Stochastic target problems and pricing under risk constraints

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Joint works with R. Elie, M. N. Dang, N. Touzi, T. N. Vu

Motivation

General setting

- ϕ : trading strategy

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- Constraint : $(X^\phi, Y_y^\phi) \in \mathcal{O}$ up to T ($\mathcal{O} : t \mapsto \mathcal{O}(t) \subset \mathbb{R}^{d+1}$)
- Price under risk constraint :

$$\inf\left\{y : \exists\, \phi \text{ s.t. } (X^\phi, Y_y^\phi) \in \mathcal{O} \text{ and } \mathbb{E}\left[G(X^\phi(T), Y_y^\phi(T))\right] \geq p\right\} .$$

Examples of dynamics : “usual” large investor model

☐ Control ϕ : predictable process with values in $U \subset \mathbb{R}^d$.

$$dX^\phi = \mu_X(X^\phi, \phi)dr + \sigma_X(X^\phi, \phi)dW$$
$$dY^\phi = \phi' \mu_X(X^\phi, \phi)dr + \phi' \sigma_X(X^\phi, \phi)dW \ .$$

☐ $\Rightarrow X^\phi =$ stocks, $Y^\phi =$ wealth, $\phi =$ number of stocks in the portfolio.

Examples of dynamics : proportional transaction costs

□ Control ϕ adapted non-decreasing process (component by component)

$$
\begin{aligned}
X^1(s) &= x^1 + \int_t^s X^1(r)\mu dr + \int_t^s X^1(r)\sigma dW_r^1 \\
X^{2,\phi}(s) &= x^2 + \int_t^s \frac{X^{2,\phi}(r)}{X^1(r)} dX^1(r) - \int_t^s d\phi_r^1 + \int_t^s d\phi_r^2 \\
Y^\phi(s) &= y + \int_t^s (1-\lambda) d\phi_r^1 - \int_t^s (1+\lambda) d\phi_r^2 \,.
\end{aligned}
$$

□ $\Rightarrow X^1 =$ stock, $X^{2,\phi} =$ value invested in the stock, $Y^\phi =$ value invested in cash

□ $\phi_t^1 =$ cumulated amount of stocks sold, $\phi_t^2 =$ cumulated amount of stocks bought.

□ $\lambda \in (0,1)$: proportional transaction cost coefficient.

Examples of dynamics : model with immediate proportional price impact

□ Control ϕ adapted non-decreasing process (component by component)

$$dX^\phi = \mu_X(X^\phi)dr + \sigma_X(X^\phi)dW + \beta_X(X^\phi)d\phi$$
$$dY^\phi = X^\phi d\phi \; .$$

□ $\Rightarrow X^\phi =$ stock, $Y^\phi =$ wealth, $d\phi =$ number of stocks bought at time t.

□ $\beta_X =$ immediate impact factor.

Examples of dynamics : model with immediate non-proportional price impact

- Control $\phi = \sum_{i \geq 1} \xi_i \mathbf{1}_{[\tau_i, \tau_{i+1})}$ adapted

$$dX^{1,\phi} = \mu_X(X^\phi)dr + \sigma_X(X^\phi)dW + \sum_{i \geq 1} \beta_X(X^\phi, \Delta\phi)\mathbf{1}_{\tau_i}$$

$$dX^{2,\phi} = \sum_{i \geq 1} \Delta\phi \mathbf{1}_{\tau_i}$$

$$dY^\phi = \sum_{i \geq 1} \beta_Y(X^\phi, \Delta\phi)\mathbf{1}_{\tau_i} \ .$$

- $\Rightarrow X^{1,\phi} =$ stock, $X^{2,\phi} =$ number of stocks in the portfolio, $Y^\phi =$ cash account, $\Delta\phi_{\tau_i} =$ number of stocks bought/sold at time τ_i.
- $\beta_X =$ immediate impact factor, $\beta_Y =$ buying/selling cost.

Other possible dynamics

- Dynamics with jumps (finance/insurance) : L. Moreau, B.

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- Any mixed control type problems.

