Stochastic target problems and pricing under risk constraints

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Joint works with R. Elie, M. N. Dang, N. Touzi, T. N. Vu
Motivation
General setting

☐ \( \phi \) : trading strategy
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- $\phi$: trading strategy
- $Y^\phi_y$: wealth process, valued in $\mathbb{R}$, initial wealth $y$
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- $Y^\phi_y$: wealth process, valued in $\mathbb{R}$, initial wealth $y$
- $X^\phi$: stocks, factors, valued in $\mathbb{R}^d$
General setting

- \( \phi \): trading strategy
- \( Y^\phi_y \): wealth process, valued in \( \mathbb{R} \), initial wealth \( y \)
- \( X^\phi \): stocks, factors, valued in \( \mathbb{R}^d \)
- Target: \( \mathbb{E} \left[ G(X^\phi(T), Y^\phi_y(T)) \right] \geq p, \ p \in \mathbb{R}, \ G : \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R} \)
General setting

- $\phi$ : trading strategy
- $Y_\phi^y$ : wealth process, valued in $\mathbb{R}$, initial wealth $y$
- $X_\phi$ : stocks, factors, valued in $\mathbb{R}^d$
- Target : $\mathbb{E}\left[ G(X_\phi(T), Y_\phi^y(T)) \right] \geq p$, $p \in \mathbb{R}$, $G : \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}$
- Constraint : $(X_\phi, Y_\phi^y) \in \mathcal{O}$ up to $T$ ($\mathcal{O} : t \mapsto \mathcal{O}(t) \subset \mathbb{R}^{d+1}$)
General setting

- $\phi$: trading strategy
- $Y_y^\phi$: wealth process, valued in $\mathbb{R}$, initial wealth $y$
- $X^\phi$: stocks, factors, valued in $\mathbb{R}^d$
- Target: $\mathbb{E}\left[G(X^\phi(T), Y_y^\phi(T))\right] \geq p$, $p \in \mathbb{R}$, $G : \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}$
- Constraint: $(X^\phi, Y_y^\phi) \in \mathcal{O}$ up to $T$ ($\mathcal{O} : t \mapsto \mathcal{O}(t) \subset \mathbb{R}^{d+1}$)
- Price under risk constraint:

$$\inf \left\{ y : \exists \phi \text{ s.t. } (X^\phi, Y_y^\phi) \in \mathcal{O} \text{ and } \mathbb{E}\left[G(X^\phi(T), Y_y^\phi(T))\right] \geq p \right\}.$$
Examples of dynamics: “usual” large investor model

- Control $\phi$: predictable process with values in $U \subset \mathbb{R}^d$.

\[
\begin{align*}
dX^\phi &= \mu_X(X^\phi, \phi)dr + \sigma_X(X^\phi, \phi)dW \\
dY^\phi &= \phi' \mu_X(X^\phi, \phi)dr + \phi' \sigma_X(X^\phi, \phi)dW .
\end{align*}
\]

- $\Rightarrow X^\phi = \text{stocks}, \ Y^\phi = \text{wealth}, \ \phi = \text{number of stocks in the portfolio}$. 
Examples of dynamics: proportional transaction costs

- Control $\phi$ adapted non-decreasing process (component by component)

\[
X^1(s) = x^1 + \int_t^s X^1(r) \mu dr + \int_t^s X^1(r) \sigma dW^1_r
\]

\[
X^{2,\phi}(s) = x^2 + \int_t^s \frac{X^{2,\phi}(r)}{X^1(r)} dX^1(r) - \int_t^s d\phi^1_r + \int_t^s d\phi^2_r
\]

\[
Y^\phi(s) = y + \int_t^s (1 - \lambda) d\phi^1_r - \int_t^s (1 + \lambda) d\phi^2_r.
\]

- $X^1 = \text{stock}$, $X^{2,\phi} = \text{value invested in the stock}$, $Y^\phi = \text{value invested in cash}$
- $\phi^1_t = \text{cumulated amount of stocks sold}$, $\phi^2_t = \text{cumulated amount of stocks bought}$.
- $\lambda \in (0, 1)$: proportional transaction cost coefficient.
Examples of dynamics : model with immediate proportional price impact

- Control $\phi$ adapted non-decreasing process (component by component)

$$dX^\phi = \mu_X(X^\phi)dr + \sigma_X(X^\phi)dW + \beta_X(X^\phi)d\phi$$

$$dY^\phi = X^\phi d\phi .$$

- $X^\phi = \text{stock}$, $Y^\phi = \text{wealth}$, $d\phi = \text{number of stocks bought at time } t$.
- $\beta_X = \text{immediate impact factor}$. 
Examples of dynamics: model with immediate non-proportional price impact

- Control $\phi = \sum_{i \geq 1} \xi_i 1_{[\tau_i, \tau_{i+1})}$ adapted

$$dX^{1,\phi} = \mu_X(X^{\phi})dr + \sigma_X(X^{\phi})dW + \sum_{i \geq 1} \beta_X(X^{\phi}, \Delta \phi) 1_{\tau_i}$$

$$dX^{2,\phi} = \sum_{i \geq 1} \Delta \phi 1_{\tau_i}$$

$$dY^{\phi} = \sum_{i \geq 1} \beta_Y(X^{\phi}, \Delta \phi) 1_{\tau_i}.$$  

- $X^{1,\phi} =$ stock, $X^{2,\phi} =$ number of stocks in the portfolio, $Y^{\phi} =$ cash account, $\Delta \phi_{\tau_i} =$ number of stocks bought/sold at time $\tau_i$.

