate M paths of stock price and Laguerre processes.
- Compute optimal exercise times by backward induction:
  1. Initialization: $\tau_N^{\pi,m} = T$, $m = 1, \dots, M$
  2. Backward induction for $i = N-1, \dots, N_\delta$, $m = 1, \dots, M$:

$$\begin{cases} \tau_i^{\pi,m} = t_i 1_{A_i^m} + \tau_{i+1}^{\pi,m} 1_{\complement A_i^m} \\ A_i^m = \left\{ \phi\left(S_{t_i}^{\pi,m}, M_{t_i}^{n,\pi,m}\right) \geq \mathbb{E}_{t_i}^M \left[\phi\left(S_{\tau_{i+1}^\pi}^\pi, M_{\tau_{i+1}^\pi}^{n,\pi} \right) \right] \right\} \end{cases}$$

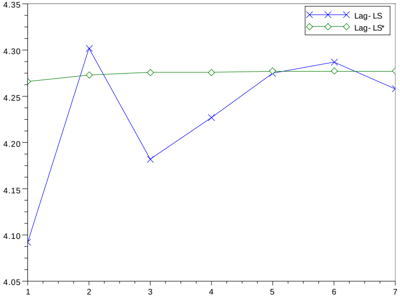
  3. Estimation of the option price at time 0:

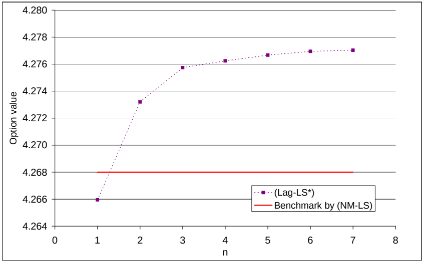
$$V_0^\pi = \frac{1}{M} \sum_{m=1}^{M} \phi\left(S_{\tau_{N_\delta}^{\pi,m}}^{\pi,m}, M_{\tau_{N_\delta}^{\pi,m}}^{n,\pi,m} \right)$$

- The conditional expectations are estimated by regression on the basis functions of state variables

- In numerical examples, we find that best results are obtained if the approximate moving average $M^{n,\pi}$ is replaced by the true moving average M^{π} in the pay-off function, while estimating the conditional expectations by regressions on $S^{\pi}, X^{0,\pi}, \dots, X^{n,\pi}$.
- The suboptimal approach often used by practitioners consists in estimating the conditional expectations by regression on (S^{π}, M^{π}) only.

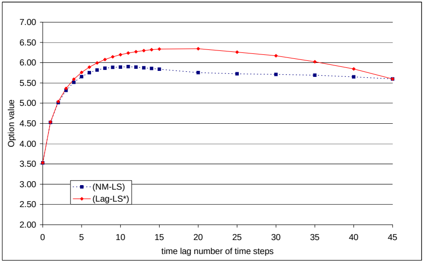
Numerical examples: convergence





Left: Laguerre approximation vs. the improved method.
Right: zoom for the improved method and the practitioner' method.

Numerical examples: delayed options



For moving average options with time delay whose pay-off depends on $X_\tau = \frac{1}{\delta}\int_{\tau-l-\delta}^{\tau-l} S_u du$, the Laguerre approximation leads to a substantial improvement compared to the practitioner's method.