Examples of constraints : super-hedging

☐ Problem :

$$v := \inf\left\{y : \exists\, \phi \text{ s.t. } (X^\phi, Y_y^\phi) \in \mathcal{O} \text{ and } \mathbb{E}\left[G(X^\phi(T), Y_y^\phi(T))\right] \geq p\right\} .$$

☐ Take

$$\mathcal{O} := \mathbb{R}^{d+1}\mathbf{1}_{[0,T)} + \mathbf{1}_{\{T\}}\{(x,y) \;:\; y \geq g(x)\} , \; G = 0 \text{ and } p = 0 .$$

☐ Super-hedging of an European option :

$$v := \inf\left\{y : \exists\, \phi \text{ s.t. } Y_y^\phi(T) \geq g(X^\phi(T))\right\} .$$

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□ Take

$$\mathcal{O} := \mathbb{R}^{d+1} \,,\; G(x, y) = \mathbf{1}_{y \geq g(x)} \text{ and } p = 1 \,.$$

□ Super-hedging of an European option :

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Examples of constraints : super-hedging of American option

□ Problem :

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□ Super-hedging of an American option :

$$v := \inf\left\{y : \exists\, \phi \text{ s.t. } Y_y^{\phi} \geq g(X^{\phi}) \text{ up to } T\right\} .$$

Examples of constraints : P&L-hedging

□ Problem :

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Take

$$\mathcal{O} := \mathbb{R}^{d+1} \;,\; G^i(x,y) = \mathbf{1}_{y-g(x)\geq -c^i} \text{ and } p^i \in (0,1] \,.$$

with

$$\mathbb{P}\left[Y_y^{\phi}(T) - g(X^{\phi}(T)) \geq -c^i \right] \geq p^i \text{ with } c^i \uparrow,\; p^i \uparrow$$

$\Rightarrow$ P&L constraint (work in progress with T. N. Vu).

Examples of constraints : shortfall-hedging

□ Problem :

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Take

$$\mathcal{O} := \mathbb{R}^{d+1} \;,\; G(x,y) = -\ell([y - g(x)]^{-}) \text{ and } p < 0 \;.$$

$\Rightarrow$ Shortfall-hedging of European option.

Examples of constraints : indifference pricing

□ Problem :

$$v := \inf \left\{ y : \exists\, \phi \text{ s.t. } (X^{\phi}, Y^{\phi}_{y}) \in \mathcal{O} \text{ and } \mathbb{E}\left[G(X^{\phi}(T), Y^{\phi}_{y}(T)) \right] \geq p \right\} .$$

Take

$$\mathcal{O} := \mathbb{R}^{d+1} ,\ G(x,y) = U(y_0 + y - g(x)) \text{ and } p := \sup_{\phi} \mathbb{E}\left[U(Y^{\phi}_{t,x,y_0}(T)) \right] .$$

⇒ Utility indifference price.

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  ($\inf_y \max_\phi \mathbb{E}\left[G(X^\phi, Y^\phi_y)\right] \geq p = v$).

□ If one can allow for high dimensions : include liquid options as assets $\Rightarrow$ automatically calibrated.

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☐ Assumption : $y' \geq y$ and $(x,y) \in \mathcal{O} \Rightarrow (x,y') \in \mathcal{O}$, $t \mapsto \mathcal{O}(t)$ is right-continuous and $G \uparrow$ in y.

The $\mathbb{P}$ – a.s. case

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- Theorem : For all ϕ and $\theta \in \mathcal{T}_{[t,T]}$:

GDP1 :

$$Z^{\phi}_{t,z} \in \mathcal{O} \text{ on } [t,T] \Rightarrow Y^{\phi}_{t,z}(\theta) \geq v(\theta, X^{\phi}_{t,x}(\theta))$$

GDP2 :

$$y < v(t,x) \Rightarrow \mathbb{P}\left[Y^{\phi}_{t,z}(\theta) \geq v(\theta, X^{\phi}_{t,x}(\theta)) \text{ and } Z^{\phi}_{t,z} \in \mathcal{O} \text{ on } [t,\theta] \right] < 1$$

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□ First introduced by Soner and Touzi for super-hedging under Gamma constraints. Extended to American type contraints : obstacle version of B. and Vu.

Constraints in expectations

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with

$$P_{t,p}(\theta) := \mathbb{E}\left[G(Z^{\phi}_{t,z}(T)) \mid \mathcal{F}_{\theta}\right]$$

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$$P_{t,p}(\theta) := \mathbb{E}\left[G(Z^{\phi}_{t,z}(T)) \mid \mathcal{F}_\theta\right] = p + \int_t^\theta \alpha_s dW_s\,,$$

if $\mathcal{F}_t = \sigma(W_s,\ s \leq t)$.