- $\beta_X =$ immediate impact factor, $\beta_Y =$ buying/selling cost.
Other possible dynamics

- Dynamics with jumps (finance/insurance): L. Moreau, B.
Other possible dynamics

- Dynamics with jumps (finance/insurance) : L. Moreau, B.
- Any mixed control type problems.
Examples of constraints: super-hedging

- Problem:

\[ v := \inf \left\{ y : \exists \phi \text{ s.t. } (X^\phi, Y_y^\phi) \in \mathcal{O} \text{ and } \mathbb{E}\left[ G(X^\phi(T), Y_y^\phi(T)) \right] \geq p \right\}. \]

- Take

\[ \mathcal{O} := \mathbb{R}^{d+1} \mathbf{1}_{[0,T]} + \mathbf{1}_T \{(x,y) : y \geq g(x)\}, \ G = 0 \text{ and } p = 0. \]

- Super-hedging of an European option:

\[ v := \inf \left\{ y : \exists \phi \text{ s.t. } Y_y^\phi(T) \geq g(X^\phi(T)) \right\}. \]
Examples of constraints: super-hedging

Problem:
\[ v := \inf \left\{ y : \exists \phi \text{ s.t. } (X^\phi, Y_y^\phi) \in \mathcal{O} \text{ and } \mathbb{E} \left[ G(X^\phi(T), Y_y^\phi(T)) \right] \geq p \right\} . \]

Take
\[ \mathcal{O} := \mathbb{R}^{d+1}, \quad G(x, y) = 1_{y \geq g(x)} \text{ and } p = 1. \]

Super-hedging of an European option:
\[ v := \inf \left\{ y : \exists \phi \text{ s.t. } Y_y^\phi(T) \geq g(X^\phi(T)) \right\} . \]
Examples of constraints: super-hedging of American option

□ Problem:

\[ v := \inf \left\{ y : \exists \phi \text{ s.t. } (X^\phi, Y^\phi_y) \in \mathcal{O} \text{ and } \mathbb{E} \left[ G(X^\phi(T), Y^\phi_y(T)) \right] \geq p \right\}. \]

\[ \mathcal{O} := \{(x, y) : y \geq g(x)\}, \; G = 0 \text{ and } p = 0. \]

□ Super-hedging of an American option:

\[ v := \inf \left\{ y : \exists \phi \text{ s.t. } Y^\phi_y \geq g(X^\phi) \text{ up to } T \right\}. \]
Examples of constraints: P&L-hedging

Problem:

\[ \nu := \inf \left\{ y : \exists \phi \text{ s.t. } (X^\phi, Y_y^\phi) \in \mathcal{O} \text{ and } \mathbb{E} \left[ G(X_Y^\phi(T), Y_y^\phi(T)) \right] \geq p \right\} . \]

Take

\[ \mathcal{O} := \mathbb{R}^{d+1}, \quad G^i(x, y) = 1_{y - g(x) \geq -c_i} \text{ and } p^i \in (0, 1). \]

with

\[ \mathbb{P} \left[ Y_y^\phi(T) - g(X^\phi(T)) \geq -c_i \right] \geq p^i \text{ with } c^i \uparrow, \quad p^i \uparrow \]

\[ \Rightarrow \text{ P&L constraint (work in progress with T. N. Vu).} \]
Examples of constraints: shortfall-hedging

Problem:

\[ v := \inf \left\{ y : \exists \phi \text{ s.t. } (X^\phi, Y_y^\phi) \in \mathcal{O} \text{ and } \mathbb{E} \left[ G(X^\phi(T), Y_y^\phi(T)) \right] \geq p \right\} . \]

Take

\[ \mathcal{O} := \mathbb{R}^{d+1}, \quad G(x, y) = -\ell([y - g(x)]^-) \text{ and } p < 0 . \]

⇒ Shortfall-hedging of European option.
Examples of constraints: indifference pricing

□ Problem:

\[ v := \inf \left\{ y : \exists \phi \text{ s.t. } (X_\phi, Y_\phi) \in \mathcal{O} \text{ and } \mathbb{E} \left[ G(X_\phi(T), Y_\phi(T)) \right] \geq p \right\}. \]

Take

\[ \mathcal{O} := \mathbb{R}^{d+1}, \ G(x, y) = U(y_0 + y - g(x)) \text{ and } p := \sup_{\phi} \mathbb{E} \left[ U(Y_{t,x,y_0}^{\phi}(T)) \right]. \]

⇒ Utility indifference price.
Aim
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- Provide a PDE characterization in the (Markovian) situations where

  - markets are incomplete
  - markets have frictions
  - models without any notion of martingale measure. Ex: WVAP guaranteed liquidation contracts.

- Based on a "risk" criteria.

- We want a direct approach:
  - one (non-linear) pricing equation
  - no-numerical inversion procedure

If one can allow for high dimensions: include liquid options as assets ⇒ automatically calibrated.
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\[(\inf_y \max_\phi \mathbb{E} \left[ G(X^\phi, Y_y^\phi) \right] \geq p = v).\]
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  - markets are incomplete
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- We want a direct approach:
  - one (non-linear) pricing equation
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    \[ \inf_y \max_\phi \mathbb{E} \left[ G(X^\phi, Y^\phi_y) \right] \geq p = v. \]
- If one can allow for high dimensions: include liquid options as assets ⇒ automatically calibrated.
Geometric Dynamic Programming

Problem extension: $Z_{t,z}^\phi = (X_{t,x}^\phi, Y_{t,x,y}^\phi)$
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\[
v(t, x, p) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^\phi \in \mathcal{O} \text{ on } [t, T] , \mathbb{E} \left[ G(Z_{t,x,y}^\phi(T)) \right] \geq p \right\} .
\]
Geometric Dynamic Programming

- Problem extension: \( Z_{t,z}^{\phi} = (X_{t,x}^{\phi}, Y_{t,x,y}^{\phi}) \)

\[ v(t, x, p) := \inf \{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^{\phi} \in \mathcal{O} \text{ on } [t, T] , \mathbb{E} \left[ G(Z_{t,x,y}^{\phi}(T)) \right] \geq p \} . \]

- Assumption: \( y' \geq y \) and \((x, y) \in \mathcal{O} \Rightarrow (x, y') \in \mathcal{O} \), \( t \mapsto \mathcal{O}(t) \) is right-continuous and \( G \uparrow \) in \( y \).
The $\mathbb{P}$ – a.s. case

- Problem extension: $Z^\phi_{t,z} = (X^\phi_{t,x}, Y^\phi_{t,x,y})$

\[ v(t, x) := \inf \left\{ y : \exists \phi \text{ s.t. } Z^\phi_{t,x,y} \in \mathcal{O} \text{ on } [t, T] \right\} . \]
The $\mathbb{P} – \text{a.s.}$ case

Problem extension: $Z_{t,z}^\phi = (X_{t,x}^\phi, Y_{t,x,y}^\phi)$

$$v(t, x) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^\phi \in \mathcal{O} \text{ on } [t, T] \right\}.$$

Theorem: For all $\phi$ and $\theta \in \mathcal{T}_{[t, T]}$:
GDP1:

$$Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, T] \Rightarrow Y_{t,z}^\phi(\theta) \geq v(\theta, X_{t,x}^\phi(\theta))$$

GDP2:

$$y < v(t, x) \Rightarrow \mathbb{P} \left[ Y_{t,z}^\phi(\theta) \geq v(\theta, X_{t,x}^\phi(\theta)) \text{ and } Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, \theta] \right] < 1$$
The $\mathbb{P} - \text{a.s. case}$

- Problem extension: $Z_{t,z}^\phi = (X_{t,x}^\phi, Y_{t,x,y}^\phi)$

\[ v(t, x) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^\phi \in \mathcal{O} \text{ on } [t, T] \right\} . \]

- **Theorem**: For all $\phi$ and $\theta \in \mathcal{T}_{[t, T]}$:
  
  **GDP1**: 
  
  \[ Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, T] \Rightarrow Y_{t,z}^\phi(\theta) \geq v(\theta, X_{t,x}^\phi(\theta)) \]

  **GDP2**: 
  
  \[ y < v(t, x) \Rightarrow \mathbb{P} \left[ Y_{t,z}^\phi(\theta) \geq v(\theta, X_{t,x}^\phi(\theta)) \text{ and } Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, \theta] \right] < 1 \]

- First introduced by Soner and Touzi for super-hedging under Gamma constraints. Extended to American type constraints: obstacle version of B. and Vu.
Constraints in expectations

Problem extension: \( Z_{t,z}^\phi = (X_{t,x}^\phi, Y_{t,x,y}^\phi) \)

\[ v(t, x, p) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^\phi \in \mathcal{O} \text{ on } [t, T], \mathbb{E} \left[ G(Z_{t,x,y}^\phi(T)) \right] \geq p \right\}. \]
Constraints in expectations

- **Problem extension**: \( Z_{t,z}^φ = (X_{t,x}^φ, Y_{t,x,y}^φ) \)

  \[ v(t, x, p) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^φ \in \mathcal{O} \text{ on } [t, T], \mathbb{E}\left[ G(Z_{t,x,y}^φ(T)) \right] \geq p \right\} . \]

- **Theorem**: For all \( φ \) and \( θ \in \mathcal{T}_{[t, T]} : \\
  \text{GDP1:} \\
  Z_{t,z}^φ \in \mathcal{O} \text{ on } [t, T] \Rightarrow Y_{t,z}^φ(θ) \geq v(θ, X_{t,x}^φ(θ), p) \) ?
Constraints in expectations

- Problem extension: $Z_{t,z}^\phi = (X_{t,x}^\phi, Y_{t,x,y}^\phi)$

$$
\nu(t, x, p) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^\phi \in \mathcal{O} \text{ on } [t, T], \mathbb{E} \left[ G(Z_{t,x,y}^\phi(T)) \right] \geq p \right\} .
$$

- Theorem: For all $\phi$ and $\theta \in \mathcal{T}_{[t, T]}$:

\[ GDP1: \]

$$
Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, T] \Rightarrow Y_{t,z}^\phi(\theta) \geq \nu(\theta, X_{t,x}^\phi(\theta), P_{t,p}(\theta))
$$

with

$$
P_{t,p}(\theta) := \mathbb{E} \left[ G(Z_{t,z}^\phi(T)) \mid \mathcal{F}_\theta \right]
$$
Constraints in expectations

Problem extension: $Z_{t,z}^\phi = (X_{t,x}^\phi, Y_{t,x,y}^\phi)$

$$v(t, x, p) := \inf \left\{ y : \exists \phi \text{ s.t. } Z_{t,x,y}^\phi \in \mathcal{O} \text{ on } [t, T], \mathbb{E}\left[G(Z_{t,x,y}(T))\right] \geq p \right\}.$$ 

Theorem: For all $\phi$ and $\theta \in \mathcal{I}_{[t, T]}$:

GDP1:

$$Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, T] \Rightarrow Y_{t,z}(\theta) \geq v(\theta, X_{t,x}(\theta), P_{t,p}(\theta))$$

with

$$P_{t,p}(\theta) := \mathbb{E}\left[G(Z_{t,z}^\phi(T)) \mid \mathcal{F}_\theta\right] = p + \int_t^\theta \alpha_s dW_s ,$$

if $\mathcal{F}_t = \sigma(W_s, s \leq t)$. 
Constraints in expectations

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\]

□ Problem reduction: For all \( \phi \):

\( Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, T] \) and \( \mathbb{E} \left[ G(Z_{t,z}^\phi(T)) \right] \geq p \)

if and only if \( \exists \alpha \) such that

\[
(Z_{t,z}^\phi, P_{t,p}^\alpha) \in \mathcal{O} \times \mathbb{R} \text{ on } [t, T] \text{ and } G(Z_{t,z}^\phi(T)) \geq P_{t,p}^\alpha(T)
\]

with

\[
P_{t,p} := \mathbb{E} \left[ G(Z_{t,z}^\phi(T)) \mid \mathcal{F} \right] = p + \int_t^T \alpha_s dW_s.
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\]

□ Can use the GDP with an increased controlled process.
PDE derivation

Previous works

• Soner and Touzi: Brownian filtration and bounded controls (apart from particular cases in finance).

Problems with Gamma constraints with Zhang, Cheridito.

• B.: Jump diffusion with bounded control and locally bounded jumps. $P_a$-a.s. criteria.

• B., Elie and Touzi: Brownian filtration with unbounded controls. Criteria in expectation (concentrating on the case of a criteria in expectation).

• B. and Vu: “American” case.

• Moreau: Extension of B., Elie and Touzi to jump diffusions.

In the following, we consider the case with controls of bounded variations types (simplification of a work with M. N. Dang).
PDE derivation

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  - Soner and Touzi: Brownian filtration and bounded controls (apart from particular cases in finance).
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The general model

- Set of controls: $L \in \mathcal{L}$ set of continuous non-decreasing $\mathbb{R}^d$-valued adapted processes $L$ s.t. $\mathbb{E} \left[ |L|_T^2 \right] < \infty$. 