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- Problem reduction : For all ϕ :

$$Z^{\phi}_{t,z} \in \mathcal{O} \text{ on } [t,T] \text{ and } \mathbb{E}\left[G(Z^{\phi}_{t,z}(T))\right] \geq p$$

if and only if $\exists\ \alpha$ such that

$$(Z^{\phi}_{t,z}, P^{\alpha}_{t,p}) \in \mathcal{O} \times \mathbb{R} \text{ on } [t,T] \text{ and } G(Z^{\phi}_{t,z}(T)) \geq P^{\alpha}_{t,p}(T)$$

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- Can use the GDP with an increased controlled process.

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- In the following, we consider the case with controls of bounded variations types (simplification of a work with M. N. Dang).

The general model

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$$v(t,x,p) := \inf\left\{y : \exists L \in \mathcal{L} \ / \ Z^L_{t,x,y} \in \mathcal{O}\ ,\ \mathbb{E}\left[G(Z^L_{t,x,y}(T))\right] \geq p\right\}$$

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□ Reduction : $\mathcal{A}$ set of predictable square integrable processes

$$\inf\left\{y : \exists(L,\alpha) \in \mathcal{L} \times \mathcal{A}\ /\ Z^L_{t,x,y} \in \mathcal{O}\ ,\ G(Z^L_{t,x,y}(T)) \geq P^{\alpha}_{t,p}(T)\right\}\ .$$

Formal derivation of the PDE

Assume that v is smooth and the inf is achieved.

For $y = v(t, x, p)$, $\exists$ (L, α) such that $Z^L_{t,z} \in \mathcal{O}$ on $[t, T]$ and $G(Z^L_{t,x,y}(T)) \geq P^{\alpha}_{t,p}(T)$.

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$$\begin{aligned}
&\left(\mu_Y(z) - \mathcal{L}^\alpha_{X,P} v(t,x,p)\right) dt \\
&\geq (\sigma_Y(z) - D_x v(t,x,p)\sigma_X(x) - D_p v(t,x,p)\alpha_t)\, dW_t \\
&+ (\beta_Y(z) - D_x v(t,x,p)\beta_X(x))\, dL_t
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$$+ \left(\beta_Y(z) - D_x v(t,x,p)\beta_X(x)\right) dL_t$$

Ok if $\quad \mu_Y(x, v(t,x,p)) - \mathcal{L}^{\alpha}_{X,P} v(t,x,p) \geq 0$

with $\sigma_Y(x, v(t,x,p)) = D_x v(t,x,p)\sigma_X(x) - D_p v(t,x,p)\alpha.$

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Or $\quad \left(\beta_Y(x, v(t,x,p)) - D_x v(t,x,p)\beta_X(x)\right)\ell > 0$

with $\ell \in \Delta_+ := \partial B_1(0) \cap \mathbb{R}^d_+$.

Formal derivation of the PDE

Set

$$
\begin{aligned}
Fv &:= \sup\left\{\mu_Y(\cdot,v) - \mathcal{L}^{\alpha}_{X,P}v,\ \alpha \in Nv\right\} \\
Gv &:= \max\left\{[\beta_Y(\cdot,v) - D_x v(t,x)\beta_X(x)]\ell,\ \ell \in \Delta_+\right\}
\end{aligned}
$$

with

$$
\begin{aligned}
Nv &:= \{\alpha : \sigma_Y(\cdot,v) = D_x v\sigma_X + D_p v\alpha\} \\
\Delta_+ &:= \mathbb{R}^d_+ \cap \partial B_1(0)\ .
\end{aligned}
$$

PDE characterization in the interior of the domain

$$
\max\{Fv\ ,\ Gv\} = 0 \text{ on } (t,x,v(t,x)) \in \mathrm{int}(D)
$$

where $D := \{(t,x,y) : (x,y) \in \mathcal{O}(t)\}$.