2

Dynamics of $Z = (X, Y) \in \mathbb{R}^d \times \mathbb{R}^d$:

$$
\begin{align*}
    dX & = \mu_X(X) \, dt + \sigma_X(X) \, dW + \beta_X(X) \, dL, \\
    dY & = \mu_Y(Z) \, dt + \sigma_Y(Z) \, dW + \beta_Y(Z) \, dL.
\end{align*}
$$

2

Problem:

$$
v(t, x, p) := \inf \left\{ y : \exists (L, \alpha) \in \mathcal{L} \times \mathcal{A} / Z_{Lt, x, y} \in \mathcal{O}, \mathbb{E} \left[ G(Z_{Lt, x, y} T) \right] \geq p \right\}
$$

2

Reduction:

A set of predictable square integrable processes

$$
v(t, x, p) := \inf \left\{ y : \exists (L, \alpha) \in \mathcal{L} \times \mathcal{A} / Z_{Lt, x, y} \in \mathcal{O}, G(Z_{Lt, x, y} T) \geq \alpha(p, t) \right\}
$$
The general model

- Set of controls: $L \in \mathcal{L}$ set of continuous non-decreasing $\mathbb{R}^d$-valued adapted processes $L$ s.t. $\mathbb{E} \left[ |L|_T^2 \right] < \infty$.

- Dynamics of $Z = (X, Y) \in \mathbb{R}^d \times \mathbb{R}$:

  \[
  dX^L = \mu_X(X^L)dr + \sigma_X(X^L)dW + \beta_X(X^L)dL \\
  dY^L = \mu_Y(Z^L)dr + \sigma_Y(Z^L)dW + \beta_Y(Z^L)dL.
  \]
The general model

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  dY^L = \mu_Y(Z^L)dr + \sigma_Y(Z^L)dW + \beta_Y(Z^L)dL.
  
- Problem:
  
  $v(t, x, p) := \inf \{y : \exists L \in \mathcal{L} / Z^L_{t,x,y} \in \mathcal{O}, \mathbb{E} [G(Z^L_{t,x,y}(T))] \geq p\}$
The general model

- Set of controls: $L \in \mathcal{L}$ set of continuous non-decreasing $\mathbb{R}^d$-valued adapted processes $L$ s.t. $\mathbb{E} \left[ |L|_{T}^2 \right] < \infty$.

- Dynamics of $Z = (X, Y) \in \mathbb{R}^d \times \mathbb{R}$:

$$
\begin{align*}
    dX^L &= \mu_X(X^L)dr + \sigma_X(X^L)dW + \beta_X(X^L)dL \\
    dY^L &= \mu_Y(Z^L)dr + \sigma_Y(Z^L)dW + \beta_Y(Z^L)dL.
\end{align*}
$$

- Problem:

$$
\nu(t, x, p) := \inf \left\{ y : \exists L \in \mathcal{L} \ / \ Z_{t,x,y}^L \in \mathcal{O} , \ \mathbb{E} \left[ G(Z_{t,x,y}^L(T)) \right] \geq p \right\}
$$

- Reduction: $\mathcal{A}$ set of predictable square integrable processes

$$
\inf \left\{ y : \exists (L, \alpha) \in \mathcal{L} \times \mathcal{A} \ / \ Z_{t,x,y}^L \in \mathcal{O} , \ G(Z_{t,x,y}^L(T)) \geq P_{t,p}^\alpha(T) \right\}.
$$
Formal derivation of the PDE

Assume that \( v \) is smooth and the inf is achieved.

For \( y = v(t, x, p) \), \( \exists (L, \alpha) \) such that \( Z^L_{t,z} \in \mathcal{O} \) on \([t, T]\) and
\[
G(Z^L_{t,x,y}(T)) \geq P^\alpha_{t,p}(T).
\]
Assume that $v$ is smooth and the inf is achieved.

For $y = v(t, x, p)$, $\exists (L, \alpha)$ such that $Z_{t,z}^L \in \mathcal{O}$ on $[t, T]$ and $G(Z_{t,x,y}^L(T)) \geq P_{t,p}^\alpha(T)$.

Then $Y_{t,z}^L(t+) \geq v(t+, X_{t,x}^L(t+), P_{t,p}^\alpha(t+))$ and
Formal derivation of the PDE

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For $y = v(t, x, p)$, $\exists (L, \alpha)$ such that $Z_{t,z}^L \in \mathcal{O}$ on $[t, T]$ and $G(Z_{t,x,y}^L(T)) \geq P_{t,p}^\alpha(T)$.

Then $Y_{t,z}^L(t+) \geq v(t+, X_{t,x}^L(t+), P_{t,p}^\alpha(t+))$ and

$$
(\mu_Y(z) - \mathcal{L}_{X,P}^\alpha v(t, x, p)) \, dt \\
\geq (\sigma_Y(z) - D_x v(t, x, p) \sigma_X(x) - D_p v(t, x, p) \alpha_t) \, dW_t \\
+ (\beta_Y(z) - D_x v(t, x, p) \beta_X(x)) \, dL_t
$$
Formal derivation of the PDE

\[
(\mu_Y(z) - \mathcal{L}^\alpha_{X,P} v(t, x, p)) \, dt \\
\geq (\sigma_Y(z) - D_x v(t, x, p)\sigma_X(x) - D_p v(t, x, p)\alpha_t) \, dW_t \\
+ (\beta_Y(z) - D_x v(t, x, p)\beta_X(x)) \, dL_t
\]
Formal derivation of the PDE

\[
(\mu_Y(z) - \mathcal{L}_{X}^{\alpha\beta} v(t, x, p)) \ dt \\
\geq (\sigma_Y(z) - D_x v(t, x, p)\sigma_X(x) - D_p v(t, x, p)\alpha_t) \ dW_t \\
+ (\beta_Y(z) - D_x v(t, x, p)\beta_X(x)) \ dL_t
\]