PDE on the space boundary $(x, y) \in \partial\mathcal{O}(t)$

Domain is

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As above it implies : either

$$\mathcal{L}_Z\delta(t, x, y) \geq 0 \text{ and } D\delta(t, x, y)\sigma_Z(x, y) = 0$$

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$$\max\{D\delta(t, x, y)\beta_z(x, y)\ell,\ \ell \in \Delta_+\} > 0 .$$

PDE on the space boundary $(x, y) \in \partial\mathcal{O}(t)$

The GDP and the need for a reflexion on the boundary leads to the definition of

$$
\begin{aligned}
N^{\rm in} v &:= \{\alpha \in Nv : D\delta(\cdot, v)\sigma_Z(\cdot, v) = 0\} \\
F^{\rm in} v &:= \sup_{\alpha \in N^{\rm in} v} \min\left\{\mu_Y(\cdot, v) - \mathcal{L}^{\alpha}_{X,P} v \ , \ \mathcal{L}_Z \delta(\cdot, v)\right\} \\
G^{\rm in} v &:= \max_{\ell \in \Delta_+} \min\left\{[\beta_Y(\cdot, v) - D_x v \beta_X]\ell \ , \ D\delta(\cdot, v)\beta_z(\cdot, v)\ell\right\}
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\end{aligned}
$$

Then, the PDE on the boundary reads

$$\max\{F^{\rm in}_0 v \ , \ G^{\rm in} v\} = 0 \ \text{ on } (t, x, v(t, x)) \in \partial D \ .$$

Example

Pricing of the WVAP-guaranteed liquidation contract

The VWAP guaranted pricing problem

- K stocks to liquidate.

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- Ensure that will guarantee a mean selling price of $\gamma \times$ the mean selling price of the market.
- What is the price of the guarantee ?

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- Risk constraint (with $\gamma \in (0,1)$)

$$X^{L,3}_{t,x} \in [\underline{\Lambda}, \overline{\Lambda}] \text{ and } \mathbb{E}\left[\ell\left(Y^L_{t,x,y}(T) - K\gamma X^{L,2}_{t,x}(T)\right)\right] \geq p\} .$$

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- Pricing function (with $\Psi(x, y) = \ell(y - \gamma K x^2)$, $\gamma > 0$)

$$v(t, x, p) := \inf\{y \geq 0 : \exists L \text{ s.t. } X^{L,3}_{t,x} \in [\underline{\Lambda}, \overline{\Lambda}] \text{ , } \mathbb{E}\left[\Psi(Z^L_{t,x,y}(T))\right] \geq p\} \text{ .}$$

PDE characterization

Proposition Under "good assumptions", v_* is a viscosity supersolution on $[0, T)$ of

$$\max\left\{F\varphi\ ,\ x^1 + x^1\beta D_{x^1}\varphi - D_{x^3}\varphi\right\} = 0 \ \text{ if } \underline{\Lambda} \le x^3 \le \overline{\Lambda}$$

and v^* is a subsolution on $[0, T)$ of

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\min\left\{\varphi\ ,\ x^1 + \beta D_{x^1}\varphi - D_{x^3}\varphi\right\} = 0 \quad & \text{if} \quad \underline{\Lambda} = x^3 \\
\min\left\{\varphi\ ,\ F\varphi\right\} = 0 \quad & \text{if} \quad x^3 = \overline{\Lambda}\ ,
\end{aligned}$$

where

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Moreover, $v_*(T, x, p) = v^*(T, x, p) = \Psi^{-1}(x, p)$.

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The "good assumptions"

- On $\underline{\Lambda}, \overline{\Lambda}$:

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□ On the loss function ℓ :

$$\exists\, \epsilon > 0 \text{ s.t. } \epsilon \leq D^{-}\ell \,,\ D^{+}\ell \leq \epsilon^{-1} \,,$$
$$\text{and } \lim_{r\to\infty} D^{+}\ell(r) = \lim_{r\to\infty} D^{-}\ell(r) \,.$$

Control on the gradients

□ Proposition v_* is a viscosity supersolution of

$$\min\{D_p\varphi - \epsilon\ ,\ (D_{x^1}\varphi - CD_p\varphi)\mathbf{1}_{x^1>0}\ ,\ -D_{x^1}\varphi + CD_p\varphi\} = 0 \quad (*)$$

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where C is continuous and depends only on x.

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where C is continuous and depends only on x.

□ Provides a control on the ratio $D_{x^1}\varphi/D_p\varphi$ in

$$F\varphi := -\mathcal{L}_X\varphi - \frac{(x^1\sigma)^2}{2}\left(|D_{x^1}\varphi/D_p\varphi|^2 D_p^2\varphi - 2(D_{x^1}\varphi/D_p\varphi)D^2_{(x^1,p)}\varphi\right)\ .$$

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- and that for $(t_n, x_n, p_n)_n$ s.t. $(t_n, x_n) \to (t,x)$:

$$\lim_{n\to\infty} v_*(t_n,x_n,p_n) = \lim_{n\to\infty} v^*(t_n,x_n,p_n) = 0 \text{ if } p_n \to -\infty \ ,$$

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☐ A little more : v is continuous in p and x^3.