**Ok if** \( \mu_Y(x, v(t, x, p)) - \mathcal{L}_{X}^{\alpha\beta} v(t, x, p) \geq 0 \)

**with** \( \sigma_Y(x, v(t, x, p)) = D_x v(t, x, p)\sigma_X(x) - D_p v(t, x, p)\alpha. \)
Formal derivation of the PDE

\[
(\mu_Y(z) - \mathcal{L}_{\alpha, p} v(t, x, p)) \, dt \\
\geq (\sigma_Y(z) - D_x v(t, x, p) \sigma_X(x) - D_p v(t, x, p) \alpha_t) \, dW_t \\
+ (\beta_Y(z) - D_x v(t, x, p) \beta_X(x)) \, dL_t
\]

**Ok if**

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\mu_Y(x, v(t, x, p)) - \mathcal{L}_{\alpha, p} v(t, x, p) \geq 0
\]

with

\[
\sigma_Y(x, v(t, x, p)) = D_x v(t, x, p) \sigma_X(x) - D_p v(t, x, p) \alpha.
\]

**Or**

\[
(\beta_Y(x, v(t, x, p)) - D_x v(t, x, p) \beta_X(x)) \ell > 0
\]

with \( \ell \in \Delta_+ := \partial B_1(0) \cap \mathbb{R}_+^d \).
Formal derivation of the PDE

Set

\[ F_v := \sup \{ \mu_Y(\cdot, v) - \mathcal{L}_{X,P}^\alpha v, \ \alpha \in Nv \} \]
\[ G_v := \max \{ [\beta_Y(\cdot, v) - D_x v(t, x)\beta_X(x)]\ell, \ \ell \in \Delta_+ \} \]

with

\[ Nv := \{ \alpha : \sigma_Y(\cdot, v) = D_x v\sigma_X + D_p v\alpha \} \]
\[ \Delta_+ := \mathbb{R}^d_+ \cap \partial B_1(0). \]

PDE characterization in the interior of the domain

\[ \max \{ F_v, Gv \} = 0 \text{ on } (t, x, v(t, x)) \in \text{int}(D) \]

where \( D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\} \).
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

Domain is

\[ D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}. \]
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

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\[ D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}. \]

Assumption: \(D \in C^{1,2}\) (or intersection of \(C^{1,2}\) domains).
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

Domain is

\[
D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}.
\]

Assumption : \(D \in C^{1,2}\) (or intersection of \(C^{1,2}\) domains).

Take \(\delta \in C^{1,2}\) such that \(\delta > 0\) in \(\text{int}(D)\), \(\delta = 0\) on \(\partial D\) and \(\delta < 0\) elsewhere.
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

Domain is

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Take \(\delta \in C^{1,2}\) such that \(\delta > 0\) in \(\text{int}(D)\), \(\delta = 0\) on \(\partial D\) and \(\delta < 0\) elsewhere.

The state constraints imposes \(d\delta(t, Z^L_{t,z}(t)) \geq 0\) if \((t, z) \in \partial D\).
PDE on the space boundary \((x, y) \in \partial \Omega(t)\)

Domain is

\[ D := \{(t, x, y) : (x, y) \in \Omega(t)\}. \]

Assumption : \(D \in C^{1,2}\) (or intersection of \(C^{1,2}\) domains).

Take \(\delta \in C^{1,2}\) such that \(\delta > 0\) in \(\text{int}(D)\), \(\delta = 0\) on \(\partial D\) and \(\delta < 0\) elsewhere.

The state constraints imposes \(d\delta(t, Z^L_{t, z}(t)) \geq 0\) if \((t, z) \in \partial D\).

As above it implies : either

\[ L_Z \delta(t, x, y) \geq 0 \text{ and } D\delta(t, x, y)\sigma_Z(x, y) = 0 \]
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

Domain is

\[ D := \{(t, x, y) : (x, y) \in \mathcal{O}(t)\}. \]

Assumption : \( D \in C^{1,2} \) (or intersection of \( C^{1,2} \) domains).

Take \( \delta \in C^{1,2} \) such that \( \delta > 0 \) in \( \text{int}(D) \), \( \delta = 0 \) on \( \partial D \) and \( \delta < 0 \) elsewhere.

The state constraints imposes \( d\delta(t, Z_{t,z}(t)) \geq 0 \) if \((t, z) \in \partial D\).

As above it implies : or

\[ \max\{D\delta(t, x, y)\beta_z(x, y)\ell, \ \ell \in \Delta_+\} > 0. \]
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

The GDP and the need for a reflexion on the boundary leads to the definition of

\[
N^{\text{in}} \nu := \{ \alpha \in N\nu : D\delta(\cdot, \nu)\sigma_Z(\cdot, \nu) = 0 \}
\]

\[
F^{\text{in}} \nu := \sup_{\alpha \in N^{\text{in}} \nu} \min \{ \mu_Y(\cdot, \nu) - \mathcal{L}^\alpha_{X,P}\nu, \mathcal{L}_Z \delta(\cdot, \nu) \}
\]

\[
G^{\text{in}} \nu := \max_{\ell \in \Delta_+} \min \{ [\beta_Y(\cdot, \nu) - D_x v \beta_X] \ell, D\delta(\cdot, \nu)\beta_z(\cdot, \nu) \ell \}
\]
PDE on the space boundary \((x, y) \in \partial \mathcal{O}(t)\)

The GDP and the need for a reflexion on the boundary leads to the definition of

\[
N^{\text{in} \nu} := \{ \alpha \in \mathcal{N} \nu : D\delta(\cdot, \nu)\sigma_Z(\cdot, \nu) = 0 \}
\]

\[
F^{\text{in} \nu} := \sup_{\alpha \in N^{\text{in} \nu}} \min \{ \mu_Y(\cdot, \nu) - \mathcal{L}_{X,P}^\alpha, \mathcal{L}_Z \delta(\cdot, \nu) \}
\]

\[
G^{\text{in} \nu} := \max_{\ell \in \Delta_+} \min \{ [\beta_Y(\cdot, \nu) - D_x v(\cdot, \nu)\beta_X] \ell, D\delta(\cdot, \nu)\beta_Z(\cdot, \nu) \ell \}
\]

Then, the PDE on the boundary reads

\[
\max\{F^{\text{in} \nu}_0, G^{\text{in} \nu}\} = 0 \text{ on } (t, x, \nu(t, x)) \in \partial D.
\]
Example

Pricing of the WVAP-guaranteed liquidation contract
The VWAP guaranteed pricing problem

☐ $K$ stocks to liquidate.
The VWAP guaranteed pricing problem

- $K$ stocks to liquidate.
- Has an impact on prices
The VWAP guaranteed pricing problem

- $K$ stocks to liquidate.
- Has an impact on prices
- Ensure that will guarantee a mean selling price of $\gamma \times$ the mean selling price of the market.
The VWAP guaranteed pricing problem

- $K$ stocks to liquidate.
- Has an impact on prices
- Ensure that will guarantee a mean selling price of $\gamma \times$ the mean selling price of the market.
- What is the price of the guarantee?
The VWAP guaranteed pricing problem

- Controls: \( L \uparrow \) adapted and continuous. \( L_t = \# \) of sold stocks.
The VWAP guaranteed pricing problem

- Controls: $L \uparrow$ adapted and continuous. $L_t = \#$ of sold stocks.
- Price dynamics:

$$dX^{L,1} = X^{L,1} \mu(X^{L,1})dt + X^{L,1} \sigma(X^{L,1})dW_t - X^{L,1} \beta(X^{L,1}(t))dL_t$$
The VWAP guaranteed pricing problem

- Controls: $L \uparrow$ adapted and continuous. $L_t = \#$ of sold stocks.
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- Cumulated gain from liquidation: $dY^L = X^{L,1} dL_t$
The VWAP guaranteed pricing problem

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  \]
- Cumulated gain from liquidation: $dY^L = X^{L,1} dL_t$
- Volume weighted market price: $dX^{L,2} = X^{L,1} d\vartheta$. 
The VWAP guaranteed pricing problem

- Controls: $L \uparrow$ adapted and continuous. $L_t = \#$ of sold stocks.
- Price dynamics:
  \[
  dX^{L,1} = X^{L,1} \mu(X^{L,1}) \, dt + X^{L,1} \sigma(X^{L,1}) \, dW_t - X^{L,1} \beta(X^{L,1}(t)) \, dL_t
  \]

- Cumulated gain from liquidation: $dY^L = X^{L,1} \, dL_t$
- Volume weighted market price: $dX^{L,2} = X^{L,1} \, d\vartheta$.
- Cumulated # of sold stocks: $X^{L,3} := L \in [\Lambda, \bar{\Lambda}] \rightarrow \{K\}$
The VWAP guaranted pricing problem

- Controls: $L \uparrow$ adapted and continuous. $L_t = \# \text{ of sold stocks.}$
- Price dynamics:

$$dX^{L,1} = X^{L,1}_t \mu(X^{L,1}_t)dt + X^{L,1}_t \sigma(X^{L,1}_t)dW_t - X^{L,1}_t \beta(X^{L,1}(t))dL_t$$

- Cumulated gain from liquidation: $dY^L = X^{L,1}_t dL_t$
- Volume weighted market price: $dX^{L,2} = X^{L,1}_t dv$.
- Cumulated # of sold stocks: $X^{L,3} := L \in [\Lambda, \bar{\Lambda}] \rightarrow \{K\}$

- Risk constraint (with $\gamma \in (0, 1)$)

$$X^{L,3}_{t,x} \in [\Lambda, \bar{\Lambda}] \text{ and } \mathbb{E} \left[ \ell \left( Y^L_{t,x,y}(T) - K \gamma X^{L,2}_{t,x}(T) \right) \right] \geq p \} .$$
The VWAP guaranteed pricing problem

- Controls: $L \uparrow$ adapted and continuous. $L_t = \#$ of sold stocks.
- Price dynamics:
  \[
dX^{L,1} = X^{L,1} \mu(X^{L,1})dt + X^{L,1} \sigma(X^{L,1})dW_t - X^{L,1} \beta(X^{L,1}(t))dL_t
  \]

- Cumulated gain from liquidation: $dY^L = X^{L,1}dL_t$
- Volume weighted market price: $dX^{L,2} = X^{L,1}d\vartheta$.
- Cumulated $\#$ of sold stocks: $X^{L,3} := L \in [\Lambda, \bar{\Lambda}] \rightarrow \{K\}$

- Pricing function (with $\Psi(x, y) = \ell(y - \gamma Kx^2), \gamma > 0$)
  \[
v(t, x, p) := \inf\{y \geq 0 : \exists L \text{ s.t. } X^{L,3}_{t,x} \in [\Lambda, \bar{\Lambda}], \mathbb{E}[\Psi(Z_{t,x,y}^{L}(T))] \geq p\}.
  \]
PDE characterization

Proposition Under “good assumptions”, \( v^* \) is a viscosity supersolution on \([0, T)\) of

\[
\max \{ F\varphi, \ x^1 + x^1 \beta D_{x^1} \varphi - D_{x^3} \varphi \} = 0 \quad \text{if} \quad \Lambda \leq x^3 \leq \bar{\Lambda}
\]

and \( v^* \) is a subsolution on \([0, T)\) of

\[
\min \{ \varphi, \max \{ F\varphi, \ x^1 + x^1 \beta D_{x^1} \varphi - D_{x^3} \varphi \} \} = 0 \quad \text{if} \quad \Lambda < x^3 < \bar{\Lambda}
\]
\[
\min \{ \varphi, \ x^1 + \beta D_{x^1} \varphi - D_{x^3} \varphi \} = 0 \quad \text{if} \quad \Lambda = x^3
\]
\[
\min \{ \varphi, F\varphi \} = 0 \quad \text{if} \quad x^3 = \Lambda,
\]

where

\[
F\varphi := -\mathcal{L}_X \varphi - \frac{(x^1 \sigma)^2}{2} \left( |D_{x^1} \varphi / D_p \varphi|^2 D_p^2 \varphi - 2(D_{x^1} \varphi / D_p \varphi) D_{(x^1, p)}^2 \varphi \right).
\]

Moreover, \( v_*(T, x, p) = v^*(T, x, p) = \Psi^{-1}(x, p) \).
Proposition Under “good assumptions”, \( v_* \) is a viscosity supersolution on \([0, T)\) of