Uniqueness

- Want a comparison resul in the class of function with the above limit and growth conditions.

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☐ We now control $D_{x^1}\varphi/D_p\varphi$.

This is not enough... If we need to penalize in x^1 (stock price) then the term $|D_{x^1}\varphi/D_p\varphi|^2 D_p^2\varphi$ will blow up as $n \to \infty$, where n comes from the usual penalisation $n|x_1^1 - x_2^1|^2$ due to the doubling of constants.

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$$\exists\ \hat{x}^1 > 0 \text{ s.t. } \mu(\hat{x}^1) \leq 0 = \sigma(\hat{x}^1) .$$

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□ Bound on the stock price...

Comparison

□ Theorem : Let U (resp. V) be a non-negative super- and subsolutions which are continuous in x^3. Assume that

$$U(t,x,p) \geq V(t,x,p) \text{ if } t = T \text{ or } x^1 \in \{0, 2\hat{x}^1\},$$

and that $\exists\ c_+ > 0$ and $c_- \in \mathbb{R}$ s.t.

$$\limsup_{(t',x',p')\to(t,x,\infty)} V(t',x',p')/p' \leq c_+ \leq \liminf_{(t',y',p')\to(t,y,\infty)} U(t',y',p')/p' \ ,$$
$$\limsup_{(t',x',p')\to(t,x,-\infty)} V(t',x',p') \leq c_- \leq \liminf_{(t',y',p')\to(t,y,-\infty)} U(t',y',p') \ .$$

If either U is a supersolution of (*) which is continuous in p, or V is a subsolution of (**) which is continuous in p, then

$$U \geq V \ .$$

Additional remarks

Optimal management under shortfall constraints

- Serves as a building block for problems of the form

$$\sup_{\phi \in \mathcal{A}_{t,z}} \mathbb{E}\left[U(X^{\phi}_{t,x}(T), Y^{\phi}_{t,z}(T))\right]$$

with $\mathcal{A}_{t,z} := \{\phi \in \mathcal{A} : Z^{\phi}_{t,z} \in \mathcal{O} \text{ on } [t, T]\}$.

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with $\mathcal{A}_{t,z} := \{\phi \in \mathcal{A} : Z^{\phi}_{t,z} \in \mathcal{O} \text{ on } [t, T]\}$.

□ Amongs to say that $Y^{\phi}_{t,z} \geq v(\cdot, X^{\phi}_{t,x})$
where $v(t,x) := \inf\left\{y : \exists\, \phi \in \mathcal{A} \text{ s.t. } Z^{\phi}_{t,z} \in \mathcal{O} \text{ on } [t, T]\right\}$,

see B., Elie and Imbert (2010).

BSDE with moment conditions

□ Look for the minimal solution (Y, Z) of

$$Y_t = Y_T + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

such that

$$\mathbb{E}[G(Y_T, \xi)] \geq p \ .$$

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$$\mathbb{E}[G(Y_T, \xi)] \geq p \, .$$

- Can use the same approach : for $\alpha \in \mathcal{A}$ set

$$Y_t^\alpha = G^{-1}(P_T^\alpha, \xi) + \int_t^T f(s, Y_s^\alpha, Z_s^\alpha)ds - \int_t^T Z_s^\alpha dW_s$$

BSDE with moment conditions

- Look for the minimal solution (Y, Z) of

$$Y_t = Y_T + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

such that

$$\mathbb{E}\left[G(Y_T, \xi)\right] \geq p \ .$$

- Can use the same approach : for $\alpha \in \mathcal{A}$ set

$$Y_t^\alpha = G^{-1}(P_T^\alpha, \xi) + \int_t^T f(s, Y_s^\alpha, Z_s^\alpha)ds - \int_t^T Z_s^\alpha dW_s$$

- The minimal solution is (formally) given by $Y = \operatorname*{essinf}_\alpha Y^\alpha$.

Optimal control vs stochastic targets

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$$w := \inf_{\phi} \mathbb{E}\left[U(X^{\phi}(T))\right]$$

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☐ Allows for a unified approach (obviously obtains -immediately- the same HJB PDE)