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\]

and \( v_* \) is a subsolution on \([0, T)\) of

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\]

\[
\min \left\{ \varphi, \, x^1 + \beta D_{x^1}\varphi - D_{x^3}\varphi \right\} = 0 \quad \text{if} \quad \Lambda = x^3
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\min \left\{ \varphi, \, F\varphi \right\} = 0 \quad \text{if} \quad x^3 = \Lambda,
\]

where

\[
F\varphi := -\mathcal{L}_x \varphi - \frac{(x^1 \sigma)^2}{2} \left( \left| D_{x^1}\varphi/D_p\varphi \right|^2 D_p^2\varphi - 2(D_{x^1}\varphi/D_p\varphi) D^2_{(x^1,p)}\varphi \right).
\]

Moreover, \( v_*(T, x, p) = v^*(T, x, p) = \Psi^{-1}(x, p) \).
On $\Lambda, \bar{\Lambda}$:

$$\Lambda, \bar{\Lambda} \in C^1, \Lambda < \bar{\Lambda} \text{ on } [0, T), \ D\Lambda, D\bar{\Lambda} \in (0, M]$$
The “good assumptions”

□ On $\Lambda, \bar{\Lambda}$:

$\Lambda, \bar{\Lambda} \in C^1$, $\Lambda < \bar{\Lambda}$ on $[0, T)$, $D\Lambda, D\bar{\Lambda} \in (0, M]$

□ On the loss function $\ell$:

$\exists \epsilon > 0$ s.t. $\epsilon \leq D^-\ell$, $D^+\ell \leq \epsilon^{-1}$, and $\lim_{r \to \infty} D^+\ell(r) = \lim_{r \to \infty} D^-\ell(r)$. 
Control on the gradients

Proposition \( v_* \) is a viscosity supersolution of

\[
\min \{ D_p \varphi - \epsilon, \ (D_{x^1} \varphi - CD_p \varphi)1_{x^1 > 0}, \ -D_{x^1} \varphi + CD_p \varphi \} = 0 \quad (*)
\]

and \( v^* \) is a viscosity subsolution of

\[
\max \{-D_p \varphi + \epsilon, \ (D_{x^1} \varphi - CD_p \varphi)1_{x^1 > 0}, \ -D_{x^1} \varphi + CD_p \varphi \} = 0 \quad (**)
\]

where \( C \) is continuous and depends only on \( x \).
Control on the gradients

□ Proposition \( v_* \) is a viscosity supersolution of

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where \( C \) is continuous and depends only on \( x \).

□ Provides a control on the ratio \( D_{x^1} \varphi / D_p \varphi \) in

\[
F \varphi := -\mathcal{L}_X \varphi - \frac{(x^1 \sigma)^2}{2} \left( |D_{x^1} \varphi / D_p \varphi|^2 D_p^2 \varphi - 2(D_{x^1} \varphi / D_p \varphi) D_{(x^1, p)}^2 \varphi \right).
\]
More controls on $v$
More controls on $\nu$

- It also implies that $\exists \eta > 0$ s.t.

$$0 \leq \nu(t, x, p) \leq \epsilon^{-1}|p - \ell(0)| + \gamma \eta (1 + |x|),$$
More controls on $v$

- It also implies that $\exists \eta > 0$ s.t.

$$0 \leq v(t, x, p) \leq \epsilon^{-1}|p - \ell(0)| + \gamma \eta (1 + |x|),$$

- and that for $(t_n, x_n, p_n)_n$ s.t. $(t_n, x_n) \to (t, x)$:

$$\lim_{n \to \infty} v^*_n(t_n, x_n, p_n) = \lim_{n \to \infty} v^*(t_n, x_n, p_n) = 0 \text{ if } p_n \to -\infty,$$

$$\lim_{n \to \infty} \frac{v^*_n(t_n, x_n, p_n)}{p_n} = \lim_{n \to \infty} \frac{v^*(t_n, x_n, p_n)}{p_n} = \frac{1}{D\ell(\infty)} \text{ if } p_n \to \infty.$$
More controls on \( v \)

\( \square \) It also implies that \( \exists \eta > 0 \) s.t.

\[
0 \leq v(t, x, p) \leq \epsilon^{-1}|p - \ell(0)| + \gamma \eta (1 + |x|)
\]

\( \square \) and that for \((t_n, x_n, p_n)_n\) s.t. \((t_n, x_n) \to (t, x)\):

\[
\lim_{n \to \infty} v^*_n(t_n, x_n, p_n) = \lim_{n \to \infty} v^*(t_n, x_n, p_n) = 0 \quad \text{if } p_n \to -\infty
\]

\[
\lim_{n \to \infty} \frac{v^*_n(t_n, x_n, p_n)}{p_n} = \lim_{n \to \infty} \frac{v^*(t_n, x_n, p_n)}{p_n} = \frac{1}{D\ell(\infty)} \quad \text{if } p_n \to \infty
\]

\( \square \) A little more : \( v \) is continuous in \( p \) and \( x^3 \).
Uniqueness

Want a comparison result in the class of function with the above limit and growth conditions.
Uniqueness

- Want a comparison result in the class of functions with the above limit and growth conditions.

- Recall that

$$F \varphi := -\mathcal{L}_X \varphi - \frac{(x^1 \sigma)^2}{2} \left( |D_{x^1} \varphi / D_p \varphi|^2 D_p^2 \varphi - 2 (D_{x^1} \varphi / D_p \varphi) D_{(x^1, p)}^2 \varphi \right).$$
Uniqueness

- Want a comparison result in the class of functions with the above limit and growth conditions.

- Recall that

\[ F_\varphi := -\mathcal{L}_X \varphi - \frac{(x_1 \sigma)^2}{2} \left( |D_{x_1} \varphi / D_p \varphi|^2 D_p^2 \varphi - 2(D_{x_1} \varphi / D_p \varphi) D_{(x_1, p)}^2 \varphi \right). \]

- We now control \( D_{x_1} \varphi / D_p \varphi \).
Uniqueness

Want a comparison result in the class of function with the above limit and growth conditions.

Recall that

$$F \phi := -\mathcal{L}_x \phi - \frac{(x^1 \sigma)^2}{2} \left( |D_{x^1} \phi / D_p \phi|^2 D_p^2 \phi - 2(D_{x^1} \phi / D_p \phi) D_{(x^1, p)}^2 \phi \right).$$

We now control $D_{x^1} \phi / D_p \phi$.

This is not enough... If we need to penalize in $x^1$ (stock price) then the term $|D_{x^1} \phi / D_p \phi|^2 D_p^2 \phi$ will blow up as $n \to \infty$, where $n$ comes from the usual penalisation $n|x_1^1 - x_2^1|^2$ due to the doubling of constants.
Uniqueness

- Want a comparison result in the class of functions with the above limit and growth conditions.

- Recall that

$$F \varphi := -\mathcal{L}_X \varphi - \frac{(x^1 \sigma)^2}{2} \left( |D_{x^1} \varphi / D_p \varphi|^2 D_p^2 \varphi - 2(D_{x^1} \varphi / D_p \varphi) D_{(x^1, p)}^2 \varphi \right).$$

- We now control $D_{x^1} \varphi / D_p \varphi$.

Assumption:

$$\exists \hat{x}^1 > 0 \text{ s.t. } \mu(\hat{x}^1) \leq 0 = \sigma(\hat{x}^1).$$
Uniqueness

- Want a comparison result in the class of function with the above limit and growth conditions.

- Recall that

\[
F\varphi := -\mathcal{L}_X \varphi - \frac{(x^1 \sigma)^2}{2} \left( \left| \frac{D_{x^1} \varphi}{D_p \varphi} \right|^2 D_p^2 \varphi - 2\left( \frac{D_{x^1} \varphi}{D_p \varphi} \right) D_{(x^1,p)}^2 \varphi \right).
\]

- We now control \( D_{x^1} \varphi / D_p \varphi \).

Assumption:

\[ \exists \hat{x}^1 > 0 \text{ s.t. } \mu(\hat{x}^1) \leq 0 = \sigma(\hat{x}^1). \]

- Bound on the stock price...
**Theorem**: Let $U$ (resp. $V$) be a non-negative super- and subsolutions which are continuous in $x^3$. Assume that

$$U(t, x, p) \geq V(t, x, p) \text{ if } t = T \text{ or } x^1 \in \{0, 2\hat{x}^1\},$$

and that $\exists c_+ > 0$ and $c_- \in \mathbb{R}$ s.t.

$$\limsup_{(t', x', p') \to (t, x, \infty)} V(t', x', p') / p' \leq c_+ \leq \liminf_{(t', y', p') \to (t, y, \infty)} U(t', y', p') / p',$$

$$\limsup_{(t', x', p') \to (t, x, -\infty)} V(t', x', p') \leq c_- \leq \liminf_{(t', y', p') \to (t, y, -\infty)} U(t', y', p').$$

If either $U$ is a supersolution of (*) which is continuous in $p$, or $V$ is a subsolution of (**) which is continuous in $p$, then

$$U \geq V.$$
Additional remarks
Serves as a building block for problems of the form

$$\sup_{\phi \in \mathcal{A}_{t,z}} \mathbb{E} \left[ U(X_{t,x}^\phi(T), Y_{t,z}^\phi(T)) \right]$$

with

$$\mathcal{A}_{t,z} := \{ \phi \in \mathcal{A} : Z_{t,z}^\phi \in \mathcal{O} \text{ on } [t, T] \} .$$
Optimal management under shortfall constraints

\[ \sup_{\phi \in \mathcal{A}_{t,z}} \mathbb{E} \left[ U(X^\phi_{t,x}(T), Y^\phi_{t,z}(T)) \right] \]

with \( \mathcal{A}_{t,z} := \{ \phi \in \mathcal{A} : Z^\phi_{t,z} \in \mathcal{O} \text{ on } [t, T] \} \).

\[ Y^\phi_{t,z} \geq v(\cdot, X^\phi_{t,x}) \]

where \( v(t, x) := \inf \left\{ y : \exists \phi \in \mathcal{A} \text{ s.t. } Z^\phi_{t,z} \in \mathcal{O} \text{ on } [t, T] \right\} \),

see B., Elie and Imbert (2010).
BSDE with moment conditions

- Look for the minimal solution \((Y, Z)\) of

\[
Y_t = Y_T + \int_t^T f(s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s
\]

such that

\[
\mathbb{E}[G(Y_T, \xi)] \geq p.
\]
BSDE with moment conditions

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\]

- Can use the same approach: for \(\alpha \in \mathcal{A}\) set

\[
Y_{t}^{\alpha} = G^{-1}(P_T^{\alpha}, \xi) + \int_t^T f(s, Y_s^{\alpha}, Z_s^{\alpha})\,ds - \int_t^T Z_s^{\alpha}\,dW_s
\]
BSDE with moment conditions

- Look for the minimal solution \((Y, Z)\) of

\[
Y_t = Y_T + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s
\]

such that

\[
\mathbb{E}[G(Y_T, \xi)] \geq p.
\]

- Can use the same approach: for \(\alpha \in \mathcal{A}\) set

\[
Y_t^\alpha = G^{-1}(P_T^\alpha, \xi) + \int_t^T f(s, Y_s^\alpha, Z_s^\alpha)ds - \int_t^T Z_s^\alpha dW_s
\]

- The minimal solution is (formally) given by \(Y = \operatorname{ess inf}_\alpha Y^\alpha\).
Consider the control problem:

\[ w := \inf_{\phi} \mathbb{E} \left[ U(X^{\phi}(T)) \right] \]
Optimal control vs stochastic targets

- Consider the control problem:
  \[ w := \inf_{\phi} \mathbb{E} \left[ U(X^{\phi}(T)) \right] \]

- Then, it can be written as a stochastic target problem
  \[ w = v := \inf \left\{ p : \exists (\phi, \alpha) \text{ s.t. } U(X^{\phi}(T)) \leq P_p^{\alpha}(T) \right\} \]

with \( P_p^{\alpha} := p + \int_0^T \alpha_s dW_s \).
Consider the control problem:

\[ w := \inf_{\phi} \mathbb{E} \left[ U(X^\phi(T)) \right] \]

Then, it can be written as a stochastic target problem

\[ w = v := \inf \left\{ p : \exists (\phi, \alpha) \text{ s.t. } U(X^\phi(T)) \leq P^\alpha_p(T) \right\} \]

with \( P^\alpha_p := p + \int_0^T \alpha_s dW_s \).

Allows for a unified approach (obviously obtains -immediately- the same HJB PDE).